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New Results on the Unimodular Equivalence of Multivariate Polynomial Matrices

Dongmei Li and Zuo Chen *

School of Mathematics and Computing Sciences, Hunan University of Science and Technology, Xiangtan 411201, China; dmli@hnust.edu.cn
* Correspondence: chenzuo98@163.com; Tel.: +86-17873780599

Abstract: The equivalence of systems is a crucial concept in multidimensional systems. The Smith normal forms of multivariate polynomial matrices play important roles in the theory of polynomial matrices. In this paper, we mainly study the unimodular equivalence of some special kinds of multivariate polynomial matrices and obtain some tractable criteria under which such matrices are unimodular equivalent to their Smith normal forms. We propose an algorithm for reducing such nD polynomial matrices to their Smith normal forms and present an example to illustrate the availability of the algorithm. Furthermore, we extend the results to the non-square case.

Keywords: multidimensional system; nD polynomial matrix; Smith normal form; unimodular equivalence

MSC: 15A06; 15A23; 13P05; 13P10

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1. Introduction

Most multidimensional (nD) systems such as dynamical control systems, distributed control systems and delay-differential systems are often represented by multivariate (nD) polynomial matrices [1–9]. The equivalence of systems is a significant concept in nD systems. From the perspective of system theory, the reduction involved must maintain the relevant system properties. It is usually valuable to simplify the given system representation to a simpler equivalent form. It is well-known that the equivalence of nD systems can be reflected by the unimodular equivalence of nD polynomial matrices. Because the Smith normal form of the polynomial matrix has good structure and properties, the unimodular equivalence plays a key role for multivariate polynomial matrices simplified to their Smith normal form. One of the purposes of reducing an nD polynomial matrix to its Smith normal form is to be capable of simplifying a corresponding system to a new system while including fewer equations and unknowns. Therefore, the problem of the unimodular equivalence for the Smith normal form and nD polynomial matrices have made great progress in the past decades.

For $1D$ polynomial matrices, the unimodular equivalence problem of a matrix to its Smith normal form is well solved [2,4]. Storey and Frost gave an example for bivariate polynomial matrices which is not unimodular equivalent to its Smith normal form [10]. For $nD (n \geq 2)$ polynomial matrices, because nD polynomial rings are not Euclidean, Euclidean division properties do not hold in such rings, which become greatly difficult in algebra. Consequently, the unimodular equivalence problem is still open. The unimodular equivalence and Smith normal form problems of several special classes of polynomial matrices have been investigated and some judgment conditions have been obtained [11–20]. For instance, Lin et al. [11] presented that a polynomial matrix $F(x) \in K^{l\times l}[x_1, x_2, \cdots, x_n]$ with $\det(F) = x_1 - f(x_2, \cdots, x_n)$ is unimodular equivalent to its Smith normal form. Furthermore, Li et al. [13] generalized the above result to a new case when $\det(F) = (x_1 - f(x_2, \cdots, x_n))^q$, where q is a positive integer. Moreover, Lu et al. [20] derived a tractable criterion under which matrix F may be unimodular equivalent to its Smith normal form $diag\{I_{l-1}, pq\}$ for $F \in K^{l\times l}[x, y]$ and $\det(F) = pq$, where $p, q \in K[x]$ are irreducible and distinct polynomials.

In this paper, we mainly study the unimodular equivalence for several classes of nD polynomial matrices and their Smith normal form. Li et al. [14] showed that a polynomial matrix $F(x) \in K^{l\times l}[x_1, x_2, \cdots, x_n]$, $\det(F) = (x_1 - f_1(x_2, \cdots, x_n))(x_2 - f_2(x_3, \cdots, x_n))$ is unimodular equivalent to its Smith normal form $diag\{I_{l-1}, \det(F)\}$ if and only if the $(l-1) \times (l-1)$ minors of $F(x)$ have no common zeros. By extending the above conclusion, we focus on the Smith normal forms of some nD polynomial matrices with special determinants. Let $F(x) \in K^{l\times l}[x_1, x_2, \cdots, x_n]$ with $\det(F) = d_1^{q_1} d_2^{q_2} = (x_1 - f_1(x_2, \cdots, x_n))^{q_1}(x_2 - f_2(x_3, \cdots, x_n))^{q_2}$, where q_1, q_2 are positive integers. We study the question as to what is the sufficient and necessary condition for the polynomial matrix $F(x)$ unimodular equivalent to its Smith normal form. Moreover, we extended the above results to the non-square case. The following problems are investigated.

Problem 1. *Let $F(x) \in K^{l\times l}[x]$ and $\det(F) = d_1^q d_2^q$, $d_1 = x_1 - f_1(x_2, \cdots, x_n)$, $d_2 = x_2 - f_2(x_3, \cdots, x_n)$, where q is a positive integer. When is the $F(x)$ unimodular equivalent to its Smith normal form*

$$S(x) = \begin{pmatrix} d_1^{r_1} d_2^{r_1} & & & \\ & d_1^{r_2} d_2^{r_2} & & \\ & & \ddots & \\ & & & d_1^{r_l} d_2^{r_l} \end{pmatrix}?$$

Problem 2. *Let $F(x) \in K^{l\times l}[x]$ and $\det(F) = (d_1^s d_2^t)^r$, $d_1 = x_1 - f_1(x_2, \cdots, x_n)$, $d_2 = x_2 - f_2(x_3, \cdots, x_n)$, where s, t are two positive integers. When is the $F(x)$ unimodular equivalent to its Smith normal form*

$$S(x) = \begin{pmatrix} I_{l-r} & & & \\ & d_1^s d_2^t & & \\ & & \ddots & \\ & & & d_1^s d_2^t \end{pmatrix}?$$

We now summarize the rest of this paper. Some basic concepts on the unimodular equivalence of a polynomial matrix, the main results of this paper and the positive answers of Problems 1 and 2 are presented in Section 2. In Section 3, we give an executable algorithm and an example to illustrate the usefulness of our method. In Section 4, we provide some concluding comments.

2. Preliminaries and Results

Let $R = K[x_1, x_2, \cdots, x_n]$ denote the set of polynomials in n variables $x_1, x_2, \cdots, x_n$ with coefficients in the field K. $R_1 = K[x_2, \cdots, x_n]$. $R^{l\times m}$ denotes the set of $l \times m$ matrices with entries from R. I_r denotes the $r \times r$ identity matrix and $0_{r\times t}$ denotes the $r \times t$ zero matrix. For convenience, we use $diag\{f_1, \cdots, f_l\}$ to denote the diagonal matrix in $R^{l\times l}$, where diagonal elements are $f_1, \cdots, f_l$, and $f_1, \cdots, f_l \in R$. In addition, we use $A(x) \sim B(x)$ to denote that A(x) is unimodular equivalent to $B(x)$. As long as the omission of parameter (x) does not lead to confusion, we omit it.

Definition 1 ([21]). *Let $F(x) \in R^{l\times m}$ with rank r, where $1 \le r \le min\{l, m\}$. For any integer k with $1 \le k \le r$, let $a_1, \cdots, a_\beta$ be all the $k \times k$ minors of $F(x)$ and denote the greatest common divisor (g.c.d.) of $a_1, \cdots, a_\beta$ by $d_k(F)$. Extracting $d_k(F)$ from $a_1, \cdots, a_\beta$ yields*

$$a_i = d_k(F) \cdot b_i,\ i = 1, \cdots, \beta.$$

The $k \times k$ reduced minors of $F(x)$ are denoted by $b_1, \cdots, b_\beta$. For simplicity, $J_k(F)$ denotes the ideal in R generated by $b_1, \cdots, b_\beta$.

Definition 2. *Let $F(x) \in R^{l\times m}$ ($l \le m$) be of rank r. The Smith normal form of $F(x)$ is defined as*

$$S = (diag\{\Phi_i\} \;\; 0_{l\times(m-l)}),$$

where

$$\Phi_i = \begin{cases} d_i/d_{i-1}, & 1 \le i \le r, \\ 0, & r < i \le m, \end{cases}$$

and let $d_0 \equiv 1$, where d_i is the greatest common divisor of the $i \times i$ minors of $F(x)$ and Φ_i satisfies the following property:

$$\Phi_1 \mid \Phi_2 \mid \cdots \mid \Phi_r.$$

Definition 3 ([22])**.** *Let $F(x) \in R^{l\times m}$ be of full row(column) rank. $F(x)$ is said to be zero left prime (zero right prime) if the $l \times l$($m \times m$) minors of $F(x)$ have no common zeros. If $F(x) \in R^{l\times m}$ is zero left prime (zero right prime), we simply say that $F(x)$ is ZLP (ZRP).*

Definition 4. *Let $F_1(x)$ and $F_2(x)$ be two matrices in $R^{l\times m}$. $F_1(x)$ and $F_2(x)$ are said to be unimodular equivalent if there exist two invertible matrices $P(x) \in R^{l\times l}$ and $Q(x) \in R^{m\times m}$ such that $F_2(x) = P(x)F_1(x)Q(x)$.*

We first provide several important lemmas, which are of great help to prove our main results.

Lemma 1 ([14])**.** *Let $F(x) \in R^{l\times m}(l \le m)$ be of rank r. If the reduced minors of $F(x)$ generate unit idea R, then there is a ZLP matrix $V(x) \in R^{(l-r)\times l}$ such that $V(x) \cdot F(x) = 0_{(l-r)\times m}$.*

Lemma 2 ([17])**.** *Let $g(x) \in R$ and $f(x) \in R_1$. If $g(f, x_2, \cdots, x_n) = 0$, then $x_1 - f(x_2, \cdots, x_n)$ is a divisor of $g(x)$.*

Lemma 3 ([17])**.** *Let $F(x), F_1(x), F_2(x) \in R^{l\times l}$, $F(x) = F_1(x) \cdot F_2(x)$. If the $(l-r) \times (l-r)$ minors of $F(x)$ have no common zeros, then the $(l-r) \times (l-r)$ minors of $F_i(x)$ ($i = 1, 2$) have no common zeros.*

In 1976, Quillen [23] and Suslin [24] proved Serre's conjecture independently, and then found a relationship between a unimodular matrix and a ZLP matrix. Now, we introduce this conclusion.

Lemma 4 ([23,24])**.** *Let $F(x) \in R^{l\times m}(l \le m)$ be a ZLP matrix. Then, there exists a unimodular matrix $H(x) \in R^{m\times m}$ such that*

$$F(x) \cdot H(x) = \begin{pmatrix} I_l & 0_{l\times(m-l)} \end{pmatrix}.$$

Lemma 5. *Let $F(x) \in R^{l\times l}$ and $\det F(x) = d_1^p d_2^q$, where $d_1 = x_1 - f_1(x_2, \cdots, x_n)$, $d_2 = x_2 - f_2(x_3, \cdots, x_n)$ and p, q are nonnegative integers.*

(1) If $d_r(F) = 1$, $J_r(F) = R$ and $d_1 | d_{r+1}(F)$, then there exists a unimodular matrix $U_1(x) \in R^{l\times l}$ such that

$$U_1(x) \cdot F(x) = \begin{pmatrix} I_r & \\ & d_1 I_{l-r} \end{pmatrix} \cdot G_1(x)$$

where $G_1(x) \in R^{l\times l}$.

(2) *If $d_r(F) = 1$, $J_r(F) = R$ and $d_2 | d_{r+1}(F)$, then there exists a unimodular matrix $U_2(x) \in R^{l \times l}$ such that*

$$U_2(x) \cdot F(x) = \begin{pmatrix} I_r & \\ & d_2 I_{l-r} \end{pmatrix} \cdot G_2(x),$$

where $G_2(x) \in R^{l \times l}$.

(3) *If $d_r(F) = 1$, $J_r(F) = R$ and $d_1 d_2 | d_{r+1}(F)$, then there exists a unimodular matrix $U_3(x) \in R^{l \times l}$ such that*

$$U_3(x) \cdot F(x) = \begin{pmatrix} I_r & \\ & d_1 d_2 I_{l-r} \end{pmatrix} \cdot G_3(x),$$

where $G_3(x) \in R^{l \times l}$.

Proof. Suppose that the $r \times r$ minors of $F(x)$ are $a_1, a_2, \cdots, a_\beta$, let $F'(x) = F(f_1, x_2, \cdots, x_n)$, and the $r \times r$ minors of $F'(x)$ are $b_1, b_2, \cdots, b_\beta$. It is obvious that $(f_1, x_2, \cdots, x_n)$ is a zero of $\det F(x)$ for every $(x_2, \cdots, x_n) \in R_1$ and $d_1 | d_{r+1}(F)$. Therefore, $rank(F'(x)) \le r$.

(1) Assume exists $(x_{20}, \cdots, x_{n0}) \in R_1$ such that

$$b_i(x_{20}, \cdots, x_{n0}) = 0, i = 1, 2, \cdots, \beta.$$

Let $x_{10} = f_1(x_{20}, \cdots, x_{n0})$, and then

$$a_i(x_{10}, x_{20}, \cdots, x_{n0}) = 0, i = 1, 2, \cdots, \beta.$$

Because $d_r(F) = 1$, $J_r(F) = R$, we have the $r \times r$ minors of $F(x)$ generate R. Leads to a contradiction. Thus, the $r \times r$ minors of $F'(x)$ generate R, $rank(F'(x)) \ge r$, and then $rank(F'(x)) = r$. By Lemma 1, there exists a ZLP matrix $T(x) \in R^{(l-r) \times l}$ such that

$$T(x) \cdot F'(x) = 0_{(l-r) \times l}.$$

By Lemma 4, a unimodular matrix $U_1(x) \in R^{l \times l}$ can be established and $T(x)$ is its last $l - r$ row. By Lemma 2, the last $l - r$ row of $U_1(x) \cdot F(x)$ has the common divisor d_1, i.e.,

$$U_1(x) \cdot F(x) = \begin{pmatrix} I_r & \\ & d_1 I_{l-r} \end{pmatrix} \cdot G_1(x).$$

(2) If $d_r(F) = 1$, $J_r(F) = R$ and $d_2 | d_{r+1}(F)$, we apply a similar method to prove that there exists a unimodular matrix $U_2(x) \in R^{l \times l}$ such that

$$U_2(x) \cdot F(x) = \begin{pmatrix} I_r & \\ & d_2 I_{l-r} \end{pmatrix} \cdot G_2(x).$$

(3) If $d_r(F) = 1$, $J_r(F) = R$ and $d_1 d_2 | d_{r+1}(F)$. Obviously, $d_1 | d_{r+1}(F)$, and then there exists a unimodular matrix $U_1(x) \in R^{l \times l}$ such that

$$U_1(x) \cdot F(x) = \begin{pmatrix} I_r & \\ & d_1 I_{l-r} \end{pmatrix} \cdot G_1(x).$$

Note that $U_1(x)$ is unimodular, assume $r \times r$ minors of $G_1(x)$ are $r_1, r_2, \cdots, r_\beta$, because the $r \times r$ minors of $F(x)$ generate unit idea R, by Lemma 3, the $r \times r$ minors of $G_1(x)$ have no common zeros and $d_{r+1}(F) = d_{r+1}(\begin{pmatrix} I_r & \\ & d_1 I_{l-r} \end{pmatrix} \cdot G_1(x))$, let $G_1(x) = \begin{pmatrix} W_1(x) \\ W_2(x) \end{pmatrix}$, where $W_1(x) \in R^{r\times l}$, $W_2(x) \in R^{(l-r)\times l}$, and then

$$\begin{pmatrix} I_r & \\ & d_1 I_{l-r} \end{pmatrix} \cdot G_1(x) = \begin{pmatrix} W_1(x) \\ d_1 \cdot W_2(x) \end{pmatrix}.$$

Note that $d_{r+1}(\begin{pmatrix} I_r & \\ & d_1 I_{l-r} \end{pmatrix} \cdot G_1(x)) = d_1 \cdot d_{r+1}(G_1(x))$ and $d_2 | d_{r+1}(F)$, and thus $d_2 | d_1 \cdot d_{r+1}(G_1(x))$, combined with $d_2 \nmid d_1$, so that $d_2 | d_{r+1}(G_1(x))$. Therefore, there exists a unimodular matrix $U_4(x) \in R^{l\times l}$ such that

$$U_4(x) \cdot G_1(x) = \begin{pmatrix} I_r & \\ & d_2 I_{l-r} \end{pmatrix} \cdot G_3'(x),$$

further, we can obtain

$$U_1(x) \cdot F(x) = \begin{pmatrix} I_r & \\ & d_1 I_{l-r} \end{pmatrix} \cdot U_4^{-1}(x) \cdot \begin{pmatrix} I_r & \\ & d_2 I_{l-r} \end{pmatrix} \cdot G_3'(x).$$

According to Lemma 2.6 in Li et al. [16], there are two unimodular matrices $U(x)$, $V(x) \in R^{l\times l}$ such that

$$\begin{pmatrix} I_r & \\ & d_1 I_{l-r} \end{pmatrix} \cdot U_4^{-1}(x) \cdot \begin{pmatrix} I_r & \\ & d_2 I_{l-r} \end{pmatrix} = U(x) \cdot \begin{pmatrix} I_r & \\ & d_1 d_2 I_{l-r} \end{pmatrix} \cdot V(x).$$

Setting $U_3(x) = U^{-1}(x) \cdot U_1(x)$, $G_3(x) = V(x) \cdot G_3'(x)$, we have

$$U_3(x) \cdot F(x) = \begin{pmatrix} I_r & \\ & d_1 d_2 I_{l-r} \end{pmatrix} \cdot G_3(x).$$

The proof is completed. □

Lemma 6 ([19]). *Let matrices $A(x), B(x) \in R^{l\times m}$, if $A(x)$ is unimodular equivalent to $B(x)$, then $d_k(A) = d_k(B)$ and $J_k(A) = J_k(B)$, where $k = 1, 2, \cdots, \min\{m, l\}$.*

Let $F\begin{pmatrix} i_1 i_2 \cdots i_t \\ j_1 j_2 \cdots j_s \end{pmatrix}$ be a $t \times s$ submatrix of $F(x)$ consisting of the i_1-*th*, i_2-*th*, $\cdots$, i_t-*th* rows and j_1-*th*, j_2-*th*, $\cdots$, j_s-*th* columns of $F(x)$.

Lemma 7. *Let $F(x) \in R^{l\times l}$ be of full row rank, $d(F) = (d_1 d_2)^q$, where $d_1 = x_1 - f_1(x_2, \cdots, x_n)$, $d_2 = x_2 - f_2(x_3, \cdots, x_n)$, and q is a positive integer. If there exist two subsets $\{i_1, i_2, \cdots, i_k\}$ and $\{j_1, j_2, \cdots, j_k\}$ of $\{1, 2, \cdots, l\}$ such that*

$$d_1 d_2 \nmid \det\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}, d_1 d_2 \mid \det\begin{pmatrix} i_1 & i_2 & \cdots & i_k & i_{k+1} \\ p_1 & p_2 & \cdots & p_k & p_{k+1} \end{pmatrix}$$

for any $i_{k+1} (i_{k+1} \neq i_1, \cdots, i_k)$ and any permutation $p_1 \cdots p_k p_{k+1}$ of $1, 2, \cdots, l$. Then, $d_1 d_2 \mid d_{k+1}(F)$.

Proof. The proof is similar to Lemma 3.6 in [19], so we omit it here. □

Lemma 8 ([19]). *Let $F(x), M(x), N(x) \in R^{l\times l}$ and $F(x) = M(x) \cdot N(x)$. For some $k (1 \le k \le l)$, if $d_k(M) = d_k(F)$, $J_k(F) = R$, then $J_k(M) = R$, $d_k(N) = 1$, $J_k(N) = R$.*

Lemma 9. *Let* $F(x), D(x), C(x) \in R^{l\times l}$, $F(x) = D(x)\cdot C(x)$, $d_i(F) = d_1^{q_i} d_2^{q_i}$, $i = 1, 2, \cdots, k+1$, *and*

$$D(x) = \begin{pmatrix} d_1^{r_1} d_2^{r_1} & & & \\ & \ddots & & \\ & & d_1^{r_k} d_2^{r_k} & \\ & & & d_1^{r_{k+1}} d_2^{r_{k+1}} I_{l-k} \end{pmatrix},$$

where $d_1 = x_1 - f_1(x_2, \cdots, x_n)$, $d_2 = x_2 - f_2(x_3, \cdots, x_n)$, $r_1 \le r_2 \le \cdots \le r_{k+1}$, $q_i = r_1 + \cdots + r_i$, $i = 1, 2, \cdots, k$. *If* $J_k(F) = R$, $q_{k+1} > r_1 + \cdots + r_{k+1}$. *Then,* $d_k(C) = 1$, $J_k(C) = R$, $d_1 d_2 \mid d_{k+1}(C)$.

Proof. By assumption $J_k(F) = R$, $q_{k+1} > r_1 + \cdots + r_{k+1}$. Because $d_k(F) = d_k(D) = (d_1 d_2)^{r_1 + \cdots + r_k}$, by Lemma 8, $d_k(C) = 1$, $J_k(C) = R$. Because

$$\det F\begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix} = (d_1 d_2)^{r_{a_1} + \cdots + r_{a_p}} \cdot \det C\begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix}.$$

(1) If $r_1 = r_2 = \cdots = r_{k+1}$, because $d_k(C) = 1$, it is obvious that there exists a $k\times k$ minor $\lambda(x)$ of $C(x)$ such that $d_1 d_2 \nmid \lambda(x)$. For any permutation $i_1 \cdots i_k i_{k+1}$ and $j_1 \cdots j_k j_{k+1}$ in $1, \cdots, l$, combined with $r_1 = r_2 = \cdots = r_{k+1}$, we have that

$$\det F\begin{pmatrix} i_1 & i_2 & \cdots & i_k & i_{k+1} \\ j_1 & j_2 & \cdots & j_k & j_{k+1} \end{pmatrix} = (d_1 d_2)^{r_1 + \cdots + r_{k+1}} \cdot \det C\begin{pmatrix} i_1 & i_2 & \cdots & i_k & i_{k+1} \\ j_1 & j_2 & \cdots & j_k & j_{k+1} \end{pmatrix}$$

Because $d_{k+1}(F) = (d_1 d_2)^{q_{k+1}}$ and $q_{k+1} > r_1 + \cdots + r_{k+1}$, we have

$$d_1 d_2 \mid \det C\begin{pmatrix} i_1 & i_2 & \cdots & i_k & i_{k+1} \\ j_1 & j_2 & \cdots & j_k & j_{k+1} \end{pmatrix}.$$

By Lemma 7, $d_1 d_2 | d_{k+1}(C)$.

(2) If there is an integer k_0 with $k_0 \le k$ such that $r_{k_0} < r_{k_0+1} = r_{k_0+2} \cdots = r_{k+1}$ or $r_k < r_{k+1}$. Because $d_k(F) = (d_1 d_2)^{r_1 + \cdots + r_k}$, there are $i_{k_0+1}, \cdots, i_k$ and $j_1, \cdots, j_k$ such that

$$d_1 d_2 \nmid \det C\begin{pmatrix} 1 & 2 & \cdots & k_0 & i_{k_0+1} & \cdots & i_k \\ j_1 & j_2 & \cdots & j_{k_0} & j_{k_0+1} & \cdots & j_k \end{pmatrix}$$

If the assertion would not hold, then we have $q_k \ge r_1 + \cdots + r_k + 1$, and this is a contradiction. For any $i_{k+1} (i_{k+1} > k_0, i_{k+1} \ne i_{k_0+1}, \cdots, i_k)$, any permutation $j_1 \cdots j_k j_{k+1}$. We have

$$\det F\begin{pmatrix} 1 & \cdots & k_0 & i_{k_0+1} & \cdots & i_{k+1} \\ j_1 & \cdots & j_{k_0} & j_{k_0+1} & \cdots & j_{k+1} \end{pmatrix} = (d_1 d_2)^{r_1 + \cdots + r_{k+1}} \cdot \det C\begin{pmatrix} 1 & \cdots & k_0 & i_{k_0+1} & \cdots & i_{k+1} \\ j_1 & \cdots & j_{k_0} & j_{k_0+1} & \cdots & j_{k+1} \end{pmatrix}$$

Because $d_{k+1}(F) = (d_1 d_2)^{q_{k+1}}$ and $q_{k+1} > r_1 + \cdots + r_{k+1}$, we have

$$d_1 d_2 | \det C\begin{pmatrix} 1 & \cdots & k_0 & i_{k_0+1} & \cdots & i_{k+1} \\ j_1 & \cdots & j_{k_0} & j_{k_0+1} & \cdots & j_{k+1} \end{pmatrix}$$

By Lemma 7, $d_1 d_2 | d_{k+1}(C)$.

□

Theorem 1. *Let* $F(x), G(x) \in R^{l\times l}$, $d_i(F) = d_1^{q_i} d_2^{q_i}$, $d_1 = x_1 - f_1(x_2, \cdots, x_n)$, $d_2 = x_2 - f_2(x_3, \cdots, x_n)$, $J_i(F) = R$, *where* q_i *are positive integers,* $i = 1, 2, \cdots, l$; *and,*

$$F(x) = \begin{pmatrix} d_1^{r_1} d_2^{r_1} & & & \\ & \ddots & & \\ & & d_1^{r_k} d_2^{r_k} & \\ & & & d_1^{r_t} d_2^{r_t} I_{l-k} \end{pmatrix} \cdot G(x),$$

where $q_0 \equiv 0$, $r_i = q_i - q_{i-1}$ *and* $i = 1, 2, \cdots, k+1$.
If $r_1 \le r_2 \le \cdots \le r_k \le r_t < r_{k+1}$, *then* $F(x)$ *is unimodular equivalent to* $M(x)$, *where*

$$M(x) = \begin{pmatrix} d_1^{r_1} d_2^{r_1} & & & \\ & \ddots & & \\ & & d_1^{r_k} d_2^{r_k} & \\ & & & d_1^{r_t+1} d_2^{r_t+1} I_{l-k} \end{pmatrix} \cdot N(x),$$

and $N(x) \in R^{l\times l}$.

Proof. It is obvious that $q_i = r_1 + r_2 + \cdots + r_i$, $i = 1, 2, \cdots, k+1$, and then $q_{k+1} = r_1 + \cdots + r_k + r_{k+1} > r_1 + \cdots + r_k + r_t$, by Lemma 9, $d_k(G) = 1$, $J_k(G) = R$, $d_1 d_2 \mid d_{k+1}(G)$. By Lemma 5, there exists a unimodular matrix $U_1(x) \in R^{l\times l}$ such that

$$U_1(x) \cdot G(x) = \begin{pmatrix} I_k & \\ & d_1 d_2 I_{l-k} \end{pmatrix} \cdot G_1(x).$$

(1) If $r_1 = r_2 = \cdots = r_k = r_t$, then

$$\begin{aligned} F(x) &= \begin{pmatrix} d_1^{r_1} d_2^{r_1} & & & \\ & \ddots & & \\ & & d_1^{r_k} d_2^{r_k} & \\ & & & d_1^{r_t} d_2^{r_t} I_{l-k} \end{pmatrix} \cdot U_1^{-1}(x) \cdot \begin{pmatrix} I_k & \\ & d_1 d_2 I_{l-k} \end{pmatrix} \cdot G_1(x) \\ &= U_1^{-1}(x) \cdot \begin{pmatrix} d_1^{r_1} d_2^{r_1} & & & \\ & \ddots & & \\ & & d_1^{r_k} d_2^{r_k} & \\ & & & d_1^{r_t+1} d_2^{r_t+1} I_{l-k} \end{pmatrix} \cdot G_1(x). \end{aligned}$$

Thus, $F(x)$ is unimodular equivalent to

$$M(x) = \begin{pmatrix} d_1^{r_1} d_2^{r_1} & & & \\ & \ddots & & \\ & & d_1^{r_k} d_2^{r_k} & \\ & & & d_1^{r_t+1} d_2^{r_t+1} I_{l-k} \end{pmatrix} \cdot G_1(x).$$

(2) If there is an integer m with $1 \le m < k$ such that $r_m < r_{m+1} = r_{m+2} \cdots = r_k = r_t$. Setting $P(x) = U_1^{-1}(x)$, let

$$P(x) = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix},$$

where $P_1 \in R^{m\times k}$, $P_2 \in R^{m\times(l-k)}$, $P_3 \in R^{(l-m)\times k}$, $P_4 \in R^{(l-m)\times(l-k)}$.
Then,

$$F(x) = \begin{pmatrix} d_1^{r_1}d_2^{r_1} & & & \\ & \ddots & & \\ & & d_1^{r_m}d_2^{r_m} & \\ & & & d_1^{r_t}d_2^{r_t}I_{l-m} \end{pmatrix} \cdot P(x) \cdot \begin{pmatrix} I_k & \\ & d_1d_2I_{l-k} \end{pmatrix} \cdot G_1(x).$$

We claim that $(P_1, d_1d_2P_2)$ is a ZLP matrix. Otherwise, the $m \times m$ minors of $(P_1, d_1d_2P_2)$ have a common zero. We compute all the $m \times m$ reduced minors of $F(x)$, because $d_m(F) = (d_1d_2)^{r_1+\cdots+r_m}$, and every $m \times m$ minor of P_1 is a factor of some $m \times m$ reduced minors of $F(x)$ and the other $m \times m$ reduced minors of $F(x)$ have a common divisor d_1d_2. Then, the $m \times m$ reduced minors of $F(x)$ have a common zero, and this contradicts that the hypothesis $J_m(F) = R$.

By Lemma 4, there exists a unimodular matrix $Q \in R^{l\times l}$ such that $(P_1, d_1d_2P_2) \cdot Q = \begin{pmatrix} I_m & 0_{m\times(l-m)} \end{pmatrix}$.

Setting $(P_3, d_1d_2P_4) \cdot Q = (P_{31}, P_{32})$, furthermore, we partition P_{31} to

$$P_{31} = (\alpha_1, \cdots, \alpha_m),$$

where $P_{31} \in R^{(l-m)\times m}$, $P_{32} \in R^{(l-m)\times(l-m)}$, $\alpha_1, \cdots, \alpha_m \in R^{(l-m)\times 1}$, and then we have

$$\begin{aligned} F(x) &= \begin{pmatrix} d_1^{r_1}d_2^{r_1} & & & \\ & \ddots & & \\ & & d_1^{r_m}d_2^{r_m} & \\ & & & d_1^{r_t}d_2^{r_t}I_{l-m} \end{pmatrix} \cdot \begin{pmatrix} P_1 & d_1d_2P_2 \\ P_3 & d_1d_2P_4 \end{pmatrix} \cdot Q \cdot Q^{-1} \cdot G_1(x) \\ &= \begin{pmatrix} d_1^{r_1}d_2^{r_1} & & & \\ & \ddots & & \\ & & d_1^{r_m}d_2^{r_m} & \\ & & & d_1^{r_t}d_2^{r_t}I_{l-m} \end{pmatrix} \cdot \begin{pmatrix} I_m & 0_{m,l-m} \\ P_{31} & P_{32} \end{pmatrix} \cdot Q^{-1} \cdot G_1(x) \\ &= \begin{pmatrix} d_1^{r_1}d_2^{r_1} & & & \\ & \ddots & & 0_{m,l-m} \\ & & d_1^{r_m}d_2^{r_m} & \\ d_1^{r_t}d_2^{r_t}\alpha_1 & \cdots & d_1^{r_t}d_2^{r_t}\alpha_m & d_1^{r_t}d_2^{r_t}P_{32} \end{pmatrix} \cdot Q^{-1} \cdot G_1(x). \end{aligned}$$

By elementary transformations, we have that $F(x)$ is unimodular equivalent to $C(x)$, where

$$C(x) = \begin{pmatrix} d_1^{r_1}d_2^{r_1} & & & \\ & \ddots & & \\ & & d_1^{r_m}d_2^{r_m} & \\ & & & d_1^{r_t}d_2^{r_t}P_{32} \end{pmatrix} \cdot Q^{-1} \cdot G_1(x),$$

In the following, we prove that $d_1d_2 \mid d_{k-m+1}(P_{32})$.

Let $e = \sum_{i=1}^{m} r_i + (k-m+1)r_t + 1$. Because $(d_1d_2)^e \mid d_{k+1}(F)$ and $F(x) \sim C(x)$, we have $(d_1d_2)^e \mid d_{k+1}(C)$. Assume W is one of all $(k-m+1)\times(k-m+1)$ submatrices of P_{32}; therefore,

$$C'(x) = \begin{pmatrix} d_1^{r_1}d_2^{r_1} & & & \\ & \ddots & & \\ & & d_1^{r_m}d_2^{r_m} & \\ & & & d_1^{r_t}d_2^{r_t}W \end{pmatrix}$$

is a $(k+1)\times(k+1)$ submatrix of $C(x)$. So, $(d_1d_2)^e \mid d_{k+1}(C')$ implies that $d_1d_2 \mid \det(W)$. It is easy to see that $d_1d_2 \mid d_{k-m+1}(P_{32})$. Then, by Lemma 5, there exists a unimodular matrix $U(x) \in R^{(l-m)\times(l-m)}$ such that

$$U(x)\cdot P_{32} = \begin{pmatrix} I_{k-m} & \\ & d_1d_2I_{l-k} \end{pmatrix} \cdot G_2(x),$$

where $G_2(x) \in R^{(l-m)\times(l-m)}$.

By some elementary transformations, we have

$$C(x) \sim \begin{pmatrix} d_1^{r_1}d_2^{r_1} & & & & \\ & \ddots & & & \\ & & d_1^{r_m}d_2^{r_m} & & \\ & & & d_1^{r_t}d_2^{r_t}I_{k-m} & \\ & & & & d_1^{r_t+1}d_2^{r_t+1}I_{l-k} \end{pmatrix} \cdot \begin{pmatrix} I_m & \\ & G_2(x) \end{pmatrix} \cdot Q^{-1}\cdot G_1(x).$$

From the transmissibility of matrix equivalent, $F(x)$ is unimodular equivalent to

$$M(x) = \begin{pmatrix} d_1^{r_1}d_2^{r_1} & & & \\ & \ddots & & \\ & & d_1^{r_k}d_2^{r_k} & \\ & & & d_1^{r_t+1}d_2^{r_t+1}I_{l-k} \end{pmatrix} \cdot N(x),$$

where $N(x) = \begin{pmatrix} I_m & \\ & G_2(x) \end{pmatrix} \cdot Q^{-1}(x)\cdot G_1(x)$.

(3) If $r_1 \le r_2 \le \cdots \le r_k < r_t$. Through the above methods, we can obtain the same conclusion. □

Theorem 2. *Let $F(x) \in R^{l\times l}$, $\det F(x) = d_1^q d_2^q$, $d_1 = x_1 - f_1(x_2,\cdots,x_n)$, $d_2 = x_2 - f_2(x_3,\cdots,x_n)$, where q is a positive integer. Then, $J_i(F) = R$ and $d_i(F) = (d_1d_2)^{q_i}$ if and only if $F(x)$ is unimodular equivalent to its Smith normal form $S(x)$, where*

$$S(x) = \begin{pmatrix} d_1^{r_1}d_2^{r_1} & & & \\ & d_1^{r_2}d_2^{r_2} & & \\ & & \ddots & \\ & & & d_1^{r_l}d_2^{r_l} \end{pmatrix}$$

and $r_i = q_i - q_{i-1}, q_0 \equiv 0, i = 1,2,\cdots,l$.

Proof. Sufficiency: Suppose that $F(x) \sim S(x) = diag\{d_1^{r_1}d_2^{r_1}, d_1^{r_2}d_2^{r_2},\cdots,d_1^{r_l}d_2^{r_l}\}$. By Lemma 6, $J_i(F) = J_i(S) = R$ and $d_i(F) = d_i(S) = (d_1d_2)^{q_i}$, where $q_i = r_1+\cdots+r_i$, $i = 1,\cdots,l$.

Necessity: Because $d_1(F) = (d_1d_2)^{r_1}$, then we have $F = (d_1d_2)^{r_1}I_l \cdot N_1$. Furthermore, we assume that $d_2(F) = (d_1d_2)^{r_1+r_2}$, by Definition 2, we have $r_2 \ge r_1$, and then we consider two cases. If $r_2 = r_1$, it is obvious that $F(x) \sim diag\{d_1^{r_1}d_2^{r_1}, d_1^{r_2}d_2^{r_2},\cdots,d_1^{r_2}d_2^{r_2}\}\cdot N_2$, where $N_2 = N_1$. If $r_2 > r_1$, by Theorem 1, we have $F(x) \sim diag\{d_1^{r_1}d_2^{r_1}, d_1^{r_1+1}d_2^{r_1+1},\cdots,d_1^{r_1+1}d_2^{r_1+1}\}\cdot N_{21}$. Repeating the preceding procedure $r_2 - r_1$ times, we obtain

$$F(x) \sim diag\{d_1^{r_1}d_2^{r_1}, d_1^{r_2}d_2^{r_2},\cdots,d_1^{r_2}d_2^{r_2}\}\cdot N_2.$$

Repeat the above steps $l-2$ times, and we have $F(x) \sim diag\{d_1^{r_1}d_2^{r_1}, d_1^{r_2}d_2^{r_2},\cdots,d_1^{r_l}d_2^{r_l}\}\cdot N$. It is clear that N is a unimodular matrix. Thus, we have that

$$F(x) \sim diag\{d_1^{r_1}d_2^{r_1}, d_1^{r_2}d_2^{r_2},\cdots,d_1^{r_l}d_2^{r_l}\}.$$

Thus, $F(x)$ is unimodular equivalent to its Smith normal form $S(x)$. □

Remark 1. *Based on Theorem 2, we give a positive answer to Problem 1. In the following, we generalize the above result to the case of a non-square matrix.*

We first give a useful lemma.

Lemma 10 ([25]). *Let $F(x) \in R^{l\times m}$ be of full row rank, and denote the greatest common divisor of all the $l \times l$ minors of $F(x)$ by d. If the $l \times l$ reduced minors of $F(x)$ generate R, then there exist $G(x) \in R^{l\times l}$ and $F_1(x) \in R^{l\times m}$ such that $F(x) = G(x)F_1(x)$, $\det G(x) = d$ and $F_1(x)$ is a ZLP matrix.*

Denote

$$A(x) = \begin{pmatrix} d_1^{r_1}d_2^{r_1} & & & \\ & d_1^{r_2}d_2^{r_2} & & \\ & & \ddots & \\ & & & d_1^{r_l}d_2^{r_l} \end{pmatrix}.$$

Theorem 3. *Let $F(x) \in R^{l\times m} (l \le m)$ have full row rank, $d_l(F) = d_1^q d_2^q$, $d_1 = x_1 - f_1(x_2, \cdots, x_n)$, $d_2 = x_2 - f_2(x_3, \cdots, x_n)$, where q is a positive integer. Then, $J_i(F) = R$, $i = 1, 2, \cdots, l$ if and only if $F(x)$ is unimodular equivalent to its Smith normal form $S(x)$, where*

$$S(x) = \begin{pmatrix} A(x) & 0_{l\times(m-l)} \end{pmatrix}.$$

Proof. Sufficiency: If $F(x)$ is unimodular equivalent to the Smith normal form $S(x)$, it is obvious that $J_i(S) = R$, $i = 1, \cdots, l$. By Lemma 6, $J_i(F) = J_i(S) = R$ for $i = 1, 2, \cdots, l$.

Necessity: According to Lemma 10, there exists a matrix $G(x) \in R^{l\times l}$ and a ZLP matrix $F_1(x) \in R^{l\times m}$ such that $F(x) = G(x)F_1(x)$, where $\det G(x) = d_1^q d_2^q$. By Lemma 8, we can obtain that $J_i(G) = R$. From Theorem 2, there exist two $l \times l$ unimodular polynomial matrices $P(x), Q(x)$ such that $G(x) = P(x)A(x)Q(x)$. Then, we have

$$F(x) = P(x)A(x)Q(x)F_1(x).$$

It is obvious that $Q(x)F_1(x)$ is also a ZLP. According to Lemma 4, there exists an $m \times m$ unimodular matrix $U_1(x)$ such that $Q(x)F_1(x)U_1(x) = \begin{pmatrix} I_l & 0_{l\times(m-l)} \end{pmatrix}$. Then, we have

$$F(x)U_1(x) = P(x)A(x)Q(x)F_1(x)U_1(x) = P(x)A(x)\begin{pmatrix} I_l & 0_{l\times(m-l)} \end{pmatrix} = P(x)S(x).$$

Therefore, $F(x)$ is unimodular equivalent to $S(x)$. □

So as to prove Problem 2, we first give a helpful lemma.

Lemma 11. *Let $U(x) \in R^{l\times l}$ be an invertible matrix, $F(x) = P_1(x) \cdot U(x) \cdot P_2(x) = diag\{I_{l-r}, pI_r\} \cdot U(x) \cdot diag\{I_{l-r}, qI_r\}$, where $p, q \in R$ satisfy $q \mid p$. Then, $F(x)$ is equivalent to $diag\{I_{l-r}, pqI_r\}$ if and only if the $(l-r) \times (l-r)$ minors of $F(x)$ generate R.*

Remark 2. *The above lemma is a generalization of Theorem 3 in Li et al. [16], so the proof is omitted here. When $p \mid q$, the Lemma still holds.*

Based on Lemma 11, we can solve Problem 2.

Theorem 4. *Let $F(x) \in R^{l\times l}$ with $\det F(x) = (d_1^s d_2^t)^r$, $d_1 = x_1 - f_1(x_2, \cdots, x_n)$, $d_2 = x_2 - f_2(x_3, \cdots, x_n)$, where s, t are positive integers. Then, all the $(l-r) \times (l-r)$ mi-*

nors of $F(x)$ generate R and $d_1^s d_2^t \mid d_{l-r+1}(F)$ if and only if $F(x)$ is unimodular equivalent to its Smith normal form

$$S(x) = \begin{pmatrix} I_{l-r} & & & \\ & d_1^s d_2^t & & \\ & & \ddots & \\ & & & d_1^s d_2^t \end{pmatrix}.$$

Proof. Sufficiency: Because $F(x)$ is unimodular equivalent to the Smith normal form $S(x)$. By Lemma 3 and Lemma 6, the $(l-r) \times (l-r)$ minors of $F(x)$ generate R and $d_1^s d_2^t \mid d_{l-r+1}(F)$.

Necessity: Without loss of generality, suppose that $1 \leq s \leq t$. Using Lemma 5 repeatedly, we have

$$F(x) \sim P_1(x)U_1(x)P_2(x)V_1(x) \cdots P_1(x)U_s(x)P_2(x)V_s(x)P_2(x)V_{s+1}(x) \cdots V_{t-1}(x)P_2(x),$$

where $P_1(x) = diag\{I_{l-r}, d_1 I_r\}$, $P_2(x) = diag\{I_{l-r}, d_2 I_r\}$, and $U_i(x), V_j(x) \in R^{l \times l}$ are unimodular matrices. According to Lemma 2.6 in Li et al. [16], we obtain

$$F(x) \sim L(x)W_1(x)L(x)W_2(x) \cdots L(x)W_s(x)P_2(x)V_{s+1}(x) \cdots V_{t-1}(x)P_2(x),$$

where $L(x) = diag\{I_{l-r}, d_1 d_2 I_r\}$ and $W_i(x) \in R^{l \times l}$ are unimodular matrices. If all the $(l-r) \times (l-r)$ minors of $F(x)$ generate R and $d_1^s d_2^t \mid d_{l-r+1}(F)$, then by Lemma 6 and Lemma 11 repeatedly we obtain that $F(x)$ is unimodular equivalent to its Smith normal form $S(x)$. □

In the following, we generalize the above result to a more general case where $F(x)$ is a non-square matrix. Denote

$$B(x) = \begin{pmatrix} I_{l-r} & & & \\ & d_1^s d_2^t & & \\ & & \ddots & \\ & & & d_1^s d_2^t \end{pmatrix}.$$

Theorem 5. *Let $F(x) \in R^{l \times m} (l \leq m)$ be of full row rank, $J_l(F) = R$, $d_l(F) = (d_1^s d_2^t)^r$, $d_1 = x_1 - f_1(x_2, \cdots, x_n)$, $d_2 = x_2 - f_2(x_3, \cdots, x_n)$, where s, t are positive integers. Then, the $(l-r) \times (l-r)$ minors of $F(x)$ generate R and $d_1^s d_2^t \mid d_{l-r+1}(F)$ if and only if $F(x)$ is unimodular equivalent to its Smith normal form*

$$S(x) = \begin{pmatrix} B(x) & 0_{l \times (m-l)} \end{pmatrix}.$$

Proof. Sufficiency: Because $F(x)$ is unimodular equivalent to $S(x)$, it is clear that the $(l-r) \times (l-r)$ minors of $S(x)$ generate R and $d_1^s d_2^t \mid d_{l-r+1}(S)$. By Lemma 6, we can obtain that the $(l-r) \times (l-r)$ minors of $F(x)$ generate R and $d_1^s d_2^t \mid d_{l-r+1}(F)$.

Necessity: According to Lemma 10, there is a matrix $G(x) \in R^{l \times l}$ and a ZLP matrix $F_1(x) \in R^{l \times m}$ such that $F(x) = G(x)F_1(x)$, where $\det G(x) = (d_1^s d_2^t)^r$. Combining with Lemma 8, we can obtain that all the $(l-r) \times (l-r)$ minors of $G(x)$ generate R and $d_1^s d_2^t \mid d_{l-r+1}(G)$. By Theorem 4, there exist two $l \times l$ unimodular polynomial matrices $P(x), Q(x)$ such that $G(x) = P(x)B(x)Q(x)$. Then, we have

$$F(x) = P(x)B(x)Q(x)F_1(x).$$

It is obvious that $Q(x)F_1(x)$ is also a ZLP matrix. According to Lemma 4, there exists an $m \times m$ unimodular matrix $U_1(x)$ such that $Q(x)F_1(x)U_1(x) = \begin{pmatrix} I_l & 0_{l\times(m-l)} \end{pmatrix}$. Then, we have

$$F(x)U_1(x) = P(x)B(x)Q(x)F_1(x)U_1(x) = P(x)B(x)\begin{pmatrix} I_l & 0_{l\times(m-l)} \end{pmatrix} = P(x)S(x).$$

Therefore, $F(x)$ is unimodular equivalent to $S(x)$. □

3. Example

In this section, we propose an executable algorithm to handle the unimodular equivalence of the matrices we discussed to their Smith normal forms. Meanwhile, we give a 3D example to illustrate the main results of this paper and the computation process of Algorithm 1.

Algorithm 1: Smith normal form.

Input: $F \in R^{l\times l}$ with $\det F = (d_1d_2)^q = (x_1 - f_1(x_2, \cdots, x_n))^q(x_2 - f_2(x_3, \cdots, x_n))^q$.
Output: $U, V \in R^{l\times l}$ are two unimodular matrices such that $F = USV$,
S is the Smith normal form of F.
1. Calculate $d_i(F)$ and $J_i(F)$, where $i = 1, \cdots, l$ such that $S = \{(d_1d_2)^{r_1}, \cdots, (d_1d_2)^{r_l}\}$.
2.**If** there exist some integers i such that $J_i(F) \neq R$ for $i = 1, \cdots, l$
Return: matrix F is not unimodular equivalent to S.
3. Extract $(d_1d_2)^{r_1}$ from every row of F, then obtain a polynomial matrix N_1 that satisfies $F = (d_1d_2)^{r_1} I_l N_1$;
4. Presume U, V are two identity matrices;
5. **When** $2 \leq i \leq l$, perform step 6; otherwise, go to step 11.
6. Check that $r_i \neq r_{i-1}$. If yes, perform step 7; otherwise, $i = i + 1$, go to step 5;
7. **For** j from 1 to $r_i - r_{i-1}$ **do**
8. Calculate two unimodular matrices U', V' and a matrix N' such that $N_1 = U' diag\{I_{i-1}, d_1d_2I_{l-i+1}\}N'V'$;
Then,
9. Calculate two unimodular matrices U'', V'' and a matrix N'' such that
$diag\{(d_1d_2)^{r_1}, \cdots, (d_1d_2)^{r_{i-1}}, (d_1d_2)^{r_{i-1}+j-1}, \cdots, (d_1d_2)^{r_{i-1}+j-1}\}U' diag\{I_{i-1}, d_1d_2I_{l-i+1}\}$
$= U'' diag\{(d_1d_2)^{r_1}, \cdots, (d_1d_2)^{r_{i-1}}, (d_1d_2)^{r_{i-1}+j}, \cdots, (d_1d_2)^{r_{i-1}+j}\}V''$;
10. $N_1 = V''N'$, $U = UU''$ and $V = V'V$;
11. $V = N_1V$;
12. **Return** U, V.

Example 1. *Consider a 3D polynomial matrix of $R^{3\times 3}$*

$$F(x,y,z) = \begin{pmatrix} 1 & -z^2 & x-y \\ x-y & a_{22} & (x-y)^2 \\ (x-y)(y-z)^2 & -z^2(x-y)(y-z)^2 & a_{33} \end{pmatrix},$$

where

$$a_{22} = (x-y)^2(y-z)^2 - (x-y)z^2,$$
$$a_{33} = (x-y)^3(y-z)^3 + (x-y)^2(y-z)^2.$$

By computing $d_1(F) = 1$, $d_2(F) = (x-y)^2(y-z)^2$, $\det F(x,y,z) = (x-y)^5(y-z)^5$. Let $d_1 = x - y$, $d_2 = y - z$. Then, calculate the reduced Gröbner bases of the ideal generated by the $i \times i$ reduced minors of $F(x,y,z)$ which is $\{1\}$, so we have that $J_i(F) = R$, $i = 1, 2, 3$. According to Theorem 2, $F(x,y,z)$ is unimodular equivalent to its Smith normal form $S(x,y,z)$, where

$$S(x,y,z) = \begin{pmatrix} 1 & & \\ & (d_1d_2)^2 & \\ & & (d_1d_2)^3 \end{pmatrix}.$$

We first consider

$$F_1(x,z,z) = \begin{pmatrix} 1 & -z^2 & x-z \\ x-z & -(x-z)z^2 & (x-z)^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, construct a unimodular matrix

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ -(x-z) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

such that

$$U_1 \cdot F_1(x,z,z) = \begin{pmatrix} 1 & -z^2 & x-z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then,

$$U_1 \cdot F = \begin{pmatrix} 1 & & \\ & d_2 & \\ & & d_2 \end{pmatrix} F_1,$$

where

$$F_1 = \begin{pmatrix} 1 & -z^2 & -y+x \\ -1 & -(y-x)^2(z-y)+z^2 & y-x \\ (y-x)(z-y) & -z^2(y-x)(z-y) & a'_{33} \end{pmatrix},$$

and $a'_{33} = (x-y)^3(y-z)^2 + (x-y)^2(y-z)$.
Then, consider F_1 *again*

$$F_1(y,y,z) = \begin{pmatrix} 1 & -z^2 & 0 \\ -1 & z^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Construct a unimodular matrix

$$U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

such that

$$U_2 \cdot F_1(y,y,z) = \begin{pmatrix} 1 & -z^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then

$$U_2 \cdot F_1 = \begin{pmatrix} 1 & & \\ & d_1 & \\ & & d_1 \end{pmatrix} F_2,$$

where

$$F_2 = \begin{pmatrix} 1 & -z^2 & -y+x \\ 0 & (y-x)(z-y) & 0 \\ -z+y & z^2(z-y) & b \end{pmatrix},$$

and $b = (x-y)^2(y-z)^2 + (x-y)(y-z)$. *Now, we have*

$$F = U_1^{-1} \begin{pmatrix} 1 & & \\ & d_2 & \\ & & d_2 \end{pmatrix} U_2^{-1} \begin{pmatrix} 1 & & \\ & d_1 & \\ & & d_1 \end{pmatrix} F_2.$$

By Lemma 2.6 in Li et al. [16], we obtain

$$\begin{pmatrix} 1 & & \\ & d_2 & \\ & & d_2 \end{pmatrix} U_2^{-1} \begin{pmatrix} 1 & & \\ & d_1 & \\ & & d_1 \end{pmatrix} = U_3 \begin{pmatrix} 1 & & \\ & d_1 d_2 & \\ & & d_1 d_2 \end{pmatrix},$$

where $U_3 = \begin{pmatrix} 1 & 0 & 0 \\ -d_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *is a unimodular matrix, then repeat the above process for* F_2*, and we have*

$$F_2 = U_4 \begin{pmatrix} 1 & & \\ & d_1 d_2 & \\ & & d_1 d_2 \end{pmatrix} F_3,$$

where

$$U_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ d_2 & 0 & 1 \end{pmatrix}, F_3 = \begin{pmatrix} 1 & -z^2 & x-y \\ 0 & 1 & 0 \\ 0 & 0 & (x-y)(y-z) \end{pmatrix}.$$

Hence,

$$F = U_1^{-1} U_3 U_4' \begin{pmatrix} 1 & & \\ & (d_1 d_2)^2 & \\ & & (d_1 d_2)^2 \end{pmatrix} F_3,$$

where

$$U_4' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ d_1 d_2^2 & 0 & 1 \end{pmatrix}.$$

It is obvious that

$$F_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & d_1 d_2 \end{pmatrix} F_4,$$

where $F_4 = \begin{pmatrix} 1 & -z^2 & x-y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *is a unimodular matrix.*

Thus,

$$\begin{aligned} F &= U_1^{-1} U_3 U_4' \begin{pmatrix} 1 & & \\ & (d_1 d_2)^2 & \\ & & (d_1 d_2)^3 \end{pmatrix} F_4 \\ &= U \begin{pmatrix} 1 & & \\ & (d_1 d_2)^2 & \\ & & (d_1 d_2)^3 \end{pmatrix} V, \end{aligned}$$

where $U = U_1^{-1} U_3 U_4'$ *and* $V = F_4$ *are unimodular matrices.*

4. Conclusions

In this paper, we considered the unimodular equivalence problem for two classes of nD polynomial matrices, and we obtained some tractable necessary and sufficient conditions that such polynomial matrices are unimodular equivalent to their Smith normal forms. Meanwhile, we designed an algorithm for simplifying such matrices to their Smith normal forms and provided an example at the end of the article to illustrate our approach. All of these are helpful for reducing nD systems.

However, the unimodular equivalence problem of many other types of multivariate polynomial matrices has not been solved, such as $F(x) \in R^{l \times l}$ with $\det(F) = d_1^{q_1} d_2^{q_2} =$

$(x_1 - f_1(x_2, \cdots, x_n))^{q_1}(x_2 - f_2(x_3, \cdots, x_n))^{q_2}$, where q_1, q_2 are two positive integers. What is the criteria for the unimodular equivalence between $F(x)$ and its Smith normal form

$$diag\{d_1^{r_1}d_2^{s_1}, d_1^{r_2}d_2^{s_2}, \cdots, d_1^{r_l}d_2^{s_l}\}.$$

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