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The Problem of Boundary Control of the Thermal Process in a Rod

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Abstract: This paper considers a rod with an insulated side surface, at the edges of which there is heat exchange with the external environment. It is assumed that the thermal process in the rod is controlled by the effect of the ambient temperature on the thermal state in the rod through its boundary temperatures. Using the technique of separation of variables and methods based on the theory of control of finite-dimensional systems, we propose a constructive approach to build the control function of the temperature conditions at the ends of the rod that change the temperature state distribution in the rod from a given initial state to a final state within a specified time interval. We have formulated the necessary and sufficient condition that the boundary control functions of the temperature modes of the rod must satisfy in order for the problem to be completely controllable under any allowable initial and final conditions. As an application of the proposed approach, we have built the temperature control functions at the ends of the rod for the first two harmonics.

Keywords: thermal process; temperature; rod; boundary control; Fourier method; complete controllability

MSC: 93C20; 93C40

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1. Introduction

In the study of controllable thermal processes, there are problems of controlling thermal processes whose mathematical models are described by partial differential equations of the parabolic type [1–16]. These control problems find their application, for example, in various fields of science and technology. In practice, there are often problems of thermal diffusion in a rod, the ends of which are exposed to variable controlled temperatures. These problems are reduced to the study of the thermal conduction equation, the boundary conditions in which are expressed through control functions [6,9–15].

The relevance of issues of developing temperature control conditions of thermal processes is well-established. A theoretical study of the above problems, as well as various formulations of problems of control and optimal control of processes described by parabolic equations, were given, in particular, in [1–16]. The papers [12–16] consider problems of control and optimal control of a thermal (parabolic) equation using distributed control, in particular, allowing taking into account of restrictions on the structure of the solution and control [16], restrictions on the control and states [13], and point restrictions for control [14]. Often, there are boundary control problems of a thermal process in which it is necessary to generate the desired temperature state within a given time interval. The problems of boundary control of the thermal process in a rod have so far not been sufficiently researched. This paper addresses the study of the boundary control of the thermal process in a rod. In the paper, we use the approaches of boundary control for distributed systems, described, in particular, in [17–19].

In this paper, we consider a homogeneous rod with an insulated side surface, at the ends of which heat exchange with the external environment takes place. It is assumed that the thermal process in the rod is controlled as follows: by changing the ambient temperatures (at the right and left ends) we influence the thermal state in the rod through its boundary temperatures. Each of these boundary temperature functions serves as a control (boundary controls).

This paper aims to develop a constructive approach to building a function of such temperature conditions at the ends of the rod that change the distribution of the temperature state in the rod from a given initial state to a final state within a given time interval. We propose a constructive approach to constructing a control function for temperature conditions at the ends of the rod, under the influence of which the distribution of the temperature state in the rod transitions into a given final state from a given initial state at a given time interval. The construction is based on the methods of separation of variables and the theory of control for finite-dimensional systems. A necessary and sufficient condition is also formulated for the complete controllability of the boundary control of the temperature conditions of the rod under any feasible initial and final conditions. To illustrate the constructiveness of the proposed approach, control functions for temperature regimes at the ends of the rod are constructed for the first two modes. We carried out a comparative analysis of the results of numerical calculations and identified patterns related to the physical properties of materials. The study continues previously reported work [16].

2. Problem Statement

We consider the thermal process in a uniform rod of length l. Let the temperature distribution in the rod be described by the function $Z(x,t)$, $0 \leqslant x \leqslant l$, $t_0 < t < T$, which conforms to the parabolic equation

$$\frac{\partial Z}{\partial t} = a\frac{\partial^2 Z}{\partial x^2},\ 0 < x < l,\ t > t_0 \tag{1}$$

subject to boundary conditions

$$Z(0,t) = \mu(t),\ Z(l,t) = \nu(t),\ t_0 \leqslant t \leqslant T. \tag{2}$$

and the initial (at $t = t_0$) and final (at $t = T$) conditions

$$Z(x,t_0) = \varphi_0(x),\ 0 \leqslant x \leqslant l, \tag{3}$$

$$Z(x,T) = \varphi_T(x),\ 0 \leqslant x \leqslant l. \tag{4}$$

In Equation (1), $a = \frac{k}{c\rho}$ is the coefficient of thermal conductivity of the rod material, ρ is the material density, c is the specific heat capacity, and k is the heat conductivity coefficient of the rod.

The considered thermal process can be stated in thermophysical terms as follows: We consider a uniform rod with an insulated side surface, at the ends of which there is heat exchange with the external environment, and the temperature of the external environment at the time t at the left end is equal to $\mu(t)$, and at the right end is equal to $\nu(t)$. It is assumed that the thermal process in the rod is controlled as follows: by changing the ambient temperatures, we thus influence the thermal state in the rod through its boundary functions $\mu(t)$ and $\nu(t)$. Each of these functions serves as a control (boundary controls).

It is assumed that the allowable controls $\mu(t)$ and $\nu(t)$ belong to $L_2(t_0,T)$. The function $Z(x,t) \in L_2(\Omega)$, where the set $\Omega = \{(x,t) : x \in [0,l],\ t \in [t_0,T]\}$, and the function $\varphi_0(x)$, $\varphi_T(x)$ belong to $L_2(0,l)$. It is also assumed that all functions are such that the following consistency conditions are satisfied:

$$\mu(t_0) = \varphi_0(0),\ \nu(t_0) = \varphi_0(l),\ \mu(T) = \varphi_T(0),\ \nu(T) = \varphi_T(l), \tag{5}$$

The problem of boundary control of the thermal process in a rod can be stated as follows.

Find such temperature conditions at the ends of the rod $\mu(t)$ and $\nu(t)$, $t_0 \leqslant t \leqslant T$ that change the temperature state distribution in the rod from a given initial state (3) to a final state (4) within a given time interval.

3. Reduction of the Problem to a Problem with Zero Boundary Conditions

Since the boundary conditions (2) are non-homogeneous, we construct the solution to Equation (1) as the sum [5]

$$Z(x,t) = W(x,t) + Y(x,t), \tag{6}$$

where $Y(x,t)$ is the function with boundary conditions

$$Y(0,t) = Y(l,t) = 0, \tag{7}$$

to be determined, and the function $W(x,t)$ is the solution to Equation (1) subject to conditions

$$W(0,t) = \mu(t),\ W(l,t) = \nu(t) \tag{8}$$

and has the form

$$W(x,t) = \mu(t) + \frac{x}{l}[\nu(t) - \mu(t)]. \tag{9}$$

Substituting (6) into (1), given (9), yields the following equation for determining the function $Y(x,t)$

$$\frac{\partial Y}{\partial t} = a\frac{\partial^2 Y}{\partial x^2} + f(x,t), 0 \leqslant x \leqslant l, t_0 \leqslant t \leqslant T. \tag{10}$$

where

$$f(x,t) = \frac{x}{l}[\dot{\mu}(t) - \dot{\nu}(t)]. \tag{11}$$

By applying the approaches reported in [16], from the initial (3) and final conditions (4), given the consistency conditions (5), we obtain that the function $Y(x,t)$ must satisfy the following initial

$$Y(x,t_0) = \varphi_0(x) - \varphi_0(0) - \frac{x}{l}[\varphi_0(l) - \varphi_0(0)], \tag{12}$$

and final

$$Y(x,T) = \varphi_T(x) - \varphi_T(0) - \frac{x}{l}[\varphi_T(l) - \varphi_T(0)] \tag{13}$$

conditions.

Thus, the solution to the original problem is reduced to the problem of controlling a thermal process described by the non-homogeneous Equation (10) with homogeneous boundary conditions (7), which is stated as follows: find such temperature conditions $\mu(t)$ and $\nu(t)$, $t_0 \leqslant t \leqslant T$ that change the temperature state as described by the non-homogeneous Equation (10) with homogeneous boundary conditions (7) from a given initial state (12) to a final state (13) within a given time interval.

4. Problem Solution

Given that the boundary conditions (7) are homogeneous, and the consistency conditions are met, the solution to Equation (10) is sought in the form

$$Y(x,t) = \sum_{k=1}^{\infty} Y_k(t) \sin\frac{\pi k}{l}x,\ Y_k(t) = \frac{2}{l}\int_0^l Y(x,t)\sin\frac{\pi k x}{l}dx. \tag{14}$$

Let us represent the functions $f(x,t)$, $\varphi_0(x)$, $\varphi_T(x)$ as Fourier series in the basis $\left\{\sin\frac{\pi k x}{l}\right\}$ $(k = 1,2,\ldots)$, and by substituting their values together with $Y(x,t)$ into Equations (10) and (11) and conditions (12) and (13), we obtain that the Fourier coefficients $Y_k(t)$ satisfy a countable number of systems of ordinary differential equations

$$\dot{Y}_k(t) + a\lambda_k Y_k(t) = f_k(t),\ \lambda_k = \left(\frac{\pi k}{l}\right)^2,\ k = 1, 2, \ldots \tag{15}$$

$$f_k(t) = -\frac{2}{\pi k}(-1)^k[\dot{\mu}(t) - \dot{\nu}(t)], \tag{16}$$

$$Y_k(t_0) = \varphi_k^{(0)} - \frac{2}{\pi k}\left[\varphi_0(0) - (-1)^k\varphi_0(l)\right], \tag{17}$$

$$Y_k(T) = \varphi_k^{(T)} - \frac{2}{\pi k}\left[\varphi_T(0) - (-1)^k\varphi_T(l)\right]. \tag{18}$$

Here, the Fourier coefficients of the function $f(x,t)$, $\varphi_0(x)$, $\varphi_T(x)$ are denoted by $f_k(t)$, $\varphi_k^{(0)}$, and $\varphi_k^{(T)}$, respectively.

The general solution to the Equation (15) with the initial condition (17) has the form [5]

$$Y_k(t) = Y_k(t_0)e^{-a\lambda_k(t-t_0)} + \int_{t_0}^{t} f_k(\tau)e^{-a\lambda_k(t-\tau)}d\tau. \tag{19}$$

Now, given the final condition (18), we obtain that the functions $f_k(\tau)$, $t_0 \leqslant \tau \leqslant T$ for each $k = 1,\ 2, \ldots$ must satisfy the following relation:

$$\int_{t_0}^{T} f_k(\tau)e^{-a\lambda_k(T-\tau)}d\tau = Y_k(T) - Y_k(t_0)e^{-a\lambda_k(T-t_0)}. \tag{20}$$

By applying the approaches reported in [16–18], we obtain that the control functions $\mu(t)$ and $\nu(t)$ for each $k = 1, 2, \ldots$ must satisfy the integral relation:

$$\int_{t_0}^{T} \mu(\tau)e^{a\lambda_k\tau}d\tau - (-1)^k \int_{t_0}^{T} \nu(\tau)e^{a\lambda_k\tau}d\tau = C_k, \tag{21}$$

$$C_k = \frac{1}{a\lambda_k}\left\{\frac{\pi k}{2}\left[Y_k(T)e^{a\lambda_k T} - Y_k(t_0)e^{a\lambda_k t_0}\right] + \left[\varphi_T(0) - (-1)^k\varphi_T(l)\right]e^{a\lambda_k T} - \left[\varphi_0(0) - (-1)^k\varphi_0(l)\right]e^{a\lambda_k t_0}\right\}.$$

Let us introduce the following notations:

$$\bar{H}_k(\tau) = \begin{pmatrix} e^{a\lambda_k\tau} & -(-1)^k e^{a\lambda_k\tau} \end{pmatrix},\quad U(\tau) = \begin{pmatrix} \mu(\tau) \\ \nu(\tau) \end{pmatrix}. \tag{22}$$

Then relation (21) is written as follows

$$\int_{t_0}^{T} \bar{H}_k(\tau)U(\tau)d\tau = C_k, k = 1, 2, \ldots. \tag{23}$$

Hence, the infinite integral relations (23) are obtained for finding the function $U(\tau)$, $\tau \in [t_0, T]$.

In practice, it is usually several first n ($k = 1, 2, \ldots, n$) of relations (23) that are chosen and the problem of control synthesis is solved by methods of the theory of control of finite-dimensional systems [1,16]. We will arrive at the solution to the problem by following this approach. Consequently, for the first n relations, from (23), we have

$$\int_{t_0}^{T} H_n(\tau)U_n(\tau)d\tau = \eta_n, \tag{24}$$

where the following notations of block matrices are introduced

$$H_n(\tau) = \begin{pmatrix} \bar{H}_1(\tau) \\ \bar{H}_2(\tau) \\ \vdots \\ \bar{H}_n(\tau) \end{pmatrix}, \quad \eta_n = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} \tag{25}$$

where the dimension of $H_n(\tau)$ is $(n \times 2)$, and the dimension of η_n is $(n \times 1)$. Here and below, the letter "n" in the lower index means "for the first n harmonics".

It follows from relation (24) that the below statement about complete controllability is valid [16–18].

Proposition 1. *The first n harmonics of the dynamic process described by* (15) *with conditions* (16)–(18) *are completely controllable, if, and only if, for any vector η_n* (25)*, we can find a control $U_n(t)$, $t \in [t_0, T]$ satisfying the condition* (24).

The control $U_n(t)$, satisfying the integral relation (24), will be represented in the form [18,19]

$$U_n(t) = H_n^T(t)S_n^{-1}\eta_n + e_n(t), \tag{26}$$

where $H_n^T(t)$ is a transposed matrix, $e_n(t)$ is a vector-valued function such that

$$\int_{t_0}^{T} H_n(t)e_n(t)dt = 0, \quad S_n = \int_{t_0}^{T} H_n(t)H_n^T(t)dt. \tag{27}$$

Here, S_n is a known matrix of dimension $(n \times n)$, for which it is assumed that $\det S_n \neq 0$.

It follows from Formula (26) that there is a set of control functions solving the boundary control problem.

5. Construction of the Solution in the Case of $n = 2$

To illustrate the above, let us construct the control functions in the case of $n = 2$. In this case, from Formula (21), we will have the following integral relations

$$\int_{t_0}^{T} \mu_2(\tau)e^{a\lambda_1\tau}d\tau + \int_{t_0}^{T} \nu_2(\tau)e^{a\lambda_1\tau}d\tau = C_1, \quad \int_{t_0}^{T} \mu_2(\tau)e^{a\lambda_2\tau}d\tau - \int_{t_0}^{T} \nu_2(\tau)e^{a\lambda_2\tau}d\tau = C_2,$$

where the constants C_1 and C_2 are determined from Formula (21).

According to (27), the matrix S_2 has the form $S_2 = \begin{pmatrix} s_{11} & 0 \\ 0 & s_{22} \end{pmatrix}$, where

$$s_{ii} = \frac{1}{a\lambda_i}(\mathrm{e}^{2a\lambda_i T} - e^{2a\lambda_i t_0}), \ i = 1, 2.$$

Let us note that $\det S_2 = s_{11}s_{22} \neq 0$. The inverse matrix has the form

$$S_2^{-1} = \frac{1}{\det S_2}\begin{pmatrix} s_{22} & 0 \\ 0 & s_{11} \end{pmatrix}.$$

From Formula (26), it follows that

$$U_2(t) = H_2^T(t)S_2^{-1}\eta_2 + e_2(t).$$

Assuming that $e_2(t) = 0$, we obtain

$$\mu_2(\tau) = \frac{1}{s_{11}s_{22}}[s_{22}C_1e^{a\lambda_1\tau} + s_{11}C_2e^{a\lambda_2\tau}],\ \nu_2(\tau) = \frac{1}{s_{11}s_{22}}[s_{22}C_1e^{a\lambda_1\tau} - s_{11}C_2e^{a\lambda_2\tau}],$$

where

$$C_1 = \frac{1}{a\lambda_1}\left\{\frac{\pi}{2}\left[Y_1(T)e^{a\lambda_1T} - Y_1(t_0)e^{a\lambda_1t_0}\right] + [\varphi_T(0) + \varphi_T(l)]e^{a\lambda_1T} - [\varphi_0(0) + \varphi_0(l)]e^{a\lambda_1t_0}\right\},$$

$$C_2 = \frac{1}{a\lambda_2}\left\{\pi\left[Y_2(T)e^{a\lambda_2T} - Y_2(t_0)e^{a\lambda_2t_0}\right] + [\varphi_T(0) - \varphi_T(l)]e^{a\lambda_2T} - [\varphi_0(0) - \varphi_0(l)]e^{a\lambda_2t_0}\right\},$$

or, in respect that (17) and (18), we have

$$C_1 = \frac{\pi}{2a\lambda_1}\left[\varphi_1^{(T)}e^{a\lambda_1T} - \varphi_1^{(0)}e^{a\lambda_1t_0}\right],\ C_2 = \frac{\pi}{a\lambda_2}\left[\varphi_2^{(T)}e^{a\lambda_2T} - \varphi_2^{(0)}e^{a\lambda_2t_0}\right].$$

Having the expressions of the function $\mu_2(\tau)$ and $\nu_2(\tau)$ from Formulas (16) and (19), we obtain an explicit expression for the function $Y_k(t)$ in the form

$$Y_k(t) = Y_k(t_0)e^{-a\lambda_k(t-t_0)} + \int_{t_0}^{t} f_k(\tau)e^{-a\lambda_k(t-\tau)}d\tau,\ k = 1,2,$$

where

$$f_1(\tau) = -\frac{4a\lambda_1C_1}{\pi s_{11}}e^{a\lambda_1\tau}, f_2(\tau) = -\frac{2a\lambda_2C_2}{\pi s_{22}}e^{a\lambda_2\tau}.$$

Substituting these expressions as an integrand and integrating, we obtain

$$Y_1(t) = Y_1(t_0)e^{-a\lambda_1(t-t_0)} - \frac{2C_1}{\pi s_{11}}e^{a\lambda_1t} + \frac{2C_1}{\pi s_{11}}e^{-a\lambda_1t}e^{2a\lambda_1t_0},$$

$$Y_2(t) = Y_2(t_0)e^{-a\lambda_2(t-t_0)} - \frac{C_2}{\pi s_{22}}e^{a\lambda_2t} + \frac{C_2}{\pi s_{22}}e^{-a\lambda_2t}e^{2a\lambda_2t_0}.$$

From (6), we obtain an explicit expression for the rod temperature function $Y_2(x,t)$, in the case of $n = 2$, in the form

$$Y_2(x,t) = Y_1(t)\sin\frac{\pi}{l}x + Y_2(t)\sin\frac{2\pi}{l}x.$$

Further, from the Formulas (6) and (9), for the temperature distribution function in the rod $Z_2(x,t), 0 \leqslant x \leqslant l, 0 < t < T$, we will have

$$Z_2(x,t) = W_2(x,t) + Y_2(x,t) = Y_1(t)\sin\frac{\pi}{l}x + Y_2(t)\sin\frac{2\pi}{l}x + \mu_2(t) + \frac{x}{l}[\nu_2(t) - \mu_2(t)].$$

6. Computational Experiment

Let us carry out numerical calculations for comparative analysis of the results and identification of regularities related to the physical properties of materials. Assume that the thermal diffusivity $a = 1$ (m^2/s), $l = 1$ (m), and $t_0 = 0$ (s), then $\lambda_1 = \pi^2$, $\lambda_2 = 4\pi^2$. The case $a = 1$ is considered as a standard (in the mathematical sense) for the purpose of comparison with the results for other values of a.

Let the following initial state be given for $t = 0$:

$$\varphi_0(x) = x^2 - x,$$

and the following final state be given for $t = T$:

$$\varphi_T(x) = 0.$$

Then, we get

$$C_1 = \frac{4}{\pi^4}, \quad C_2 = 0, \quad s_{11} = \frac{e^{2\pi^2 T} - 1}{\pi^2}, \quad s_{22} = \frac{e^{8\pi^2 T} - 1}{4\pi^2},$$

$$\mu_2(t) = \nu_2(t) = \frac{4e^{\pi^2 t}}{\pi^2(e^{2\pi^2 T} - 1)}, \quad f_1(t) = -\frac{16e^{\pi^2 t}}{\pi\left(e^{2\pi^2 T} - 1\right)}, \quad f_2(t) = 0,$$

$$Y_1(t) = -\frac{8(e^{2\pi^2 T} + e^{2\pi^2 t} - 2)e^{-\pi^2 t}}{\pi^3\left(e^{2\pi^2 T} - 1\right)}, \quad Y_2(t) = 0,$$

$$Y_1(t_0) = -\frac{8}{\pi^3}, \quad Y_2(t_0) = 0, \quad Y_1(T) = -\frac{16(e^{\pi^2 T} - e^{-\pi^2 T})}{\pi^3(e^{2\pi^2 T} - 1)}, \quad Y_2(T) = 0,$$

$$Y_2(x,t) = -\frac{8(e^{2\pi^2 T} + e^{2\pi^2 t} - 2)e^{-\pi^2 t}}{\pi^3(e^{2\pi^2 T} - 1)} \sin \pi x, \quad W_2(x,t) = \frac{4e^{\pi^2 t}}{\pi^2(e^{2\pi^2 T} - 1)},$$

$$Z_2(x,t) = -\frac{8(e^{2\pi^2 T} + e^{2\pi^2 t} - 2)e^{-\pi^2 t}}{\pi^3(e^{2\pi^2 T} - 1)} \sin \pi x + \frac{4e^{\pi^2 t}}{\pi^2(e^{2\pi^2 T} - 1)},$$

$$Z_2(x,0) = -\frac{8 \sin \pi x}{\pi^3} + \frac{4}{\pi^2\left(e^{2\pi^2 T} - 1\right)},$$

$$Z_2(x,T) = -\frac{16e^{-\pi^2 T}}{\pi^3} \sin \pi x + \frac{4e^{\pi^2 T}}{\pi^2(e^{2\pi^2 T} - 1)}.$$

Figure 1 shows the behavior of the obtained function of temperature distribution in the rod.

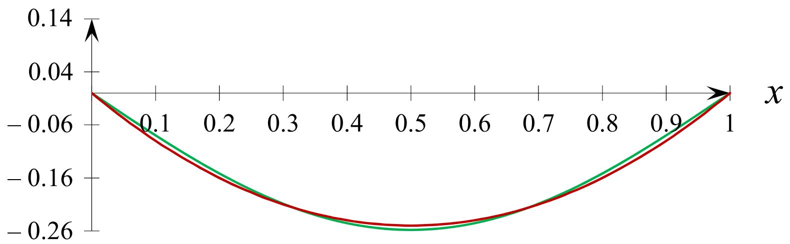

Figure 1. Graphs of functions $\varphi_0(x)$ (red) and $Z_2(x,3)$ (green).

Table 1 shows the estimation of the deviation of these functions

$$\varepsilon(t_j) = \max_{0 \le x \le 1} \left| Z_2(x,t_j) - \varphi_j(x) \right|, j = 0,1, \ t_1 = T,$$

and comparative analysis of the obtained results.

Table 1. Deviation estimates for $a = 1$.

$T = 1$	$T = 3$
$\varepsilon(0) = 0.00801$ $\varepsilon(T) = 2 \times 10^{-5}$	$\varepsilon(0) = 0.00801$ $\varepsilon(T) = 6 \times 10^{-14}$

Let the material of the rod be copper with a coefficient of thermal diffusivity (25 °C) $a = 1.11 \times 10^{-4}$ (m^2/s). Denote $\alpha = \pi^2 a = 1.09553 \times 10^{-3}$, then, we have:

$$C_1 = \frac{36036.036}{\pi^4}, \quad C_2 = 0, \quad s_{11} = 912.803(e^{2\alpha T} - 1), \quad s_{22} = 228.201(e^{8\alpha T} - 1),$$

$$\mu_2(t) = \nu_2(t) = \frac{0.405e^{\alpha t}}{e^{2\alpha T} - 1}, \quad f_1(t) = -\frac{0.000565e^{\alpha t}}{e^{2\alpha T} - 1}, \quad f_2(t) = 0,$$

$$Y_1(t) = -\frac{0.258(e^{2\alpha T} + e^{2\alpha t} - 2)e^{-\alpha t}}{e^{2\alpha T} - 1}, \quad Y_2(t) = 0,$$

$$Y_1(t_0) = -\frac{8}{\pi^3}, \quad Y_2(t_0) = 0, \quad Y_1(T) = -\frac{0.516(e^{\alpha T} - e^{-\alpha T})}{e^{2\alpha T} - 1}, \quad Y_2(T) = 0,$$

$$Y_2(x,t) = -\frac{0.258(e^{2\alpha T} + e^{2\alpha t} - 2)e^{-\alpha t}}{e^{2\alpha T} - 1} \sin \pi x, \quad W_2(x,t) = \frac{0.405e^{\alpha t}}{e^{2\alpha T} - 1},$$

$$Z_2(x,t) = -\frac{0.258(e^{2\alpha T} + e^{2\alpha t} - 2)e^{-\alpha t}}{e^2\alpha T - 1} \sin \pi x + \frac{0.405e^{\alpha t}}{e^{2\alpha T} - 1},$$

$$Z_2(x,0) = -0.258 \sin \pi x + \frac{0.405}{e^{2\alpha T} - 1},$$

$$Z_2(x,T) = -0.516e^{-\alpha T} \sin \pi x + \frac{0.405e^{\alpha T}}{e^{2\alpha T} - 1}.$$

Figures 2 and 3 show the behavior of the obtained function of temperature distribution in the rod.

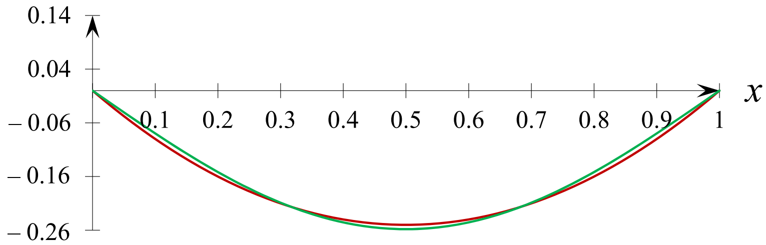

Figure 2. Graphs of functions $\varphi_0(x)$ (red) and $Z_2(x, 4500)$ (green).

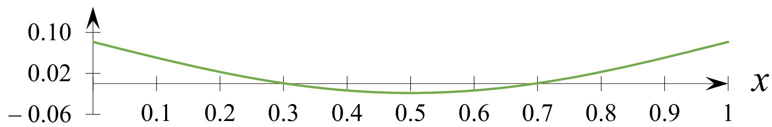

Figure 3. Graphs of functions $\varphi_T(x) = 0$ and $Z_2(x, 1500)$ (green).

Table 2 shows the deviation estimate of these functions:

$$\varepsilon(t_j) = \max_{0 \le x \le 1} |Z_2(x,t_j) - \varphi_j(x)|, \quad j = 0,1; \quad t_1 = T,$$

and a comparative analysis of the obtained results for copper.

Table 2. Deviation estimates for $a = 1.11 \times 10^{-4}$.

$T = 300$	$T = 900$	$T = 1500$	$T = 4500$
$\varepsilon(0) = 0.44661$	$\varepsilon(0) = 0.07617$	$\varepsilon(0) = 0.02638$	$\varepsilon(0) = 0.01066$
$\varepsilon(T) = 0.60561$	$\varepsilon(T) = 0.17565$	$\varepsilon(T) = 0.08140$	$\varepsilon(T) = 0.00293$

Let now the material of the rod be iron with thermal diffusivity $a = 2.3 \times 10^{-5}$ (m^2/s). Denote $\alpha = \pi^2 a = 2.27001 \times 10^{-4}$, then, we have:

$$C_1 = \frac{173913.043}{\pi^4}, \quad C_2 = 0, \quad s_{11} = 4405.269(e^{2\alpha T} - 1), \quad s_{22} = 1101.317(e^{8\alpha T} - 1),$$

$$\mu_2(t) = \nu_2(t) = \frac{0.405e^{\alpha t}}{e^{2\alpha T} - 1}, \quad f_1(t) = -\frac{0.000117e^{\alpha t}}{e^{2\alpha T} - 1}, \quad f_2(t) = 0,$$

$$Y_1(t) = -\frac{0.258(e^{2\alpha T} + e^{2\alpha t} - 2)e^{-\alpha t}}{e^{2\alpha T} - 1}, \quad Y_2(t) = 0,$$

$$Y_1(t_0) = -\frac{8}{\pi^3}, \quad Y_2(t_0) = 0, \quad Y_1(T) = -\frac{0.516(e^{\alpha T} - e^{-\alpha T})}{e^{2\alpha T} - 1}, \quad Y_2(T) = 0,$$

$$Y_2(x,t) = -\frac{0.258(e^{2\alpha T} + e^{2\alpha t} - 2)e^{-\alpha t}}{e^{2\alpha T} - 1} \sin \pi x, \quad W_2(x,t) = \frac{0.405e^{\alpha t}}{e^{2\alpha T} - 1},$$

$$Z_2(x,t) = -\frac{0.258(e^{2\alpha T} + e^{2\alpha t} - 2)e^{-\alpha t}}{e^{2\alpha T} - 1} \sin \pi x + \frac{0.405e^{\alpha t}}{e^{2\alpha T} - 1},$$

$$Z_2(x,0) = -0.258 \sin \pi x + \frac{0.405}{e^{2\alpha T} - 1},$$

$$Z_2(x,T) = -0.516e^{-\alpha T} \sin \pi x + \frac{0.405e^{\alpha T}}{e^{2\alpha T} - 1}.$$

As can be seen from the formulas obtained at the end of Section 5, the difference between the corresponding formulas for copper and iron materials is only in the value α.

Figure 4 shows the behavior of the obtained function of temperature distribution in the rod.

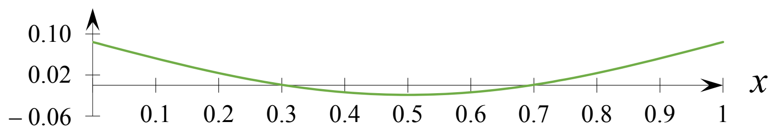

Figure 4. Graphs of functions $\varphi_T(x) = 0$ and $Z_2(x, 7100)$ (green).

Table 3 shows the deviation estimate of these functions:

$$\varepsilon(t_j) = \max_{0 \le x \le 1} \left| Z_2(x, t_j) - \varphi_j(x) \right|, \quad j = 0, 1; \quad t_1 = T,$$

and a comparative analysis of the obtained results for iron.

Table 3. Deviation estimates for $a = 2.3 \times 10^{-5}$.

$T = 900$	$T = 1500$	$T = 4500$	$T = 7100$
$\varepsilon(0) = 0.81364$	$\varepsilon(0) = 0.42595$	$\varepsilon(0) = 0.07100$	$\varepsilon(0) = 0.02745$
$\varepsilon(T) = 0.98502$	$\varepsilon(T) = 0.58378$	$\varepsilon(T) = 0.16766$	$\varepsilon(T) = 0.08423$

Thus, the analysis of the results of the computational experiment confirms the following regularity of the physical properties of the rod material: the lower the thermal diffusivity of the material, the more time is required for boundary control, which provides the desired deviation ϵ.

So, using the proposed approach, given $n = 2$, we have constructed explicit expressions of the thermal process control function solving the problem stated above and the explicit expression of the corresponding rod temperature distribution function.

7. Conclusions

For a rod with an insulated side surface, we have proposed a constructive approach to building a function of such temperature conditions at the ends of the rod that change the distribution of the temperature state in the rod from a given initial state to a final state within a given time interval. The proposed approach relies on the method of separation of variables and methods of the theory of control of finite-dimensional systems. With the aid of the Fourier method, the proposed approach can be applied to constructing control of thermal conditions for other non-uniform thermal processes. This attests to the practical significance of the results obtained. The results obtained can be used in the design of boundary control of thermal processes.

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