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On Extendibility of Evolution Subalgebras Generated by Idempotents

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Abstract: In the present paper, we examined the extendibility of evolution subalgebras generated by idempotents of evolution algebras. The extendibility of the isomorphism of such subalgebras to the entire algebra was investigated. Moreover, the existence of an evolution algebra generated by arbitrary idempotents was also studied. Furthermore, we described the tensor product of algebras generated by arbitrary idempotents and found the conditions of the tensor decomposability of four-dimensional S-evolution algebras. This paper's findings shed light on the field of algebraic structures, particularly in studying evolution algebras. By examining the extendibility of evolution subalgebras generated by idempotents, we provide insights into the structural properties and relationships within these algebras. Understanding the isomorphism of such subalgebras and their extension allows a deeper comprehension of the overall algebraic structure and its behaviour.

Keywords: evolution algebra; idempotent; subalgebra

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1. Introduction

In the theory of non-associative algebras, several classes, such as baric, evolution, Bernstein, train, stochastic, etc. algebras, are located in the intersection of abstract algebra and biology [1–3]. The study of these algebras has addressed several problems in population genetics theory [3]. We emphasize that the origin of population genetics problems first appeared in the work of Bernstein [4], where evolution operators describing genetic algebras were explored (see [3,5,6]). On the other hand, Tian [7–9] introduced different types of evolution algebras that have a dynamic nature. These kinds of algebras are non-associative (see [10]). Later on, evolution algebras appeared in several genetic law models [10–15]. Moreover, relations between evolution algebras and other branches of mathematics have studied in many papers (see, for example, [16–21]).

From the definition of evolution algebra, one can canonically associate weighted digraphs, which identify the algebra. Hence, algebraic tools are used to investigate certain features of digraphs [8,22,23]. In most investigations, evolution algebras were taken as nilpotent [22,24–30]. A few papers have been devoted to non-nilpotent evolution algebras [31–33]. Therefore, in [34], the exploration of Volterra evolution algebras was initiated; these are related to genetic Volterra algebras [35]. Furthermore, in [36], we studied S-evolution algebras which are more general than Volterra ones. In the mentioned paper, solvability, simplicity, semisimlicity, and the structure of enveloping algebras using the attached graph were carried out. In [37], we introduced an entropy of Markov evolution algebras, allowing us to treat their isomorphism with entropy. The reader is referred to [38] for a review on the recent development of evolution algebras.

Remarkably, a subalgebra and an ideal of a population's genetic algebra can be understood as a subpopulation and a dominating subpopulation concerning mating. On the other hand, to understand the structure of subalgebras, it is essential to explore idempotent elements of evolution algebras. In general, the existence of idempotent elements for a given evolution algebra is an open problem [39]. Therefore, in the present paper, we first studied some evolution algebras that have idempotent elements. Furthermore, the extendibility of subalgebras generated by idempotent elements of some S-evolution algebras was investigated. Consequently, the question of the extendibility of isomorphism was also established. One of the main aims of present paper was to construct algebras with idempotent elements and study when these kinds of algebras become evolution algebras; this construction allows the production of evolution algebras that have an idempotent element while, in general, this kind of evolution algebra may not exist (see [39]). Our research provides an advantage in studying evolution algebras by addressing the challenging task of their classification. Rather than approaching the classification problem directly, we focused on the isomorphism of subalgebras generated by idempotents. This approach simplifies the classification process by leveraging the isomorphism of subalgebras as a means to understand the isomorphism of the entire algebra. By examining the isomorphism of subalgebras, we gained insights into the overall structure and properties of the algebra, making the classification task more manageable and efficient.

The current paper is organized as follows. Section 2 provides some basic properties of S-evolution algebras. Section 3 deals with idempotent elements of some S-evolution algebras. Furthermore, the extendibility of subalgebras generated by those idempotent elements was examined. In Section 4, we construct low dimensional algebras whose basis is idempotent elements and investigate their evolution algebraic structure. Here, we stress that the obtained algebras do not belong to the S-evolution algebra class. Furthermore, it is not solvable and not nilpotent. Finally, Section 5 is devoted to the specific properties of the tensor product of S-evolution algebras.

2. S-Evolution Algebras

Assume that $\mathcal{E}$ is a vector space over a field $\mathbb{K}$ equipped with binary operation $\cdot$ and a basis $B := \{e_1, e_2, \ldots, e_n\}$. A triple $(\mathcal{E}, \cdot, B)$ is called an *evolution algebra* if $e_i \cdot e_j = 0,\ i \neq j$, $e_i \cdot e_i = \sum_{k=1}^{n} a_{ik} e_k,\ i \geq 1$. The collection B is referred to as a *natural basis*. Moreover, the corresponding matrix $A = (a_{ij})_{i,j=1}^{n}$ is called a *structural matrix* of $\mathcal{E}$, with respect to B.

From the definition of evolution algebra, one infers that it is commutative. Moreover, one has $rankA = \dim(\mathcal{E} \cdot \mathcal{E})$. In what follows, unless otherwise stated, the field $\mathbb{K}$ is assumed to be algebraically closed with zero characteristic.

We recall that [36] a matrix $A = (a_{ij})_{i,j=1}^{n}$ is said to be *S-matrix* if

(i) $a_{ii} = 0$ for every $i \in \{1, \ldots, n\}$;
(ii) $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$ $(i \neq j)$.

An evolution algebra $\mathcal{E}$ is called an *S-evolution algebra* if its structural matrix is an S-matrix. If a structural matrix is skew-symmetric, then the corresponding evolution algebra is called *Lotka–Volterra (or Volterra) evolution algebra* [34,40]. One can see that any Lotka–Volterra algebras form a class of S-evolution algebras. This was one of the motivation behind introducing S-evolution algebras [36].

By $\mathfrak{E}_2(n)$ and $\mathfrak{E}_2^*(n)$ (here n is even), we will denote the set of all S-evolution algebras whose structural matrices have the following form, respectively:

$$A = \begin{pmatrix} \begin{bmatrix} 0 & a_1 \\ a_2 & 0 \end{bmatrix} & 0 & \cdots & 0 \\ 0 & \begin{bmatrix} 0 & a3 \\ a_4 & 0 \end{bmatrix} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \begin{bmatrix} 0 & a_{n-1} \\ a_n & 0 \end{bmatrix} \end{pmatrix} \tag{1}$$

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix} \\ 0 & 0 & \cdots & \begin{bmatrix} 0 & b_3 & \\ b_4 & 0 & 0 \end{bmatrix} & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \begin{bmatrix} 0 & b_{n-3} \\ b_{n-2} & 0 \end{bmatrix} & \cdots & 0 & 0 \\ \begin{bmatrix} 0 & b_{n-1} \\ b_n & 0 \end{bmatrix} & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{2}$$

where $a_k \neq 0$, $b_k \neq 0$ for all $k \in \{1, \ldots, n\}$.

Theorem 1. *Let $\mathcal{E}_1 = \langle e_k : 1 \leq k \leq n \rangle$ and $\mathcal{E}_2 = \langle f_k : 1 \leq k \leq n \rangle$ belong to $\mathfrak{E}_2(n)$. Then, $\mathcal{E}_1 \cong \mathcal{E}_2$ if and only if*

$$\begin{aligned} f_{2k-1} &= A_k C e_{\pi(2k-1)} \\ f_{2k} &= B_k \bar{C} e_{\pi(2k)}; \end{aligned}$$

here, $\pi \in S_n$ such that if $\pi(k) = m$ then $\pi(k+1) = m+1$, and

$$A_k = \sqrt[3]{\left(\frac{b_{2k}}{a_{\pi(2k)}}\right)\left(\frac{b_{2k-1}}{a_{\pi(2k-1)}}\right)^2}, \tag{3}$$

$$B_k = \sqrt[3]{\left(\frac{b_{2k}}{a_{\pi(2k)}}\right)^2\left(\frac{b_{2k-1}}{a_{\pi(2k-1)}}\right)}, \tag{4}$$

$$C \in \{1, \lambda, \bar{\lambda}\},\ \lambda = \frac{-1+\sqrt{-3}}{2}, \tag{5}$$

where $1 \leq k \leq n/2$.

Proof. Assume that the mapping $\psi : \mathcal{E}_1 \to \mathcal{E}_2$ is an isomorphism, then

$$f_i = \sum_{k=1}^{n} \psi_{ik} e_k.$$

For any $i \neq j$, one has

$$f_i f_j = \sum_{k=1}^{n} \psi_{ik} \psi_{jk} e_k^2 = 0.$$

Due to $e_k^2 \neq 0$ for all $1 \leq k \leq n$, and e_k^2 are linearly independent, we infer that $\psi_{ik}\psi_{jk} = 0$. Let $\pi \in S_n$. Then,

$$f_m = \psi_{m\pi(m)} e_{\pi(m)}.$$

Hence,

$$\begin{cases} f_{2k-1} &= \psi_{2k-1,\pi(2k-1)}e_{\pi(2k-1)} \\ f_{2k} &= \psi_{2k,\pi(2k)}e_{\pi(2k)}. \end{cases} \tag{6}$$

Let us consider f_{2k-1}^2 and f_{2k}^2. Then,

$$\begin{cases} f_{2k-1}^2 &= a_{\pi(2k-1)}\psi_{2k-1,\pi(2k-1)}^2 e_{\pi(2)} \\ f_{2k}^2 &= a_{\pi(2k)}\psi_{2k,\pi(2k)}^2 e_{\pi(2k-1)}. \end{cases} \tag{7}$$

On the other hand,

$$\begin{cases} f_{2k-1}^2 &= b_{2k-1}\psi_{2k,\pi(2k)}^2 e_{\pi(2k)} \\ f_{2k}^2 &= b_{2k}\psi_{2k-1,\pi(2k-1)}^2 e_{\pi(2k-1)}. \end{cases} \tag{8}$$

Comparing the equations of systems (7) and (8), one finds

$$\begin{cases} a_{\pi(2k-1)}\psi_{2k-1,\pi(2k-1)}^2 &= b_{2k-1}\psi_{2k,\pi(2k)}^2 \\ a_{\pi(2k)}\psi_{2k,\pi(2k)}^2 &= b_{2k}\psi_{2k-1,\pi(2k-1)}^2. \end{cases} \tag{9}$$

Solving the system (9) for $\psi_{2k-1,\pi(2k-1)}$, one gets

$$\left(\frac{a_{\pi(2k)}}{b_{2k}}\right)\left(\frac{a_{\pi(2k-1)}}{b_{2k-1}}\right)^2 \psi_{2k-1,\pi(2k-1)}^3 - 1 = 0. \tag{10}$$

The solutions of (10) are $\psi_{2k-1,\pi(2k-1)} = CA_k$ and $\psi_{2k,\pi(2k)} = B_k\bar{C}$, where A_k, B_k and C are given by (3)–(5).

Conversely, the isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2$, can be performed by the following change of basis

$$\begin{aligned} f_{2k-1} &= A_k C e_{\pi(2k-1)} \\ f_{2k} &= B_k \bar{C} e_{\pi(2k)}. \end{aligned}$$

This completes the proof. □

3. Idempotents of S-Evolution Algebras

In this section, we describe the set of all idempotent elements of algebras belonging to $\mathfrak{E}_2(n)$. Recall that an element $x \in \mathcal{E}$ is called *idempotent* if $x \cdot x = x$.

Theorem 2. *Let $\mathcal{E}$ belong to $\mathfrak{E}_2(n)$. Then, the idempotent elements of $\mathcal{E}$ have the following form:*

$$p = \sum_{i=1}^{\frac{n}{2}} \frac{K_i}{\sqrt[3]{a_{2i-1}^2 a_{2i}}} e_{2i-1} + \sum_{i=1}^{\frac{n}{2}} \frac{\bar{K}_i}{\sqrt[3]{a_{2i}^2 a_{2i-1}}} e_{2i},$$

where $K_i \in \{0, 1, \lambda, \bar{\lambda}\}$.

Proof. To find the idempotent, we have to solve $p^2 = p$, let $p = \sum_{i=1}^{n} \alpha_i e_i$; then, one can rewrite p as follows:

$$p = \sum_{i=1}^{\frac{n}{2}} \alpha_{2i-1}e_{2i-1} + \sum_{i=1}^{\frac{n}{2}} \alpha_{2i}e_{2i}.$$

Consider

$$p^2 = \sum_{i=1}^{\frac{n}{2}} \alpha_{2i-1}^2 e_{2i-1}^2 + \sum_{i=1}^{\frac{n}{2}} \alpha_{2i}^2 e_{2i}^2 = \sum_{i=1}^{\frac{n}{2}} \alpha_{2i-1}^2 a_{2i-1} e_{2i} + \sum_{i=1}^{\frac{n}{2}} \alpha_{2i}^2 a_{2i} e_{2i-1}.$$

Comparing the last two equations, we have the following system:

$$\alpha_{2i-1}^2 a_{2i-1} = \alpha_{2i}, \ \ \alpha_{2i}^2 a_{2i} = \alpha_{2i-1}.$$

Solving this system, we have the following cubic equation, $a_{2i}a_{2i-1}^2\alpha_{2i-1}^3 - 1 = 0$. Then, one can easily find $\alpha_{2i-1} = \frac{K_i}{\sqrt[3]{a_{2i-1}^2 a_{2i}}}$, $\alpha_{2i} = \frac{\bar{K}_i}{\sqrt[3]{a_{2i}^2 a_{2i-1}}}$, where $K_i \in \{0, 1, \lambda, \bar{\lambda}\}$, where, as before, $\lambda = \frac{-1+\sqrt{-3}}{2}$. Hence,

$$p = \sum_{i=1}^{\frac{n}{2}} \frac{K_i}{\sqrt[3]{a_{2i-1}^2 a_{2i}}} e_{2i-1} + \sum_{i=1}^{\frac{n}{2}} \frac{\bar{K}_i}{\sqrt[3]{a_{2i}^2 a_{2i-1}}} e_{2i},$$

where $K_i \in \{0, 1, \lambda, \bar{\lambda}\}$. $\square$

Remark 1. *We emphasize that, for each i, and any choice of K_i in the expression*

$$p = \sum_{i=1}^{\frac{n}{2}} \frac{K_i}{\sqrt[3]{a_{2i-1}^2 a_{2i}}} e_{2i-1} + \sum_{i=1}^{\frac{n}{2}} \frac{\bar{K}_i}{\sqrt[3]{a_{2i}^2 a_{2i-1}}} e_{2i},$$

one can get the idempotent element.

The following corollary describes the idempotent elements of an S-evolution algebra whose structural matrix is given as a one block matrix in (1).

Corollary 1. *Let $\mathcal{E}_1$ be an S-evolution algebra whose structural matrix is one of the block matrix in (1) say k. Then the idempotent elements of $\mathcal{E}_1$ are as follows:*

$$\begin{aligned} p_1^{(k)} &= \frac{1}{\sqrt[3]{(a_{2k})(a_{2k-1})^2}} e_{2k-1} + \frac{1}{\sqrt[3]{(a_{2k})^2(a_{2k-1})}} e_{2k}, \\ p_2^{(k)} &= \frac{\lambda}{\sqrt[3]{(a_{2k})(a_{2k-1})^2}} e_{2k-1} + \frac{\bar{\lambda}}{\sqrt[3]{(a_{2k})^2(a_{2k-1})}} e_{2k}, \\ p_3^{(k)} &= \frac{\bar{\lambda}}{\sqrt[3]{(a_{2k})(a_{2k-1})^2}} e_{2k-1} + \frac{\lambda}{\sqrt[3]{(a_{2k})^2(a_{2k-1})}} e_{2k}, \end{aligned}$$

where, as before, $\lambda = \frac{-1+\sqrt{-3}}{2}$, $1 \le k \le n/2$.

The proof of the corollary can be readily deduced from Theorem (2) and Remark (1).

Remark 2. *Here, we stress the following points:*

(i) *The idempotents $p_{k_i}^{(i)}$ and $p_{k_j}^{(j)}$ $(i \neq j)$ are orthogonal, i.e., $p_{k_i}^{(i)} p_{k_j}^{(j)} = 0$ for any $k_i, k_j \in \{1, 2, 3\}$;*

(ii) *For each $k \in \{1, \ldots, n/2\}$* any pair of *$\{p_1^{(k)}, p_2^{(k)}, p_3^{(k)}\}$ is linearly independent;*

(iii) *For each $k \in \{1, \ldots, n/2\}$, the set $\{p_1^{(k)}, p_2^{(k)}, p_3^{(k)}\}$ is linearly dependent;*

(iv) *Each set $\{p_1^{(k)}\}_{k=1}^{n/2}$, $\{p_2^{(k)}\}_{k=1}^{n/2}$, $\{p_3^{(k)}\}_{k=1}^{n/2}$ itself consists of linearly independent elements.*

Let us consider a subalgebra generated by those orthogonal idempotent elements of evolution algebra $\mathcal{E} \in \mathfrak{E}_2(n)$. To define it, let us pick a collection $\kappa = \{k_i \in \{1, 2, 3\} : 1 \le i \le \frac{n}{2}\}$. Now, define

$$M_{\mathcal{E}}^{(\kappa)} := alg\Big\langle p_{k_i}^{(i)} : \ 1 \le i \le \frac{n}{2} \Big\rangle. \tag{11}$$

Here, $p_{k_i}^{(i)}$ is an idempotent element corresponding to the block matrix i in (1). We note that from each block we take only one idempotent element. Furthermore, the number of subalgebras that can be constructed as in (11) equal $3^{\frac{n}{2}}$. However, the following proposition shows that any different choice of such subalgebras is isomorphic.

Proposition 1. *Let $\kappa = \{k_i \in \{1,2,3\} : 1 \le i \le \frac{n}{2}\}$, $\bar{\kappa} = \{k_i \in \{1,2,3\} : 1 \le i \le \frac{n}{2}\}$ be two different collections. Assume that $M_{\mathcal{E}}^{(\kappa)}$ and $M_{\mathcal{E}}^{(\bar{\kappa})}$ are the corresponding subalgebras defined by (11). Then, $M_{\mathcal{E}}^{(\kappa)} \cong M_{\mathcal{E}}^{(\bar{\kappa})}$.*

The proof is straightforward. Hence, it is omitted.

Due to Proposition 1, in what follows, we will consider the following subalgebra:

$$M_{\mathcal{E}}^{(\mathbf{1})} := alg\left\langle p_1^{(i)} : 1 \le i \le n/2 \right\rangle, \tag{12}$$

where $\mathbf{1} = (\underbrace{1,\ldots,1}_{\frac{n}{2}})$.

Theorem 3. *Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{E}_2(n)$, such that $(\mathcal{E}_1, \{e_i\}_{i=1}^n)$ and $(\mathcal{E}_2, \{f_i\}_{i=1}^n)$. Assume that $M_{\mathcal{E}_1}^{(\mathbf{1})} = alg\left\langle p_1^{(i)} : 1 \le i \le n/2 \right\rangle$, and $M_{\mathcal{E}_2}^{(\mathbf{1})} = alg\left\langle q_1^{(j)} : 1 \le j \le n/2 \right\rangle$ are two subalgebras of $\mathcal{E}_1$ and $\mathcal{E}_2$ defined by (12), respectively. Then, any isomorphism from $M_{\mathcal{E}_1}^{(\mathbf{1})}$ into $M_{\mathcal{E}_2}^{(\mathbf{1})}$ can be extended to an isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2$.*

Proof. We first notice that idempotent elements have the following forms (see Corollary 1):

$$\begin{aligned}
p_1^{(i)} &= \frac{1}{\sqrt[3]{a_{2i-1}^2 a_{2i}}} e_{2i-1} + \frac{1}{\sqrt[3]{a_{2i-1} a_{2i}^2}} e_{2i} \\
q_1^{(j)} &= \frac{1}{\sqrt[3]{b_{2j-1}^2 b_{2j}}} f_{2j-1} + \frac{1}{\sqrt[3]{b_{2j-1} b_{2j}^2}} f_{2j}.
\end{aligned}$$

Since $M_{\mathcal{E}_1}^{(\mathbf{1})}$ and $M_{\mathcal{E}_2}^{(\mathbf{1})}$ are isomorphic, there exists a bijective mapping, say ϕ from $M_{\mathcal{E}_1}^{(\mathbf{1})}$ onto $M_{\mathcal{E}_1}^{(\mathbf{1})}$, which can be defined by

$$\phi(q_1^{(j)}) = \sum_{k=1}^{\frac{n}{2}} \alpha_{jk} p_1^{(k)}.$$

Now, for any $s \neq r$, we have $\phi(q_1^{(s)} q_1^{(r)}) = 0$, which implies that $\sum_{k=1}^{\frac{n}{2}} \alpha_{sk}\alpha_{rk} p_1^{(k)} = 0$. The linear independence of the set $\{p_1^{(k)} : 1 \le k \le \frac{n}{2}\}$ yields that $\alpha_{sk}\alpha_{rk} = 0$ for any $1 \le s \neq r \le \frac{n}{2}$. Hence, $\phi(q_1^{(s)}) = \alpha_{s\sigma(s)} p_1^{(\sigma(s))}$, for some permutation $\sigma \in S_{\frac{n}{2}}$. On the other hand, the equality $\phi((q_1^{(s)})^2) = (\phi(q_1^{(s)}))^2$ implies that $\alpha_{s\sigma(s)} = 1$. Consequently, $\phi(q_1^{(s)}) = p_1^{\sigma(s)}$. Let us consider

$$\phi(q_1^{(j)}) = p_1^{\sigma(j)} = \frac{1}{\sqrt[3]{a_{2\sigma(j)-1}^2 a_{2\sigma(j)}}} e_{2\sigma(j)-1} + \frac{1}{\sqrt[3]{a_{2\sigma(j)-1} a_{2\sigma(j)}^2}} e_{2\sigma(j)}.$$

Define a permutation π of $\{1,2,\ldots,n\}$ by

$$\pi(2j-1) = 2\sigma(j) - 1, \ \pi(2j) = 2\sigma(j).$$

Hence,

$$\phi(q_1^{(j)}) = p_1^{\sigma(j)} = \frac{1}{\sqrt[3]{a^2_{\pi(2j-1)}a_{\pi(2j)}}}e_{\pi(2j-1)} + \frac{1}{\sqrt[3]{a_{\pi(2j-1)}a^2_{\pi(2j)}}}e_{\pi(2j)}.$$

By Theorem 1, the isomorphism, say ψ between $\mathcal{E}_1$ and $\mathcal{E}_2$, can be chosen as follows:

$$\begin{aligned}\psi(f_{2j-1}) &= Ae_{\pi(2j-1)}\\ \psi(f_{2j}) &= Be_{\pi(2j)},\end{aligned}$$

where, as before,

$$A_j = \sqrt[3]{\left(\frac{b_{2j}}{a_{\pi(2j)}}\right)\left(\frac{b_{2j-1}}{a_{\pi(2j-1)}}\right)^2},$$

$$B_j = \sqrt[3]{\left(\frac{b_{2j}}{a_{\pi(2j)}}\right)^2\left(\frac{b_{2j-1}}{a_{\pi(2j-1)}}\right)}.$$

Moreover, we have

$$\begin{aligned}\psi(q_1^{(j)}) &= \frac{1}{\sqrt[3]{b^2_{2j-1}b_{2j}}}\psi(f_{2j-1}) + \frac{1}{\sqrt[3]{b_{2j-1}b^2_{2j}}}\psi(f_{2j})\\ &= \frac{1}{\sqrt[3]{b^2_{2j-1}b_{2j}}}A_je_{2\pi(j)-1} + \frac{1}{\sqrt[3]{b_{2j-1}b^2_{2j}}}B_je_{2\pi(j)}\\ &= \frac{1}{\sqrt[3]{a^2_{\pi(2j-1)}a_{\pi(2j)}}}e_{\pi(2j-1)} + \frac{1}{\sqrt[3]{a_{\pi(2j-1)}a^2_{\pi(2j)}}}e_{\pi(2j)}\\ &= \frac{1}{\sqrt[3]{a^2_{2\sigma(j)-1}a_{2\sigma(j)}}}e_{2\sigma(j)-1} + \frac{1}{\sqrt[3]{a_{2\sigma(j)-1}a^2_{2\sigma(j)}}}e_{2\sigma(j)}\\ &= p_1^{(\sigma(j))},\end{aligned}$$

which yields $\phi = \psi$ for all $x \in M^{(\mathbf{1})}_{\mathcal{E}_1}$; this completes the proof. $\square$

In fact, for a linear subspace $\mathcal{E}_1$ of an evolution algebra $\mathcal{E}$, the notion of a subalgebra of $\mathcal{E}$ is different than that of the usual one, since the definition of the evolution algebra depends on a natural basis [39].

Let $\mathcal{E}$ be an evolution algebra and $\mathcal{E}_1$ be a subspace of $\mathcal{E}$. If $\mathcal{E}_1$ has a natural basis $\{e_i : i \in \Lambda_1\}$, which can be extended to a natural basis $\{e_j : j \in \Lambda\}$ of $\mathcal{E}$, then $\mathcal{E}_1$ is called an *evolution subalgebra*, where Λ_1 and Λ are index sets and Λ_1 is a subset of Λ (see [9], for details).

Define

$$\mathrm{Supp}(p) := \left\{k : \alpha_k \neq 0,\ p = \sum_{k=1}^{n}\alpha_k e_k\right\}.$$

In what follows, for the sake of simplicity, we always assume that $\dim(\mathcal{E}) = \dim(\mathcal{E}^2)$. It is important to note that, if this condition is not satisfied, i.e., if the dimension of $\mathcal{E}$ is not equal to the dimension of $\mathcal{E}^2$, then several cases may arise and it becomes difficult to cover all of these cases comprehensively. The analysis and classification of evolution subalgebras, as well as the extendibility of isomorphisms, become more complex in such scenarios.

Proposition 2. *Let $\mathcal{E}$ be an evolution algebra with $dim(\mathcal{E}) = \dim(\mathcal{E}^2)$. Let p and q be two orthogonal idempotents. Then, $Supp(p) \cap Supp(q) = \emptyset$.*

Proof. Due to $dim(\mathcal{E}) = \dim(\mathcal{E}^2)$, the set $\{e_1^2, e_2^2, \ldots, e_n^2\}$ is linearly independent. Suppose that there is $s \in \text{Supp}(p) \cap \text{Supp}(q)$. Then, from $pq = 0$ together with

$$p = \sum_{k \in \text{Supp}(p)} \alpha_k e_k, \quad q = \sum_{m \in \text{Supp}(q)} \alpha_m e_m,$$

one finds $\alpha_s^2 e_s^2 = 0$, which implies $\alpha_s^2 = 0$. Hence, we get a contradiction with our assumption. □

Example 1. *Let $\mathcal{E}$ be a four dimensional S-evolution algebra that has the structural matrix*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0. \end{pmatrix}$$

The idempotents of this algebra are (see Theorem 1):

$$p_1 = e_1 + e_2, \ p_2 = \lambda e_1 + \bar{\lambda} e_2, p_3 = \bar{\lambda} e_1 + \lambda e_2$$

$$p_4 = e_3 + e_4, \ p_5 = \lambda e_3 + \bar{\lambda} e_4, p_6 = \bar{\lambda} e_3 + \lambda e_4.$$

Let us consider subalgebra $M_{\mathcal{E}}$, generated by orthogonal idempotents. We may choose $M_{\mathcal{E}}$ as

$$M_{\mathcal{E}} = alg\langle p_1, p_4 \rangle.$$

Now, we are going to show that this algebra has a natural basis. Let $\{w_1, w_2\}$ be a natural basis of $M_{\mathcal{E}}$; then,

$$\begin{aligned} w_1 &= Ap_1 + Bp_4 \\ w_2 &= Cp_1 + Dp_4. \end{aligned}$$

From $w_1 w_2 = 0$, we have $ACp_1 + BDp_2 = 0$. Hence, we may assume that

$$w_1 = Ap_1, \ w_2 = Dp_4.$$

Next, suppose that $\{w_1, w_2, w_3, w_4\}$ is a natural basis of $\mathcal{E}$; then,

$$\begin{aligned} w_1 &= Ap_1 \\ w_2 &= Dp_4 \\ w_3 &= \gamma_{11} e_1 + \gamma_{12} e_2 + \gamma_{13} e_3 + \gamma_{14} e_4 \\ w_4 &= \gamma_{21} e_1 + \gamma_{22} e_2 + \gamma_{23} e_3 + \gamma_{24} e_4, \end{aligned}$$

with $det(F) \neq 0$ where

$$F = \begin{bmatrix} A & A & 0 & 0 \\ 0 & 0 & D & D \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24}. \end{bmatrix}$$

However, $w_1 w_3 = 0$ yields that $\gamma_{11} = \gamma_{12} = 0$ and $w_2 w_3 = 0$ implies $\gamma_{13} = \gamma_{14} = 0$. So, $w_3 = 0$. Hence, $\{w_1, w_2, w_3, w_4\}$ is not a natural basis of $\mathcal{E}$, which means that $\{w_1, w_2\}$ is not an extendible basis. Thus, $M_{\mathcal{E}}$ is not an extendible evolution subalgebra.

In what follows, by $M_{\mathcal{E}}$ we denote the subalgebra of $\mathcal{E}$ generated by all orthogonal idempotents of $\mathcal{E}$. It is important to note that $\mathcal{E}$ is an evolution algebra, which may or may not be an S-evolution algebra.

Theorem 4. *Let $(\mathcal{E}, \{e_i\}_{i=1}^n)$ be an evolution algebra and $M_{\mathcal{E}}$ be the subalgebra generated by all orthogonal idempotents of $\mathcal{E}$, then the following statements hold true:*

(i) If $dim(M_{\mathcal{E}}) = dim(\mathcal{E})$, then $M_{\mathcal{E}}$ is a trivially extendible evolution subalgebra;
(ii) If $dim(M_{\mathcal{E}}) < dim(\mathcal{E})$, and for each idempotent $p \in M_{\mathcal{E}}$ one has $|Supp(p)| = 1$, then $M_{\mathcal{E}}$ is an extendible evolution subalgebra;
(iii) If $dim(M_{\mathcal{E}}) < dim(\mathcal{E})$ and $|Supp(p)| > 1$ for some idempotent $p \in M_{\mathcal{E}}$, then $M_{\mathcal{E}}$ is not an extendible evolution subalgebra.

Proof. Assume that $M_{\mathcal{E}} = Alg\langle p_1, p_2, \dots p_m\rangle$.

(i) Since $m = dim(\mathcal{E})$, then one can find that $\{f_1, f_2, \dots f_m\}$ with $f_i = p_i$ is an extendible natural basis;

(ii) Assume that $dim(M_{\mathcal{E}}) < dim(\mathcal{E})$. We need to show that $M_{\mathcal{E}}$ has a natural basis. Assume that $\{f_1, f_2, \dots, f_m\}$ is a natural basis for $M_{\mathcal{E}}$, then $f_i = \sum_{k=1}^m \alpha_{ik} p_k$. By $f_i f_j = 0$, one finds $\alpha_{ik}\alpha_{jk} = 0$ for any $i \neq j$, Hence, we may assume without loss of generality that $f_i = \alpha_{ii} p_i$ is a natural basis of $M_{\mathcal{E}}$. Next, we show that $\{f_1, f_2, \dots, f_m, \dots, f_n\}$ is a natural basis of $\mathcal{E}$, where

$$\begin{cases} f_i = \alpha_{ii} p_i,\ 1 \le i \le m \\ f_j = \sum_{k=1}^n \beta_{jk} e_k,\ m+1 \le j \le n. \end{cases} \tag{13}$$

Consider $f_i f_j = 0$, where $1 \le i \le m, m+1 \le j \le n$. Then, $f_j = \sum_{k \notin \mathrm{Supp}(f_i)} \beta_{jk} e_k$, $m+1 \le j \le n$. Then, $|\mathrm{Supp}(f_j)| \le n - m$. Now, for any r, s with $r \neq s$, $m+1 \le r, s \le n$, one has

$$f_r f_s = \sum_{l \notin \mathrm{Supp}(f_i)} \beta_{rl}\beta_{sl} e_l^2 = 0,$$

which yields $\beta_{rl}\beta_{sl} = 0$ for any $m+1 \le r \neq s \le n$. Therefore, for any $m+1 \le j \le n$, we get $|\mathrm{Supp}(f_j)| = 1$. Hence, $\{f_1, f_2, \dots, f_m, \dots, f_n\}$ is a natural basis of $\mathcal{E}$;

(iii) By (ii) $\{f_1, f_2, \dots, f_m\}$ is a natural basis of $M_{\mathcal{E}}$. Consider $f_i f_j = 0$, then

$$f_j = \sum_{k \notin \mathrm{Supp}(f_i)} \beta_{jk} e_k,\ m+1 \le j \le n.$$

So,

$$|\mathrm{Supp}(f_j)| = n - |\cup_{i=1}^m \mathrm{Supp}(f_i)| < n - m.$$

Then $m+1 \le l \le n$ exist such that $|\mathrm{Supp}(f_l)| = 0$; this means that $f_l = 0$ for some $m+1 \le l \le n$. Hence, $\{f_1, f_2, \dots, f_m\}$ is not an extendible basis. □

Theorem 5. *Let $(\mathcal{E}_1, \{e_i\}_{i=1}^n)$ and $(\mathcal{E}_1, \{f_i\}_{i=1}^n)$ be two evolution algebras. Assume that $M_{\mathcal{E}_1} = alg\langle p_i : 1 \le i \le m\rangle$, $M_{\mathcal{E}_2} = alg\langle q_i : 1 \le i \le m\rangle$, are two extendible evolution subalgebras generated by all orthogonal idempotent elements of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Then, the isomorphism between $M_{\mathcal{E}_1}$ and $M_{\mathcal{E}_2}$ can be extendible to an isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2$.*

Proof. Assume that $dim(\mathcal{E}_1) = dim(M_{\mathcal{E}_1})$, then $M_{\mathcal{E}_1} = \mathcal{E}_1$. In this case, there is nothing to prove.

Suppose that $dim(M_{\mathcal{E}_1}) < dim(\mathcal{E}_1)$ and let ϕ be an isomorphism between $M_{\mathcal{E}_1}$ and $M_{\mathcal{E}_2}$ given by:

$$\phi(p_i) = \sum_{k=1}^m \alpha_{ik} q_k.$$

Using the orthogonality property for any $1 \le i \neq j \le m$, one finds

$$p_i p_j = \sum_{k=1}^n \alpha_{ik}\alpha_{jk} q_k = 0.$$

This implies $\alpha_{ik}\alpha_{jk} = 0$ for any $1 \le i \ne j \le m$. Thus,

$$\phi(p_i) = \alpha_{i\pi(i)} q_{\pi(i)}, \ \pi \in S_m.$$

Due to

$$\phi(p_i^2) = \phi(p_i p_i) = \phi(p_i)$$

one gets $\alpha_{i\pi(i)} = 1$, which yields

$$\phi(p_i) = q_{\pi(i)}$$

for some permutation π of $\{1, \dots, m\}$.

Since $M_{\mathcal{E}_1}, \ M_{\mathcal{E}_2}$ are extendible evolution subalgebras, then

$$\{p_1, \dots p_m, e_{m+1}, \dots, e_n\}$$

and

$$\{q_1, \dots q_m, f_{m+1}, \dots, f_n\}$$

are the natural basis of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively.

Let us construct an isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2$ as follows:

$$\begin{cases} \psi(p_i) = q_{\pi(i)} : 1 \le i \le \ m \\ \psi(e_j) = f_{\bar{\pi}(j)} : m+1 \le j \le n, \end{cases}$$

where $\bar{\pi}$ is a permutation of $\{m+1, \dots, n\}$. Hence, $\psi = \phi$ for all $x \in M_{\mathcal{E}_1}$. This completes the proof. $\square$

Remark 3. *From the above given construction, we infer that an isomorphism could be constructed in many ways.*

Proposition 3. *Let $\mathcal{E}$ be an n-dimensional S-evolution algebra and let p be any nonzero idempotent, then $|Supp(p)| > 1$.*

Proof. Suppose that $p = \alpha_i e_i$ is a nonzero idempotent, then $p^2 = \alpha_i^2 e_i^2 = p = \alpha_i e_i$. Using the definition of the structural matrix of an S-evolution algebra, one finds $e_i^2 = \sum_{i \ne j=1}^{n} a_{ij} e_j$; then, $\alpha_i^2 \sum_{i \ne j=1}^{n} a_{ij} e_j = \alpha_i e_i$. Thus, $\alpha_i = 0$. Therefore, $p = 0$, which is a contradiction. Hence, $|\mathrm{Supp}(p)| > 1$. $\square$

Corollary 2. *Let $\mathcal{E}$ be an n-dimensional S-evolution algebra and let $M_{\mathcal{E}}$ be defined as above. If $dim(M_{\mathcal{E}}) < dim(\mathcal{E})$, then $M_{\mathcal{E}}$ is not an evolution subalgebra.*

Proof. The proof immediately follows from Proposition 3 and (iii) of Theorem 4. $\square$

4. Construct Evolution Algebra from Given Idempotent Elements

In this section, we construct a low dimensional algebra generated by idempotent elements. Namely,

$$M := Alg\langle p_i : \ 1 \le i \le 3 \rangle. \tag{14}$$

Assume that the table of multiplication of this algebra is defined as follows:

$$p_i p_j = \begin{cases} p_{\phi_{ij}} & : i \ne j \\ p_i & : i = j, \end{cases} \tag{15}$$

where each of $\{p_i\}$ are linearly independent and $\phi_{ij} : I \times I \to I, \ I = \{1, 2, 3\}$.

Now, we are going to study when M becomes an evolution algebra. To answer to this question, we need the next auxiliary fact.

Proposition 4. *Let $\phi_{ij} : I \times I \to I$ be given, where $I = \{1,2,3\}$. Then, the following statements hold true:*

(i) *If ϕ_{ij} is surjective such that $\phi_{ij} = k,\ k \notin \{i,j\}$, then the set $\{p_i :\ 1 \le i \le 3\}$ is linearly dependent;*

(ii) *If ϕ_{ij} is surjective such that $\phi_{ij} = k,\ k \in \{i,j\}$, then the set $\{p_i :\ 1 \le i \le 3\}$ is linearly independent;*

(iii) *If ϕ_{ij} is not surjective, then the set $\{p_i :\ 1 \le i \le 3\}$ is linearly independent.*

Proof. (i). If ϕ_{ij} is surjective with $\phi_{ij} = k,\ k \notin \{i,j\}$ and we assume that

$$\sum_{j=1}^{3} \lambda_j p_j = 0, \tag{16}$$

then by multiplying both sides of (16) by p_i, one gets

$$p_i\left(\sum_{j=1}^{3} \lambda_j p_j\right) = \sum_{j=1}^{3} \lambda_j p_{\phi_{i,j}} = 0.$$

Hence, we obtain the following system:

$$\lambda_1^2 = \lambda_2\lambda_3,\ \lambda_2^2 = \lambda_1\lambda_3,\ \lambda_3^2 = \lambda_1\lambda_2.$$

The solution of the above system is $\lambda_1 = \lambda_2 = \lambda_3$. Plugging these values into (16) and assuming $\lambda_1 \neq 0$, we get $p_1 + p_2 + p_3 = 0$. Hence, the set $\{p_i :\ 1 \le i \le 3\}$ is linearly dependent;

(ii). If ϕ_{ij} is surjective with $\phi_{ij} = k,\ k \in \{i,j\}$, then without loss of generality, we may assume that $\phi_{12} = 1,\ \phi_{13} = 3,\ \phi_{23} = 2$. Now, let us assume that

$$\sum_{j=1}^{3} \lambda_j p_j = 0. \tag{17}$$

Now, multiplying both sides of (17) by p_i, we find

$$p_i\left(\sum_{j=1}^{3} \lambda_j p_j\right) = \sum_{j=1}^{3} \lambda_j p_{\phi_{i,j}} = 0.$$

Hence, one gets

$$(\lambda_1 + \lambda_2)p_1 + \lambda_3 p_3 = 0,\ \lambda_1 p_1 + (\lambda_2 + \lambda_3)p_2 = 0,\ \lambda_2 p_2 + (\lambda_1 + \lambda_3)p_3 = 0.$$

It is not difficult to find that the solution of the last system is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence, the set $\{p_i :\ 1 \le i \le 3\}$ is linearly independent.

(iii). Assume that ϕ_{ij} is not surjective, here we consider two cases as follows:

Case 1. If $|\phi_{ij}| = 1$. Without loss of generality, we can assume that $\phi_{ij} = 1$. Let

$$\sum_{j=1}^{3} \lambda_j p_j = 0.$$

Consider now,

$$p_2 \sum_{j=1}^{n} \lambda_j p_j = (\lambda_1 + \lambda_3)p_1 + \lambda_2 p_2 = 0.$$

Since each of p_i are linearly independent, then $\lambda_1 = -\lambda_3,\ \lambda_2 = 0$. On the other hand,

$$p_3 \sum_{j=1}^{n} \lambda_j p_j = (\lambda_1 + \lambda_2)p_1 + \lambda_3 p_3 = 0.$$

Again using the linear independence of p_i, then $\lambda_1 = -\lambda_2 = 0,\ \lambda_3 = 0$. Hence, the set $\{p_i :\ 1 \leq i \leq 3\}$ is linearly independent.

Case 2. If $|\phi_{ij}| = 2$, then, without loss of generality, we may assume that $\phi_{ij} \in \{1, 2\}$ such that $\phi_{12} = 2,\ \phi_{13} = 1,\ \phi_{23} = 2$. The other choices in the same manner will give the same result. Suppose that

$$\sum_{j=1}^{3} \lambda_j p_j = 0.$$

Now, let us consider

$$p_1 \sum_{j=1}^{n} \lambda_j p_j = (\lambda_1 + \lambda_3)p_1 + \lambda_2 p_2 = 0.$$

The linear independence of p_i implies $\lambda_1 = -\lambda_3,\ \lambda_2 = 0$. On the other hand,

$$p_3 \sum_{j=1}^{n} \lambda_j p_j = (\lambda_1 + \lambda_2)p_1 + \lambda_3 p_1 = 0,$$

which again by the linear independence of p_i, one gets $\lambda_1 = -\lambda_2 = 0,\ \lambda_3 = 0$. Hence, the set $\{p_i :\ 1 \leq i \leq 3\}$ is linearly independent.

This completes the proof. $\square$

Theorem 6. *Let M be an algebra defined by (14) and let $\phi_{lm}\colon I \times I \to I,\ I = \{1, 2, 3\}$ be surjective. Then the following statements hold true:*

(i) M is an evolution algebra if $\phi_{lm} = k$, where $k \notin \{l, m\}$;
(ii) M is not an evolution algebra if $\phi_{lm} = k$, where $k \in \{l, m\}$.

Proof. (i). Assume that $\phi_{lm} = k$, where $k \notin \{l, m\}$. By (i) of Proposition 4, we infer that the algebra M is two-dimensional. Let us suppose that $\{w_1, w_2\}$ is a natural basis for M. Then, we have the following change of basis:

$$w_i = \sum_{k=1}^{2} \alpha_{ik} p_k$$

with $det(\alpha_{ij})_{i,j=1}^{2} \neq 0$. Then, M is an evolution algebra if and only if $w_1 w_2 = 0$, which implies

$$\alpha_{11}\beta_{11} = 0,\ \ \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21} + \alpha_{12}\beta_{22} = 0.$$

The solution of the last system is

$$\alpha_{11} = 1,\ \alpha_{12} = -\alpha_{11},\ \alpha_{21} = 0,\ \alpha_{22} = 1.$$

So, we have the following two-dimensional evolution algebra:

$$\begin{aligned} w_1 &= p_1 - p_2 \\ w_2 &= p_2. \end{aligned}$$

Thus, M is an evolution algebra;

(ii). Assume that $\phi_{lm} = k$, $k \notin \{l, m\}$. Then, by (ii) of Proposition 4, M is thee-dimensional. Let us suppose that $\{w_1, w_2, w_3\}$ is a natural basis for M. Then, we have the following change of basis:

$$w_i = \sum_{k=1}^{3} \alpha_{ik} p_k$$

with $det(\alpha_{ij})_{i,j=1}^{3} \neq 0$. Then, M is an evolution algebra if and only if $w_i w_j = 0$, which implies that

$$\alpha_{ik}\alpha_{jk} + \alpha_{il}\alpha_{jm} + \alpha_{im}\alpha_{jl} = 0.$$

Assume that $l = k$, then the above system becomes as follows:

$$\alpha_{ik}\alpha_{jk} + \alpha_{ik}\alpha_{jm} + \alpha_{im}\alpha_{jk} = 0.$$

One can easily find that the solutions of the last system always have the property $det(\alpha_{ij})_{i,j=1}^{3} = 0$. A similar result is $m = k$. Hence, in this case, M is not an evolution algebra. □

Theorem 7. *Let M be an algebra defined by* (14) *and let ϕ_{lm} not be surjective, then M is a three-dimensional evolution algebra.*

Proof. Here, we shall consider two cases:

Case 1. If $|\phi_{lm}| = 1$. Assume that $\phi_{lm} = q$, $q \in \{1, 2, 3\}$. Let w_i, $i = \overline{1,3}$ be a natural basis for M. Then, we have the following change of basis:

$$w_i = \sum_{k=1}^{3} \alpha_{ik} p_k.$$

with $det(\alpha_{ik})_{i,k=1}^{3} \neq 0$ if M is an evolution algebra if and only if $w_i w_j = 0$ for any $i \neq j$, $i, j = \overline{1,3}$. Now, consider $w_i w_j$; then, we have the following system:

$$\sum_{k=1}^{3} \alpha_{ik}\alpha_{jk} pk + \sum_{l,m=1}^{3} \left(\alpha_{il}\alpha_{jm} + \alpha_{im}\alpha_{jl}\right) p_{\phi_{lm}} = 0, \; l \neq m. \tag{18}$$

Rewriting the above system, one has

$$\alpha_{iq}\left(\sum_{k=1}^{3} \alpha_{jk}\right) + \sum_{1=r\neq q}^{3} \alpha_{ir}\left(\sum_{1=r\neq s}^{3} \alpha_{js}\right) = 0$$
$$\alpha_{qt}\alpha_{tt} = 0, \; t \neq q, \; 1 \leq t \leq 3.$$

If $q = 1$, then the solution of the last system is the following one: $\alpha_{11} = \alpha_{21} = \alpha_{31} = 1$, $\alpha_{13} = \alpha_{32} = -1$ and the remaining values are zero. Hence,

$$\begin{aligned} w_1 &= p_1 - p_3 \\ w_2 &= p_1 \\ w_3 &= p_1 - p_2. \end{aligned}$$

If $q = 2$, then the solution of the above system is as follows: $\alpha_{12} = \alpha_{22} = \alpha_{32} = 1$, $\alpha_{13} = \alpha_{33} = -1$ and the remaining values are zero. Hence,

$$\begin{aligned} w_1 &= p_2 - p_3 \\ w_2 &= p_2 \\ w_3 &= p_1 - p_2. \end{aligned}$$

If $q = 3$, then the solution of the above system is as follows $\alpha_{12} = \alpha_{23} = \alpha_{31} = 1$, $\alpha_{13} = \alpha_{32} = -1$ and the remaining values are zero. Hence,

$$\begin{aligned} w_1 &= p_2 - p_3 \\ w_2 &= p_3 \\ w_3 &= p_1 - p_2. \end{aligned}$$

Hence, in this case, M is an evolution algebra.

Case 2. If $|\phi_{lm}| = 2$. In this case, we have several possibilities of ϕ_{lm}. We consider one possible case, and other cases can be proceeded in the same manner. Assume that $\phi_{12} = \phi_{13} = 1,\ \phi_{23} = 2$. Let $w_i,\ i = \overline{1,3}$ be a natural basis for M. Then, one can write

$$w_i = \sum_{k=1}^{3} \alpha_{ik} p_k.$$

with $det(\alpha_{ik})_{i,k=1}^{3} \neq 0$. Then, M is an evolution algebra if and only if $w_i w_j = 0$ for any $i \neq j,\ i,j = \overline{1,3}$. Simple calculations yield that $\alpha_{11} = \alpha_{21} = \alpha_{32} = 1$, $\alpha_{12} = \alpha_{33} = -1$ and the remaining values are zero. Hence,

$$\begin{aligned} w_1 &= p_1 - p_2 \\ w_2 &= p_1 \\ w_3 &= p_2 - p_3. \end{aligned}$$

This completes the proof. □

Now, we are going to study the structure of the algebra M.

Theorem 8. *Let M be an algebra defined by (14), then the following statements are true:*

(i) If ϕ_{lm} is surjective with $\phi_{lm} = k,\ k \notin \{l, m\}$, then M is isomorphic to E_1 with the following table of multiplication:

$$e_1^2 = e_1,\ e_2^2 = e_2;$$

(ii) If ϕ_{lm} is not surjective, then M is isomorphic to E_2 with the following table of multiplication:

$$e_1^2 = e_1,\ e_2^2 = e_2,\ e_3^2 = e_3.$$

Proof. (i). Let us first find

$$\begin{aligned} w_1^2 &= (p_1 - p_2)^2 = p_1 - p_2 = w_1 \\ w_2^2 &= p_2^2 = p_2 = w_2. \end{aligned}$$

A simple change of basis yields that this algebra is isomorphic to E_1;

(ii). If ϕ_{lm} is not surjective, then

$$\begin{aligned} w_1^2 &= (p_1 - p_3)^2 = p_3 - p_1 = -w_1 \\ w_2^2 &= p_1^2 = p_1 = w_2 \\ w_3^2 &= (p_1 - p_2)^2 = p_2 - p_1 = -w_3. \end{aligned}$$

A simple change of basis yields that this algebra is isomorphic to E_2.

This completes the proof. □

Remark 4. *We stress the following points:*

- *The algebras E_1 and E_2 are not isomorphic;*
- *Both algebras E_1 and E_2 are not solvable, hence are not nilpotent.*

5. Tensor Product of S-Evolution Algebras

In this section, we investigate the relation between the set of idempotent elements of given two-dimensional S-evolution algebras and the set of idempotent elements of their tensor product. Let us first define the structure matrix of the tensor product of finite dimensional evolution algebras.

Definition 1 ([41]). *Suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are two finite dimensional evolution algebras (over the field $\mathbb{K}$) with a natural basis $B_1 = \{e_1, \ldots, e_N\}$ and $B_2 = \{f_1, \ldots, f_M\}$, respectively. Assume that $A_{B_1} = (a_{ij})$ and $A_{B_2} = (b_{km})$ are the structure matrices associated to $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Then, the structure matrix of the evolution algebra $\mathcal{E}_1 \otimes \mathcal{E}_2$ relative to the basis $B_1 \otimes B_2 = \{e_1 \otimes f_1, \ldots, e_1 \otimes f_M, \ldots, e_N \otimes f_1, \ldots, e_N \otimes f_N\}$ is the Kronecker product of A_{B_1} and A_{B_2}, i.e., $A_{B_1 \otimes B_2} = A_{B_1} \otimes A_{B_2}$.*

Remark 5. *We notice that the multiplication of $\mathcal{E}_1 \otimes \mathcal{E}_2$ in the basis $B_1 \otimes B_2$ is defined as follows:*

$$(e_i \otimes f_j)^2 = \sum_{j=1}^{N} a_{ik} \left[\sum_{m=1}^{M} b_{jm} (e_i \otimes f_j) \right]. \tag{19}$$

Definition 2. *An evolution algebra $\mathcal{E}$ is tensor decomposable if it is isomorphic to $\mathcal{E}_1 \otimes \mathcal{E}_2$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are evolution algebras with $\dim(\mathcal{E}_1), \dim(\mathcal{E}_2) > 1$. Otherwise, $\mathcal{E}$ is said to be tensor indecomposable.*

Proposition 5. *Let $\mathcal{E}$ be an S-evolution algebra and $\mathcal{E}$ be tensor decomposable. Then, $\mathcal{E} \cong \mathcal{E}_1 \otimes \mathcal{E}_2$ with at least one of $\mathcal{E}_1,\ \mathcal{E}_2$ is an S-evolution algebra.*

Proof. Since $\mathcal{E}$ is tensor decomposable, then $\mathcal{E} \cong \mathcal{E}_1 \otimes \mathcal{E}_2$ with $\dim(\mathcal{E}_1), \dim(\mathcal{E}_2) > 1$. Suppose that $\mathcal{E}_1 = \langle e_i : 1 \le i \le N \rangle$, and $\mathcal{E}_2 = \langle f_j : 1 \le j \le M \rangle$ are evolution algebras. Assume that $(a_{ij})_{i,j=1}^{N}$, $(b_{ij})_{i,j=1}^{M}$, $(c_{ij})_{i,j=1}^{N\times M}$ are the structural matrices of $\mathcal{E}_1,\ \mathcal{E}_2,\ \mathcal{E}$, respectively. Consider

$$(e_i \otimes f_j)^2 = \sum_{j=1}^{N} a_{ik} \left[\sum_{m=1}^{M} b_{jm} (e_i \otimes f_j) \right]. \tag{20}$$

However, if the structural matrix of $\mathcal{E}_1 \otimes \mathcal{E}_2$ is an S-matrix, then from the above equation one finds $a_{ii} b_{jj} = 0$; this implies either $a_{ii} = 0$ or $b_{jj} = 0$. We may assume that $a_{ii} = 0$. Next, let $a_{is} \neq 0$ for some $1 \le i \neq s \le N$; then, due to the isomorphism between $\mathcal{E}_1 \otimes \mathcal{E}_2$ and $\mathcal{E}$, one finds $a_{si} \neq 0$. Hence, $\mathcal{E}_1$ is an S-evolution algebra. $\square$

Theorem 9. *Any four-dimensional S-evolution algebra is tensor decomposable if it is isomorphic to the following S-evolution algebras:*

$$\begin{aligned} E_1 : e_1^2 &= e_4,\ e_2^2 = e_3,\ e_3^2 = e_2,\ e_4^2 = e_1. \\ E_2 : e_1^2 &= e_3 + a e_4,\ e_2^2 = b e_3 + e_4,\ e_3^2 = e_1 + a e_2,\ e_4^2 = b e_1 + e_2. \end{aligned}$$

Proof. Let $\mathcal{E}$ be tensor decomposable, then $\mathcal{E} \cong \mathcal{E}_1 \otimes \mathcal{E}_2$ such that with $\dim(\mathcal{E}_1), \dim(\mathcal{E}_2) > 1$. Using Proposition 5, we have either $\mathcal{E}_1$ or $\mathcal{E}_2$ is an S-evolution algebra. Let us assume that $\mathcal{E}_1$ is an S-evolution algebra with the following table of multiplication:

$$\mathcal{E}_1 : e_1^2 = a e_2\ \ e_2^2 = b e_1.$$

After simple scaling, we can write the table of multiplication of the above algebra as

$$\mathcal{E}_1 : e_1^2 = e_2\ \ e_2^2 = e_1.$$

Now, assume that $\mathcal{E}_2$ is an S-evolution algebra, then the table of multiplication of $\mathcal{E}_2$ is as follows:

$$\mathcal{E}_2 : f_1^2 = f_2 \;\; f_2^2 = f_1.$$

Then, $\mathcal{E}$ is decomposable in this case if $\mathcal{E} \cong \mathcal{E}_1 \otimes \mathcal{E}_2$ but $\mathcal{E}_1 \otimes \mathcal{E}_2$ has the following table of multiplication.

$$\mathcal{E}_1 \otimes \mathcal{E}_2 : (e_1 \otimes f_1)^2 = e_2 \otimes f_2,\ (e_1 \otimes f_2)^2 = e_2 \otimes f_1,\ (e_2 \otimes f_1)^2 = e_1 \otimes f_2,\ (e_2 \otimes f_2)^2 = e_1 \otimes f_1.$$

Clearly, this algebra isomorphic to E_1.

Now, assume that $\mathcal{E}_2$ is not an S-evolution algebra, then its table of multiplication is as follows:

$$\mathcal{E}_2 : f_1^2 = af_1 + bf_2 \;\; f_2^2 = cf_1 + df_2.$$

After simple scaling, we can rewrite the table of multiplication of $\mathcal{E}_2$ as

$$\mathcal{E}_2 : f_1^2 = f_1 + bf_2 \;\; f_2^2 = cf_1 + f_2.$$

Then, $\mathcal{E}$ is decomposable in this case if $\mathcal{E} \cong \mathcal{E}_1 \otimes \mathcal{E}_2$. However, $\mathcal{E}_1 \otimes \mathcal{E}_2$ has the following table of multiplication:

$$\begin{aligned} \mathcal{E}_1 \otimes \mathcal{E}_2 : \quad & (e_1 \otimes f_1)^2 = (e_2 \otimes f_1) + b(e_2 \otimes f_2),\ (e_1 \otimes f_2)^2 = c(e_2 \otimes f_1) + (e_2 \otimes f_2), \\ & (e_2 \otimes f_1)^2 = (e_1 \otimes f_1) + b(e_1 \otimes f_2),\ (e_2 \otimes f_2)^2 = c(e_1 \otimes f_1) + (e_1 \otimes f_2). \end{aligned}$$

This algebra is isomorphic to E_2. □

Theorem 10. *Consider the E_1 and E_2 evolution algebras given in Theorem 8. Then, the following statements hold true:*

(i) *$E_1 \otimes E_1 \cong E$, where E is a four-dimensional evolution algebra with the following table of multiplication:*
$$g_i^2 = g_{\pi(i)},\ \pi \in S_4;$$

(ii) *$E_1 \otimes E_2 \cong E$, where E is a six-dimensional evolution algebra with the following table of multiplication:*
$$g_i^2 = g_{\pi(i)},\ \pi \in S_6;$$

(iii) *$E_2 \otimes E_2 \cong E$, where E is a nine-dimensional evolution algebra with the following table of multiplication:*
$$g_i^2 = g_{\pi(i)},\ \pi \in S_9.$$

Proof. (i) Now, let us consider $E_1 \otimes E_1$; then, the table of multiplication is as follows:

$$(e_i \otimes f_j)^2 = e_i \otimes f_j,\ 1 \leq i, j \leq 2.$$

This algebras is isomorphic to E with table of multiplication

$$g_k^2 = g_k,\ 1 \leq k \leq 4.$$

Clearly, $\tilde{E} \cong E$. Hence, $E_1 \otimes E_1 \cong E$. The statements of (ii) and (iii) can be proceeded by the similar argument. This completes the proof. □

6. Conclusions

This research contributes significantly to the field of algebraic structures, particularly to the study of evolution algebras. By examining the extendibility of evolution subalgebras generated by idempotents, we have gained insights into the structural properties and relationships within these algebras. The investigation of the extendibility of isomorphism from subalgebras to the entire algebra sheds light on the overall algebraic structure and its behaviour. Moreover, the study of the existence of evolution algebras generated by arbitrary idempotents adds to our understanding of the algebraic landscape. Additionally, the description of the tensor product of algebras generated by arbitrary idempotents and the determination of the conditions for tensor decomposability of four-dimensional S-evolution algebras further enrich the knowledge in this area. Overall, this research has significant importance for advancing our comprehension of evolution algebras and their structural properties.

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