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Multiple-Criteria Heuristic Rating Estimation

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Abstract: One of the most widespread multi-criteria decision-making methods is the Analytic Hierarchy Process (AHP). AHP successfully combines the pairwise comparisons method and the hierarchical approach. It allows the decision-maker to set priorities for all ranked alternatives. But what if, for some of them, their ranking value is known (e.g., it can be determined differently)? The Heuristic Rating Estimation (HRE) method proposed in 2014 tried to bring the answer to this question. However, the considerations were limited to a model only considering a few criteria. This work analyzes how HRE can be used as part of the AHP hierarchical framework. The theoretical considerations are accompanied by illustrative examples showing HRE as a multiple-criteria decision-making method.

Keywords: pairwise comparisons; analytic hierarchy process; heuristic ranking estimation; MCDM; AHP; HRE

MSC: 91B06; 90B50

1. Introduction

We constantly face the need to make decisions, which inevitably involve comparisons. The easiest way is to compare one thing to another—objects in a pair. We are accustomed to making such comparisons daily. Take the market, for example, where people try to choose the heavier box of cookies (or the lighter bicycle in a bike store). By comparing the weight of one object in our hand with the weight of another, we can easily estimate the relative "importance" of two similar-looking things. Often we can make the right choice without relying on external tools to indicate weight. In practice, we often have to deal with much more complicated comparisons than assessing the usefulness of products during shopping. There is usually no natural way to make an accurate comparison, and the object does not have a pre-assigned "weight". Moreover, many different factors often need to be evaluated and compared. The pairwise comparison (PC) method comes to the rescue in such a case. It allows us to use our innate ability to compare alternatives, enabling us to deal with these challenges more effectively. When creating a decision model based on pairwise comparisons of alternatives, one of the problems we face is determining the nature of the objects under study. Some may be tangible, i.e., they can be measured and weighed; others are intangible and difficult to decide on by direct measurement [1]. For example, available methods such as AHP cannot deal with this problem effectively. Hence, in AHP, we prefer to deal with relative expert-based assessments even if we consider distances between cities [2].

The Heuristic Rating Estimation (HRE) method is an attempt to deal with this problem. The HRE was initially defined as two alternative methods (additive and multiplicative) for calculating the ranking [3,4]. In this article, we extend this model to include a hierarchy that allows multiple criteria for evaluating alternatives to be considered. The HRE calculation procedure is based on the same concept as EVM (Eigenvalue Method) [5] and GMM (Geometric Mean Method) [6]. That is, the priority of a given alternative is equal to the weighted

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average value of the priorities of the other alternatives [7]. This compatibility allows the original HRE idea to be easily combined with these methods within one hierarchical model (Example 2). Thus, naturally, HRE becomes a multi-criteria method that complements AHP in cases where some alternatives have measurable or previously known ratings.

The presented work is a follow-up to [3,4]. The preliminary section introduces the reader to the issues related to pairwise comparisons and priority-deriving methods, such as EVM and GMM. The HRE calculation procedure is described in (Section 4). The next section (Section 5) fits HRE into a multi-criteria hierarchical model used in the context of AHP. The theoretical description is supplemented by practical examples showing the applications of HRE (Example 1) and its integration with AHP (Example 2).

A list of the important symbols used in the work can be found in Table 1 below.

Table 1. A list of the important symbols.

Symbol	Description
A	set of alternatives
A_K	set of alternatives with a priori known priorities (in HRE)
A_U	set of alternatives with a priori unknown priorities (in HRE)
a_i	i-th alternative
C	pairwise comparisons matrix
c_{ij}	comparison of the i-th and j-th alternatives
$w(a_i)$	weight (priority) of the i-th alternative
w	vector of priority weights
CI	Saaty's consistency index
$\mathscr{K}$	Koczkodaj's inconsistency index
M	auxiliary matrix (in the additive HRE method)
b	constant terms vector (in the additive HRE method)
N	auxiliary matrix (in the geometric HRE method)
d	constant terms vector (in the geometric HRE method)
μ	logarithmized priority vector (in the geometric HRE method)

2. Related Works

The pairwise comparisons (PC) method has a centuries-old history. The first considerations of its use come from a thirteenth-century treatise by Ramon Lull, Ars Electionis [8,9]. Later, it was the subject of research by Condorcet, Fechner, Thurston and others [10–13]. Currently, pairwise comparisons are a valuable source of preferential information in many decision-making methods, including TOPSIS [14], PROMETHEE [15], MACBETH [16], BWM [17] or multiple-criteria sorting [18]. One of the best known methods of multiple-criteria decision making based on pairwise comparisons is the Analytic Hierarchy Process (AHP) [7,19]. The method defined by Saaty in 1977 [5] quickly gained popularity and became widely used in many areas of application, including the performance assessment of employees [20], country competitiveness [21], supplier segmentation [22], portfolio management [23], or sustainability [24].

This method competes with other decision-making methods in many areas, including logistics optimization [25,26], IT management [27,28], military applications [29] or such exotic applications as analyzing the possibility of oil exploration in the Arctic National Wild refuge [19] (pp. 89–100).

More application examples can be found in the reviews [30–32]. It also quickly gained opponents who very often rightly pointed out the imperfections of the method [33–35]. In addition to a range of applications, AHP has seen numerous extensions and modifications.

In particular, various variants of the method have been developed using the representation of uncertain knowledge, including fuzzy sets [36–38], intervals [39], and gray numbers [40].

3. Preliminaries

3.1. Pairwise Comparisons

Let $A = \{a_1, \ldots, a_n\}$ be a set of alternatives and $C = [c_{ij}]$ be a pairwise comparisons (PC) matrix, such that $c_{ij} \in \mathbb{R}_+$ for $i, j \in \{1, .., n\}$. A single element c_{ij} corresponds to a direct comparison of the i-th and j-th alternatives. A single entry c_{ij} has a quantitative meaning. For example, when an expert decides that the a_i alternative is twice more preferred than a_j, then c_{ij} takes the value 2. As a result, the elements of the diagonal are ones, i.e., C takes the form

$$C = \begin{bmatrix} 1 & c_{12} & \cdots & c_{1n} \\ c_{21} & 1 & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & 1 \end{bmatrix}.$$

Based on the information stored in C, a priority vector is calculated. For the purpose of the article, we will denote it as $[w(a_1), \ldots, w(a_n)]^T$ where w is the function $w : A \to \mathbb{R}_+$. We say that a_i is more preferred than a_j (denoted as $a_i \succ a_j$) if $w(a_i) > w(a_j)$. Similarly, a_i is considered as equally preferred as a_j (denoted as $a_i \sim a_j$) when $w(a_i) = w(a_j)$.

In the ideal case $c_{ij} = w(a_i)/w(a_j)$, i.e., the results of comparisons provided by experts perfectly match the computed ranking. In practice, however, we may expect only that $c_{ij} \approx w(a_i)/w(a_j)$. The reason for this inequality is data inconsistency, which is most often the result of expert errors.

Definition 1. *A PC matrix C is called consistent if*

$$c_{ik} = c_{ij}c_{jk}, \textit{ for every } i, j, k = 1, \ldots, n \tag{1}$$

Every PC matrix is said to be inconsistent if it is not a consistent PC matrix.

Definition 2. *A PC matrix C is called reciprocal if*

$$c_{ij} = \frac{1}{c_{ji}} \textit{ for every } i, j = 1, ..., n \tag{2}$$

Although the reciprocity condition is natural, the use of non-reciprocal matrices can also be found in the literature [7,41]. It is easily seen that every consistent PC matrix is also reciprocal, but not reversely.

One of the essential issues in the PC methods concerns the question of how the priority vector needs to be determined. The answer is not straightforward because the expert's judgments may not be consistent (For a consistent matrices, each ranking method will lead to the same priority vector [7] (p. 96)). There are many methods of priority vector estimation in the literature [42]. Two of the most popular prioritization methods are the Eigenvector Method (EVM) [5] and the Geometric Mean Method (GMM) [6].

In EVM, the priorities of alternatives are determined by the real and positive solution of $Cw = \lambda_{max}w_{max}$, where λ_{max} is the principal eigenvalue (spectral radius) of C, and w_{max} is an appropriate eigenvector. The vector of priorities w is assumed to be w_{max} rescaled so that the sum of its components is 1.

In GMM, the priority of an individual a_i is calculated as the geometric mean of the i-th row of C. The sum of components also rescales the created ranking vector. Both EVM and GMM have been repeatedly debated and analyzed, and both methods have their proponents and opponents [43,44].

Depending on how inconsistent the PC matrix is, calculating the priority vector reflects the real preferences of respondents. To determine the degree to which PC matrices are inconsistent, appropriate indices are used. The most popular one comes from Saaty [5] and is defined as:

$$CI = \frac{\lambda_{max} - n}{n-1}, \tag{3}$$

where λ_{max} denotes the principal eigenvalue of the $n \times n$ matrix C. Another interesting inconsistency indicator comes from Koczkodaj [45]. For the $n \times n$ PC matrix C, it is defined as:

$$\mathscr{K}(C) \overset{df}{=} \max_{\substack{i,j,k \in \{1,\ldots,n\} \\ i \neq j,\, j \neq k,\, k \neq i}} \left\{ \min\left\{ \left| 1 - \frac{c_{ij}}{c_{ik}c_{kj}} \right|, \left| 1 - \frac{c_{ik}c_{kj}}{c_{ij}} \right| \right\} \right\}. \tag{4}$$

In general, the more inconsistent the matrix C, the larger the inconsistency indices. When the value of the inconsistency index is considerable, the ranking results are not credible. Thus, in most cases, it is essential to keep the inconsistency as small as possible. More about different inconsistency indices can be found in [46].

3.2. Analytic Hierarchy Process

A single PC matrix allows us to compare alternatives with respect to only one criterion at a time. In practice, however, there is often a need to take more than one criterion into account. For example, when buying a car, we do not just choose a "better" vehicle, but also pay attention to its price, terms of warranty, and so on. A framework that allows the use of pairwise comparisons as a multi-criteria method is provided by the Analytic Hierarchy Process (AHP). Saaty proposed the method in 1977 [5] following J. R. Miller idea published in 1966 [47]. using PC matrices hierarchy for processing a large number of criteria. AHP is able to handle multiple criteria by defining a hierarchy (a tree) of alternatives and criteria. Hence, at the bottom of the hierarchical tree defining the decision model, alternatives are compared with respect to the lowest level criteria, then those criteria are compared against each other with respect to the higher level of criteria, and so on. Although the AHP hierarchy can be freely expanded, the most popular models consist of three layers: alternatives $A = \{a_1, \ldots, a_n\}$, criteria $S = \{s_1, \ldots, s_m\}$ and the root node of the hierarchy called "goal" or "purpose".

As an example of a hierarchical AHP decision model, let us consider the problem of selecting a company manager. Suppose there are three candidates for this position: Andrew, Benjamin and Christopher. When considering these alternatives, we take into account their experience in managerial positions, education and interpersonal skills, as well as the ability to work under pressure. We compare the candidates in terms of these criteria. The one with the best results in our comparison will become the CEO of the company.

Going into details, according to the AHP method, we create a separate PC matrix for each criterion. Thus, $C^{(s)}$ is a PC matrix corresponding to the mutual comparisons of all three candidates with regard to the criterion s. There are four criteria for assessing candidates: experience (ex), education (ed), interpersonal skills (is), and stress resistance (sr). Hence, for example, $C^{(ex)}$ corresponds to the comparisons of our candidates in terms of their experience. We assume that the following four matrices were prepared as a result of the expert assessment.

$$C^{(ex)} = \begin{bmatrix} 1 & 2 & 4 \\ \frac{1}{2} & 1 & 2 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}, \; C^{(ed)} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{8} \\ 2 & 1 & \frac{1}{4} \\ 8 & 4 & 1 \end{bmatrix},$$

$$C^{(is)} = \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 2 & 1 & 6 \\ \frac{1}{3} & \frac{1}{6} & 1 \end{bmatrix}, \; C^{(sr)} = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 2 & 1 & 2 \\ 1 & \frac{1}{2} & 1 \end{bmatrix}$$

For each considered matrix $C^{(s)}$, we calculate, using EVM [5], the priority vector. These are

$$w^{(ex)} = \left[\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\right]^T, \; w^{(ed)} = \left[\frac{1}{11}, \frac{2}{11}, \frac{8}{11}\right]^T,$$

$$w^{(is)} = \left[\frac{3}{10}, \frac{6}{10}, \frac{1}{10}\right]^T, \; w^{(sr)} = \left[\frac{1}{4}, \frac{2}{4}, \frac{1}{4}\right]^T.$$

When calculating the final priority vector, we must take into account the fact that the criteria may contribute to a different extent to achieving the goal. For this reason, we compare them in pairs. These comparisons form one more 4×4 PC matrix $\widehat{C} = [\widehat{c}_{ij}]$ where $\widehat{c}_{ij}$ denotes the result of individual comparisons between the criteria s_i and s_j, where $s_1, \ldots, s_4$ mean (ex), (ed), (is), and (sr), correspondingly.

$$\widehat{C} = \begin{bmatrix} 1 & 4 & 2 & 8 \\ \frac{1}{4} & 1 & \frac{1}{2} & 2 \\ \frac{1}{2} & 2 & 1 & 4 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{4} & 1 \end{bmatrix}.$$

The priority vector $\widehat{w}$ for $\widehat{C}$ is

$$\widehat{w} = \left[\frac{8}{15}, \frac{2}{15}, \frac{4}{15}, \frac{1}{15}\right]^T.$$

Thus, according to the expert opinion, the first criterion; experience, has the priority 8/15, education 2/15, interpersonal skills 4/15, and stress resistance–1/15. The priorities of criteria determine the degree to which the comparisons of alternatives affect the overall score. Thus, they become weights scaling the results of direct comparisons between alternatives (the final priority vector is a linear combination of vectors $w^{(s)}$ where the scaling factors come from $\widehat{w}$). The overall ranking w is given as:

$$w = \sum_{j=1}^{m} \hat{w}_j w^{(s_j)}.$$

In our case, we have:

$$w = \hat{w}_1 w^{(ex)} + \hat{w}_2 w^{(ed)} + \hat{w}_3 w^{(ch)} + \hat{w}_4 w^{(ag)} =$$

$$= \frac{8}{15}\left[\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\right]^T + \frac{2}{15}\left[\frac{1}{11}, \frac{2}{11}, \frac{8}{11}\right]^T + \frac{4}{15}\left[\frac{3}{10}, \frac{6}{10}, \frac{1}{10}\right]^T + \frac{1}{15}\left[\frac{1}{4}, \frac{2}{4}, \frac{1}{4}\right]^T \approx$$

$$\approx [0.414, 0.370, 0.216]^T.$$

The above priority vector translates to the observation that Andrew is slightly more preferred to be a company manager than Benjamin, and both are more preferred than Christopher.

4. Heuristic Rating Estimation

When comparing alternatives, sometimes, we already know priorities for some of them. In such a case, we can use them and not ask experts for unnecessary comparisons, saving time and money. This observation gave rise to the Heuristic Rating Estimation (HRE) method [3,4]. In HRE, the set of alternatives A is composed of two disjoint subsets: A_K-alternatives for which the final priorities are known, and A_U-alternatives for which priority weights need to be estimated (also referred as to unknown alternatives). Knowledge about the priority values of elements in A_K combined together with pairwise comparisons of all the elements from $A = A_K \cup A_U$ allow us to calculate priorities for A_U. Hence, A_K can be considered as the set of references that provides a benchmark for new alternatives in A_U. For the sake of simplicity, in this study we will denote $A_U = \{a_1, ..., a_k\}$ and

$A_K = \{a_{k+1}, ..., a_n\}$. If $a_i, a_j \in A_K$, then we will assume that $w(a_i)$ and $w(a_j)$ are known. Therefore, the value c_{ij} is also known and equals $c_{ij} = w(a_i)/w(a_j)$, so querying experts about c_{ij} is also not necessary. The comparisons matrix C in HRE takes the form:

$$C = \begin{bmatrix} 1 & \cdots & c_{1,k} & c_{1k+1} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ c_{k,1} & \cdots & 1 & c_{k,k+1} & \cdots & c_{k,n} \\ c_{k+1,1} & \cdots & c_{k+1,k} & 1 & \cdots & w(a_{k+1})/w(a_n) \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,k} & w(a_n)/w(a_{k+1}) & \cdots & 1 \end{bmatrix}.$$

The above matrix and vector of reference values $[w(a_{k+1}), ..., w(a_n)]^T$ allow us to calculate the complete ranking for A (Figure 1).

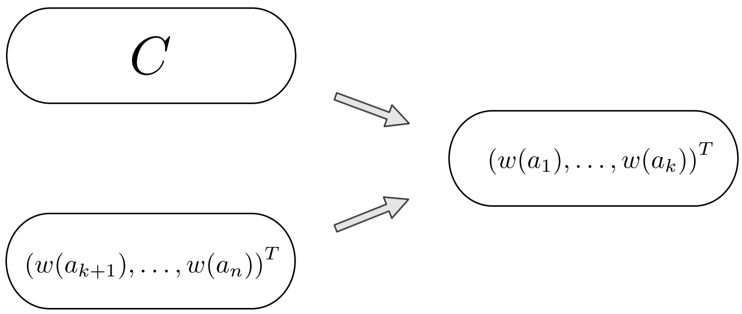

Figure 1. How does the HRE priority-deriving method work?

The HRE method has been designed as an extension of AHP, where part of the ranking is already known. Therefore, it is based on the same premise that

$$w(a_i) \approx c_{ij} w(a_j). \tag{5}$$

Thus, as in AHP, we can request that the priority of the i-th alternative $w(a_i)$ is a weighted average of priorities of all the other alternatives [7], i.e.,

$$w(a_i) = \frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} w(a_j), \quad \text{for } i = 1, \ldots, k. \tag{6}$$

This leads to a matrix equation

$$Mw = b, \tag{7}$$

where M is given as

$$M = \begin{bmatrix} 1 & -\frac{1}{n-1} c_{1,2} & \cdots & -\frac{1}{n-1} c_{1,k} \\ -\frac{1}{n-1} c_{2,1} & \ddots & \cdots & -\frac{1}{n-1} c_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n-1} c_{k,1} & -\frac{1}{n-1} c_{k,2} & \cdots & 1 \end{bmatrix},$$

the constant term vector,

$$b = \begin{bmatrix} \frac{1}{n-1} c_{1,k+1} w(a_{k+1}) + ... + \frac{1}{n-1} c_{1,n} w(a_n) \\ \frac{1}{n-1} c_{2,k+1} w(a_{k+1}) + ... + \frac{1}{n-1} c_{2,n} w(a_n) \\ \vdots \\ \frac{1}{n-1} c_{k,k+1} w(a_{k+1}) + ... + \frac{1}{n-1} c_{k,n} w(a_n) \end{bmatrix}$$

and the ranking vector for unknown alternatives is given as:

$$w = [w(a_1), ..., w(a_k)]^T.$$

The solution of (7)-if it exists and is acceptable, forms the desired vector of priorities. It has been shown [48] that for a small value of $\mathscr{K}(C)$ and sufficiently large set of the reference alternatives A_K the feasible solution of (7) always exists.

In HRE, as in AHP, many methods for calculating the ranking can be defined. The GMM equivalent in AHP is geometric HRE [4]. Thus, starting from (5), we may request

$$w(a_i) = \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} w(a_j) \right)^{\frac{1}{n-1}} \quad \text{for } i = 1, \ldots, n. \tag{8}$$

After raising both sides of (8) to the power $n-1$, we obtain the non-linear equation system

$$\begin{array}{c} w^{n-1}(a_1) = c_{1,2}w(a_2) \cdot c_{1,3}w(a_3) \cdot \ldots\ldots\ldots \cdot c_{1,n}w(a_n) \\ w^{n-1}(a_2) = c_{2,1}w(a_1) \cdot c_{2,3}w(a_3) \cdot \ldots\ldots\ldots \cdot c_{2,n}w(a_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w^{n-1}(a_k) = c_{k,1}w(a_1) \cdot c_{k,2}w(a_2) \cdot \ldots \cdot c_{k,n-1}w(a_{n-1}) \end{array}. \tag{9}$$

Thanks to the logarithmic transformation, the above is equivalent to the linear equation system

$$N\mu = d, \tag{10}$$

where N is a $k \times k$ auxiliary matrix in the form

$$N = \begin{bmatrix} (n-1) & -1 & \cdots & -1 \\ -1 & \ddots & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & (n-1) \end{bmatrix},$$

d is a constant term vector

$$d = \begin{bmatrix} \lg(c_{1,2}c_{1,3} \cdot \ldots \cdot c_{1,k}c_{1,k+1}w(a_{k+1}) \cdot \ldots \cdot c_{1,n}w(a_n)) \\ \lg(c_{2,1}c_{2,3} \cdot \ldots \cdot c_{2,k} \cdot c_{2,k+1}w(a_{k+1}) \cdot \ldots \cdot c_{2,n}w(a_n)) \\ \vdots \\ \lg(c_{k,1}c_{k,2} \cdot \ldots \cdot c_{k,k-1} \cdot c_{k,k+1}w(a_{k+1}) \cdot \ldots \cdot c_{k,n}w(a_n)) \end{bmatrix},$$

and

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \vdots \\ \mu_k \end{bmatrix} = \begin{bmatrix} \lg w(a_1) \\ \vdots \\ \vdots \\ \lg w(a_k) \end{bmatrix}.$$

The solution μ of (10) induces the weight vector w—a solution of the original non-linear problem (9)

$$w = [\exp \mu_1, ..., \exp \mu_k]^T.$$

The feasible solution of (10) always exists and is optimal [4].

5. Multiple-Criteria Heuristic Rating Estimation Method

Sometimes when we want to compare a few objects or select several items from many others, we follow different criteria with varying degrees of importance. Moreover, we may have preliminary information about specific things. For example, when choosing a school for a child, parents pay attention to the level of education and the percentage of pupils entering the next school, the educational system, location, social group from which the pupils come, or additional activities offered by the school. Specific information about some of the considered schools can be known, although the set of known criteria may differ in each school. As another example, consider a company planning to expand its business by offering some new goods for sale in addition to existing ones. Before introducing new products or services, it is necessary to conduct market research and then select the most-profitable products/services. The company has data on the currently offered goods (i.e., sales volume, profitability, and popularity), only the data on the goods considered for introduction to the market are unknown. During pairwise comparisons, the different criteria could be taken into account.

The cases described above have two distinctive features: some of the alternatives considered have initially known priorities, and all the alternatives are compared with respect to more than one criterion. This observation leads us to the idea of the hierarchical HRE method. Such an approach would reduce the multi-criteria problem to a series of simple pairwise comparisons and calculate the weights of concepts related to individual criteria. Finally, the priorities of the alternatives are determined. Moreover, based on previously known estimations, we would be able to estimate actual values of different decision options. Hence, the meaning of the priority is not only relative but real, expressing the current value or unit. In general, using a hierarchy is nothing new. One of the first hierarchical methods —AHP—was introduced by Saaty and is still one of the most popular decision-making methods. The structure of the hierarchical HRE is shown in Figure 2.

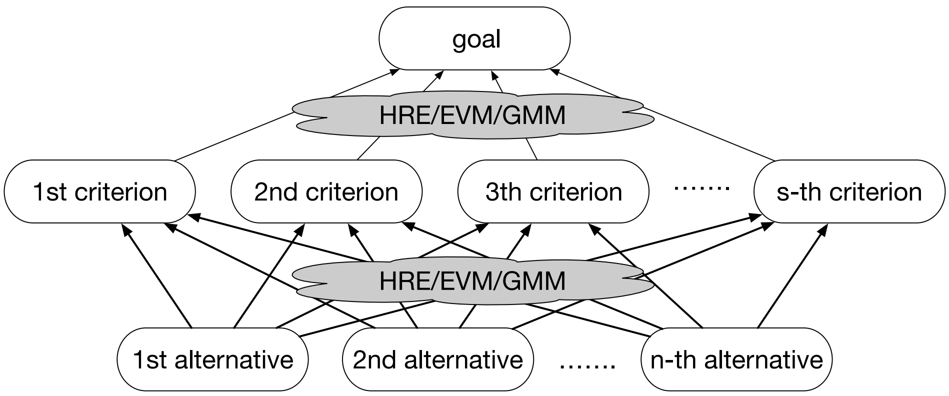

Figure 2. The structure of hierarchical HRE.

The primary data for the hierarchical HRE method are two sets: a set of criteria $Q = \{q_1, .., q_s\}$ and a set of alternatives $A = \{a_1, ..., a_n\}$. For each criterion q_t $(t \in \{1, 2, \ldots, s\})$ we construct a PC matrix $C^{(q_t)} = [c_{ij}^{(t)}]$ containing comparisons of elements from A. For this purpose, we establish a set of reference alternatives (with known priorities) $A_K^{(q_t)}$, and a set of unknown alternatives (i.e., with unknown priorities) $A_U^{(q_t)}$. Obviously, those sets for each $t \in \{1, 2, \ldots, s\}$, must satisfy the following conditions $A = A_K^{(q_t)} \cup A_U^{(q_t)}$, $A_K^{(q_t)} \cap A_U^{(q_t)} = \emptyset$. It is worth noting that the set of initially known alternatives and set of unknown alternatives may differ among criteria.

We construct the PC matrix $C^{(q_t)}$ based on pairwise comparisons of alternatives, where at least one is unknown. Then, we apply the HRE prioritization method for the matrix $C^{(q_t)}$. More specifically, we construct the following linear equation:

$$M^{(q_t)} w^{(q_t)} = b^{(q_t)}, \tag{11}$$

where $w^{(q_t)}$ is a priority vector of alternatives $a_i \in A_U^{(q_t)}$, $M^{(q_t)}$ is a $k_t \times k_t$ matrix, where k_t means the number of known alternatives ($k_t = \left|A_K^{(q_t)}\right|$) in the form

$$M^{(q_t)} = \begin{bmatrix} 1 & -\frac{1}{n-1}c_{1,2}^{(t)} & \cdots & -\frac{1}{n-1}c_{1,k_t}^{(t)} \\ -\frac{1}{n-1}c_{2,1}^{(t)} & \ddots & \cdots & -\frac{1}{n-1}c_{1,k_t}^{(t)} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n-1}c_{k_t,1}^{(t)} & -\frac{1}{n-1}c_{k_t,2}^{(t)} & \cdots & 1 \end{bmatrix}$$

and the constant term vector is given as

$$b^{(q_t)} = \begin{bmatrix} \frac{1}{n-1}c_{1,k_t+1}^{(t)} w_K^{(q_t)}(a_{k_t+1}^{(t)}) + \ldots + \frac{1}{n-1}c_{1,n}^{(t)} w_K^{(q_t)}(a_n^{(t)}) \\ \frac{1}{n-1}c_{2,k_t+1}^{(t)} w_K^{(q_t)}(a_{k_t+1}^{(t)}) + \ldots + \frac{1}{n-1}c_{2,n}^{(t)} w_K^{(q_t)}(a_n^{(t)}) \\ \vdots \\ \frac{1}{n-1}c_{k_t,k_t+1}^{(t)} w_K^{(q_t)}(a_{k_t+1}^{(t)}) + \ldots + \frac{1}{n-1}c_{k_t,n}^{(t)} w_K^{(q_t)}(a_n^{(t)}) \end{bmatrix}, \tag{12}$$

where $w_K^{(q_t)}$ is a priority vector for alternatives from $A_K^{(q_t)}$. We calculate the priority vector $w^{(q_t)}$ by solving the Equation (11).

In the geometric version of the HRE approach [4] for each criterion q_t and the appropriate PC matrix $C^{(q_t)}$, we construct the following linear equation:

$$N^{(q_t)} \mu^{(q_t)} = d^{(q_t)}, \tag{13}$$

where

$$N^{(q_t)} = \begin{bmatrix} (n-1) & -1 & \cdots & -1 \\ -1 & \ddots & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & (n-1) \end{bmatrix}$$

is a matrix of dimensions $k_t \times k_t$, ($k_t = \left|A_K^{(q_t)}\right|$) and

$$d^{(q_t)} = \begin{bmatrix} log\left(c_{1,2}^{(t)} c_{1,3}^{(t)} \cdot \ldots \cdot c_{1,k_t}^{(t)} c_{1,k_t+1}^{(t)} w(a_{k_t+1}^{(t)}) \cdot \ldots \cdot c_{1,n}^{(t)} w(a_n^{(t)})\right) \\ log\left(c_{2,1}^{(t)} c_{2,3}^{(t)} \cdot \ldots \cdot c_{2,k_t}^{(t)} c_{2,k_t+1}^{(t)} w(a_{k_t+1}^{(t)}) \cdot \ldots \cdot c_{2,n}^{(t)} w(a_n^{(t)})\right) \\ \vdots \\ log\left(c_{k_t,1}^{(t)} c_{k_t,2}^{(t)} \cdot \ldots \cdot c_{k_t,k_t-1}^{(t)} c_{k_t,k_t+1}^{(t)} w(a_{k_t+1}^{(t)}) \cdot \ldots \cdot c_{k_t,n}^{(t)} w(a_n^{(t)})\right) \end{bmatrix},$$

and $\mu^{(q_t)}$ is a logarithmized priority vector of alternatives $a_i \in A_U^{(q_t)}$. We compute the vector $\mu^{(q_t)}$ by solving (13) and then we calculate the vector $w_U^{(q_t)} = \left(w_U^{(q_t)}(a_1^{(t)}), \ldots,\ w_U^{(q_t)}(a_{k_t}^{(t)})\right)$ using the dependency

$$\mu^{(q_t)} = \left(\log w_U^{(q_t)}(a_1^{(t)}), \ldots, \log w_U^{(q_t)}(a_{k_t}^{(t)})\right) \text{ for every } a_i \in A_U^{(q_t)}. \tag{14}$$

Based on the priorities given in $w_U^{(q_t)}$, we create an n dimensional vector $w^{(q_t)}$, being a catenation of a priori known and just calculated priorities:

$$w^{(q_t)} = \left[w_U^{(q_t)}, w_K^{(q_t)}\right]^T.$$

Similarly, as in AHP, when calculating the final priority vector, we must take into account the fact that the criteria $q_1, ..., q_s$ may contribute at different extents to the goal. Thus, we construct a PC matrix $\widehat{C}$, in which every single entry $\widehat{c}_{kl}$ corresponds to the result of comparison q_k against q_l. Then (using the HRE approach), we calculate the priority vector

$$\widehat{w} = [\widehat{w}(q_1), \ldots, \widehat{w}(q_s)]^T.$$

The final vector of weights is a linear combination of vectors $w^{(q_t)}$ where scaling factors come from $\widehat{w}$, i.e.,

$$w = \sum_{t=1}^{s} \widehat{w}(q_t) w^{(q_t)}. \tag{15}$$

Note that if an additive approach is used, the existence of a solution to each HRE Equation (7) used must be verified.

6. Numerical Examples

The HRE method can be used independently or with other strategies for calculating priorities for pairwise comparison matrices. In the first example, we will show the use of HRE in a model with three criteria and seven alternatives. In the second, the HRE approach is combined with the EVM method in a single hierarchical multi-criteria model.

Example 1. *A company, managed by Mr. Smith, runs a sports facility consisting of a sports swimming pool, gym and fitness club. The company is developing dynamically. Therefore, Mr. Smith intends to expand the facility with two further investments. He considers a bowling alley, a professional massage parlor, a trampoline fitness point, or a recreational pool. He would spend a similar amount of money on each project. However, to increase profitability, Mr. Smith intends to conduct a market survey first. Market research will be carried out based on several criteria: $Q = \{pr$ — average monthly income (profitability), du — durability of the equipment (how many months it will serve), and pop — possible increase in the popularity of a given form of spending free time (also as a result of advertising campaigns)$\}$. A group of management professionals (from now on referred to as experts) will apply the HRE hierarchical approach to the analysis of the obtained data. Let $A = \{a_1$ — a bowling alley, a_2 — a professional massage salon, a_3 — a trampoline fitness point, a_4 — a recreational pool, a_5 — a sports pool, a_6 − gym, and a_7 — fitness club$\}$. The set of reference alternatives is the same for each criterion $A_K = \{a_5, a_6, a_7\}$.*

The first considered criterion is profitability (pr). For objects from the reference set, the values $w^{(pr)}(a_5) = 20$ thous., $w^{(pr)}(a_6) = 12$ thous., $w^{(pr)}(a_7) = 9$ thous. represent the average monthly income from the previous year. Considering profitability, experts create a PC matrix $C^{(pr)}$. Each c_{ij} in the matrix $C^{(pr)}$ corresponds to the relative profitability of a_i with respect to a_j. We calculate the values c_{ij} for $i, j \in \{5, 6, 7\}$, $i \neq j$ as $\frac{w(a_i)}{w(a_j)}$, therefore

$$C^{(pr)} = \begin{bmatrix} 1 & \frac{2}{3} & 2 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{3}{2} \\ \frac{3}{2} & 1 & 2 & \frac{2}{3} & \frac{1}{2} & 1 & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{3} & \frac{1}{4} & 1 & \frac{2}{3} \\ 2 & \frac{3}{2} & 3 & 1 & \frac{2}{3} & \frac{3}{2} & 1 \\ 2 & 2 & 4 & \frac{2}{3} & 1 & \frac{20}{12} & \frac{20}{9} \\ 1 & 1 & 1 & \frac{3}{2} & \frac{12}{20} & 1 & \frac{20}{9} \\ \frac{2}{3} & \frac{1}{2} & \frac{3}{2} & 1 & \frac{9}{20} & \frac{9}{20} & 1 \end{bmatrix}.$$

In order to estimate the vector $w^{(pr)}$, we use the HRE method, so according to (11) we construct the matrix $M^{(pr)}$ and vector $b^{(pr)}$

$$M^{(pr)} = \begin{bmatrix} 1 & -\frac{1}{n-1}c_{1,2} & -\frac{1}{n-1}c_{1,3} & -\frac{1}{n-1}c_{1,4} \\ -\frac{1}{n-1}c_{2,1} & 1 & -\frac{1}{n-1}c_{2,3} & -\frac{1}{n-1}c_{2,4} \\ -\frac{1}{n-1}c_{3,1} & -\frac{1}{n-1}c_{3,2} & 1 & -\frac{1}{n-1}c_{3,4} \\ -\frac{1}{n-1}c_{4,1} & -\frac{1}{n-1}c_{4,2} & -\frac{1}{n-1}c_{4,3} & 1 \end{bmatrix}$$

$$b^{(pr)} = \begin{bmatrix} \frac{1}{n-1}c_{1,5}w^{(pr)}(a_5) + \frac{1}{n-1}c_{1,6}w^{(pr)}(a_6) + \frac{1}{n-1}c_{1,7}w^{(pr)}(a_7) \\ \frac{1}{n-1}c_{2,5}w^{(pr)}(a_5) + \frac{1}{n-1}c_{2,6}w^{(pr)}(a_6) + \frac{1}{n-1}c_{2,7}w^{(pr)}(a_7) \\ \frac{1}{n-1}c_{3,5}w^{(pr)}(a_5) + \frac{1}{n-1}c_{3,6}w^{(pr)}(a_6) + \frac{1}{n-1}c_{3,7}w^{(pr)}(a_7) \\ \frac{1}{n-1}c_{4,5}w^{(pr)}(a_5) + \frac{1}{n-1}c_{4,6}w^{(pr)}(a_6) + \frac{1}{n-1}c_{4,7}w^{(pr)}(a_7) \end{bmatrix}.$$

The Equation (11) $M^{(pr)}w^{(pr)} = b^{(pr)}$, written numerically, takes the form

$$\begin{bmatrix} 1 & -0.111 & -0.333 & -0.083 \\ -0.25 & 1 & -0.333 & -0.111 \\ -0.083 & -0.083 & 1 & -0.056 \\ -0.333 & -0.25 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} w^{(pr)}(a_1) \\ w^{(pr)}(a_2) \\ w^{(pr)}(a_3) \\ w^{(pr)}(a_4) \end{bmatrix} = \begin{bmatrix} 5.917 \\ 6.667 \\ 3.833 \\ 6.722 \end{bmatrix}$$

After solving the equation, we obtain the following weights $w^{(pr)}(a_1) = 11.164$, $w^{(pr)}(a_2) = 13.667$, $w^{(pr)}(a_3) = 6.863$, $w^{(pr)}(a_4) = 17.292$, the values of which represent the estimated average monthly income for the objects a_1, a_2, a_3, a_4. After normalization, the resulting weights are

$$w^{(pr)} = [\ 0.124 \quad 0.152 \quad 0.076 \quad 0.192 \quad 0.222 \quad 0.133 \quad 0.1\ ]^T.$$

According to the second considered criterion (du), we compare objects in terms of the durability of the equipment. The values c_{ij} in the matrix $C^{(du)}$ indicate how many times the durability of the equipment of the object a_i is better than the durability of the equipment of the object a_j. For reference alternatives, the values $w^{(du)}(a_5) = 72$, $w^{(du)}(a_6) = 24$, and $w^{(du)}(a_7) = 36$ represent the average number of months the appliance has been used in the last few years. The PC matrix $C^{(du)}$ prepared by the experts has the form

$$C^{(du)} = \begin{bmatrix} 1 & 3 & 2 & 1 & \frac{1}{2} & 2 & \frac{3}{2} \\ \frac{1}{3} & 1 & \frac{3}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} & 1 & \frac{1}{3} & \frac{1}{4} & \frac{2}{3} & \frac{1}{2} \\ 1 & 3 & 3 & 1 & \frac{4}{5} & 2 & 2 \\ 2 & 4 & 4 & \frac{5}{4} & 1 & \frac{72}{24} & \frac{72}{36} \\ \frac{1}{2} & 2 & \frac{3}{2} & \frac{1}{2} & \frac{24}{72} & 1 & \frac{24}{36} \\ \frac{2}{3} & 2 & 2 & \frac{1}{2} & \frac{36}{72} & \frac{36}{24} & 1 \end{bmatrix}.$$

Using HRE for the matrix $C^{(du)}$, similarly to the first criterion (see (11) and (12)), we obtain the matrix $M^{(du)}$ and vector $b^{(du)}$:

$$M^{(du)} = \begin{bmatrix} 1 & -0.5 & -0.333 & -0.167 \\ -0.056 & 1 & -0.25 & -0.056 \\ -0.083 & -0.111 & 1 & -0.056 \\ -0.167 & -0.5 & -0.5 & 1 \end{bmatrix},$$

$$b^{(du)} = \begin{bmatrix} 23 \\ 8 \\ 8.667 \\ 29.6 \end{bmatrix}.$$

Solving the equation $M^{(du)}w^{(du)} = b^{(du)}$ gives values that indicate the estimated average durability in months:

$$w^{(du)}(a_1) = 47.183,\ w^{(du)}(a_2) = 18.119,\ w^{(du)}(a_3) = 17.688,\ w^{(du)}(a_4) = 55.367,$$

Scaling the solution vector gives

$$w^{(du)} = [\ 0.174 \quad 0.067 \quad 0.065 \quad 0.205 \quad 0.266 \quad 0.089 \quad 0.133\ ]^T.$$

Analysis of the third criterion—the possibility of increasing the popularity of activities offered by individual facilities (pop)*—the values* c_{ij} *in the matrix* $C^{(pop)}$ *mean how many times the popularity of the object* a_i *can increase in relation to the increase in the popularity of the object* a_j*. The reference alternatives have the values* $w^{(pop)}(a_5) = 5$, $w^{(pop)}(a_6) = 20$, $w^{(pop)}(a_7) = 25$*, and represent the average percentage increase in popularity in recent years. In this case, the PC-matrix* $C^{(pop)}$ *proposed by the experts is*

$$C^{(pop)} = \begin{bmatrix} 1 & 1 & \frac{1}{5} & 3 & 3 & 2 & 2 \\ 1 & 1 & \frac{1}{3} & 3 & 3 & 2 & 2 \\ 5 & 3 & 1 & 4 & 5 & 3 & 3 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} & 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{5} & \frac{2}{3} & 1 & \frac{5}{20} & \frac{5}{25} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 2 & \frac{20}{5} & 1 & \frac{20}{25} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 2 & \frac{25}{5} & \frac{25}{20} & 1 \end{bmatrix}.$$

The auxiliary matrix $M^{(pop)}$ *and the vector* $b^{(pop)}$ *derived from the matrix* $C^{(pop)}$ *are*

$$M^{(pop)} = \begin{bmatrix} 1 & -0.167 & -0.033 & -0.5 \\ -0.167 & 1 & -0.056 & -0.5 \\ -0.833 & -0.5 & 1 & -0.667 \\ -0.056 & -0.056 & -0.042 & 1 \end{bmatrix},$$

$$b^{(pop)} = \begin{bmatrix} 17.5 \\ 17.5 \\ 26.667 \\ 5 \end{bmatrix}.$$

Using the HRE method to solve the equation $M^{(pop)}w^{(pop)} = b^{(pop)}$*, we obtain the following values:*

$$w^{(pop)}(a_1) = 31.459, \; w^{(pop)}(a_2) = 32.93, \; w^{(pop)}(a_3) = 77.21, \; w^{(pop)}(a_4) = 11.794,$$

These numbers represent the estimated average growth in popularity of objects a_1, a_2, a_3, a_4*. After normalization, the solution vector of the weights is equal:*

$$w^{(du)} = [\; 0.155 \quad 0.162 \quad 0.38 \quad 0.058 \quad 0.025 \quad 0.098 \quad 0.123 \;]^T.$$

Since in the considered case the weights of the criteria are known (in the past, the company has encountered comparable cases and has used similar weights to address this set of criteria) and equal

$$\hat{w} = [\; 0.5 \quad 0.2 \quad 0.3 \;]^T.$$

the final priorities of individual objects are calculated according to the schema (15):

$$w(a_i) = 0.5 \cdot w^{(pr)}(a_i) + 0.2 \cdot w^{(du)}(a_i) + 0.3 \cdot w^{(pop)}(a_i).$$

So the final weight vector is equal

$$w = [\; 0.143 \quad 0.138 \quad 0.165 \quad 0.154 \quad 0.172 \quad 0.114 \quad 0.114 \;]^T,$$

which means that investing in a trampoline fitness point and a recreational pool may be slightly more profitable than any of the other two new facilities.

Example 2. *Mr. Kowalski became interestedin collecting porcelain. He paid particular attention to the cups. Mr. Kowalski visited a local antique store and found five exciting items. He would* *like to buy two of them, which are the most valuable. So, he asked our art appraiser's office for help in this matter. Unfortunately, there is no universal algorithm to value porcelain step by step. Moreover, the valuation of porcelain is complex and unmeasurable due to the peculiarities of different markets. Identical cups can have different prices depending on the region where they are offered. For example, Polish collectors are fond of German products, which undervalue Polish porcelain. At the same time, the English are proud of their wares and are eager to buy them, and they also value Chinese porcelain. Therefore, when analyzing trends in the porcelain market, we should also consider the place of purchase and ask an expert to evaluate the various criteria by which they make their purchasing decision. Let us assume that Mr. Kowalski lives in Poland, so we will take the values of the different criteria according to the Polish market and help him make the right decision. Our market research shows that the final value of mugs here depends on several factors.*

The most straightforward and obvious rule is that the more difficult it is to obtain products from a specific series, period, or manufacturer, the higher the price it receives. Due to the fragility of porcelain, whole sets are more expensive than single pieces. Some units have special status: they are almost unattainable on the market, are valued as "unlimited", and are usually auctioned at exorbitant prices. The products from the first period of operation of the factory in Meissen, Germany, run by the "father of European porcelain", Friedrich Böttger, are a good example. Moreover, products of a particular manufacturer may be more in demand at one time and less at another.

The second criterion is the reputation of the manufacturer. It is closely influenced by the history of a given factory, the quality of the porcelain produced there, or its uniqueness. It is also worth remembering that reputation is tied to the current fashion—it depends on time and location. For example: in Germany, Meissen porcelain is the most valued, while in Poland, collectors like Rosenthal more.

Traces, called signatures, can often be found on porcelain. They define its origin, age, model name, and sometimes the project's author. Some signatures even specify the type of porcelain the item is made of. Thanks to the signature, we can estimate the value of the product. Usually, the older it is, the more valuable it is. The first characters were discovered on Chinese porcelain from the 14th century. The well-known collector's rarities include, among others, unique items from the Ming Dynasty. There are a maximum of a dozen of them in museum collections worldwide. They are almost nonexistent at auctions; they reach exorbitant prices if they appear. In Europe, the first porcelain products were made in Saxony in the 18th century during the reign of Augustus II and were marked with letters and numbers. Soon, other manufacturers started putting their signatures on the bottom of dishes. Depending on the creation time, the signatures differ in color and how they were made (painting, printing, decal). For example, the markings also changed within one factory with a change of ownership. On their basis, the experienced collector can accurately determine the creation time.

Ancient porcelain is exceptionally fragile, which, of course, increases its value. However, in the event of damage, even porcelain from an excellent manufacturer is worthless. Therefore, before buying, you should evaluate the condition of a single piece of porcelain. An ideal copy will be a product with no acquired or primary defects. By primary defects, we mean any flaws visible in the product that are traces of an ineffective porcelain forming process - stains, painting, or decals errors. Reputable manufacturers sold only first-class products. For example, all defective products are considered broken at Rosenthal factories, so they do not end up on the market. Unlike acquired defects, primary defects usually do not influence the product's price. Acquired defects affecting porcelain price include nicks, sticking together, painting with other paint, or visible spider web shape cracks.

The quality of porcelain is primarily determined by the appropriate selection of raw materials, especially a large amount of kaolin. Good porcelain is hard, durable, and scratch-resistant. It requires manual work as machines do not provide a top-quality finish. Therefore, each item is processed precisely and accurately by hand in good factories. It guarantees the highest quality and timeless beauty of the final product. Ornaments on porcelain also determine its quality. The most valuable is hand-decorated porcelain. Often, ornaments are made of noble materials—real gold or, for example, platinum.

Based on the above information, we will compare the value of considered porcelain cups against five criteria: factory, period of creation, uniqueness, state of preservation and quality. Let

$$Q = \{man, per, un, st, qua\}$$

denote a set of significant criteria, where man, per, un, st, qua *mean factory, period of creation, uniqueness, state of preservation and quality, respectively.*

Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ *denote a considered collection of cups. First, we compare the cups in terms of the first criterion. Depending on the reputation of the factory, each cup is assigned a value from 1 to 10, where 1 denotes the lowest possible score. All cups come from different factories. We know three of them (for the alternatives* a_3*,* a_4 *and* a_5*) and we can assign values to them based on our research (e.g., the indicated values may be the average list price of cups produced by the indicated manufacturers expressed in tens of dollars). They are* $w^{(man)}(a_3) = 8.7$*,* $w^{(man)}(a_4) = 4.2$*,* $w^{(man)}(a_5) = 7.2$*, respectively. As it is the first time that we are dealing with the products of manufacturers from which the cups* a_1 *and* a_2 *come from, we will apply the HRE procedure. As a result, we obtain the matrix:*

$$C^{(man)} = \begin{bmatrix} 1 & 2 & \frac{1}{3} & 3 & 1 \\ \frac{1}{2} & 1 & \frac{1}{5} & 1 & \frac{1}{2} \\ 3 & 5 & 1 & \frac{8.7}{4.2} & \frac{8.7}{7.2} \\ \frac{1}{3} & 1 & \frac{4.2}{8.7} & 1 & \frac{4.2}{7.2} \\ 1 & 2 & \frac{7.2}{8.7} & \frac{7.2}{4.2} & 1 \end{bmatrix}.$$

Since the weight of cups a_3*,* a_4 *and* a_5 *are known, the pairs* (a_3, a_4)*,* (a_3, a_5) *and* (a_4, a_5) *are not evaluated, but* $c_{ij}^{(man)}$ *is calculated as* $\frac{w^{(man)}(i)}{w^{(man)}(j)}$ $(3 \leq i < j \leq 5)$*. To estimate the expected values of factories A and B, we follow the HRE procedure. Since* $A_U^{((man)} = \{a_1, a_2\}$ *and* $A_K^{(man)} = \{a_3, a_4, a_5\}$*, the matrix* $M^{(man)}$ *and the constant term vector* $b^{(man)}$ *have the form (see (11))*

$$M^{(man)} = \begin{bmatrix} 1 & -\frac{1}{n-1}c_{1,2} \\ -\frac{1}{n-1}c_{2,1} & 1 \end{bmatrix},$$

$$b^{(man)} = \begin{bmatrix} \frac{1}{n-1}c_{1,3}w^{(man)}(a_3) + \frac{1}{n-1}c_{1,4}w^{(man)}(a_4) + \frac{1}{n-1}c_{1,5}w^{(man)}(a_5) \\ \frac{1}{n-1}c_{2,3}w^{(man)}(a_3) + \frac{1}{n-1}c_{2,4}w^{(man)}(a_4) + \frac{1}{n-1}c_{2,5}w^{(man)}(a_5) \end{bmatrix}.$$

Hence, numerically,

$$M^{(man)} = \begin{pmatrix} 1 & -0.476 \\ -0.119 & 1 \end{pmatrix}, \; b^{(man)} = \begin{bmatrix} 5.675 \\ 2.385 \end{bmatrix}.$$

As $M^{(man)}w^{(man)} = b^{(man)}$ *has an admissible solution (i.e., real and positive), the ranking vector* $w^{(man)}$ *takes the form*

$$w^{(man)} = [\; 7.22 \quad 3.244 \quad 8.7 \quad 4.2 \quad 7.2 \;]^T.$$

Hence, according to experts' judgments $w^{(man)}(a_1) = 7.22$ *and* $w^{(man)}(a_2) = 3.244$*. After rescaling*

$$w^{(man)} = [\; 0.236 \quad 0.106 \quad 0.284 \quad 0.137 \quad 0.235 \;]^T.$$

The second considered criterion applies to the date the cup was manufactured. Based on the signatures placed on the cups a_1*,* a_3 *and* a_5*, we can read the years of production and assign the appropriate number of points according to age. The older the cup, the more points it becomes. Finally, we receive the results* $w^{(per)}(c_1) = 8$*,* $w^{(per)}(c_3) = 10$*,* $w^{(per)}(c_5) = 5$*. The cup* a_2 *has no signature and the signature of the cup* a_4 *does not include the year, so in order to establish the period of creation* *for these two cups we ask an expert for help. In the case of this criterion* $A_U^{(per)} = \{a_2, a_4\}$ *and* $A_K^{(per)} = \{a_1, a_3, a_5\}$*. As a result of the expert's assessment, we obtain the matrix:*

$$C^{(per)} = \begin{bmatrix} 1 & 2 & \frac{8}{10} & 4 & \frac{8}{5} \\ \frac{1}{2} & 1 & \frac{1}{3} & 2 & 1 \\ \frac{10}{8} & 3 & 1 & 5 & \frac{10}{5} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{5} & 1 & \frac{1}{2} \\ \frac{5}{8} & 1 & \frac{5}{10} & 2 & 1 \end{bmatrix}.$$

In order to obtain the values of the priority vector $w^{(per)}$ *for cups* a_2 *and* a_4*, it is enough to create the matrix* $M^{(per)}$ *and vector* $b^{(per)}$ *using the HRE method and then solve the equation* $M^{(per)}w^{(per)} = b^{(per)}$*. The matrix* $M^{(per)}$ *can be obtained from the matrix* $C^{(per)}$ *by removing the columns and rows corresponding to known alternatives from the set* $A_K^{(per)}$ *and multiplying values out of a diagonal by* $-\frac{1}{n-1}$*. The values removed only from the columns (but not from the rows) are used to form the vector* $b^{(per)}$*. Thus,*

$$M^{(per)} = \begin{bmatrix} 1 & -\frac{1}{n-1}c_{2,4} \\ -\frac{1}{n-1}c_{4,2} & 1 \end{bmatrix},$$

$$b^{(per)} = \begin{bmatrix} \frac{1}{n-1}c_{2,1}w^{(per)}(a_1) + \frac{1}{n-1}c_{2,3}w^{(per)}(a_3) + \frac{1}{n-1}c_{2,5}w^{(per)}(a_5) \\ \frac{1}{n-1}c_{4,1}w^{(per)}(a_1) + \frac{1}{n-1}c_{4,3}w^{(per)}(a_3) + \frac{1}{n-1}c_{4,5}w^{(per)}(a_5) \end{bmatrix},$$

and the equation $M^{(per)}w^{(per)} = b^{(per)}$ *takes the shape*

$$\begin{pmatrix} 1 & -0.498 \\ -0.124 & 1 \end{pmatrix} \begin{bmatrix} w^{(per)}(a_2) \\ w^{(per)}(a_4) \end{bmatrix} = \begin{pmatrix} \frac{37}{12} \\ \frac{13}{8} \end{pmatrix}.$$

The solution of the above equation is the following vector:

$$w^{(per)} = \begin{bmatrix} 8 & 4.15 & 10 & 2.14 & 5 \end{bmatrix}^T.$$

Thus, in the category "period of creation", the expected number of points on a scale from 1 to 10 for cups a_2 *and* a_4 *are* 4.15 *and* 2.14*, respectively. After normalization,*

$$w^{(per)} = \begin{bmatrix} 0.273 & 0.141 & 0.341 & 0.073 & 0.171 \end{bmatrix}^T.$$

The third considered criterion concerns uniqueness, although this time the set of reference alternatives is empty. Thus, once again, we ask an expert to compare in pairs the profitability of purchasing a given cup in terms of its uniqueness on the market. His assessments form the following PC matrix:

$$C^{(un)} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & 2 & 1 & \frac{1}{2} \\ 2 & \frac{1}{2} & 1 & 1 & \frac{1}{2} \\ 2 & 1 & 1 & 1 & \frac{1}{2} \\ 3 & 2 & 2 & 2 & 1 \end{bmatrix}.$$

In such a case, we can use EVM (Section 3.1), i.e., solve the equation $C^{(un)}w = \lambda_{max}w$*. Hence, after appropriate rescaling of the vector* w*, we obtain*

$$w^{(un)} = \begin{bmatrix} 0.097 & 0.214 & 0.161 & 0.182 & 0.345 \end{bmatrix}^T.$$

The condition of the cup depends mainly on two factors: primary damage and acquired damage. Therefore, the experts assess all our alternatives against two sub-criteria: level of primary damage and level of acquired damage. As a result of comparing the degree of primary damage, we obtain the following matrix:

$$C^{(pd)} = \begin{pmatrix} 1 & 9 & 9 & \frac{8}{3} & \frac{5}{3} \\ \frac{1}{9} & 1 & \frac{4}{9} & \frac{1}{9} & \frac{1}{8} \\ \frac{1}{9} & \frac{9}{4} & 1 & \frac{1}{9} & \frac{1}{8} \\ \frac{3}{8} & 9 & 9 & 1 & 2 \\ \frac{3}{5} & 8 & 8 & \frac{1}{2} & 1 \end{pmatrix}.$$

The comparisons of acquired damage are as follows:

$$C^{(ad)} = \begin{pmatrix} 1 & \frac{1}{9} & \frac{1}{9} & \frac{4}{5} & \frac{1}{9} \\ 9 & 1 & \frac{3}{4} & 9 & \frac{6}{5} \\ 9 & \frac{4}{3} & 1 & 9 & \frac{9}{4} \\ \frac{5}{4} & \frac{1}{9} & \frac{1}{9} & 1 & \frac{1}{6} \\ 9 & \frac{5}{6} & \frac{4}{9} & 6 & 1 \end{pmatrix}.$$

The priority vectors for the $C^{(pd)}$ and $C^{(ad)}$ matrices calculated using EVM are as follows:

$$w^{(pd)} = \begin{bmatrix} 0.416 & 0.029 & 0.04 & 0.289 & 0.223 \end{bmatrix}^T,$$

and

$$w^{(ad)} = \begin{bmatrix} 0.033 & 0.301 & 0.39 & 0.038 & 0.235 \end{bmatrix}^T.$$

Comparing the significance of both types of damage, the expert found that the acquired damage is two times more important than the primary damage. Hence, the ranking of both damage significance (ds) sub-criteria is as follows:

$$w^{(ds)} = \begin{bmatrix} w^{(ds)}(pd) \\ w^{(ds)}(ad) \end{bmatrix} = \begin{bmatrix} 0.333 \\ 0.666 \end{bmatrix}.$$

The state of preservation for the i-th cup is calculated as

$$w^{(st)}(a_i) = w^{(ds)}(pd) \cdot w^{(pd)}(a_i) + w^{(ds)}(ad) \cdot w^{(ad)}(a_i),$$

for $i = 1, \ldots, 5$, hence, finally the assessment of the state of the cups' preservation is as follows:

$$w^{(st)} = \begin{bmatrix} 0.16 & 0.211 & 0.27 & 0.12 & 0.23 \end{bmatrix}^T.$$

When assessing the quality of porcelain, the expert considered its resistance to scratching. In the case of two cups a_1 and a_2, it was known and calculated as the average number of significant scratches among a similar class of products appearing at online auctions in the last five years. Thus, $A_U^{(qua)} = \{a_3, a_4, a_5\}$ and $A_K^{(qua)} = \{a_1, a_2\}$, where $w^{(qua)}(a_1) = 5.7$ and $w^{(qua)}(a_2) = 2.4$. As a result of comparing the quality of alternatives by an expert, we obtain the matrix

$$C^{(qua)} = \begin{pmatrix} 1 & \frac{5.7}{2.4} & 9 & 3 & \frac{7}{5} \\ \frac{2.4}{5.7} & 1 & 9 & \frac{9}{5} & \frac{8}{3} \\ \frac{1}{9} & \frac{1}{9} & 1 & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{3} & \frac{5}{9} & 9 & 1 & \frac{6}{5} \\ \frac{5}{7} & \frac{3}{8} & 9 & \frac{5}{6} & 1 \end{pmatrix}.$$

Based on the HRE approach, the auxiliary matrix and the constant term vector are

$$M^{(qua)} = \begin{pmatrix} 1 & -0.0259 & -0.0259 \\ -2.104 & 1 & -0.28 \\ -2.104 & -0.194 & 1 \end{pmatrix}, \ b^{(qua)} = \begin{pmatrix} 0.225 \\ 0.808 \\ 1.242 \end{pmatrix}.$$

Solving the HRE equation $M^{(qua)}w = b^{(qua)}$ *reveals that the expected number of scratches for* a_3, a_4 *and* a_5 *are* $0.344, 2.2$ *and* 2.39. *This allows us to form the ranking vector*

$$\widehat{w}^{(qua)} = \begin{bmatrix} 5.7 & 2.4 & 0.344 & 2.2 & 2.39 \end{bmatrix}^T.$$

However, since the ranking values represent the expected number of scratches, it means that the higher the value, the worse the product. To change this, we will raise each of the elements of $\widehat{w}^{(qua)}$ *to the* -1 *power and then normalize it. As a result, we obtain the vector:*

$$w^{(qua)} = \begin{bmatrix} 0.04 & 0.0954 & 0.665 & 0.104 & 0.0955 \end{bmatrix}^T.$$

It is worth noting that, thanks to the last transformation, if the i-th alternative is two times worse than the j-th one according to $\widehat{w}^{(qua)}$*, then according to* $w^{(qua)}$*, it is two times better than the j-th alternative, etc. The obtained vector is a ranking of the cups against the porcelain quality criterion.*

In order to establish the weights of criteria ($q_1 = man$, $q_2 = per$, $q_3 = un$, $q_4 = st$, $q_5 = qua$) *the expert constructed a PC-matrix, in which* $\widehat{c}_{ij}$ *denotes how many times the criterion* q_i *is more preferred than the criterion* q_j.

$$\widehat{C} = \begin{bmatrix} 1 & 1 & \frac{3}{2} & 3 & 3 \\ 1 & 1 & \frac{3}{2} & 3 & 3 \\ \frac{2}{3} & \frac{2}{3} & 1 & 2 & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & 1 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & 1 & 1 \end{bmatrix}$$

Then, using EVM again, we obtain a priority vector for the criteria:

$$\hat{w} = \begin{bmatrix} 0.3 & 0.3 & 0.2 & 0.1 & 0.1 \end{bmatrix}^T.$$

The final weights of individual cups $a_1, \ldots, a_5$ *are calculated according to the well-known schema:*

$$w(a_i) = 0.3w^{(man)}(a_i) + 0.3w^{(per)}(a_i) + 0.2w^{(un)}(a_i) + 0.1w^{(st)}(a_i) + 0.1w^{(qua)}(a_i).$$

Thus, we obtain

$$w = \begin{bmatrix} 0.1922 & 0.1478 & 0.3138 & 0.1222 & 0.2237 \end{bmatrix}^T.$$

The recommendation presented to Mr. Kowalski puts in the first place the cup a_3 *with the rank* 0.3138*, followed by* a_5 *with the rank* 0.2237*, then* a_1, a_2 *and* a_4 *with the ranking values* 0.1922, 0.1478 *and* 0.1222.

7. Discussion and Summary

We demonstrate the hierarchical HRE as a multiple-criteria decision-making method based on the same assumptions found in EVM or GMM. Unlike AHP, HRE allows the use of actual measured values. HRE fills a gap by combining tangible and intangible criteria within a single hierarchical model. The ranking obtained in HRE maintains the same scales on the input and output of the priority deriving procedure, e.g., if the reference values are expressed in a particular currency, the vector of weights in the output contains the "valuation" of the alternatives for that specific currency. This property makes HRE an exciting proposition for anyone who needs a well-defined scale in the output of a ranking procedure. Combining HRE with other methods can relativize the outcome and make the loss of a "unit" at the output of the ranking procedure. Future research will explore uncertain knowledge representations and hybrid multi-criteria models, providing practical and theoretical insights. Guidelines will be developed to facilitate decision-making models using HRE.

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