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Travelling Waves in the Ring of Coupled Oscillators with Delayed Feedback

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Abstract: We studied travelling waves in N nonlinear differential equations with a delay and large parameter. This system is important because it can be regarded as a phenomenological model of N-coupled neuron-like oscillators with delay. The problem of the existence of travelling-wave-type solutions was reduced to the study of the dynamics of an auxiliary equation with two delays. Using a special asymptotic method for the large parameter we proved that this equation has a relaxation cycle, studied its properties (amplitude, period and asymptotics) and found the sufficient stability conditions. Based on this periodic solution the travelling waves of the initial model were constructed.

Keywords: multiple delays; relaxation oscillations; travelling wave; rotating wave; asymptotics; large parameter

MSC: 34K13, 34K25

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1. Introduction

Differential equations with delay arise in many applied problems in biology, physics, medicine and ecology [1–7].

A single equation of the form

$$\dot{x} + \nu x = \lambda F(x(t - T_1)), \tag{1}$$

where ν, λ, and T_1 are positive parameters and F is a nonlinear function plays an important role in radiophysical and biological applications. For example, in radiophysics this model simulates a generator with a first-order low-pass filter and delayed feedback [8,9]. Such generators are used in the manufacture of D-amplifiers, sonars and noise radars [8]. Moreover, model (1) simulates a biological process where the single state variable x decays at a present rate ν proportional to x, and is produced at a rate dependent on its the value at some time in the past [10]. Such processes arise in many areas in biology, such as in population biology, neurophysiology and metabolic regulation [10]. If the function F is compactly supported and λ is large enough equation (1) has a relaxation cycle [11]. Note, that despite the fact that only the first derivative is present in the equation (there is no second derivative), model (1) has oscillatory regimes for many nonlinear functions F. (See, for example, [2,8–11]).

In this paper we considered a unidirectionally coupled ring

$$\dot{x}_k + x_k = \lambda F(x_k(t - T_1)) + \gamma(x_{k-1} - x_k), \quad k = 1, \dots, N, \quad x_0 \equiv x_N, \tag{2}$$

of N oscillators of the form (1) with parameter $\nu = 1$. Here, parameter λ is positive and sufficiently large ($\lambda \gg 1$), delay time T_1 and coupling parameter γ are positive, and feedback function F is a compactly supported positive function:

$$F(y) = \begin{cases} f(y), & \text{if } y \in [-p, p], \\ 0, & \text{if } y < -p \text{ or } y > p, \end{cases}$$

where p is some positive constant, and $f(y)$ is a piece-wise continuous and bounded function on the segment $y \in [-p, p]$ and $f(y) > 0$ for all $y \in (0, p]$. In the segment $y \in [-p, 0]$ it may change its sign, but it can't be zero on any interval of the nonzero length.

The study of the dynamics of coupled oscillators is of great interest because they simulate a lot of processes in different areas of science; for example, they simulate the work of the heart or neurons in physiology and the rotation of generators in electrical grids in physics [12–14].

In the papers of [15,16] the nonlocal dynamics of model (2) when $N = 2$, $\lambda \gg 1$ and an asymptotically small γ were studied. It was shown that studying the existence and stability of the periodic solutions of this infinite-dimensional problem may be reduced to studying the dynamics of constructed finite-dimensional mappings, which were constructed for different orders of smallness of parameter γ on λ. The asymptotics of the periodic solutions of the initial model were found, and multistability was proven.

The nonlocal dynamics of the ring of diffusion-coupled oscillators

$$\dot{x}_k + x_k = \lambda F(x_k(t - T_1)) + \gamma(x_{k-1} - 2x_k + x_{k+1}), \quad k = 1, \ldots, N, \quad x_0 \equiv x_N, x_{N+1} \equiv x_1, \tag{3}$$

under condition $\lambda \gg 1$ was studied [17,18]. It was proved that when $\gamma > 0$ all oscillators were synchronized, and when $-\frac{1}{4} < \gamma < 0$ and N was even two-cluster synchronization was observed. When $-\frac{1}{4} < \gamma < 0$ and $N = 3$, different periodic inhomogeneous regimes were found, and there were non-regular oscillations.

In this paper we studied the existence and properties of the travelling waves

$$x_k(t) = x(t - (k-1)T_2), \quad k = 1, \ldots, N, \tag{4}$$

where T_2 is some phase lag of model (2).

The existence of the travelling waves in the rings of coupled oscillators was studied in the papers of [19,20]. Note that for the case of unidirectionally coupled oscillators, solution (4) is often called "a rotating wave" (see [21,22]).

The function $x(t)$ from (4) must satisfy the equation with two delays

$$\dot{x} + x = \lambda F(x(t - T_1)) + \gamma(x(t - T_2) - x). \tag{5}$$

We showed that Equation (5) has a periodic solution with period P. Moreover, we proved that it is exponentially orbitally stable in the phase space $C_{[-T,0]}$, where $T = \max(T_1, T_2)$ under some additional requirements on the function f.

Condition $x_0 \equiv x_N$ lead us to

$$T_2 N = nP \tag{6}$$

for some integer n. Therefore, if equality (6) is true, then our initial model (2) has a travelling wave solution (4).

Furthermore, if we consider the ring of N oscillators with delayed coupling

$$\dot{x}_k + x_k = \lambda F(x_k(t - T_1)) + \gamma(x_{k-1}(t - T_2) - x_k), \quad k = 1, \ldots, N, \quad x_0 \equiv x_N \tag{7}$$

instead of model (2), then its homogeneous regime is the solution to Equation (5).

In the paper, we proved the existence of several coexisting travelling waves of model (2) in the case of large numbers N (N of the order $O(\ln \lambda)$). These solutions are relaxation spike-like regimes. In addition, we proved that all solutions of system (2) with positive initial conditions were positive for all $t \in [0, +\infty)$. That is why we can say that this system may be regarded as a phenomenological model of N-coupled neurons [23].

It is important to mention, that models (2), (5) and (7) are rather complicated. At present, there are no analytical methods to study their dynamics on the semiaxis $t \in [0, +\infty)$ for any arbitrary values of parameters λ, γ, T_1, and T_2. If we study the dynamics of these models numerically, we take several concrete functions F, but we cannot enumerate all infinite sets of compactly supported functions F and draw justified conclusions about the qualitative behavior of a model with an arbitrarily compact supported nonlinearity. That is why we analyzed the nonlocal dynamics of this model under the assumption that λ was a large parameter.

This assumption allowed us to use a special analytical method to study the dynamic properties of the solutions to models (2), (5) and (7) on the entire semiaxis $t \in [0, +\infty)$. Let's describe the essence of this method in the simplest case of a model with one delay. First, we selected a special set of initial conditions S from phase space $C_{[-T,0]}$, where T is the delay time of the model. Then we integrated the model using the method of steps [24]. On the first step $t \in [0, T]$ function $u(t - T)$ is a known function, the initial condition $\varphi(t) \in S$. That is why, on this segment, we considered our initial equation with delay as an ordinary differential equation with inhomogeneity depending on $\varphi(t)$. We found asymptotics of the solution to this equation at $\lambda \to +\infty$. After that, we considered our model on the segment $t \in [T, 2T]$ as an ordinary differential equation with inhomogeneity depending on $u(t - T)$., which is known from the previous step, and constructed the asymptotics to this model at $\lambda \to +\infty$. We did the same several times and then proved that for every initial function $\varphi \in S$ there existed a moment t_0 such that the solution to our model with this initial condition returned to the set S. This meant that there existed a Poincaré operator Π of translation along the trajectories such that $\Pi S \subset S$. That is why using the Schauder fixed-point theorem [25] we concluded that there was a fixed point $\varphi_* \in S$ of the operator Π: $\Pi\varphi_* = \varphi_*$. Therefore, if we took the function φ_* as the initial condition to the equation with delay, then we derived a periodic solution to this equation.

Note, that this analytical method is rather general and may be applied to various systems of differential equations with delay, including mathematical models of radiophysical devices.

The paper is organized is as follows. In Section 2 we prove the positiveness of solutions to system (2) and Equation (5) with positive initial conditions. In Section 3 we state and prove some properties of the linear part of Equation (5). In Section 4 we construct asymptotics of the relaxation solutions to Equation (5) and prove that there exists a relaxation cycle of this equation. In Section 5 we give the sufficient conditions of the stability of this cycle. In Section 6 we discuss the conditions for the existence of the travelling wave solutions to model (2). In Section 7 we give some generalizations about the results and draw conclusions.

2. Positiveness of Solutions

In this section we prove the positiveness of all solutions to models (2) and (5) with positive initial conditions.

First, let's prove the positiveness of all solutions to an equation with two delays (5) with positive initial conditions.

Lemma 1. *Let $\varphi(t) \in C_{[-T,0]}$ and $\varphi(t) > 0$ for all $t \in [-T, 0]$, where $T = \max(T_1, T_2)$. Then solution $x(t)$ to the Equation (5) with initial conditions $x(t) = \varphi(t)$ for $t \in [-T, 0]$ satisfies inequality $x(t) > 0$ for all $t > 0$.*

Proof. Initial condition $\varphi(t)$ is positive on the segment $t \in [-T, 0]$; therefore, $x(0) > 0$ and $x(t - T_2) > 0$ for all $t \in [0, T_2]$. Function F is non-negative for all values of its argument.

Suppose there exists a moment $t = t_0 > 0$ such that $x(t_0) = 0$ and $x(t) > 0$ for all $t \in [0, t_0)$. Then from Formula (5) we obtain

$$\dot{x}(t_0) = \lambda F(x(t_0 - T_1)) + \gamma x(t_0 - T_2) > 0.$$

However, if for all $t < t_0$ the inequality $x(t) > 0 = x(t_0)$ holds, then $\dot{x}(t_0)$ should be negative. This contradiction completes the proof. □

The next Lemma states that all solutions of system (2) with positive initial conditions are positive. This result is important because the research model may be regarded as a phenomenological model of coupled neurons only if its solutions with positive initial conditions stay positive on the whole semiaxis $t \in [0, +\infty)$.

Lemma 2. *Let $\varphi_k(t) \in C_{[-T_1,0]}$ and $\varphi_k(t) > 0$ for all $t \in [-T_1, 0]$ and $k = 1, \ldots, N$. Then solution $x(t) = (x_1(t), \ldots, x_N(t))$ to the system (2) with initial conditions $x_k(t) = \varphi_k(t)$ for $t \in [-T_1, 0]$, $k = 1, \ldots, N$ satisfies inequalities $x_k(t) > 0$ for all $t > 0$.*

Proof. The proof of this Lemma is very similar to the proof of Lemma 1. Let $t_0 > 0$ be the first moment when $x_m(t_0) = 0$ for some m. Thus, $x_k(t) > 0$ for all $t \in [-T_1, t_0)$ and $k = 1, \ldots, N$. From (2), we obtain

$$x_m(t_0) = x_m(0)e^{-(1+\gamma)t_0} + \int_0^{t_0} e^{-(1+\gamma)(t-s)} \Big(\lambda F(x_m(s - T_1)) + \gamma x_{m-1}(s) \Big) \, ds.$$

The value $x_m(t_0)$ is positive because $x_m(0) > 0$ and inequalities $F(x) \geq 0$ and $x_{m-1}(s) > 0$ hold for all $t \in [0, t_0)$. This contradiction completes the proof. □

3. Some Properties of the Linear Equation

In this section, we establish several important properties of the linear part of Equation (5). Consider the linear part of Equation (5)—a differential equation with one delay

$$\dot{x} = -(1+\gamma)x + \gamma x(t - T_2), \tag{8}$$

where γ and T_2 are positive parameters. The solutions and stability of Equation (8) are described by the roots of the characteristic equation

$$\mu = -(1+\gamma) + \gamma e^{-\mu T_2}. \tag{9}$$

Let's denote as μ_* a root of Equation (9) with a real part that is maximal from all roots of this equation.

Lemma 3. *Root μ_* of Equation (9) is real and negative. Its multiplicity is equal to one.*

Proof. Let's represent μ in the form $\mu = \text{Re}\,\mu + i\text{Im}\,\mu$. Then if we separate the real and imaginary parts of the Equation (9), we obtain the following system of equations

$$\text{Re}\,\mu + \gamma + 1 = \gamma \exp(-\text{Re}\,\mu T_2) \cos\big(-\text{Im}\mu T_2\big), \tag{10}$$
$$\text{Im}\,\mu = \gamma \exp(-\text{Re}\,\mu T_2) \sin\big(-\text{Im}\mu T_2\big). \tag{11}$$

Consider the right and left sides of Equation (10) as two functions of the argument $\text{Re}\mu$ taking the value of $\text{Im}\mu$ as a parameter. Let $h_1(\text{Re}\,\mu) = \text{Re}\,\mu + \gamma + 1$ and $h_2(\text{Re}\,\mu) = \gamma \exp(-\text{Re}\,\mu T_2) \cos\big(-\text{Im}\mu T_2\big)$. Then $h_1(0) = \gamma + 1 > \gamma \geq h_2(0)$. If $\text{Re}\mu > 0$, then $h_1(\text{Re}\mu) > h_1(0)$ and $h_2(\text{Re}\mu) < \gamma < h_1(0)$. Therefore, all roots $\text{Re}\mu$ of Equation (10) are less than zero; thus, $\text{Re}\mu_* < 0$.

Let $\cos(-\text{Im}\mu T_2) = 1$. Then Equation (10) has a root $\text{Re}\,\mu = \mu_0 < 0$, where μ_0 satisfies the equation $\mu_0 + \gamma + 1 = \gamma \exp(-\mu_0 T_2)$. This equation has only one root because the left side an is increasing function and the right is a decreasing function. The multiplicity of this root is equal to one because the derivative on the left is positive whereas the derivative on the right is negative). If $\cos(-\mathrm{Im}\mu T_2) < 1$, then $h_2(\mathrm{Re}\,\mu) < \gamma\exp(-\mathrm{Re}\,\mu T_2)$; therefore, all roots of Equation (10) with $\cos(-\mathrm{Im}\mu T_2) < 1$ are less then μ_0.

Let's prove that μ_0 is the root of the system (10) and (11). If $\cos(-\mathrm{Im}\mu T_2) = 1$, then $\sin(-\mathrm{Im}\mu T_2) = 0$; hence, Equation (11) in this case is equivalent to equation $\mathrm{Im}\mu = 0$. Consequently, the real value μ_0 is the root of the system (10) and (11). Moreover, this root has the maximal real part from all roots of this system, there are no another complex roots with the same real part, and this root has a multiplicity equal to 1. □

Lemma 4. *Let $\varphi(t) \in C_{[t_0-T_2,t_0]}$ and $\varphi(t) > 0$ for $t \in [t_0 - T_2, t_0]$. Then the solution to Equation (8) with initial condition $\varphi(t)$ is positive for all $t > t_0$.*

Proof. This follows from Lemma 1 with $F \equiv 0$. □

Next, we recall some important results from the monograph [24] on the form of the solutions to differential equations with delay.

Lemma 5 ([24])**.** *The solution to Equation (8) with initial condition $x(t) = \varphi(t)$ for $t \in [t_0 - T_2, t_0]$ is*

$$x(t) = L(t-t_0)\varphi,$$

where $L(t-t_0)$ is a linear compact operator such that for any $\mu_* < \mu_0 < 0$ there exists c such that $|L(t-t_0)\varphi| \le \tilde{c}e^{\mu_0(t-t_0)}||\varphi||_{T_2} = ce^{\mu_0 t}||\varphi||_{T_2}$, where $||\varphi||_{T_2} = \max\limits_{s\in[-T_2,0]} |\varphi(t_0+s)|$.

Lemma 6 (Variation-of-constants formula [24])**.** *Let $X(t)$ be the fundamental solution to Equation (8); i.e., solution with initial condition*

$$X(t) = \begin{cases} 0, & t < 0; \\ 1, & t = 0. \end{cases}$$

Then the solution to the linear inhomogeneous equation

$$\dot{x} = -(1+\gamma)x + \gamma x(t-T_2) + h(t)$$

with initial conditions $x(t) = \varphi(t)$ for $t \in [t_0 - T_2, t_0]$ is

$$x(t) = L(t-t_0)\varphi + \int\limits_{t_0}^{t} X(t-s)h(s)\,ds.$$

Lemma 7. *Let $\varphi(t) > 0$ for all $t \in [t_0 - T_2, t_0]$. Then there exists positive x_0, c, c', and M such that for all $t \ge t_0$ solution to Equation (8) with initial conditions $x(t) = \varphi(t)$ ($t \in [t_0 - T_2, t_0]$) is*

$$x(t) = x_0 e^{\mu_*(t-t_0)}(1 + w(t)), \quad w(t) = Q(t-t_0)\varphi,$$

where $Q(t-t_0)$ is a compact linear operator such that

$$|w(t)| = |Q(t-t_0)\varphi| \le \tilde{c}e^{-M(t-t_0)}||\varphi||_{T_2} = ce^{-Mt}||\varphi||_{T_2} \quad \text{and} \quad |\dot{w}(t)| \le c'e^{-Mt}||\varphi||_{T_2},$$

where $||\varphi||_{T_2} = \max\limits_{s\in[-T_2,0]} |\varphi(t_0+s)|$.

Proof. The representation of $x(t)$ and inequality $|w(t)| \leq ce^{-Mt}||\varphi||_{T_2}$ follows from [24] (see Chapter 7). The value of x_0 is determined by the formula

$$x_0 = x(t_0) + \int_{-T_2}^{0} x(s+t_0)e^{-\mu_*(s+T_2)}\, ds > 0. \tag{12}$$

To prove the inequality $|\dot{w}(t)| \leq c'e^{-Mt}||\varphi||_{T_2}$ substitute the formula for $x(t)$ into Equation (8), and after simplification we have

$$\dot{w}(t) = -(1+\gamma+\mu_*)w + \gamma e^{-\mu_* T_2} w(t-T_2),$$

so

$$|\dot{w}(t)| \leq |1+\gamma+\mu_*|ce^{-Mt}||\varphi||_{T_2} + \gamma e^{(M-\mu_*)T_2} ce^{-Mt}||\varphi||_{T_2} = c'e^{-Mt}||\varphi||_{T_2}.$$

□

Lemma 8. *Let $0 < \varphi(t_0) \leq p$ and $0 < \varphi(t) \leq \frac{1+\gamma}{\gamma} p$ for all $t \in [t_0 - T_2, t_0]$. Then the solution $x(t)$ to Equation (8) with initial condition $x(t) = \varphi(t)$ ($t \in [t_0 - T_2, t_0]$) is less than or equal to p for all $t \geq t_0$.*

Proof. If Equation (8) is a linear inhomogeneous ODE, its solution has the form

$$x(t) = x(t_0)e^{-(1+\gamma)(t-t_0)} + \gamma \int_{t_0}^{t} e^{-(1+\gamma)(t-\tau)} x(\tau - T_2)\, d\tau. \tag{13}$$

The Second term in this formula can be estimated on the segment $t \in [t_0, t_0 + T_2]$

$$\gamma \int_{t_0}^{t} e^{-(1+\gamma)(t-\tau)} x(\tau - T_2)\, d\tau \leq \gamma \frac{1+\gamma}{\gamma} p \int_{t_0}^{t} e^{-(1+\gamma)(t-\tau)}\, d\tau = p(1 - e^{-(1+\gamma)(t-t_0)}). \tag{14}$$

Thus, on the segment $t \in [t_0, t_0 + T_2]$

$$x(t) \leq x(t_0)e^{-(1+\gamma)(t-t_0)} + p(1 - e^{-(1+\gamma)(t-t_0)}) \leq p,$$

and $0 < x(t_0 + T_2) \leq p$. Then, if we replace t_0 by $t_0 + T_2$ in Formulas (13) and (14) we obtain the estimation of $0 < x(t) \leq p$ on the segment $t \in [t_0 + T_2, t_0 + 2T_2]$.

Acting like this, by induction we obtain estimation $0 < x(t) \leq p$ for all $t \geq t_0$. □

Lemma 9. *Let λ be sufficiently large and $x(t) = \lambda\varphi(t)$ for all $t \in [t_0 - T_2, t_0]$, where $\varphi(t)$ is a positive continuous function. Then there exists t_1 such that $x(t_1) = p$, $x(t) > p$ for all $t \in [t_0, t_1)$ and for all $t \in [t_1 - T_2, t_1]$*

$$p\frac{1+\gamma}{\gamma} \geq x(t) \geq p.$$

Proof. Using Lemma 7 we get $x(t) = \lambda e^{\mu_*(t-t_0)} x_0(1 + w(t))$, where x_0 is determined by (12); $w(t) = Q(t - t_0)\varphi$; and $|w(t)| \leq ce^{-Mt}||\varphi||_{T_2}$ for some positive c and M. Solving equation $x(t) = p$ we find $t_1 = t_0 + |\mu_*^{-1}| \ln \lambda(1 + o(1))$.

Thus, the values $t \in [t_1 - T_2, t_1]$ are large enough, so for such t

$$x(t) = pe^{\mu_*(t-t_1)}(1 + o(1)).$$

Since μ_* is a negative root of Equation (9), then $e^{-\mu_* T_2} = \mu_*\gamma^{-1} + \frac{1+\gamma}{\gamma} < \frac{1+\gamma}{\gamma}$.

Let $t \in [t_1 - T_2, t_1]$, then

$$x(t) = pe^{\mu_*(t-t_1)}(1+o(1)) < p(1+o(1))e^{-\mu_* T_2} < p\frac{1+\gamma}{\gamma}.$$

Thus, Lemma is proved. □

4. Relaxation Oscillations

In this section we prove that Equation (5) has a relaxation cycle, construct its asymptotics and study its main properties.

Let T be the maximum of delays ($T = \max(T_1, T_2)$) and T_m be the minimum of delays ($T_m = \min(T_1, T_2)$).

Consider the following set of initial conditions:

$$S = \{\varphi(t) \in C_{[-T,0]} : \varphi(0) = p, \varphi(t) \geq p \, \forall t \in [-T, 0], \frac{1+\gamma}{\gamma} p \geq \varphi(t) \forall t \in [-T_2, 0]\}.$$

Examples of S are shown in Figure 1.

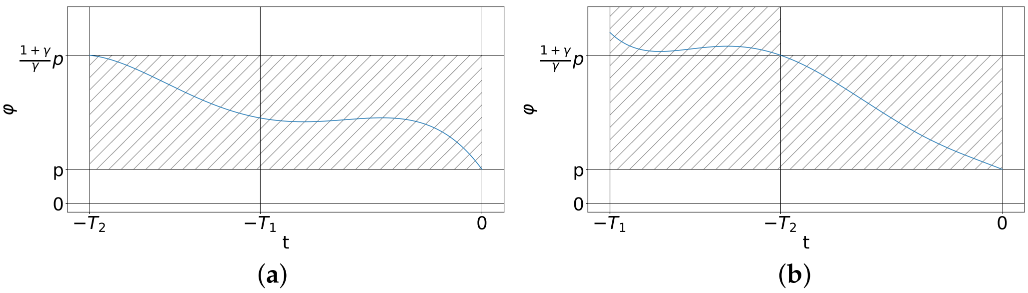


Figure 1. Examples of set S for (**a**) $T_1 < T_2$; (**b**) $T_2 < T_1$.

Denote as $x_\varphi(t)$ a solution to Equation (5) with initial condition $\varphi \in S$. Let's construct an asymptotic approximation of solution $x_\varphi(t)$.

1. Let $t \in [0, T_1]$. For such t inequality $x(t - T_1) > p$ holds, so $F(x(t - T_1)) \equiv 0$ and Equation (5) has the form of a linear differential equation with one delay (8). Therefore, solution $x_\varphi(t)$ satisfies the formula

$$x_\varphi(t) = pe^{-(1+\gamma)t} + \gamma \int_0^t e^{-(1+\gamma)(t-\tau)}\varphi(\tau - T_2)d\tau. \quad (15)$$

From Lemma 8 we find that $x_\varphi(t) \leq p$ for all $t \in [0, T_1]$.

2. Let $t \in [T_1, 2T_1]$. For these values of t, the inequality $x_\varphi(t - T_1) < p$ holds; therefore, $F(x_\varphi(t - T_1)) = f(x_\varphi(t - T_1)) > 0$.

By Lemma 6 solution $x_\varphi(t)$ on this segment becomes asymptotically large. It is of the order $O(\lambda)$ at $\lambda \to +\infty$. Let's write out the exact solution formulas. First, consider Equation (5) on the segment $t \in [T_1, T_1 + T_m]$.

In Equation (5) is considered to be an ordinary differential equation, then its solution has the form

$$x_\varphi(t) = x(T_1)e^{-(1+\gamma)(t-T_1)} + \gamma \int_{T_1}^t e^{-(1+\gamma)(t-\tau)}x_\varphi(\tau - T_2)d\tau + \lambda \int_{T_1}^t e^{-(1+\gamma)(t-\tau)}F(x_\varphi(\tau - T_1))d\tau. \quad (16)$$

It follows from Formula (16), Lemma 8, and conditions $f > 0$ and $\lambda \gg 1$ that there exists a positive value δ such that $x(T_1 + \delta) = p$ and $\delta = o(1)$. On the interval $t \in (T_1, T_1 + T_m]$, the solution $x_\varphi(t)$ satisfies the formula

$$x_\varphi(t) = \lambda \int_{T_1}^{t} e^{-(1+\gamma)(t-\tau)} f(x_\varphi(\tau - T_1)) d\tau (1 + o(1)). \quad (17)$$

If $T_m < T_1$, then on segment $t \in [T_1 + T_m, 2T_1]$ the following asymptotic formula for solution $x_\varphi(t)$ holds:

$$x_\varphi(t) = \gamma \int_{T_1+T_m}^{t} e^{-(1+\gamma)(t-\tau)} x_\varphi(\tau - T_2) d\tau + \lambda \int_{T_1}^{t} e^{-(1+\gamma)(t-\tau)} f(x_\varphi(\tau - T_1)) d\tau (1 + o(1)). \quad (18)$$

Note, that the first term on the right side of (18) is of the order $O(\lambda)$. Since $x_\varphi(t - T_2) > 0$ (see Lemma 1), $f(x_\varphi(t - T_1)) > 0$ on the segment $t \in [T_1 + T_m, 2T_1]$, and $2T_1 = O(1)$, then on the segment $t \in [T_1 + T_m, 2T_1 + \delta]$ solution $x_\varphi(t)$ has the order $O(\lambda)$.

Figure 2a shows the solution of Equation (5) on a segment $[0, 2T_1]$, and Figure 2b shows an enlarged part in the neighbourhood of T_1.

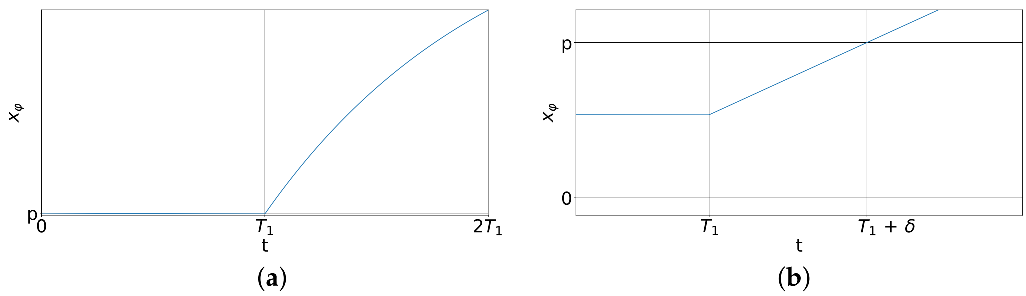


Figure 2. Solution of the Equation (5) for (**a**) $t \in [0, 2T_1]$; (**b**) t in neighbourhood of T_1.

3. Let $t \geq 2T_1 + \delta$. Then Equation (5) is the linear differential equation of the form (8) while solution $x_\varphi(t - T_1) > p$. From Lemma 7, its solution is

$$x_\varphi(t) = \lambda e^{\mu_*(t-2T_1-\delta)} x_0 (1 + w(t)), \quad |w(t)| \leq c e^{-Mt} \max_{s \in [-T_2, 0]} |\lambda^{-1} x_\varphi(2T_1 + \delta + s)|. \quad (19)$$

Here $x_0, c, M > 0$. This function tends to zero; therefore, for some $P > 2T_1 + \delta$ equality $x_\varphi(P) = p$ holds. Moreover,

$$P = |\mu_*^{-1}| \ln \lambda + 2T_1 + |\mu_*^{-1}|(\ln p - \ln x_0) - |\mu_*^{-1}| \ln(1 + w(P)) + \delta = |\mu_*^{-1}| \ln \lambda (1 + o(1)). \quad (20)$$

By Lemma 9 on the segment $t \in [P - T_2, P]$, the solution satisfies the inequality $p\frac{1+\gamma}{\gamma} \geq x_\varphi(t) \geq p$ and on the segment $t \in [P - T, P]$ condition $x(t) \geq p$ holds; therefore, $x_\varphi(t + P) \in S$. Moreover, the solution $x_\varphi(t + P)$ for $t \in [-T, 0]$ satisfies the following inequality:

$$x_\varphi(t + P) = \lambda e^{\mu_*(t+P-2T_1-\delta)} x_0 (1 + w(t + P)) = e^{\mu_* t} p \frac{(1 + w(t + P))}{(1 + w(P))} \leq \frac{2 + 2\gamma}{2 + 2\gamma + \mu_*} p e^{\mu_* t}. \quad (21)$$

Consider a set S_0:

$$S_0 = \{\varphi(t) \in C_{[-T,0]} : \varphi(0) = p, p \leq \varphi(t) \leq \frac{2 + 2\gamma}{2 + 2\gamma + \mu_*} p e^{\mu_* t} \text{ for } t \in [-T, 0]\}.$$

Note that S_0 is a non-empty bounded closed convex subset of S. In Figure 3 the examples of S_0 are shown.

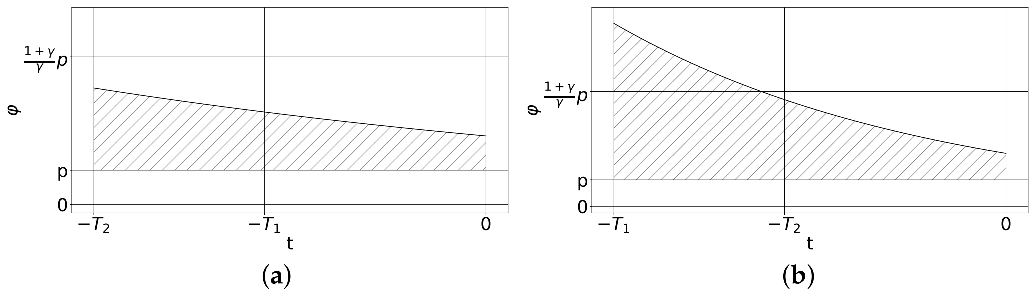


Figure 3. Example of set S_0 for (**a**) $T_1 < T_2$; (**b**) $T_2 < T_1$.

Thus, we have proved that the Poincaré operator Π maps the set S to S_0. Since S_0 is a subset of S, then Π maps S_0 to S_0.

Since S_0 is a non-empty bounded closed convex set and Π is a compact operator (see [24]), we may use the Schauder fixed-point theorem [25]. As a result, we find that $\varphi_* \in S_0 \subset S$ such that $\Pi\varphi_* = \varphi_*$; therefore, solution $x_{\varphi_*}(t)$ is a periodic solution.

Thus, we obtain the following statement.

Theorem 1. *For all sufficiently large λ, Equation (5) has a periodic solution $x_*(t)$ with initial conditions from the set S_0 with the period $P = 2T_1 + |\mu_*^{-1}| \ln \lambda(1 + o(1))$ and amplitude $O(\lambda)$. Asymptotics of this solution are*

$$x_*(t) = pe^{\mu_* t}(1 + o(1)) \tag{22}$$

on the segment $t \in [0, T_1]$,

$$x_*(t) = \lambda \int_{T_1}^{t} e^{-(1+\gamma)(t-\tau)} f(pe^{\mu_*(\tau - T_1)}) d\tau (1 + o(1)) \tag{23}$$

on the segment $t \in (T_1, T_1 + T_m]$,

$$x_*(t) = \lambda \int_{T_1}^{t} e^{-(1+\gamma)(t-\tau)} f(pe^{\mu_*(\tau - T_1)}) d\tau (1 + o(1)) + \gamma \int_{T_1+T_m}^{t} e^{-(1+\gamma)(t-\tau)} x_*(\tau - T_2) d\tau \tag{24}$$

on the segment $t \in (T_1 + T_m, 2T_1]$,

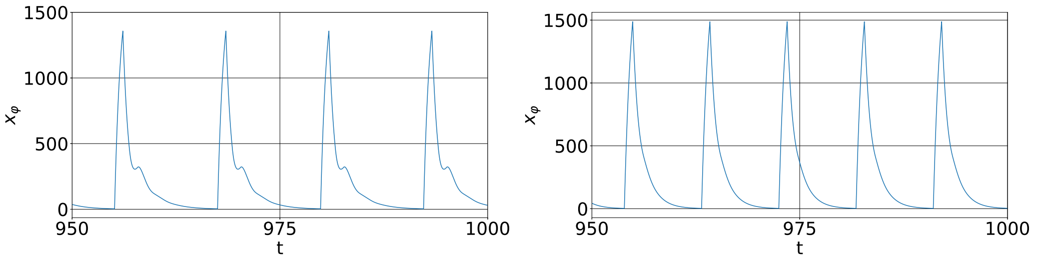
$$x_*(t) = \left(x_*(2T_1) + \int_{-T_2}^{0} x_*(2T_1 + s) e^{-\mu_*(s+T_2)} ds \right) e^{\mu_*(t - 2T_1)} (1 + w(t) + o(1)) \tag{25}$$

on the segment $t \in (2T_1, P]$, where $|w(t)| \le ce^{-Mt}$ and $c > 0$, $M > 0$.

Proof. We have proven that the compact Poincaré operator Π maps the non-empty bounded closed convex set S_0 to its pre-compact subset. That is why by the Schauder theorem there exists a fixed point of this operator $\varphi_*(t) \in S_0$ [25]. From Formula (21) we obtained asymptotics of this fixed point: $\varphi_*(t) = pe^{\mu_* t}(1 + o(1))$ at $\lambda \to +\infty$.

If we get $\varphi_*(t)$ as an initial condition to Equation (5), then we obtain a periodic solution of Equation (5). Denote it as $x_*(t)$. For all of the solutions of Equation (5) with initial conditions from the set S Formulas (15) and (17)–(19) hold. Hence, if we substitute $\varphi_*(t)$ into them and take into account Formula (12), then we obtain Formulas (22)–(25). □

Examples of periodic solutions to Equation (5) are shown in the Figure 4.



(a) **(b)**

Figure 4. Solution of the equation with two delays (5). Parameters: (**a**) $\lambda = 500$, $\gamma = 0.38$, $p = 5$, $T_1 = 1$, $T_2 = 1.77$, $f(x) = 5$; (**b**) $\lambda = 500$, $\gamma = 0.15$, $p = 5$, $T_1 = 1$, $T_2 = 1.33$, $f(x) = 5$.

Note, that asymptotic Formula (25) contains a function $w(t)$ that satisfies an exponential estimation $|w(t)| \le ce^{-Mt}$, where $c > 0$, $M > 0$. On the interval $t \in (2T_1, 2T_1 + O(1))$ this function can make a significant contribution to the asymptotics of the solution. For example, the second (smaller) spikes on the period of the function $x_*(t)$ on the Figure 4a are described by this function.

Theorem 1 does not say anything about the stability of the constructed periodic relaxation solution. In the next section we show the sufficient conditions under which this cycle is exponentially orbitally stable.

5. The Stability of Relaxation Cycle

Let's discuss the stability of relaxation cycle from Theorem 1. To prove sufficient stability conditions, we need the following auxiliary result:

Theorem 2. *Let $f(x) > 0$ be Lipschitz continuous on the segment $[0, p]$:*

$$|f(x) - f(y)| \le c_f |x - y|, \quad \forall x, y \in [0, p],$$

Then the Poincaré operator Π for the Equation (5) is a contraction operator on the set S_0.

Proof. Denote as $||x|| = \max\limits_{t \in [-T,0]} |x(t)|$—norm in the space $C_{[-T,0]}$ and $||x||_{T_2} = \max\limits_{t \in [-T_2,0]} |x(t)|$—norm in the space $C_{[-T_2,0]}$.

We know that $\Pi(S_0) \subset S_0$; therefore, to prove the Theorem it is enough to prove that there exists $0 < q < 1$ such that for any $\varphi, \psi \in S_0$ the inequality

$$||\Pi\varphi - \Pi\psi|| \le q||\varphi - \psi||$$

holds.

Let φ and ψ be two functions from set S_0, so let's construct solutions to Equation (5) $x_\varphi(t)$ and $x_\psi(t)$ with initial conditions φ and ψ respectively.

We need to repeat all constructions of this section.

1. Let $t \in [0, T_1]$. For these t, Equation (5) takes form of linear differential Equation (8). From Lemma 5 we found that its solutions for each $t \in [0, T_1]$ satisfied the

$$|x_\varphi(t) - x_\psi(t)| = |L(t)(\varphi - \psi)| \le c_1 e^{\mu_0 t} ||\varphi - \psi|| \le c_1 ||\varphi - \psi||,$$

where c_1 is some positive constant, and $\mu_* < \mu_0 < 0$.

2. Consider $x_\varphi(t)$ for $t \in [T_1, 2T_1 + \delta_\varphi]$ and $x_\psi(t)$ for $t \in [T_1, 2T_1 + \delta_\psi]$. As before, denote as δ_φ and δ_ψ the first positive roots of equations $x_\varphi(T_1 + \delta_\varphi) = p$, and $x_\psi(T_1 + \delta_\psi) = p$, respectively. From Formula (16) and inequality $f(p) > 0$, the values of δ_φ and δ_ψ are $O(\lambda^{-1})$ at $\lambda \to +\infty$. Without any loss of generality, we assumed that δ_φ is less than δ_ψ.

By Lemma 6 we may represent solutions as follows:

$$x_\varphi(t) = L(t-T_1)x_\varphi^{T_1} + \lambda \int\limits_{T_1}^{t} X(t-s) f(x_\varphi(s-T_1))\, ds,$$

$$x_\psi(t) = L(t-T_1)x_\psi^{T_1} + \lambda \int\limits_{T_1}^{t} X(t-s) f(x_\psi(s-T_1))\, ds,$$

where $X(t)$ is the fundamental solution to (8). Here x_φ^τ and x_ψ^τ are functions $x_\varphi(\tau+t)$ and $x_\psi(\tau+t)$ for $t \in [-T_2, 0]$. Note that $L(t-T_1)x_\varphi^{T_1} = L(t)\varphi$ and $L(t-T_1)x_\psi^{T_1} = L(t)\psi$. From this formula, the difference between solutions on the segment $[T_1, 2T_1]$ satisfies the inequalities

$$|x_\varphi(t) - x_\psi(t)| \le |L(t)(\varphi-\psi)| + \lambda \int_{T_1}^{t} |X(t-s)(f(x_\varphi(s-T_1)) - f(x_\psi(s-T_1)))|\, ds \le$$
$$c_1||\varphi-\psi|| + \lambda c_f c_2 \int\limits_{T_1}^{t} |x_\varphi(t-T_1) - x_\psi(t-T_1)|\, ds \le (c_1 + c_3\lambda(t-T_1))||\varphi-\psi||. \quad (26)$$

Here $c_2 = \max\limits_{t\in[0,2T_1]} |X(t)|$ and $c_3 = c_f c_2 c_1$ are positive constants. In particular,

$$|x_\varphi(T_1+\delta_\psi) - x_\psi(T_1+\delta_\psi)| \le (c_1 + \lambda c_3 \delta_\psi)||\varphi-\psi|| \le c_4||\varphi-\psi||.$$

In the same way inequality (26) can be generalized to the segment $t \in [2T_1, 2T_1 + \min(\delta_\varphi, \delta_\psi)]$.

Let's estimate $\delta_\psi - \delta_\varphi$. Using (16), we get

$$\begin{aligned} 0 = p - p = x_\varphi(T_1+\delta_\varphi) - x_\psi(T_1+\delta_\psi) = \\ x_\varphi(T_1+\delta_\varphi) - x_\psi(T_1+\delta_\varphi) + x_\psi(T_1+\delta_\varphi) - x_\psi(T_1+\delta_\psi) = \\ x_\varphi(T_1+\delta_\varphi) - x_\psi(T_1+\delta_\varphi) + \dot{x}_\psi(T_1+\theta)(\delta_\varphi - \delta_\psi). \end{aligned} \quad (27)$$

Here θ is a value between δ_ψ and δ_φ. It is asymptotically small and positive, so $x_\psi(\theta) = p + o(1) < p$. Let's estimate $\dot{x}_\psi(T_1+\theta)$.

$$\dot{x}_\psi(T_1+\theta) = -(1+\gamma)x_\psi(T_1+\theta) + \gamma x_\psi(T_1+\theta-T_2) + \lambda f(x_\psi(\theta)) \ge c_5\lambda,$$

where $c_5 = \frac{1}{2}f(p) > 0$. Thus from (27),

$$|\delta_\varphi - \delta_\psi| = |\dot{x}_\psi(T_1+\theta)|^{-1}|x_\varphi(T_1+\delta_\varphi) - x_\psi(T_1+\delta_\varphi)| \le c_\delta\lambda^{-1}||\varphi-\psi||, \quad c_\delta = 2c_4c_5^{-1} > 0.$$

Here we also note that for all $t \in [-T, 0]$

$$\lambda c_{max} \ge x_\varphi(2T_1+\delta_\varphi+t), \quad \lambda c_{max} \ge x_\psi(2T_1+\delta_\psi+t), \quad (28)$$

where

$$c_{max} = 2T_1 \max_{t\in[0,2T_1]} |X(t)| \max_{x\in[0,p]} f(x),$$

and

$$x_\varphi(2T_1+\delta_\varphi) \ge \lambda c_6, \quad x_\psi(2T_1+\delta_\psi) \ge \lambda c_6,$$

where $c_6 = \frac{1}{2}T_1 \min\limits_{t\in[0,2T_1]} |X(t)| \min\limits_{x\in[0,p]} f(x)$.

3. Consider each solution on the time interval t from $2T_1 + \delta_\varphi$ and $2T_1 + \delta_\psi$ till the moment when it returns to the set S_0. By Lemma 7, both solutions can be represented in the form

$$x_\varphi(t) = \lambda x_1 e^{\mu_*(t-2T_1-\delta_\varphi)}(1 + w_\varphi(t)), \quad x_\psi(t) = \lambda x_2 e^{\mu_*(t-2T_1-\delta_\psi)}(1 + w_\psi(t)),$$

where $w_\varphi(t) = \lambda^{-1}Q(t - 2T_1 - \delta_\varphi)x_\varphi^{2T_1+\delta_\varphi}$ and $w_\psi(t) = \lambda^{-1}Q(t - 2T_1 - \delta_\psi)x_\psi^{2T_1+\delta_\psi}$. For short, we denote $Q(t - 2T_1 - \delta_\varphi)$ as $Q_\varphi(t)$ and denote $Q(t - 2T_1 - \delta_\psi)$ as $Q_\psi(t)$.

Values λx_1 and λx_2 are determined by formula (12); therefore

$$\begin{aligned}\lambda|x_1 - x_2| \le |x_\varphi(2T_1 + \delta_\varphi) - x_\psi(2T_1 + \delta_\varphi)| + |x_\psi(2T_1 + \delta_\varphi) - x_\psi(2T_1 + \delta_\psi)| + \\ \Big|\int_{-T_2}^{0} e^{-\mu_*(s+T_2)}(x_\varphi(s + 2T_1 + \delta_\varphi) - x_\psi(s + 2T_1 + \delta_\varphi))\,ds\Big| + \\ \Big|\int_{-T_2}^{0} e^{-\mu_*(s+T_2)}(x_\psi(s + 2T_1 + \delta_\varphi) - x_\psi(s + 2T_1 + \delta_\psi))\,ds\Big| \le \lambda c_7||\varphi - \psi|| + c_8\lambda|\delta_\varphi - \delta_\psi|.\end{aligned}$$

Here were used that for $t \in [T_1, 2T_1 + \delta_\psi]$ from (28) and equality (5) it follows that

$$|\dot{x}_\psi(t)| \le (1 + \gamma)|x_\psi(t)| + \gamma|x_\psi(t - T_2)| + \lambda f(x_\psi(t - T_1)) \le \lambda c_9, \tag{29}$$

where $c_9 = (1 + 2\gamma)c_{max} + \max_{x\in[0,p]} f(x)$.

Therefore, there exists $c_x > 0$ such that

$$|x_1 - x_2| \le c_x||\varphi - \psi||.$$

In addition, from Formula (12) it follows that $x_{1,2} \ge c_6 > 0$.

Let P_φ and P_ψ be the first moments of time such that $P_\varphi > 2T_1$, $P_\psi > 2T_1$, $x_\varphi(P_\varphi) = p$, and $x_\psi(P_\psi) = p$. They are represented by the Formula (20). Let's estimate the difference $P_\varphi - P_\psi$:

$$\begin{aligned}|P_\varphi - P_\psi| \le |\mu_*^{-1}||\ln x_1 - \ln x_2| + |\mu_*^{-1}||\ln(1 + w_\varphi(P_\varphi)) - \ln(1 + w_\psi(P_\psi))| + |\delta_\psi - \delta_\varphi| \le \\ (|\mu_*^{-1}|c_6^{-1}c_x + c_\delta\lambda^{-1})||\varphi - \psi|| + c_{10}|w_\varphi(P_\varphi) - w_\psi(P_\psi)|.\end{aligned}$$

To estimate $|w_\varphi(P_\varphi) - w_\psi(P_\psi)|$ consider a more general inequality for $t \in [-T, 0]$

$$\begin{aligned}|w_\varphi(P_\varphi + t) - w_\psi(P_\psi + t)| \le |w_\varphi(P_\varphi + t) - w_\psi(P_\varphi + t)| + |w_\psi(P_\varphi + t) - w_\psi(P_\psi + t)| \le \\ \lambda^{-1}|Q_\varphi(P_\varphi + t)(x_\varphi^{2T_1+\delta_\varphi} - x_\psi^{2T_1+\delta_\psi})| + \lambda^{-1}|Q_\psi(P_\varphi + t - \delta_\varphi + \delta_\psi)x_\psi^{2T_1+\delta_\psi} - Q_\psi(P_\varphi + t)x_\psi^{2T_1+\delta_\psi}| + |\dot{w}_\psi(\theta)||P_\varphi - P_\psi| \le \\ \lambda^{-1}|Q_\varphi(P_\varphi + t)(x_\varphi^{2T_1+\delta_\varphi} - x_\psi^{2T_1+\delta_\varphi})| + \lambda^{-1}|Q_\varphi(P_\varphi + t)(x_\psi^{2T_1+\delta_\varphi} - x_\psi^{2T_1+\delta_\psi})| + |w_\psi(P_\varphi + t - \delta_\varphi + \delta_\psi) - w_\psi(P_\varphi + t)| + \\ |\dot{w}_\psi(\theta)||P_\varphi - P_\psi| \le ce^{-M(P_\varphi+t)}c_{11}||\varphi - \psi|| + c\lambda^{-1}e^{-M(P_\varphi+t)}|\dot{x}_\psi(\theta_1)||\delta_\varphi - \delta_\psi| + |\dot{w}_\psi(\theta_2)||\delta_\varphi - \delta_\psi| + |\dot{w}_\psi(\theta)||P_\varphi - P_\psi|.\end{aligned} \tag{30}$$

Here $c_{11} = 2c_3T_1$, θ_1 is a time moment between $2T_1 + \delta_\varphi$ and $2T_1 + \delta_\psi$, θ_2—between $P_\varphi + t$ and $P_\varphi + t - \delta_\varphi + \delta_\psi$, and θ is a time moment between $P_\varphi + t$ and $P_\psi + t$.

From Formula (29) $|\dot{x}_\psi(\theta_1)| \le \lambda c_9$. By Lemma 7 and inequality (28) $|\dot{w}_\psi(t)| \le \lambda^{-1}c'e^{-Mt}||x_\psi^{2T_1+\delta_\psi}|| \le c'c_{max}e^{-Mt}$ and, since θ_2 and θ are large enough, $|\dot{w}_\psi(\theta_2)|$ and $|\dot{w}_\psi(\theta)|$ are small, so $1 - c_{10}|\dot{w}_\psi(\theta)| > 1/2$. Then we have

$$(1 - c_{10}|\dot{w}_\psi(\theta)|)|P_\varphi - P_\psi| \le \left(|\mu_*^{-1}|c_6^{-1}c_x + c_\delta\lambda^{-1} + cc_{10}(c_{11} + c_9c_\delta\lambda^{-1})e^{-MP_\varphi} + c_{10}c'c_\delta c_{max}\lambda^{-1}e^{-M\theta_2}\right)||\varphi - \psi||,$$

so there exists $c_P > 0$ such that

$$|P_\varphi - P_\psi| \le c_P ||\varphi - \psi||.$$

Moreover, from (30) it follows that there exists $c_w > 0$ such that the next inequality is true:

$$||w_\varphi(P_\varphi) - w_\psi(P_\psi)|| \le e^{-MP_\varphi} c_w ||\varphi - \psi||.$$

Represent solutions x_φ and x_ψ in the form

$$x_\varphi(t + P_\varphi) = \lambda e^{\mu_*(t+P_\varphi-2T_1-\delta_\varphi)} x_1(1 + w_\varphi(t + P_\varphi)) = x_\varphi(P_\varphi) e^{\mu_* t} \frac{1 + w_\varphi(t + P_\varphi)}{1 + w_\varphi(P_\varphi)} = p e^{\mu_* t} \frac{1 + w_\varphi(t + P_\varphi)}{1 + w_\varphi(P_\varphi)}$$

and

$$x_\psi(t + P_\psi) = p e^{\mu_* t} \frac{1 + w_\psi(t + P_\psi)}{1 + w_\psi(P_\psi)}.$$

Let $t \in [-T, 0]$. For these t let's estimate the difference

$$|x_\varphi(t + P_\varphi) - x_\psi(t + P_\psi)| \le p e^{\mu_* t} \left| \frac{1 + w_\varphi(t + P_\varphi)}{1 + w_\varphi(P_\varphi)} - \frac{1 + w_\psi(t + P_\psi)}{1 + w_\psi(P_\psi)} \right|.$$

Here

$$\left| \frac{1 + w_\varphi(t + P_\varphi)}{1 + w_\varphi(P_\varphi)} - \frac{1 + w_\psi(t + P_\psi)}{1 + w_\psi(P_\psi)} \right| = \left| \frac{(w_\varphi(t + P_\varphi) - w_\psi(t + P_\psi))(1 + w_\psi(P_\psi)) + (1 + w_\psi(t + P_\psi))(w_\psi(P_\psi) - w_\varphi(P_\varphi))}{(1 + w_\varphi(P_\varphi))(1 + w_\psi(P_\psi))} \right| \le$$
$$2|w_\varphi(t + P_\varphi) - w_\psi(t + P_\psi)| + 2|w_\psi(P_\psi) - w_\varphi(P_\varphi)| \le$$
$$4||w_\varphi^{P_\varphi} - w_\psi^{P_\psi}|| \le 4c_w e^{-MP_\varphi} ||\varphi - \psi|| = 4c_w \lambda^{M/\mu_*}(1 + o(1))||\varphi - \psi||.$$

Here, as before, we denote as w_φ^τ and w_ψ^τ functions $w_\varphi(\tau + t)$ and $w_\psi(\tau + t)$, respectively, for $t \in [-T_2, 0]$.

So for all $t \in [-T, 0]$

$$|x_\varphi(t + P_\varphi) - x_\psi(t + P_\psi)| \le 5c_w p e^{-\mu_* T} \lambda^{M/\mu_*} ||\varphi - \psi||.$$

Let $q = 5c_w p e^{-\mu_* T} \lambda^{M/\mu_*}$. Since $M > 0$, $\mu_* < 0$ then for sufficiently large λ the value q is less than 1 and is small enough. Therefore, for all $\varphi, \psi \in S_0$

$$||\Pi\varphi - \Pi\psi|| \le q||\varphi - \psi||.$$

Thus, $\Pi : S_0 \to S_0$ is a contraction operator. The theorem is proved. □

Theorem 3. *Let F satisfy conditions from Theorem 2. Then for all sufficiently large values of λ, Equation (5) has a unique periodic solution with initial conditions from S_0. This solution is exponentially orbitally stable. Asymptotics of this solution are given in the Theorem 1.*

Proof. Due to the contraction mapping principle, operator Π has a unique fixed point $\varphi_* \in S_0$ that corresponds to the periodic solution $x_*(t)$ of (5). For any $\varphi \in S_0$, the sequence $\Pi^n\varphi$ tends to the φ_* exponentially fast, so solution $x_*(t)$ is exponentially orbitally stable. □

6. Travelling Waves

Let's discuss the existence of travelling wave solutions in the system of N-coupled oscillators (2).

Theorem 4. *Let λ be large enough and N be a large integer of the order $O(\ln\lambda)$, i.e., $N = c(\lambda)\ln\lambda$, where $c(\lambda)$ is a positive bounded function such that N is an integer. Let $n > 0$ be an integer and $nc^{-1}(\lambda) < \ln(1+\frac{1}{\gamma})$. Then there exist positive $T_{2,1}, T_{2,2}, \ldots, T_{2,n}$ such that system (2) has at least n travelling wave solutions in the form*

$$x_{k,m} = x_*^m(t-(k-1)T_{2,m}), \quad m = 1,\ldots,n, \quad k = 1,\ldots,N,$$

where $x_*^m(t)$ is a periodic relaxation solution to Equation (5) from Theorem 1 for corresponding $T_{2,m}$.

Proof. From Theorem 1 for any T_2 we may construct a periodic solution $x_*(t)$ to Equation (5) with period $P(T_2)$. We need T_2 such that condition (6) holds. Thus, we need to prove the existence of at least n roots of equation

$$T_2 N = mP(T_2)$$

for integers $m = 1,\ldots,n$. We know that $P(T_2) = -\frac{1}{\mu_*}\ln\lambda(1+o(1))$. To determine T_2, therefore, we have the equation

$$-\mu_* T_2 = \frac{m}{c(\lambda)}(1+o(1)). \tag{31}$$

The product $|\mu_* T_2|$ takes values in the interval $(0, \ln(1+\frac{1}{\gamma}))$; therefore, Equation (31) has n solutions $T_2 = T_{2,m}$ $(m = 1,\ldots,n)$ if and only if $nc^{-1}(\lambda) < \ln(1+\frac{1}{\gamma})$. □

Figures 5 and 6 show examples of stable travelling waves for some parameter values. In the Figure 5 there is solution to the system (2) for $\lambda = 500$, $\gamma = 0.38$, $p = 5$, $T_1 = 1$, $f(x) \equiv 5$, $N = 7$. Here, $\ln\lambda \approx 6.21$, so, $c(\lambda) \approx 1.13$ and $c^{-1}(\lambda) \approx 0.88 < 1.29 \approx \ln(1+\gamma^{-1})$. Thus, conditions of the Theorem 4 are fulfilled. In this case $T_2 \approx 1.77$. The corresponding periodic solution $x_*(t)$ to Equation (5) is in Figure 4a. In Figure 6 there is solution of the system (2) for $\lambda = 500$, $\gamma = 0.15$, $p = 5$, $T_1 = 1$, $f(x) \equiv 5$, $N = 7$. As in Figure 5 $c^{-1}(\lambda) \approx 0.88$. It is less than $2.03 \approx \ln(1+\gamma^{-1})$. In this case $T_2 \approx 1.33$. The corresponding periodic solution $x_*(t)$ to the Equation (5) is in Figure 4b.

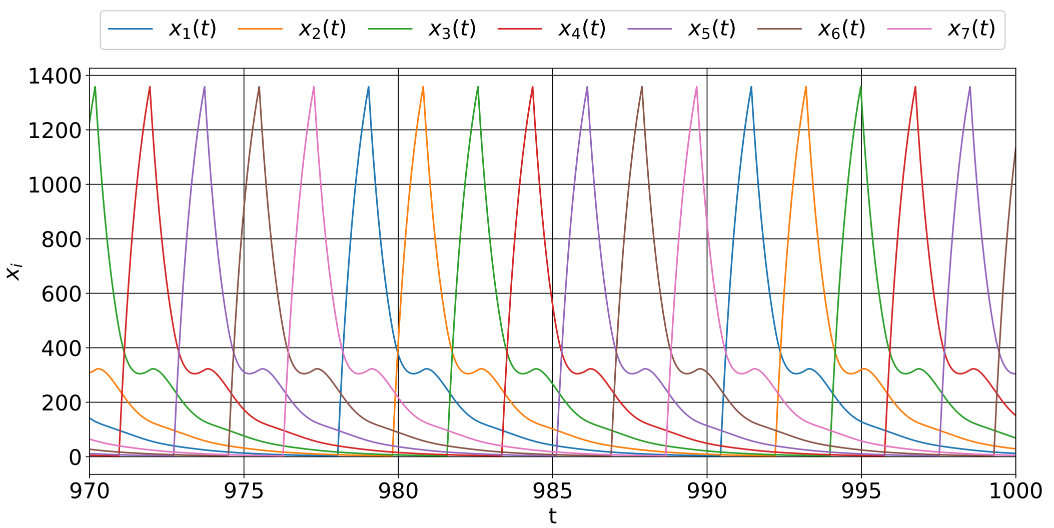


Figure 5. Example of travelling wave solution to (2). Parameters: $N = 7$; $\lambda = 500$; $\gamma = 0.38$; $p = 5$; $T_1 = 1$; $f(x) = 5$.

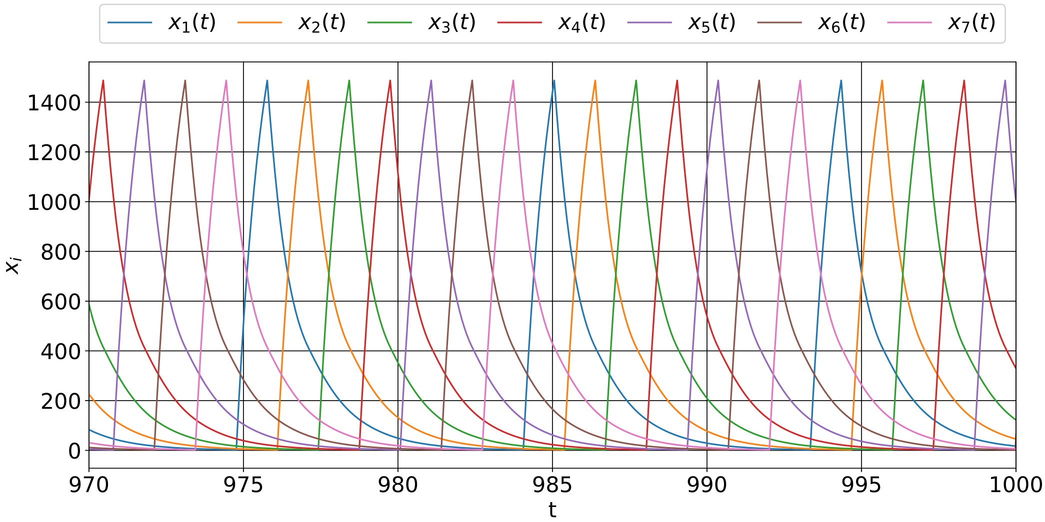


Figure 6. Example of travelling wave solution to (2). Parameters: $N = 7$; $\lambda = 500$; $\gamma = 0.15$; $p = 5$; $T_1 = 1$; $f(x) = 5$.

Note, that system (2) has a homogeneous relaxation cycle. We may treat it as a travelling wave corresponding to $T_2 = 0$. Asymptotics of this cycle coincided with the asymptotics of the homogeneous relaxation cycle of the system (3). For asymptotics of this cycle, see [18]. Note that this cycle does not depend on γ. An example of such a solution is in Figure 7.

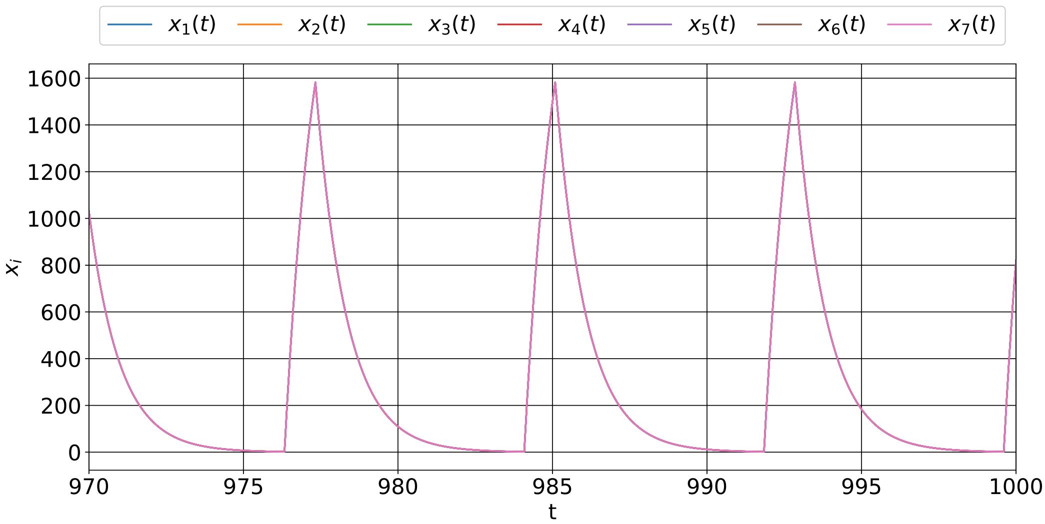


Figure 7. Example of travelling wave solution (homogeneous cycle) of (2). Parameters: $N = 7$; $\lambda = 500$; $p = 5$; $T_1 = 1$; $f(x) = 5$.

7. Discussion and Conclusions

We studied the periodic solutions to equation with two delays and compactly supported nonlinearity (5) under condition $\lambda \gg 1$. We chose a set of initial conditions and proved that translation along the trajectories operator Π mapped this set to itself. We proved that Π had a fixed point, and the solution to Equation (5) corresponding to this fixed point was periodic. This cycle had a large period of the order $O(\ln \lambda)$, and an amplitude of the order $O(\lambda)$. With additional requirements for the function $f(x)$ we proved that Π is a contraction operator and a found cycle is exponentially orbitally stable.

This cycle was used to construct travelling waves of the system of N-coupled neuron-like oscillators (2) when N was asymptotically large (on the order of $O(\ln \lambda)$).

The results allowed for some generalizations.

1. The function $f(x)$ may take negative values on the segment $x \in [0, p]$. The only reason we required positiveness of function $f(x)$ on the segment $x \in [0, p]$ while constructing asymptotics was to guarantee the positiveness of the solution on the interval $[T_1, 2T_1]$. Although for the positiveness of solution, it was sufficient to demand the positiveness of integral.

$$\int_{T_1}^{t} e^{-(1+\gamma)(t-\tau)} f(pe^{\mu_*(\tau - T_1)}) d\tau > 0. \quad (32)$$

Therefore, under condition (32) (instead of condition $f(x) > 0$ on the $x \in [0, p]$) there also existed a relaxation cycle of Equation (5).

2. If we required that $f(x) < 0$ for $x \in [-p, 0)$ and that $f(x)$ was Lipschitz continuous on the segment $[-p, 0]$ and took negative initial conditions and repeated all the constructions and proofs in the paper then we obtained an exponentially orbitally stable negative relaxation cycle. Therefore, under these conditions we obtained multistability in Equation (5): positive and negative exponentially orbitally stable relaxation cycles coexisted. In Figure 8, the exponentially orbitally stable negative relaxation cycle of the model (5) is shown.

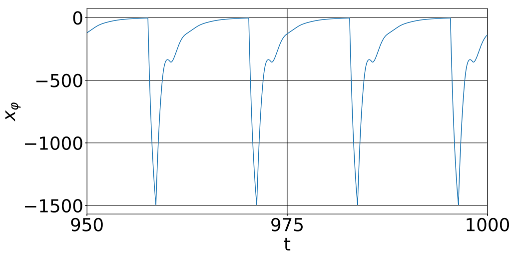


Figure 8. Solution of the equation with two delays (5). $\lambda = 550$; $\gamma = 0.38$; $p = 5$; $T_1 = 1$; $T_2 = 1.77$; $f(x) = -5$.

3. All previous results were given for sufficiently large values of λ. However, if λ were not large enough, the method still worked: the fixed points and cycles of the operator Π corresponded to the periodic solutions of (5).

In Figure 9 there is an example of a complex periodic solution of (5) for $\lambda = 500$.

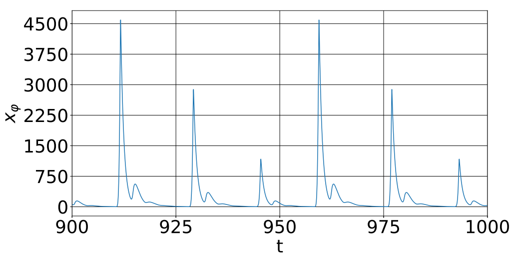


Figure 9. Solution of the equation with two delays (5). $\lambda = 500$; $\gamma = 0.3$; $p = 5$; $T_1 = 1$; $T_2 = 3$; $f(x) = 5$.

Note that this solution was not predicted by Theorem 1. This is not a contradiction because increasing λ leads to the disappearance of this cycle, which exists when λ is not large enough.

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