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Application of STEM Technologies on the Example of the Problem of a Thread with a Load

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Abstract: This paper demonstrates the application of STEM technologies using the example of the problem of a thread with a load. As a training problem, parameter finding of a flexible inextensible thread with a load sliding along it, as well as with a fixed load, is considered. A mathematical model of a thread with a load is created without using simplifying assumptions such as a negligibly small mass of the thread, a thread with a small sag or a parabolic approximation of the thread form. When solving the problem, methods of analysis from various branches of mathematics, theoretical mechanics, physics, history of mathematics, as well as various tools of computer mathematical packages (Mathcad), are used. The reasonability of considering such tasks from the perspective of teaching effectiveness for first-year students in engineering as well as in physics and mathematics is proved. Promising educational tasks based on the investigated problem are considered.

Keywords: STEM education; catenary curve; thread with load

MSC: 97-11

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1. Introduction

The application of science, technology, engineering and mathematics (STEM) technologies has become a key area of educational process development in the last decades, especially in science and engineering education [1–4]. STEM technology is a rather new educational trend that describes an integrative approach to learning. Thereby, the scientific disciplines are not studied in isolation but rather in relationship to each other to solve the problem at hand. This increases interdisciplinary connections and intensifies the cognitive aspect of learning. The main factor promoting the use of STEM technologies is clearly the extensive computational possibilities of modern mathematical computer software such as Mathcad, Matlab, Mathematica, etc. [5–7].

Thus, quantum computing ideas have been proposed in literature [1], as well as applications of screencasts and simulations [8]. The development of a high-school physics unit and associated assessment tasks were described [3]. The original technology of training samples used for constructing hybrid neural networks were considered [7]. Studies [2,9] have been devoted to the training of teaching staff for conducting STEM classes.

This paper, on the one hand, follows the trends set in [3,4], which propose ready-made scenarios for STEM classes. On the other hand, a new educational aspect was considered which is not completely taken into account in regular STEM technologies, namely a rational and meaningful choice of the problems to be solved with students using such technologies. Several aspects which form the essence of STEM education should be considered. The problem has to be relevant to the specialty and have a practical application in real engineering construction or physical processes. Moreover, it is necessary to demonstrate a variety of mathematical instruments, as well as capabilities of computer software tools providing practical implementation of such instruments. Finally, in terms of influence on the cognitive process and motivational qualities of students [10], the most effective tasks seem to be illustrative with simple physical interpretations.

In this paper, a problem of parameter estimation for a flexible inextensible thread with a load was solved. It meets all the mentioned requirements. On one hand, this long-term problem is rather well known and studied both for applied and educational purposes [11–20]. On the other hand, it allows the application of various analysis techniques, application of different mathematical fields, theoretical mechanics, physics, history of mathematics and demonstration of a wide range of computer software tools (in this case, Mathcad). This provides an opportunity to achieve the main goal of STEM technologies—to enhance learning efficiency. This problem can be presented to first-year engineering and physics-mathematics students.

Thus, a new educational approach to STEM technologies is demonstrated in this paper. In addition to the educational aspects, the scientific originality of this paper is the examination of a more accurate mathematical model of a thread with a load without using simplifying assumptions such as a negligibly small mass of the thread, a thread with a small sag or an parabolic approximation of the thread shape, which are often accepted in engineering calculations [21–25].

The paper has the following structure:

- Composition and analysis of a mathematical model of a flexible inextensible thread with a freely sliding (frictionless) load on it;
- Composition and analysis of a mathematical model of the flexible inextensible thread with an attached load;
- Results of testing the proposed approach;
- Analysis of educational tasks resulting from the mentioned problems.

2. Materials and Methods

In this work, the following research and educational methods were used [26–28]. Methods of mathematical modeling were used in compiling models of a flexible inextensible thread with both a sliding and fixed load. At the same time, an educational method known as a problem statement [26], in which the teacher sets the direction of scientific search for students, as well as an explanatory and illustrative method [26] (consideration of graphs, drawings), were implemented.

Further consideration of mathematical models took place with the application of mathematical analysis methods, specifically the study of derivative concepts, integral concepts, formulas of a curve gravity center, etc. Methods of theoretical mechanics related to the concepts of potential energy, as well as the balance of forces and moments, were applied to compose equations describing mathematical models. A heuristic method of education [26] was used to allow students to participate in discussions and to create mathematical apparatus for further research.

Further, for the decision of the received algebraic equations, a method of computer modeling in Mathcad was involved. Methods of linear algebra and numerical solutions of algebraic equations, as well as methods of optimization, were used in the decision process. Here, a reproductive method of education [26] was implemented, where students followed the instructions.

Finally, during the review of the obtained results of the calculations and graphical material, an explanatory and illustrative method [26] were used, as well as a comparative analysis method, which encouraged students to draw the required conclusions.

Such a combination of research and educational methods is one of the manifestations of STEM learning.

3. Results and Discussion

3.1. Mathematical Model of a Thread with a Freely Sliding Load

Let us consider a common engineering structure, such as a hanging cable car which moves along a cable fixed between the supports of different heights (Figure 1).

Figure 1. Cable car [29].

Then, we will consider a simplified physical model of such a structure: a load (a bead) slides without friction along a flexible inextensible thread hanging from supports of different heights (Figure 2). The students may be asked to think about the question: at which point in the thread will it stop? What would the shape of the thread look like?

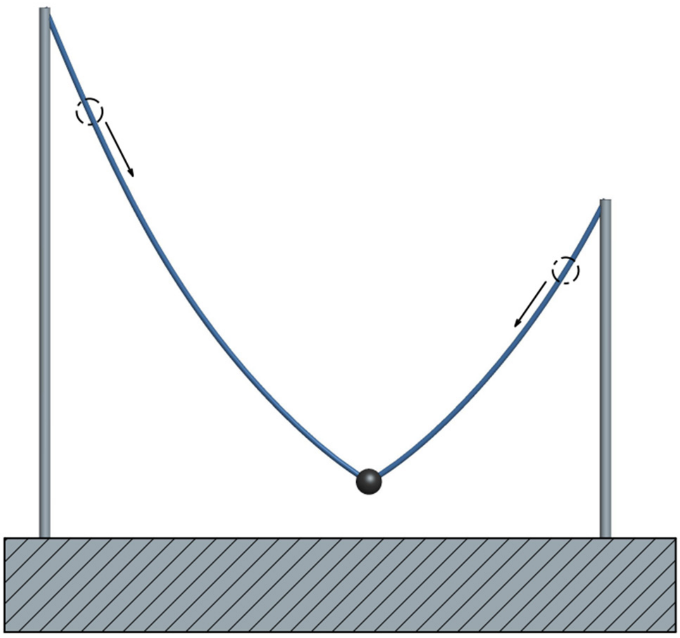


Figure 2. Physical model of a thread with a sliding bead.

When the students have finished their hypotheses, let us move on to creating a mathematical model.

An absolute flexible and inextensible thread has length S and specific mass m_c. The thread is suspended at a height h_1 on the left and at a height h_2 on the right side. The horizontal distance between the suspension points is L. A bead (material point) of mass M can slide along the thread without friction. The question is what forces will act on the thread, how it will sag and what will be the abscissa x_b and ordinate h_b of the bead when it stops (Figure 3). Figure 3 shows the raw data in green and the main unknowns in yellow.

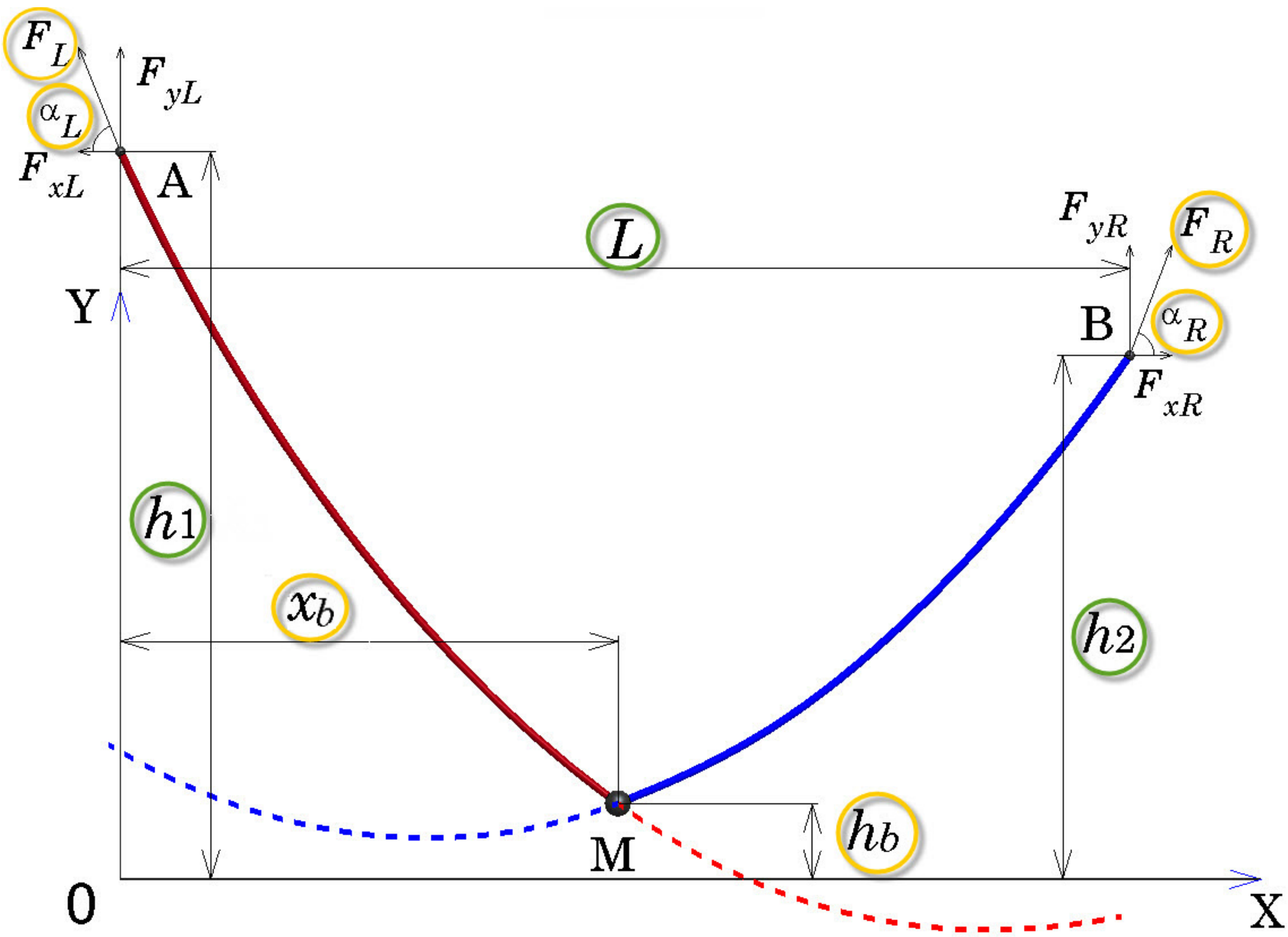


Figure 3. Mathematical model of a thread with a sliding bead.

Before answering the proposed questions, it is important to discuss with students the fact that a flexible thread without a load sags in the shape of a catenary curve [24]:

$$y(x) = a \cdot ch\left(\frac{x}{a}\right), \tag{1}$$

where a is the parameter which determines the "steepness" of the catenary.

This parameter has a certain physical meaning: the ratio of the horizontal force projection stretching the thread at any place to the specific gravity of the thread [24]. The students' attention should be focused on the fact that the parameter a is not dimensionless but has the dimension $[a]$ = meters.

Here, it is appropriate to mention the history of catenary curve discovery, in particular the names of mathematicians Leibniz, Huygens and Bernoulli [24,30]. It is also important to note that the parabola formula [22,23] (Figure 4) is used as a simplification for engineering calculations in the case of small, sagged threads:

$$y(x) = \frac{x^2}{a}. \tag{2}$$

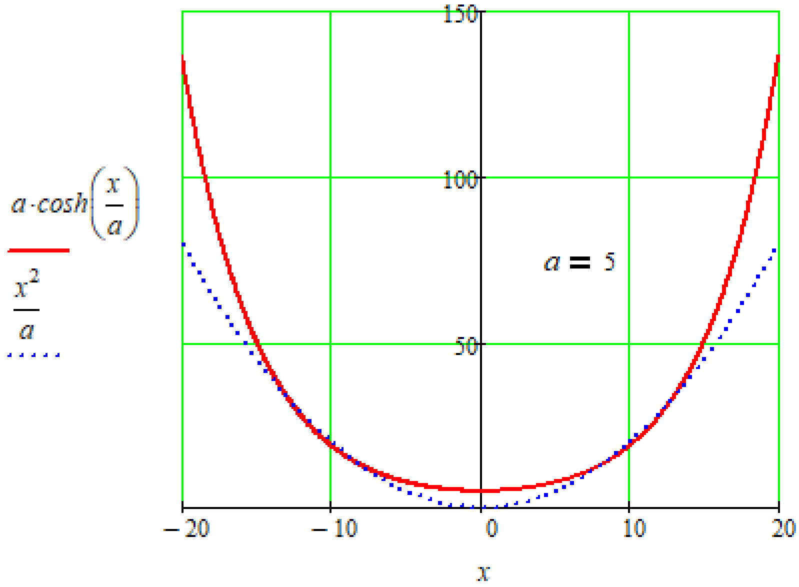


Figure 4. A catenary and its comparison with a parabola x^2/a.

How would the shape of the thread sag shown in Figure 4 change in the presence of a sliding load (bead) (Figure 3)?

With a concentrated load, the shape of the thread will be determined by the two sections AM and MB (see Figure 3). Both of them are parts of two imaginary catenary curves suspended at heights h_1 and h_2, respectively, and intersecting at point M (see dashed lines in Figure 3). Thus, the problem is reduced to determination of the two catenary equations described above.

In order to describe these catenary curves Formula (1) in modified form, (3) is used, where parameters x_0 and h have been added, which are abscissa and ordinate, respectively, of the minimum of the catenary graph (for understanding the problem it is recommended to ask students to determine values of x_0 and h for the catenary in Figure 4):

$$y(x, a, x_0, h) = h + a \cdot \left(ch\left(\frac{x - x_0}{a} \right) - 1 \right). \tag{3}$$

Using the theory of mathematical analysis, the students are then given the corresponding formulas for the derivative of the catenary $y'(x, a, x_0)$ on the coordinate x, its length $s(x_1, x_2, a, x_0)$, and the coordinates of its gravity center $x_{cb}(x_1, x_2, a, x_0)$, $y_{cb}(x_1, x_2, a, x_0,\ h_0)$ [31]:

$$\begin{aligned} y'(x, a, x_0) &= sh\left(\tfrac{x - x_0}{a}\right), \\ s(x_1, x_2, a, x_0) &= \int_{x_1}^{x_2} \sqrt{1 + y'(x, a, x_0)}dx, \\ x_{cb}(x_1, x_2, a, x_0) &= \frac{\int_{x_1}^{x_2} x\sqrt{1 + y'(x, a, x_0)}dx}{s(x_1, x_2, a, x_0)}, \\ y_{cb}(x_1, x_2, a, x_0,\ h) &= \frac{\int_{x_1}^{x_2} y(x, a, x_0, h) \cdot \sqrt{1 + y'(x, a, x_0)}dx}{s(x_1, x_2, a, x_0)}. \end{aligned} \tag{4}$$

Then, the equations of the two catenary sections described above are sought in the form $y(x, a_L, x_{0L}, h_L)$, $y(x, a_R, x_{0R}, h_R)$, where (a_L, x_{0L}, h_L), (a_R, x_{0R}, h_R) are parameters characterizing the "left" and "right" sections of catenary curves, respectively (red and blue plots, respectively, in Figure 3).

Let students note that the variables $x_1,\ x_2$, introduced in Equation (4), are to define the abscissa of the beginning and the end of the two above-described sections of catenary curves: 0 and x_b; x_b and L.

Following STEM principles, now let us move from mathematical analysis to using the basics of theoretical mechanics, that is, the fact that the mechanical system in rest takes a position with minimal potential energy [32–34].

The potential energy $PE(x_b, a_L, x_{0L}, h_L, a_R, x_{0R}, h_R)$ of a thread with a bead is the sum of three quantities: the potential energy of the "left" part of the thread, the potential energy of the suspended bead and the potential energy of the "right" part of the thread:

$$PE(x_b, a_L, x_{0L}, h_L, a_R, x_{0R}, h_R) = g\cdot\begin{pmatrix} s(0, x_b, a_L, x_{0L})\cdot m_c\cdot y_{cb}(0, x_b, a_L, x_{0L},\ h_L)+ \\ +M\cdot y(x_b, a_L, x_{0L}, h_L)+ \\ +s(x_b, L, a_R, x_{0R})\cdot m_c\cdot y_{cb}(x_b, L, a_R, x_{0R},\ h_R) \end{pmatrix}. \quad (5)$$

Proceeding in the STEM framework, one proceeds to the solution of the optimization problem. The solution of the problem on the sag of a thread with a bead reduces to finding a minimum of potential energy with constraints: one needs to find values of bead abscissa x_b and catenary parameters (a_L, x_{0L}, h_L), (a_R, x_{0R}, h_R), at which the objective function $PE(x_b, a_L, x_{0L}, h_L, a_R, x_{0R}, h_R)$ will have a minimum value.

The following conditions act as constraints:

- The "left" part of the thread is fixed at height h_1;
- The "right" part of the thread is fixed at height h_2;
- The "left" and "right" parts of the thread meet at the point with abscissa x_b;
- The sum of lengths of "left" and "right" parts of the thread is equal to initial length S:

$$\begin{gathered} y(0, a_L, x_{0L}, h_L) = h_1, \\ y(L, a_R, x_{0R}, h_R) = h_2, \\ y(x_b, a_L, x_{0L}, h_L) = y(x_b, a_R, x_{0R}, h_R), \\ s(0, x_b, a_L, x_{0L}) + s(x_b, L, a_R, x_{0R}) = S. \end{gathered} \quad (6)$$

Mathcad was used to solve this optimization problem. It is expected that students are competent in Mathcad and know how to perform the simplest computational operations. The solving of the problem will allow them to learn more complex operators, such as the *Minimize* search operator.

It should be noted that the application of software tools within STEM makes it possible to demonstrate to students the solution of a rather complex nonlinear programming problem, avoiding the description of the mathematical mechanism for solving such problems (this would make it unacceptable for undergraduates to consider such a problem).

Let us draw students' attention to such an important point as the choice of initial values of the sought parameters $x_b, a_L, x_{0L}, h_L, a_R, x_{0R}, h_R$ for numerical minimum search. This point is relevant for most numerical procedures implemented in Mathcad, as an unsuccessful choice leads to an inability to find a numerical solution in the software application [35]. Let us demonstrate a useful educational iterative data fitting technique. The shape of a thread with a suspended load is not known, but the shape of the same thread without a load is available as a catenary (3) with parameters a, x_0, h. These parameters can be easily computed in Mathcad by solving a system of three algebraic equations:

$$\begin{cases} y(0, a, x_0, h) = h_1, \\ y(L, a, x_0, h) = h_2, \\ s(0, L, a, x_0) = S. \end{cases} \quad (7)$$

Note that the numerical solution of system (7) also requires the specification of initial values. However, there are only three such values in this case, and their choice is very intuitive. For example, x_0 can be chosen as half of the support distance $L/2$, and h as half of the smaller support height $h_2/2$. As for the parameter a, a simple empirical choice is sufficient.

The listing of the program for solving system (7) with *Find* is shown in Figure 5 (here and below we use Mathcad Prime 6.0).

$$y(x, a, h, x_0) := h + a \cdot \left(\cosh\left(\frac{x - x_0}{a}\right) - 1\right)$$

$$s(x_1, x_2, a, h, x_0) := \int_{x_1}^{x_2} \sqrt{1^2 + \left(1 \cdot \frac{d}{dx} y(x, a, h, x_0)\right)^2}\, dx$$

$$a := 2\ \mathbf{m} \qquad h := 5\ \mathbf{m} \qquad x_0 := 5\ \mathbf{m}$$

$$h_1 = y(0\ \mathbf{m}, a, h, x_0) \qquad h_2 = y(L, a, h, x_0)$$

$$S = s(0\ \mathbf{m}, L, a, h, x_0)$$

$$\begin{bmatrix} a \\ h \\ x_0 \end{bmatrix} := \mathbf{Find}(a, h, x_0) = \begin{bmatrix} 4.171 \\ 3.466 \\ 8.346 \end{bmatrix} \mathbf{m}$$

Figure 5. Solution of the system (7) in Mathcad.

The calculations use following input data for the constants: $L = 15$ m, $S = 25$ m, $h_1 = 15$ m, $h_2 = 10$ m, $m_c = 0.1$ kg/m, $M = 1$ kg. The choice of thread length S for given values L, h_1, h_2 is not random and should be greater than minimum $S_{min} = \sqrt{L^2 - (h_2 - h_1)^2} = 15.81$ m, determined by simple geometric property (students should derive this relation by themselves).

Students should also note that the length function $s(x_1, x_2, a, h, x_0)$ formally has an additional parameter h, although it does not actually depend on it (see Formula (4)).

Then, the obtained solution of system (7) obtained can be used as initial values to find the minimum of potential energy: value a—as starting values for a_L, a_R , value x_0 —for x_b, x_{0L}, x_{0R}, and value h—for h_L, h_R.

The minimum search procedure in Mathcad can be directly handled by the *Minimize* operator. The listing of the corresponding program is shown in Figure 6.

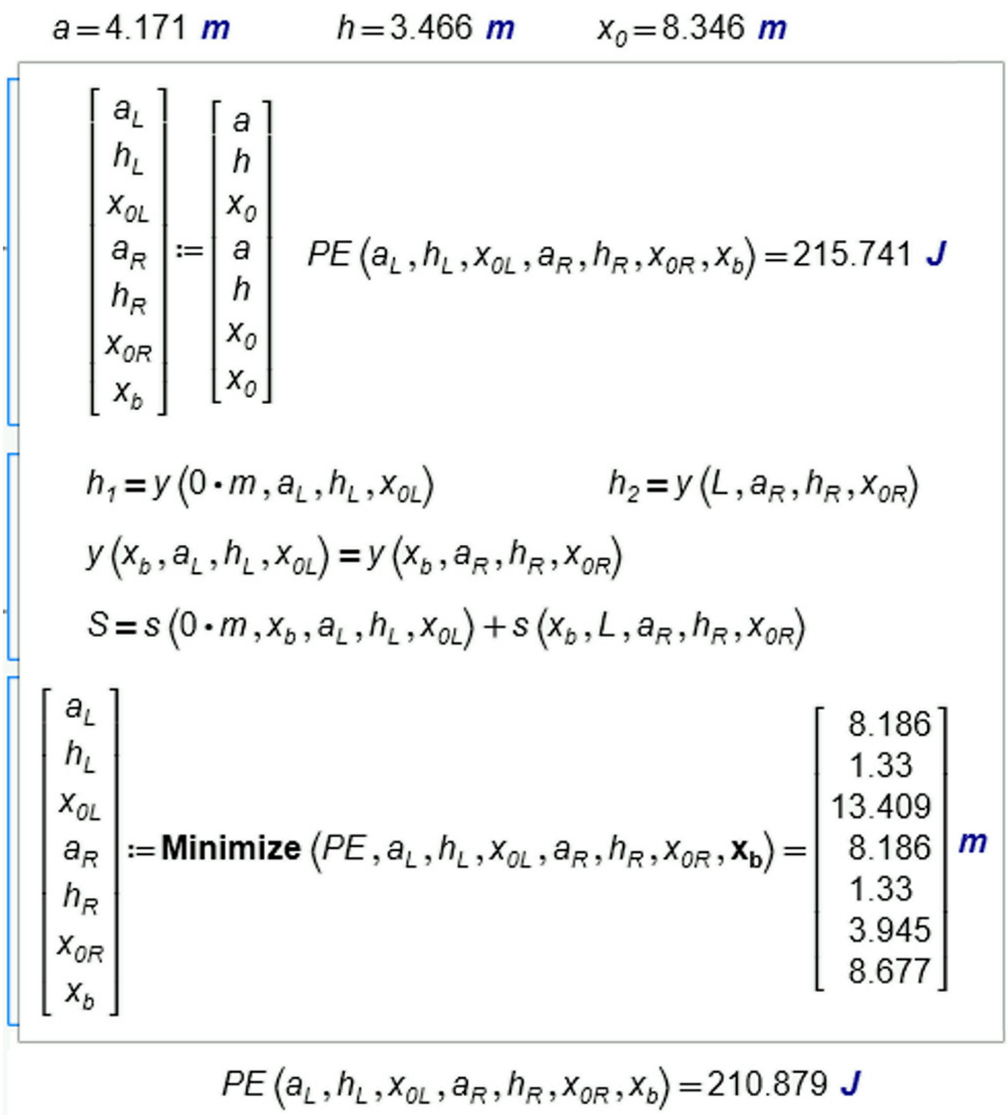


Figure 6. Searching for minimum of potential energy in Mathcad for the case of a sliding bead.

Based on the obtained parameter values, a graph of the thread form by its two parts corresponding to functions $y(x, a_L, x_{0L}, h_L), y(x, a_R, x_{0R}, h_R)$ can be plotted. However, for educational purposes, the results should be initially analyzed with students. It should be noted that the values of parameters a_L, a_R are the same (as well as the values of h_L, h_R), and the values of x_{0L}, x_{0R} are symmetric with respect to x_b:

$$\Delta = |x_b - x_{0L}| = |x_b - x_{0R}|. \tag{8}$$

The resulting equality of the parameters a_L, a_R is not by chance and follows from their physical meaning—the ratio of the horizontal projection modulus of the force stretching the thread at any point to the specific gravity of the thread:

$$a_L = a_R = \frac{|F_{xL}|}{g \cdot m_c} = \frac{|F_{xR}|}{g \cdot m_c}. \tag{9}$$

The modulus of horizontal projection of the tension force is the same at all points of the thread, so the parameters a_L, a_R are the same for both catenary curves.

Based on this explanation, the students can hypothesize what the desired graph of the shape of the thread would look like qualitatively and fix the proposed variants without presenting the graph obtained from Mathcad calculations.

Returning to the questions introduced at the beginning of Section 3.1, the forces (and their projections) acting on the thread at the suspension points and the values of angles between the thread and the OX axis at the left (α_L) and right (α_R) suspension points remain to be found. At this point, our focus returned to theoretical mechanics, and we drew up a balance of forces.

At the suspension points, the thread is under the action of the horizontal (F_{xL} and F_{xR}) and vertical (F_{yL} and F_{yR}) projections (Figure 3) of the F_L (L—left,) and F_R (R—right) forces.

Figure 3 does not show the other three forces (the gravity of the load, the gravity of the "left" part of the thread and the gravity of the "right" part) acting down vertically, as they are evenly distributed along the thread. However, these are taken into account in the calculations below.

Finding the angles α_L and α_R is not difficult given the parameters found for the catenary curves (Figure 6). Here, it is useful to remember the geometric meaning of the derivative and to invite the students to calculate these angles themselves:

$$\begin{aligned} \alpha_L &= arctg(y'(0, a_L, x_{0L}, h_L)), \\ \alpha_R &= arctg(y'(L, a_R, x_{0R}, h_R)). \end{aligned} \tag{10}$$

If the angles between the thread and the axis OX at the suspension points on the left (α_L) and on the right (α_R) are known, then it is easy to calculate the forces F_L and F_R by solving a system of equations, which represent the equalities of the horizontal and vertical force projections:

$$\begin{cases} F_{xL} = F_{xR} \\ F_{yL} + F_{yR} = g \cdot m_c \cdot S + g \cdot M. \end{cases} \tag{11}$$

or, which is equivalent to:

$$\begin{cases} F_L \cdot \cos(\alpha_L) = F_R \cdot \cos(\alpha_R) \\ F_L \cdot \sin(\alpha_L) + F_R \cdot \sin(\alpha_R) = g \cdot m_c \cdot S + g \cdot M. \end{cases} \tag{12}$$

Students can now see how to solve systems of equations symbolically (analytically) in Mathcad. Figure 7 shows the use of the *solve* operator for this purpose.

$$\begin{bmatrix} F_{xL} = F_{xR} \\ F_{yL} + F_{yR} = G \end{bmatrix} \qquad G = g \cdot m_c \cdot S + g \cdot M$$

$$\begin{bmatrix} F_L \cdot \cos(\alpha_L) = F_R \cdot \cos(\alpha_R) \\ F_L \cdot \sin(\alpha_L) + F_R \cdot \sin(\alpha_R) = G \end{bmatrix} \xrightarrow{solve, F_L, F_R}$$

$$\rightarrow \begin{bmatrix} \dfrac{G \cdot \cos(\alpha_R)}{\cos(\alpha_R) \cdot \sin(\alpha_L) + \cos(\alpha_L) \cdot \sin(\alpha_R)} & \dfrac{G \cdot \cos(\alpha_L)}{\cos(\alpha_R) \cdot \sin(\alpha_L) + \cos(\alpha_L) \cdot \sin(\alpha_R)} \end{bmatrix}$$

Figure 7. Solution in Mathcad for the equation system of forces' balance at the suspension points.

It is also possible to use linear algebra techniques in another way [31].

The system (12) is linear in the form $Tx = v$, where T is the square matrix of the coefficients for the unknowns, x is the vector of the unknowns and v is the vector of free terms. In order to analyze and solve this system of equations in Mathcad, some special tools can be applied (Figure 8). The Mathcad built-in function rank determines the ranks of the main and extended matrices. Since there are two unknowns, the system has only one solution. This solution can be found from the vector multiplication of the inverted matrix T and the vector v.

$$T \cdot x = v \qquad T = \begin{bmatrix} \cos(\alpha_L) & -\cos(\alpha_R) \\ \sin(\alpha_L) & \sin(\alpha_R) \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ G \end{bmatrix}$$

$$\text{rank}\left(\begin{bmatrix} \cos(\alpha_L) & -\cos(\alpha_R) \\ \sin(\alpha_L) & \sin(\alpha_R) \end{bmatrix}\right) \to 2 \quad \text{rank}\left(\begin{bmatrix} \cos(\alpha_L) & -\cos(\alpha_R) & 0 \\ \sin(\alpha_L) & \sin(\alpha_R) & G \end{bmatrix}\right) \to 2$$

$$\begin{bmatrix} \cos(\alpha_L) & -\cos(\alpha_R) \\ \sin(\alpha_L) & \sin(\alpha_R) \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ G \end{bmatrix} \to \begin{bmatrix} \dfrac{G \cdot \cos(\alpha_R)}{\cos(\alpha_R) \cdot \sin(\alpha_L) + \cos(\alpha_L) \cdot \sin(\alpha_R)} \\ \dfrac{G \cdot \cos(\alpha_L)}{\cos(\alpha_R) \cdot \sin(\alpha_L) + \cos(\alpha_L) \cdot \sin(\alpha_R)} \end{bmatrix}$$

Figure 8. Analysis and solution of the system of linear algebraic equations in Mathcad.

Finally, when all the preliminary calculations are done, the students can observe the graphs of the shape of the thread with the freely sliding load together with the calculated values (Figure 9).

Figure 9 shows three graphs for different mass values:

- Figure 9a for an initial value of M = 1 kg;
- Figure 9b for a zero value of M = 0 kg (no load);
- Figure 9c for the case when the mass of the load (M = 1000 kg) is significantly greater than the mass of the thread (2.5 kg).

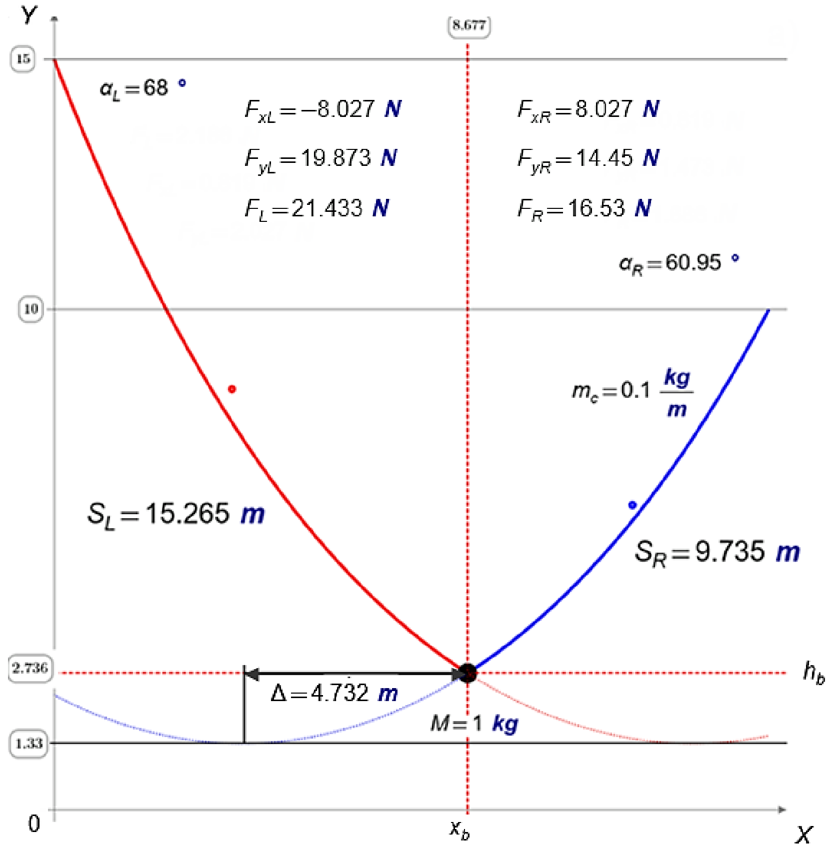
Students have the opportunity to analyze the graphs and compare them with the hypotheses proposed earlier. Consider the following facts:

- The sections of the graphs of Figure 9, which are between the level h_b of the load suspension and the level h_2, are symmetrical about the vertical axis drawn through x_b;
- The diagram Figure 9b (without load) looks like a normal catenary, while the diagram Figure 9c looks like a “funicular polygon” that is often used in engineering calculations [36];
- The red and blue dots on the graphs represent the centers of gravity of both sections of the thread.

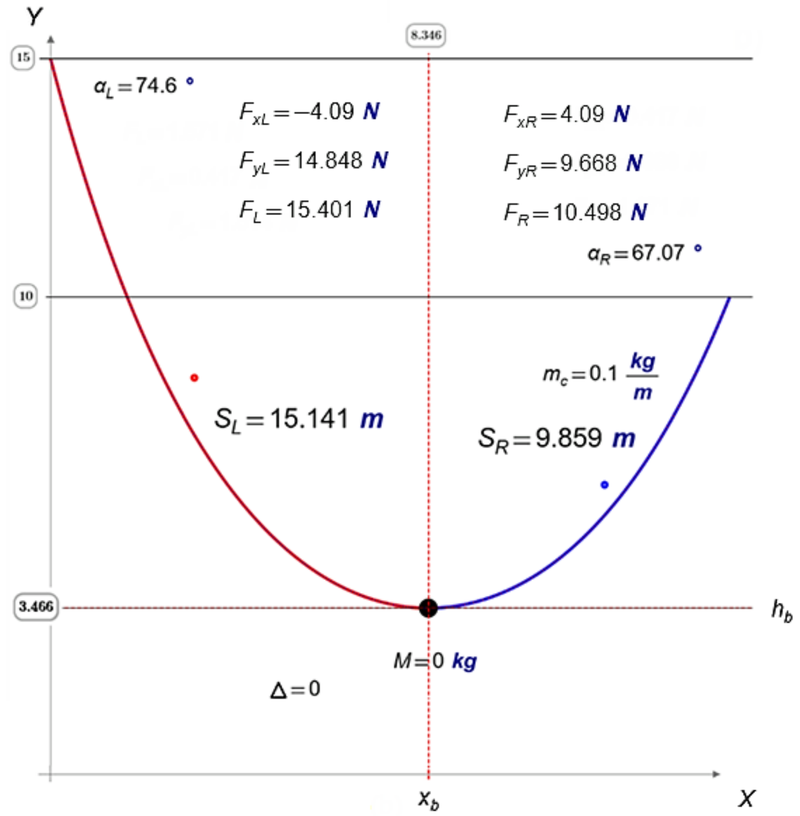
Based on the obtained values of the catenary parameters a_L, a_R (Figure 6) and their physical sense (9), as well as the geometrical property (10), the tension force $F(x)$ and its projections $F_x(x)$, $F_y(x)$ can be calculated for any point of the thread:

$$\begin{gathered} F_x(x) = g \cdot m_c \cdot \begin{cases} a_L, \; if \; x \leq x_b \\ a_R, \; if \; x > x_b \end{cases}, \\ F_y(x) = F_x(x) \cdot \begin{cases} y'(x, a_L, x_{0L}, h_L), \; if \; x \leq x_b \\ y'(x, a_R, x_{0R}, h_R), \; if \; x > x_b \end{cases}, \\ F(x) = \sqrt{F_x(x)^2 + F_y(x)^2}. \end{gathered} \tag{13}$$

It should be explained to students that, according to the tradition of theoretical mechanics, such a calculation is usually illustrated by a graph of the tension force of a thread (Figure 10).

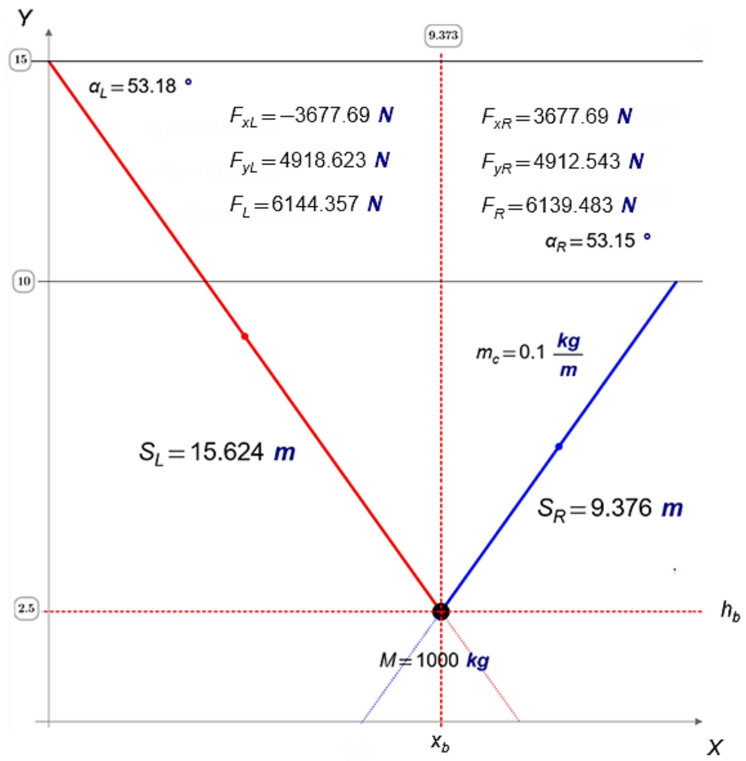


(a)



(b)

Figure 9. *Cont.*



(**c**)

Figure 9. Graphs of the shape of the thread with a freely sliding load with mass M = 1 kg (**a**), M = 0 kg (**b**), M = 1000 kg (**c**).

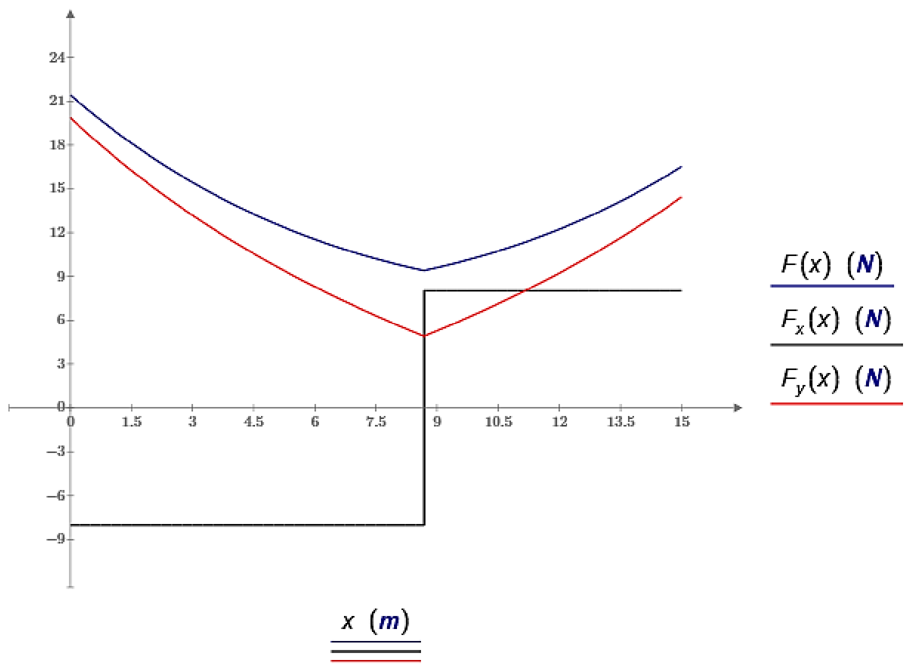


Figure 10. Tension force diagram for the case of a sliding bead with mass M = 1 kg.

So, we solved the problem together with the students by answering all the questions posed at the beginning of Section 3.1: we identified the shape of the thread, the location of the load and the forces acting on the thread.

3.2. Mathematical Model of a Thread with a Fixed Load

Consider the more complex case where a load (a bead) is attached to a thread so that, after the thread is suspended on supports of different heights, it is at a certain distance x_b from the left support.

An example of an engineering solution of this kind is the barrage balloons on power lines [37] (Figure 11).

Figure 11. Barrage balloons on power lines [38].

The physical and mathematical interpretation of this case is shown in Figures 12 and 13, respectively. As in Section 3.1, Figure 13 shows the raw data in green and the values to be calculated in yellow.

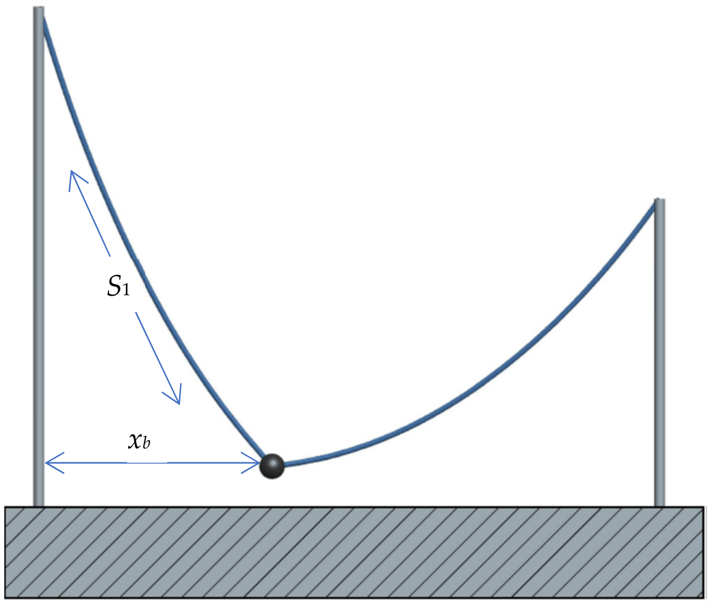


Figure 12. Physical model of a thread with a fixed bead.

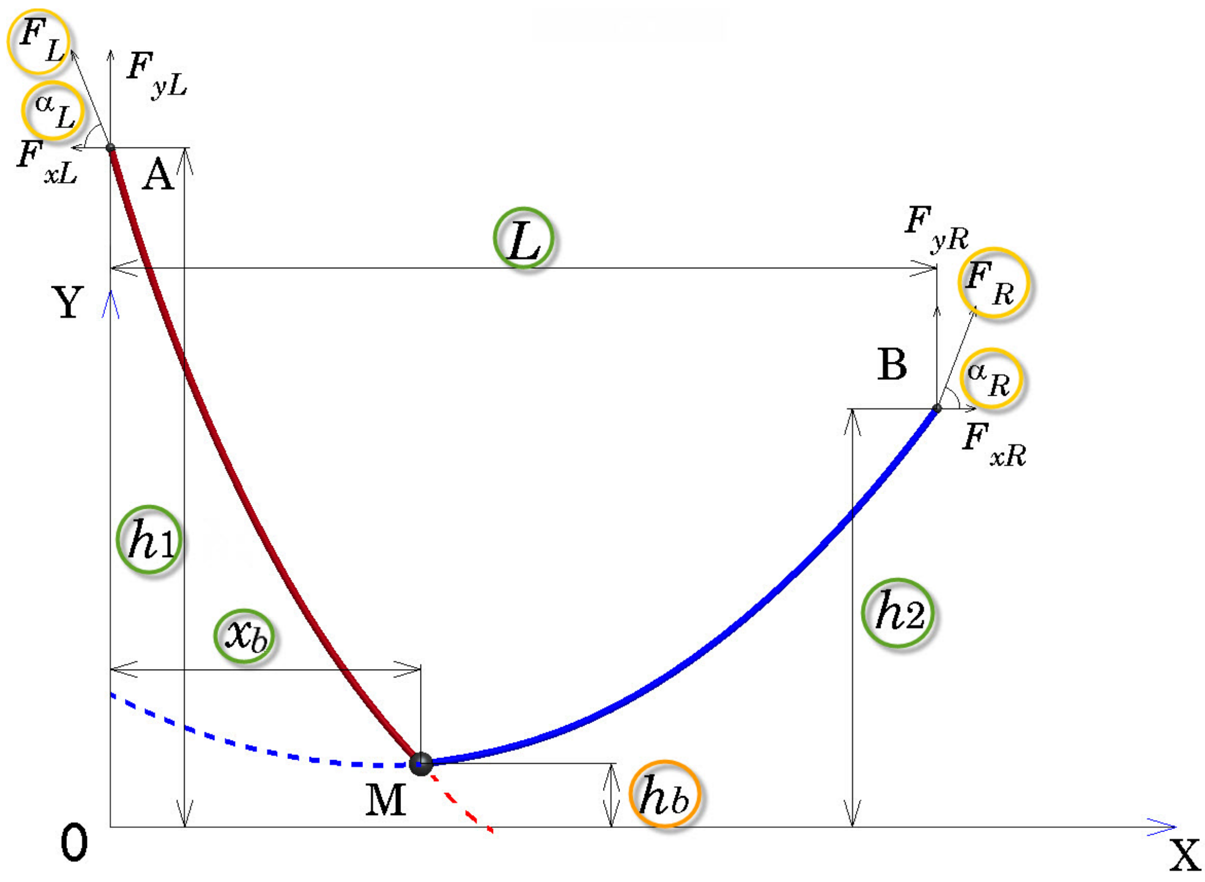


Figure 13. Mathematical model of a thread with a fixed bead.

Just as in Section 3.1, the problem is reduced to finding the parameters (a_L, x_{0L}, h_L), (a_R, x_{0R}, h_R) of two catenary curves converging to a point with abscissa x_b which is given in this case. Equation (6) also remains valid.

Since, in this case, the value of x_b is known, it is possible to dispense with a simpler mathematical apparatus than the search for a minimum, namely an algebraic system of six equations with six unknown parameters (a_L, x_{0L}, h_L), (a_R, x_{0R}, h_R).

In addition to four Equation (6), two more equations are added: condition (9) for the equality of parameters a_L, a_R and balance condition for vertical projection of tension forces F_{yL}, F_{yR}—the second equation of system (11) written using (13). Thus, the required system of equations is obtained:

$$\begin{cases} y(0, a_L, x_{0L}, h_L) = h_1, \\ y(L, a_R, x_{0R}, h_R) = h_2, \\ y(x_b, a_L, x_{0L}, h_L) = y(x_b, a_R, x_{0R}, h_R), \\ s(0, x_b, a_L, x_{0L}) + s(x_b, L, a_R, x_{0R}) = S, \\ a_L = a_R, \\ g \cdot m_c \cdot a_L \cdot |y'(0, a_L, x_{0L}, h_L)| + g \cdot m_c \cdot a_R \cdot |y'(L, a_R, x_{0R}, h_R)| = g \cdot m_c \cdot S + g \cdot M. \end{cases} \tag{14}$$

The listing of the program for solving system (14) using the *Find* operator is shown in Figure 14. The calculation uses the same input data as in Section 3.1 and x_b = 7 m.

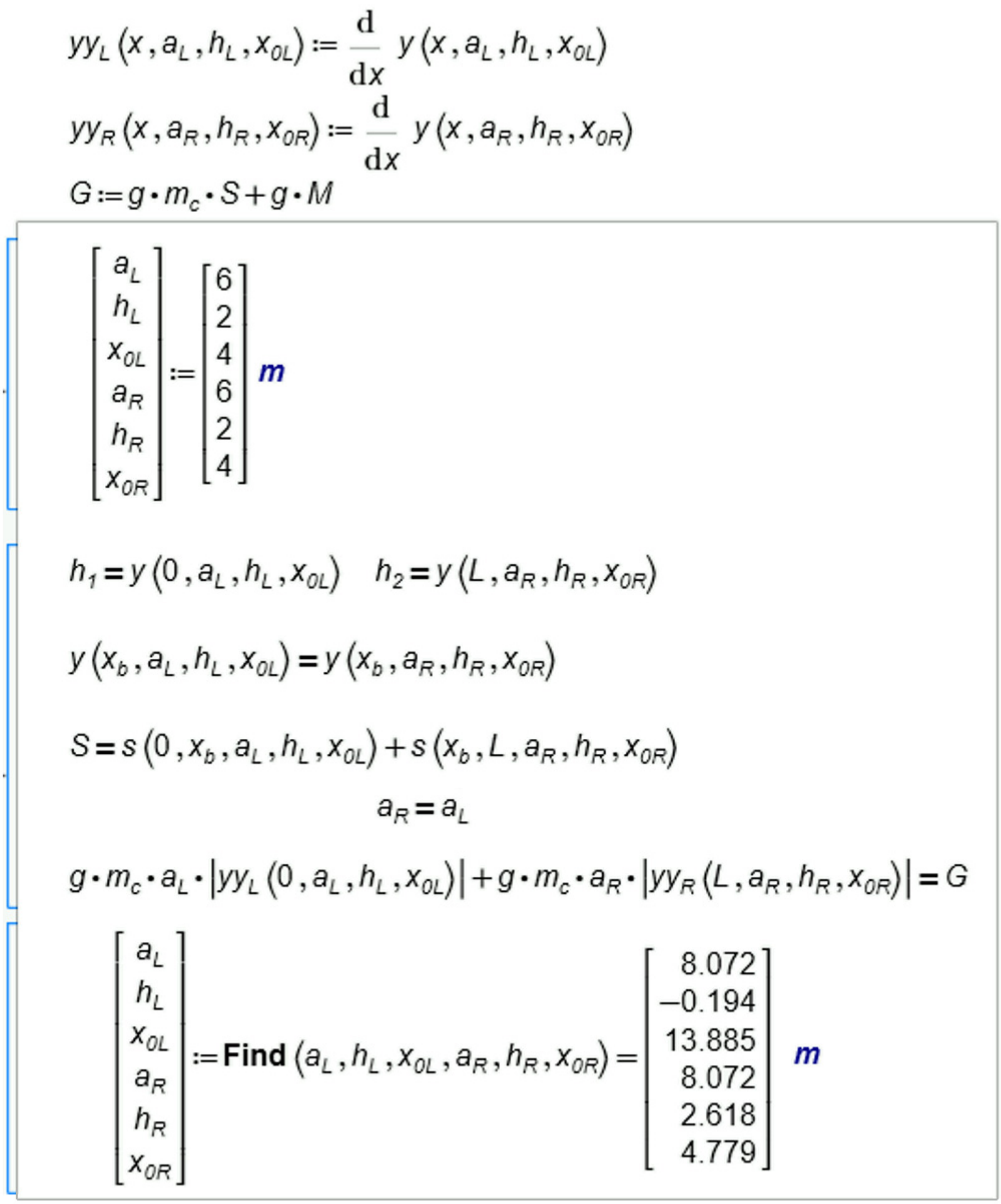


Figure 14. Searching for the solution in Mathcad for the case of a fixed bead.

To verify that the solution found is correct, the balance of the force moments at the point where the bead is fixed was calculated (the sum of the clockwise force moments with respect to the point where the bead is fixed is equal to the sum of the counterclockwise force moments) [34]:

$$\begin{aligned} &F_{xR}(h_2 - h_b) + g \cdot m_c \cdot s(x_b, L, a_R, x_{0R}) \cdot (x_{cb}(x_b, L, a_R, x_{0R}) - x_b) + F_{yL} \cdot x_b = \\ &= F_{xL} \cdot (h_1 - h_b) + g \cdot m_c \cdot s(0, x_b, a_L, x_{0L}) \cdot (x_b - x_{cb}(0, x_b, a_L, x_{0L})) + F_{yL} \cdot x_b = \\ &= 253.187\ \mathrm{N \cdot m} \end{aligned} \quad (15)$$

The graph of the thread shape with a fixed load position, based on the results of the calculation, is shown in Figure 15 (the red and blue dots in the graphs indicate the centers of gravity of both sections of thread).

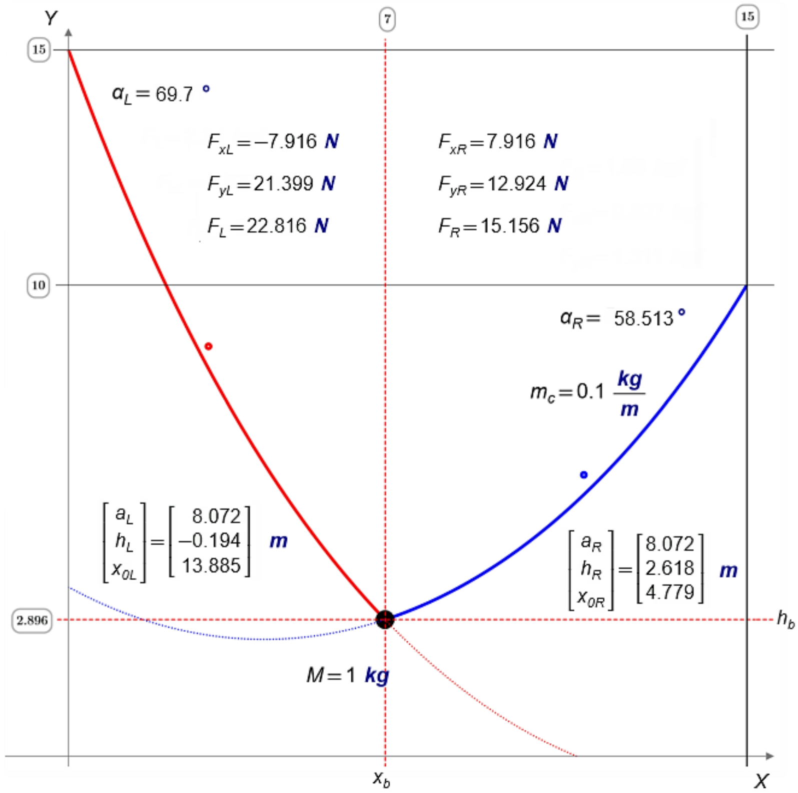


Figure 15. Solution of the problem with a fixed bead.

Analysis of Figure 15 shows that, in this case, there is no symmetry of the catenary sections as in the case of the sliding bead (Figure 9). The corresponding force diagram for the thread tension is shown in Figure 16.

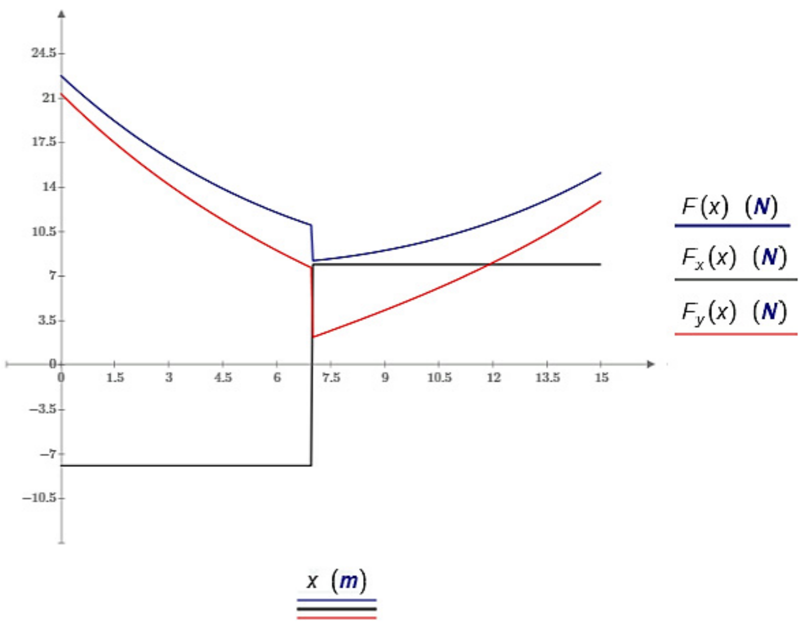


Figure 16. Tension force diagram for a fixed bead with mass $M = 1$ kg.

The next practical point to discuss with the students is as follows. Students may be asked: at what distance S_1 from the left end of the thread should the bead initially be secured so that, when the thread is suspended, it will be at the required distance from the left-hand support x_b (Figure 12)? The following simple and straightforward way for the students is suggested. Based on the above calculations (Figure 14), we can determine at what distance $S_1 = s(0, x_b, a_L, x_{0L})$ from the left end of the thread the bead will end up depending on the value of x_b. So, for example, with $x_b = 5$ m, $S_1 = 12.46$ m; with $x_b = 7$ m, $S_1 = 14.07$ m; with $x_b = 9$ m, $S_1 = 15.49$ m. Thus, it is possible iteratively to find the desired initial value of the bead fixing point.

So, we have analyzed the case of a suspended thread with a fixed load with the students.

Note that a similar educational problem was solved in [21], where a laboratory experiment to experimentally study the equilibrium of a cable under the action of a finite number of vertical, parallel, concentrated and external forces was considered. However, in [21], the emphasis was placed on experimental measurements of the parameters under the simplifying assumption of a negligibly small weight of the thread and, as a consequence, the shape of the thread as a "funicular polygon". Whereas, in this paper, a full mathematical model of the thread with a load was constructed without simplifications, and a solution close to the analytical one was numerically found.

3.3. Result of the Approach Tests

The educational STEM technology considered in this paper is an integral part of the experimental program conducted in recent decades at the Department of Theoretical Bases of Heat Engineering of National Research University Moscow Power Engineering Institute. Within the framework of the "Information Technology" academic discipline, physical and engineering problems demonstrating practical application of mathematical analysis and theoretical mechanics concepts are considered. Besides, the systems of algebraic and differential equations arising in the process of solving such problems are successfully solved by students with the help of computer mathematical packages.

In particular, the problem of a thread with a load considered in the work allows students to understand in practice what "derivative" and "integral" concepts are needed to solve the problem and to calculate the length of a curve and its center of gravity. Later in the course of theoretical mechanics, students also use this knowledge and skillset.

In addition, the application of Mathcad within the framework of STEM technologies makes it possible to demonstrate to students the solution of a rather complex nonlinear programming problem, avoiding the theoretical description of the mathematical apparatus for solving such problems (this would make it unacceptable for younger students to consider such a problem).

Such an opportunity to obtain a quick and clear (due to graphical material) result of the problem solution increases students' motivation and stimulates their interest in scientific activity.

The analysis of academic performance [39] in disciplines "Mathematical Analysis" and "Theoretical Mechanics" in groups of students taught using the proposed STEM technology showed that the performance increased by about 20% compared to the study groups with the traditional form of teaching. This proves the effectiveness of the proposed STEM technology.

3.4. Prospective Problems Related to the Problem to Be Solved

The considered problem has a number of promising directions both educationally and scientifically. Here are some of the possible tasks that could be considered with students.

1. Solve the problem of a thread with a load using a system of equations describing balances of forces and moments of forces, rather than through the minimization of potential energy.
2. Hang two (or three or more) fixed beads of different masses from the thread.

3. Find the ratio of the length of the thread without load and the distance between the suspension points (under conditions of equal height of the suspension supports), under which the force of tension of the thread at any point will not exceed the given limit value.
4. Calculate the cross-sectional area of the thread with and/or without load at which the specific tensile force of the thread is a constant value along its length.
5. Calculate the cross-sectional area of the thread without load at which the form of its sag is close to parabola (hyperbola, ellipse arc).
6. Replace the thread with a rubber band whose degree of stretching obeys Hooke's law. Take into account the change of linear mass of the rubber band during stretching.

Such problems can be considered within the framework of using the innovative STEM technologies described in this paper.

4. Conclusions

The paper demonstrates an innovative STEM technology of the educational process based on the problem of finding the shape of a thread with both a sliding and fixed load. The educational technology was designed for first-year engineering and physics and mathematics students.

The mathematical model of a thread with a load was explored without making simplifying assumptions such as a negligibly small mass of a thread, a thread with a small sag or a parabolic approximation of the shape of the thread.

The solution of the problem showed, in a student-friendly form, the interrelation of different fields of mathematics, the history of mathematics, theoretical mechanics, physics, as well as mathematical and computer modeling. Aspects to be discussed with students were explained in detail.

Diverse tools of computer mathematical package Mathcad Prime 6.0 were demonstrated.

Attention was given to the features of the numerical and analytical methods of the solution of algebraic equations in the Mathcad environment.

Calculation results were accompanied by graphic material, as well as practical examples of engineering structures.

Prospective problems related to the problem being solved, which may be considered within the framework of using STEM technologies in the educational process of engineering and physics and mathematics specialties, were indicated.

The considered STEM technology can be used for the professional training of teachers in the field of STEM technologies, which is directly related to engineering design [2].

The obtained refined mathematical model can be used in scientific and engineering calculations related to the design of suspended structures (a suspended structure), power lines, etc.

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References

1. How, M.-L. Advancing Multidisciplinary STEM Education with Mathematics for Future-Ready Quantum Algorithmic Literacy. *Mathematics* **2022**, *10*, 1146. [CrossRef]
2. Mohamad Hasim, S.; Rosli, R.; Halim, L.; Capraro, M.M.; Capraro, R.M. STEM Professional Development Activities and Their Impact on Teacher Knowledge and Instructional Practices. *Mathematics* **2022**, *10*, 1109. [CrossRef]
3. Touitou, I.; Schneider, B.; Krajcik, J. Incorporating Mathematical Thinking and Engineering Design into High School STEM Physics: A Case Study. In *Integrated Approaches to STEM Education. Advances in STEM Education*; Springer: Cham, Switzerland, 2020. [CrossRef]
4. Ochkov, V. *2^5 Problems for STEM Education*; Chapman and Hall/CRC: London, UK, 2020.
5. Brenner, A.; Shacham, M.; Cutlip, B. Applications of mathematical software packages for modeling and simulations in environmental engineering education. *Environ. Model. Softw.* **2005**, *20*, 1307–1313. [CrossRef]
6. Costa, S.A.; Danesi, M.; Martinovic, D. The Information Age, Mathematics, and Mathematics Education. In *Mathematics (Education) in the Information Age. Mathematics in Mind*; Costa, S.A., Danesi, M., Martinovic, D., Eds.; Springer: Cham, Switzerland, 2020; pp. 1–14. [CrossRef]
7. Smirnov, E.; Dvoryatkina, S.; Martyushev, N.; Shcherbatykh, S. Software Package to Support Students' Research Activities in the Hybrid Intellectual Environment of Mathematics Teaching. *Mathematics* **2023**, *11*, 952. [CrossRef]
8. Ellermeijer, T.; Tran, T. Stem, Inquiry Practices and Technology in Physics Education. In *Fundamental Physics and Physics Education Research*; Springer: Cham, Switzerland, 2021; pp. 127–161. [CrossRef]
9. Martinez, B.L.; Sweeder, R.D.; VandenPlas, J.R.; Herrington, D.G. Improving conceptual understanding of gas behavior through the use of screencasts and simulations. *Int. J. STEM Educ.* **2021**, *8*, 5. [CrossRef]
10. Wiley, D. Learning Objects: Difficulties and Opportunities. 2003. Available online: https://www.opencontent.org/docs/lo_do.pdf (accessed on 30 April 2023).
11. Feng, X.; Gao, S.; Song, Y.; Hu, Z.; Chen, L.; Liang, T. Static and Dynamic Analysis of Conductor Rail with Large Cross-Sectional Moment of Inertia in Rigid Catenary Systems. *Energies* **2023**, *16*, 1810. [CrossRef]
12. Liu, Z.; Song, Y.; Han, Y.; Wang, H.; Zhang, J.; Han, Z. Advances of research on high-speed railway catenary. *J. Mod. Transport.* **2018**, *26*, 1–23. [CrossRef]
13. Belendez, T.; Neipp, C.; Belendez, A. Static study of a homogeneous cable under the action of its own weight: Catenary. *Rev. Esp. Fis.* **2001**, *4*, 38–42.
14. Nedev, S. The catenary—An ancient problem on a computer screen. *Eur. J. Phys.* **2000**, *21*, 451–457. [CrossRef]
15. Qin, J.; Ba, Y.; Ding, Y.; Bai, J.; Zhang, H. Newton Iteration Method for Analysis of Suspension Cable. In Proceedings of the 2015 International Conference on Modeling, Simulation and Applied Mathematics, Phuket, Thailand, 23–24 August 2015; pp. 350–354. [CrossRef]
16. Qina, J.; Chen, J.; Qiao, L.; Wan, J.; Xia, Y. Catenary Analysis and Calculation Method of Track Rope of Cargo Cableway with Multiple Beads. *MATEC Web Conf.* **2016**, *82*, 01008. [CrossRef]
17. Hussein, H.; Gouttefarde, M.; Pierrot, F. Static Modeling of Sagging Cables with Flexural Rigidity and Shear Forces. In *Advances in Robot Kinematics 2018. ARK 2018. Springer Proceedings in Advanced Robotics*; Lenarcic, J., Parenti-Castelli, V., Eds.; Springer: Cham, Switzerland, 2019; Volume 8. [CrossRef]
18. Fatehi, M.H.; Eghtesad, M.; Amjadifard, R. Modelling and Control of an Overhead Crane System with a Flexible Cable and Large Swing Angle. *J. Low Freq. Noise Vib. Act. Control* **2014**, *33*, 395–410. [CrossRef]
19. Mareno, A.; English, L.Q. The stability of the catenary shapes for a hanging cable of unspecified length. *Eur. J. Phys.* **2009**, *30*, 97. [CrossRef]
20. Rosenblat, G.M. On the balance of an inextensible heavy thread on a cone or sphere. *Mech. Solids* **2021**, *56*, 349–367. [CrossRef]
21. Belendez, T.; Neipp, C.; Belendez, A. Cables Under Concentrated Beads: A Laboratory Project for an Engineering Mechanics Course. *Int. J. Eng. Educ.* **2003**, *2*, 272–281.
22. Ren, W.-X.; Huang, M.-G.; Hu, W.-H. A parabolic cable element for static analysis of cable structures. Engineering Computations. *Int. J. Comput. Aided Eng.* **2019**, *25*, 366–384. [CrossRef]
23. Hatibovic, A.; Kadar, P.; Morva, G. Comparison of the Length of the Catenary Curve and its Parabolic Approximation in the Span of an Overhead Line. In Proceedings of the 2020 IEEE 3rd International Conference and Workshop in Óbuda on Electrical and Power Engineering (CANDO-EPE), Budapest, Hungary, 18–19 November 2020. [CrossRef]
24. Merkin, D. *Introduction to Mechanics of Flexible Thread*; Nauka: Moscow, Russia, 1980. (In Russian)
25. Attaway, S.; Beverly, J.; Mills, M.; Weber, E.; Belknap, E.; Attaway, N. Track Lines and Guiding Lines: Forces Based on Point Loads in a Catenary. In Proceedings of the International Technical Rescue Symposium, Albuquerque, NM, USA, 8–10 November 2013. [CrossRef]
26. Pavlovych, L.; Bilous, I. Concept of Methods of Education, Their Classification. *Sci. Eur.* **2019**, *41*, 41. Available online: https://cyberleninka.ru/article/n/concept-of-methods-of-education-their-classification (accessed on 31 May 2023).
27. Lessani, A.; Suraya, A.; Abu Bakar, K. Comparison of new mathematics teaching methods with traditional method. *People Int. J. Soc. Sci.* **2017**, *3*, 1285–1297. [CrossRef]
28. Voskoglou, M. Comparing Teaching Methods of Mathematics at University Level. *Educ. Sci.* **2019**, *9*, 204. [CrossRef]

29. Thomas Vogel. «Cable Car». 21 August 2019. Available online: https://unsplash.com/photos/E5_D8ymwVpI (accessed on 30 May 2023).
30. Stillwell, J. *Mathematics and Its History*; Springer: Cham, Switzerland, 2010.
31. Smith, G. *Introductory Mathematics: Algebra and Analysis*; Springer: London, UK, 1998. [CrossRef]
32. Toklu, Y.; Bekdas, G.; Temur, R. Analysis of cable structures through energy minimization. *Struct. Eng. Mech.* **2017**, *62*, 749–758. [CrossRef]
33. Wu-jun, C.; Du, G.-S.; Ren, X.-Q. Minimization iteration procedure of potential energy and structural analysis for the tension cable-strut structure. *Chin. J. Comput. Mech.* **2010**, *27*, 1001–1005.
34. Bedford, A.; Fowler, W. *Engineering Mechanics: Statics*; Addison Wesley: Boston, MA, USA, 1996.
35. Ochkov, V.; Vasileva, I.; Orlov, K.; Chudova, J.; Tikhonov, A. Visualization in Mathematical Packages When Teaching with Information Technologies. *Mathematics* **2022**, *10*, 3413. [CrossRef]
36. Markou, A.; Ruan, G. Graphic statics: Projective funicular polygon. *Structures* **2022**, *41*, 1390–1396. [CrossRef]
37. Bladyko, Y. Mechanical Calculation of Flexible Wires of Overhead Lines with Aerial Barrage Balls in Different Climatic Regimes. *Energetika Proc. CIS High. Educ. Inst. Power Eng. Assoc.* **2019**, *62*, 24–36. [CrossRef]
38. Barrage Balloons on Power Lines. Available online: https://masterok.livejournal.com/6412928.html (accessed on 30 May 2023).
39. Beshah, B. Students' Performance Evaluation Using Statistical Quality Control. *Int. J. Sci. Adv. Technol.* **2012**, *2*, 75–79.