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Existence and Uniqueness of Non-Negative Solution to a Coupled Fractional q-Difference System with Mixed q-Derivative via Mixed Monotone Operator Method

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Abstract: In this paper, we study a nonlinear Riemann-Liouville fractional a q-difference system with multi-strip and multi-point mixed boundary conditions under the Caputo fractional q-derivative, where the nonlinear terms contain two coupled unknown functions and their fractional derivatives. Using the fixed point theorem for mixed monotone operators, we constructe iteration functions for arbitrary initial value and acquire the existence and uniqueness of extremal solutions. Moreover, a related example is given to illustrate our research results.

Keywords: the coupled Riemann-Liouville fractional q-difference system; the Caputo fractional q-derivative boundary conditions; mixed monotone operator

MSC: 34B18; 26A33; 34B27

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1. Introduction

Fractional calculus come into people's view in 1695 [1], extending the traditional integral calculus concept to the whole field of real numbers. It is originally of great significance in many areas [2]. As we all know, in the research process of these fields, equations need to be established to describe the specific change process. On the other hand, fractional calculus has the nice property of being able to accurately describe these processes with genetic and memory traits. Therefore, the fractional differential equation has gradually become the focus of people's research. At the same time, the existence analysis, uniqueness analysis, stability analysis of solutions in fractional differential equation become an important research direction. Many scholars have studied them in recent years, and readers can refer to the literature [3–13].

At the beginning of the twentieth century, the appearance of Quantum Mechanics promoted the generation and development of Quantum calculus (q-calculus). F. H. Jackson made the first complete study of q-calculus [14,15]. Later W. A. Al-Salam [16] and R. P. Agarwal [17] proposed the basic concepts and properties of fractional q-calculus. Q-calculus has been an important bridge between mathematics and physics since its birth. It plays an extremely important role in quantum physics, spectral analysis and dynamical systems [18–21]. In recent years, q-calculus has also been increasingly used in engineering [22,23]. With the application and development of fractional differential equation and the extensive research and application of q-calculus in mathematics, physics and other fields, the study of fractional q-difference equation has become a topic of widespread concern. Increased experts begin to pay attention to the theoretical research of fractional q-difference equation [24–32].

In 2019, the authors [27] studied the boundary value problem of the following mixed fractional q-difference by the Guo–Krasnoselskii's fixed point theorem and the Banach contraction mapping principle:

$$\begin{cases} D_q^{\alpha} u(t) + f(t, u(t)) = 0, & t \in [0,1], \\ u(0) = {}^c D_q^{\beta} u(0) = {}^c D_q^{\beta} u(1), \end{cases}$$

where $0 < \beta \leqslant 1, 2 < \alpha < 2 + \beta$, D_q^{α}, ${}^c D_q^{\beta}$ are the Riemann–Liouville fractional q-derivative and Caputo fractional q-derivative of order α, β.

In [30], utilizing the monotone iterative approach, the authors are considered with the fractional q-difference system involing four-point boundary conditions:

$$\begin{cases} D_q^{\alpha} u(t) + f(t, v(t)) = 0, & t \in (0,1), \\ D_q^{\beta} v(t) + f(t, u(t)) = 0, & t \in (0,1), \\ u(0) = 0, \ u(1) = \gamma_1 u(\eta_1), \\ v(0) = 0, \ v(1) = \gamma_2 v(\eta_2), \end{cases}$$

where $1 < \beta \leqslant \alpha \leqslant 2, 0 < \eta_1, \eta_2 < 1, 0 < \gamma_1 \eta_1^{\alpha-1} < 1$ and $0 < \gamma_2 \eta_2^{\beta-1} < 1$.

In [9], in view of the method of mixed monotone operators, the conclusion of the existence and uniqueness of solutions for the following coupled system is drawn:

$$\begin{cases} D_{0^+}^{\alpha} x(\tau) + f_1\left(\tau, x(\tau), D_{0^+}^{\eta} x(\tau)\right) + g_1(\tau, y(\tau)) = 0, \\ D_{0^+}^{\beta} y(\tau) + f_2(\tau, y(\tau), D_{0^+}^{\gamma} y(\tau)) + g_2(\tau, x(\tau)) = 0, \\ \tau \in (0,1), \quad n-1 < \alpha, \beta < n, \\ x^{(i)}(0) = y^{(i)}(0) = 0, \quad i = 0, 1, 2, \ldots, n-2, \\ \left[D_{0^+}^{\xi} y(\tau)\right]_{\tau=1} = k_1(y(1)), \quad \left[D_{0^+}^{\zeta} x(\tau)\right]_{\tau=1} = k_2(x(1)), \end{cases}$$

where the integer number $n > 3$ and $1 \leqslant \gamma \leqslant \xi \leqslant n-2, 1 \leqslant \eta \leqslant \zeta \leqslant n-2$, f_1, f_2: $[0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+, g_1, g_2$: $[0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$and k_1, k_2: $\mathbb{R}^+ \to \mathbb{R}^+$are continuous functions, $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ represent the Riemann-Liouville derivatives.

There are many ways to deal with the boundary value problem, such as monotone iteration techniques, the Banach contraction mapping principle and so on. In these methods, the constraints are often stringent, one of which is that completely continuity of the operator must be proved and the proving process is often very complicated. However, The mixed monotone operators have relatively loose requirements and only need to prove some properties like the upper bound. Now, there has been some literature using it to prove the existence and uniqueness of solutions, see [9–13,31,32]. Therefore, this paper also intends to introduce this method to prove the corresponding conclusion.

Motivated by the above mentioned papers, we investigate the following coupled nonlinear fractional q-difference system:

$$\begin{cases} D_q^{\alpha_1} u(t) + f_1(t, u(t), v(t), D_q^{\gamma_1} u(t), D_q^{\gamma_2} v(t)) = 0, & t \in (0,1), \\ D_q^{\alpha_2} v(t) + f_2(t, u(t), v(t), D_q^{\gamma_1} u(t), D_q^{\gamma_2} v(t)) = 0, & t \in (0,1), \end{cases} \tag{1}$$

subject to the multi-strip and multi-point mixed boundary conditions:

$$\begin{cases} u(0) = 0, \ v(0) = 0, \\ {}^c D_q^{\alpha_1 - 1} u(1) = \sum_{i=1}^{m} \lambda_{1i} I_q^{\beta_{1i}} v(\xi_i) + \sum_{j=1}^{n} b_{1j} v(\eta_j), \\ {}^c D_q^{\alpha_2 - 1} v(1) = \sum_{i=1}^{m} \lambda_{2i} I_q^{\beta_{2i}} u(\xi_i) + \sum_{j=1}^{n} b_{2j} u(\eta_j), \end{cases} \tag{2}$$

where ${}^{c}D_q^{\alpha}$, D_q^{α} are the Caputo fractional q-derivative and Riemann-Liouville fractional q-derivative of order α respectively, and $I_q^{\beta_{ki}}$ is the Riemann-Liouville fractional q-integral of order β_{ki}, for $k = 1, 2$ and $i = 1, 2, \dots, m$. $\alpha_1, \alpha_2 \in [1, 2]$, $\gamma_1, \gamma_2 \in [0, 1]$; $\lambda_{1i}, \lambda_{2i} \in [0, +\infty)$, $\beta_{1i}, \beta_{2i} \in (0, +\infty)$, $\xi_i \in [0, 1]$, for $i = 1, 2, \dots, m$; $b_{1j}, b_{2j} \in [0, +\infty)$, $\eta_j \in [0, 1]$, for $j = 1, 2, \dots, n$.

The system has the following four main characteristics: First, it is based on q-calculus, so it is closely connected with Physics and has practical research significance. Second, there are two types of derivatives (the Caputo and Riemann-Liouville fractional q-derivative) in the system, which is more in line with the complex conditions of the real world. Third, the unknown functions $u(t)$ and $v(t)$ in the system influence each other which can have better practical applications. Fourth, the nonlinear part, $f_1$and f_2, contain two derivative operators $D_q^{\gamma_1}, D_q^{\gamma_2}$. Due to the complexity of the model, it is difficult to find the Green's function and its upper and lower bounds. After that, we choose the mixed monotone operator method to get the existence and uniqueness of non-negative solution to our system. Compared with the monotone iterative method for the requirements of fixed initial values, mixed monotone operators do not need to prove complete continuity, and there are no restrictions for initial values, that is, the arbitrarily initial value works. Therefore, the mixed monotone method is more widely applicable.

This article is arranged as the following aspects: In Section 2, some fundamental definitions and lemmas are introduced. Moreover, some crucial results and their proofs are discussed. In Section 3, we set out the main conclusion: the existence and uniqueness results of non-negative solutions. At last, an example is given to illustrate our result.

2. Preliminaries

For the reader's convenience, we list some important definitions of q-calculus. On the other hand, there are also basic notion and lemmas for the proof which will be used in the next section.

Let $(E, \|\cdot\|)$ be a Banach space, partially ordered by a cone $P \subset E$. In other words, $x \preceq y$ if and only if $y - x \in P$. We use θ to represent the zero element of E. We consider a cone P to be normal if there exists a constant $M > 0$ such that for all $x, y \in E$, $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant M\|y\|$; on this condition M is called the normality constant of P. Giving $h > \theta$, we denote by P_h the set $P_h = \{x \in E |\ \exists \lambda, \mu > 0 : \lambda h \leqslant x \leqslant \mu h\}$.

Definition 1 ([33,34]). *$A : P \times P \to P$ is said to be a mixed monotone operator of $A(x, y)$ is increasing in x and decreasing in y, i.e., for $x_i, y_i \in P (i = 1, 2)$, $x_1 \leqslant x_2$, $y_1 \geqslant y_2$ implies that $A(x_1, y_1) \leqslant A(x_2, y_2)$. Element $x \in P$ is called a fixed point of A if $A(x, x) = x$.*

Lemma 1 ([35]). *Let P a normal cone of a real Banach space E. Also, let $A : P \times P \to P$ be a mixed monotone operator. Assume that*

(A_1) *there exists $h \in P$ with $h \neq \theta$ such that $A(h, h) \in P_h$;*

(A_2) *for any $u, v \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (0, 1]$ such that $A(tu, t^{-1}v) \geqslant \varphi(t)A(u, v)$.*

Then operator A has a unique fixed point x^* in P_h. Moreover, for any initial $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \cdots,$$

one has $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

For $q \in (0, 1)$ and $a \in \mathbb{R}$, define

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

The q-analogue of the power function is

$$(a-b)_q^{(0)}=1,$$

$$(a-b)_q^{(k)}=\prod_{i=0}^{k-1}(a-bq^i),\ \ k\in\mathbb{N},a,b\in\mathbb{R}.$$

Generally, if $\alpha\in\mathbb{R}$, there is

$$(a-b)_q^{(\alpha)}=a^\alpha\prod_{i=0}^{\infty}\frac{a-bq^i}{a-bq^{i+\alpha}}\ .$$

It is clearly that $a^{(\alpha)}=a^\alpha$ for $b=0$ and $0^{(\alpha)}=0$ for $\alpha\geq 0$.

The q-Gamma function is given by

$$\Gamma_q(x)=\frac{(1-q)^{(x-1)}}{(1-q)^{x-1}},\ \ x\in\mathbb{R}\backslash\{0,-1,-2,\dots\},$$

then we have $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$.

For $x,y>0$, we have

$$B_q(x,y)=\int_0^1 t^{x-1}(1-qt)^{(y-1)}d_qt, \tag{3}$$

especially,

$$B_q(x,y)=\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}.$$

The q-derivative of a function f is defined by

$$(D_qf)(x)=\frac{f(qx)-f(x)}{(q-1)x},$$

$$(D_qf)(0)=\lim_{x\to 0}(D_qf)(x),$$

and the q-derivative of higher order by

$$(D_q^0f)(x)=f(x),$$

$$(D_q^nf)(x)=D_q(D_q^{n-1}f)(x),\ \ n\in\mathbb{N}.$$

The q-integral of a function f defined on the interval $[0,b]$ is given by

$$(I_qf)(x)=\int_0^x f(s)d_qs=x(1-q)\sum_{k=0}^{\infty}f(xq^k)q^k,\ \ x\in[0,b].$$

If $a\in[0,b]$ and f is defined in the interval $[0,b]$, then its integral from a to b is defined by

$$\int_a^b f(s)d_qs=\int_0^b f(s)d_qs-\int_0^a f(s)d_qs,$$

and similarly the q-integral of higher order is given by

$$(I_q^0f)(x)=f(x),$$

$$(I_q^nf)(x)=I_q(I_q^{n-1}f)(x),\ \ n\in\mathbb{N}.$$

Definition 2 ([17]). *Let $\alpha \geqslant 0$ and f be a real function defined on a certain interval $[a,b]$. The Riemann-Liouville fractional q-integral of order α is defined by*

$$(I_q^0 f)(t) = f(t),$$

$$(I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s) d_q s, \ \ \alpha > 0.$$

Definition 3 ([17]). *The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geqslant 0$ of a continuous and differential function f on the interval $[a,b]$ is given by*

$$(D_q^0 f)(t) = f(t),$$

$$(D_q^\alpha f)(t) = (D_q^l I_q^{l-\alpha} f)(t), \ \ \alpha > 0,$$

where l is the smallest integer greater than or equal to α.

Definition 4 ([17]). *Let $\alpha \geqslant 0$, and the Caputo fractional q-derivatives of f be defined by*

$$({}^c D_q^\alpha f)(t) = (I_q^{l-\alpha} D_q^l f)(t),$$

where l is the smallest integer greater than or equal to α.

Lemma 2 ([36]). *Let $\alpha, \beta \geqslant 0$ and $f : [a,b] \to \mathbb{R}$ be a continuous function defined on $[a,b]$ and its derivative exist. Then the following formulas hold:*

$$(D_q^\alpha I_q^\alpha f)(t) = f(t),$$

$$(I_q^\alpha I_q^\beta f)(t) = (I_q^{\alpha+\beta} f)(t).$$

Lemma 3 ([36]). *Let $\alpha > 0$ and p be a positive integer. Then the following equality holds:*

$$(I_q^\alpha D_q^p f)(t) = (D_q^p I_q^\alpha f)(t) - \sum_{k=0}^{p-1} \frac{t^{\alpha-p+k}}{\Gamma_q(\alpha - p + k + 1)} (D_q^k f)(0).$$

Lemma 4 ([36,37]). *Let $\alpha > 0$ and $n = [\alpha] + 1$. Then we have*

$$(I_q^\alpha \, {}^c D_q^\alpha f)(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_0, c_1, \ldots, c_{n-1}$ are some constants.

For convenience, we denote

$$\begin{cases} l_1 = \dfrac{1}{\Gamma_q(\alpha_2)} \left[\displaystyle\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})} \int_0^{\xi_i} (\xi_i - qs)^{(\beta_{1i}-1)} s^{\alpha_2 - 1} d_q s + \sum_{j=1}^n b_{1j} \eta_j^{\alpha_2 - 1} \right], \\ l_2 = \dfrac{1}{\Gamma_q(\alpha_1)} \left[\displaystyle\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})} \int_0^{\xi_i} (\xi_i - qs)^{(\beta_{2i}-1)} s^{\alpha_1 - 1} d_q s + \sum_{j=1}^n b_{2j} \eta_j^{\alpha_1 - 1} \right]. \end{cases} \tag{4}$$

The following assumptions are introduced for analysis:

(F_1) $1 \leqslant \alpha_k \leqslant 2$, $\beta_{ki} > 0$, for $k = 1,2$ and $i = 1,2,\ldots,m$;
(F_2) $0 \leqslant \eta_j, \xi_i \leqslant 1, \lambda_{1i}, \lambda_{2i} \geqslant 0$, $b_{1j}, b_{2j} \geqslant 0$, for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$;
(F_3) $1 - l_1 l_2 > 0$, where l_1, l_2 are defined by (4);
(F_4) $f_k : [0,1] \times [0,+\infty)^4 \to [0,+\infty)$ is continuous ($k = 1,2$).

A corresponding linear differential system with BVP (1) and (2) is considered, and the expression of the corresponding Green's functions are established.

Lemma 5. *Assume that* (F_1)–(F_3) *hold. For* $h_1, h_2 \in C(0,1)$, *the fractional differential system*

$$\begin{cases} D_q^{\alpha_1} u(t) + h_1(t) = 0, & t \in (0,1), \\ D_q^{\alpha_2} v(t) + h_2(t) = 0, & t \in (0,1), \end{cases} \tag{5}$$

with boundary conditions (2) has an integral representation

$$\begin{cases} u(t) = \int_0^1 K_1(t,qs)h_1(s)d_qs + \int_0^1 H_1(t,qs)h_2(s)d_qs, \\ v(t) = \int_0^1 K_2(t,qs)h_2(s)d_qs + \int_0^1 H_2(t,qs)h_1(s)d_qs, \end{cases} \tag{6}$$

where

$$\begin{aligned} K_1(t,qs) &= g_1(t,qs) + \frac{l_1 t^{\alpha_1-1}}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{2i}-1)} g_1(\tau,qs)d\tau + \sum_{j=1}^n b_{2j} g_1(\eta_j,qs)\right], \\ H_1(t,qs) &= \frac{t^{\alpha_1-1}}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{1i}-1)} g_2(\tau,qs)d_q\tau + \sum_{j=1}^n b_{1j} g_2(\eta_j,qs)\right], \end{aligned} \tag{7}$$

$$\begin{aligned} K_2(t,qs) &= g_2(t,qs) + \frac{l_2 t^{\alpha_2-1}}{\Gamma_q(\alpha_2)(1-l_1l_2)}\left[\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{1i}-1)} g_2(\tau,qs)d\tau + \sum_{j=1}^n b_{1j} g_2(\eta_j,qs)\right], \\ H_2(t,qs) &= \frac{t^{\alpha_2-1}}{\Gamma_q(\alpha_2)(1-l_1l_2)}\left[\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{2i}-1)} g_1(\tau,qs)d_q\tau + \sum_{j=1}^n b_{2j} g_1(\eta_j,qs)\right], \end{aligned} \tag{8}$$

and for k = 1, 2,

$$g_k(t,qs) = \frac{1}{\Gamma_q(\alpha_k)}\begin{cases} t^{\alpha_k-1} - (t-qs)^{(\alpha_k-1)}, & 0 \leqslant qs \leqslant t \leqslant 1, \\ t^{\alpha_k-1}, & 0 \leqslant t \leqslant qs \leqslant 1. \end{cases} \tag{9}$$

Proof. According to Lemma 3, the Equation (5) can be reduced to the following equivalent integral equations:

$$\begin{cases} u(t) = -I_q^{\alpha_1}h_1(t) + c_{11}t^{\alpha_1-1} + c_{12}t^{\alpha_1-2}, \\ v(t) = -I_q^{\alpha_2}h_2(t) + c_{21}t^{\alpha_2-1} + c_{22}t^{\alpha_2-2}, \end{cases} \tag{10}$$

where $c_{11}, c_{12}, c_{21}, c_{22}$ are constants.

From $u(0) = v(0) = 0$, we obtain $c_{12} = c_{22} = 0$. By using Lemma 4, we get

$$\begin{cases} {}^cD_q^{\alpha_1-1}u(t) &= -{}^cD_q^{\alpha_1-1}I_q^{\alpha_1}h_1(t) + c_{11}\,{}^cD_q^{\alpha_1-1}t^{\alpha_1-1} = -I_qh_1(t) + c_{11}[\alpha_1-1]_q I_q^{2-\alpha_1}t^{\alpha_1-2}, \\ {}^cD_q^{\alpha_2-1}v(t) &= -{}^cD_q^{\alpha_2-1}I_q^{\alpha_2}h_2(t) + c_{21}\,{}^cD_q^{\alpha_2-1}t^{\alpha_2-1} = -I_qh_2(t) + c_{21}[\alpha_2-1]_q I_q^{2-\alpha_2}t^{\alpha_2-2}. \end{cases} \tag{11}$$

Then from (3) we get

$$\begin{cases} {}^cD_q^{\alpha_1-1}u(1) = -I_qh_1(1) + c_{11}[\alpha_1-1]_q I_q^{2-\alpha_1}1 = -\int_0^1 h_1(s)d_qs + c_{11}\Gamma_q(\alpha_1), \\ {}^cD_q^{\alpha_2-1}v(1) = -I_qh_2(1) + c_{21}[\alpha_2-1]_q I_q^{2-\alpha_2}1 = -\int_0^1 h_2(s)d_qs + c_{21}\Gamma_q(\alpha_2). \end{cases} \tag{12}$$

From the rest of the condition of (2), it can be obtained that

$$\begin{cases} c_{11} = \dfrac{1}{\Gamma_q(\alpha_1)}\left[\sum\limits_{i=1}^{m}\lambda_{1i}I_q^{\beta_{1i}}v(\xi_i) + \sum\limits_{j=1}^{n}b_{1j}v(\eta_j) + \int_0^1 h_1(s)d_qs\right], \\ c_{21} = \dfrac{1}{\Gamma_q(\alpha_2)}\left[\sum\limits_{i=1}^{m}\lambda_{2i}I_q^{\beta_{2i}}u(\xi_i) + \sum\limits_{j=1}^{n}b_{2j}u(\eta_j) + \int_0^1 h_2(s)d_qs\right]. \end{cases} \tag{13}$$

Further, we can reduce (10) to

$$\begin{cases} u(t) = \dfrac{t^{\alpha_1-1}}{\Gamma(\alpha_1)}\left[\sum\limits_{i=1}^{m}\dfrac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i - qs)^{(\beta_{1i}-1)}v(s)d_qs + \sum\limits_{j=1}^{n}b_{1j}v(\eta_j)\right] + \int_0^1 g_1(t,qs)h_1(s)d_qs, \\ v(t) = \dfrac{t^{\alpha_2-1}}{\Gamma(\alpha_2)}\left[\sum\limits_{i=1}^{m}\dfrac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i - qs)^{(\beta_{2i}-1)}u(s)d_qs + \sum\limits_{j=1}^{n}b_{2j}u(\eta_j)\right] + \int_0^1 g_2(t,qs)h_2(s)d_qs, \end{cases} \tag{14}$$

where $g_k(t,qs)(k = 1,2)$ are introduced by (9). Then we can get

$$\begin{aligned} &\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i - qs)^{(\beta_{1i}-1)}v(s)d_qs + \sum_{j=1}^{n}b_{1j}v(\eta_j) \\ &= \frac{1}{\Gamma_q(\alpha_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i - qs)^{(\beta_{1i}-1)}s^{\alpha_2-1}d_qs + \sum_{j=1}^{n}b_{1j}\eta_j^{\alpha_2-1}\right] \\ &\quad\cdot\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i - qs)^{(\beta_{2i}-1)}u(s)d_qs + \sum_{j=1}^{n}b_{2j}u(\eta_j)\right] \\ &\quad+ \sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{1i}-1)}\int_0^1 g_2(\tau,qs)h_2(s)d_qsd_q\tau + \sum_{j=1}^{n}b_{1j}\int_0^1 g_2(\eta_j,qs)h_2(s)d_qs. \end{aligned} \tag{15}$$

Moreover, we have

$$\begin{aligned} &\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i - qs)^{(\beta_{2i}-1)}u(s)d_qs + \sum_{j=1}^{n}b_{2j}u(\eta_j) \\ &= \frac{1}{\Gamma_q(\alpha_1)}\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i - qs)^{(\beta_{2i}-1)}s^{\alpha_1-1}d_qs + \sum_{j=1}^{n}b_{2j}\eta_j^{\alpha_1-1}\right] \\ &\quad\cdot\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i - qs)^{(\beta_{1i}-1)}v(s)d_qs + \sum_{j=1}^{n}b_{1j}v(\eta_j)\right] \\ &\quad+ \sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{2i}-1)}\int_0^1 g_1(\tau,qs)h_1(s)d_qsd_q\tau + \sum_{j=1}^{n}b_{2j}\int_0^1 g_1(\eta_j,qs)h_1(s)d_qs. \end{aligned} \tag{16}$$

Combining (15) and (16), it can be seen that

$$
\begin{aligned}
&\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-qs)^{(\beta_{1i}-1)}v(s)d_qs+\sum_{j=1}^{n}b_{1j}v(\eta_j)\\
=&\frac{1}{1-l_1l_2}\Bigg[l_1\Bigg(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\int_0^1 g_1(\tau,qs)h_1(s)d_qsd_q\tau+\sum_{j=1}^{n}b_{2j}\int_0^1 g_1(\eta_j,qs)h_1(s)d_qs\Bigg)\\
&+\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\int_0^1 g_2(\tau,qs)h_2(s)d_qsd_q\tau+\sum_{j=1}^{n}b_{1j}\int_0^1 g_2(\eta_j,qs)h_2(s)d_qs\Bigg],\\
&\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-qs)^{(\beta_{2i}-1)}u(s)d_qs+\sum_{j=1}^{n}b_{2j}u(\eta_j)\\
=&\frac{1}{1-l_1l_2}\Bigg[l_2\Bigg(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\int_0^1 g_2(\tau,qs)h_2(s)d_qsd_q\tau+\sum_{j=1}^{n}b_{1j}\int_0^1 g_2(\eta_j,qs)h_2(s)d_qs\Bigg)\\
&+\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\int_0^1 g_1(\tau,qs)h_1(s)d_qsd_q\tau+\sum_{j=1}^{n}b_{2j}\int_0^1 g_1(\eta_j,qs)h_1(s)d_qs\Bigg],
\end{aligned}
\tag{17}
$$

where l_k $(k=1,2)$ is defined by (4). According to (14) and (17), we can get

$$
\begin{aligned}
u(t)=&\int_0^1 g_1(t,qs)h_1(s)d_qs\\
&+\frac{t^{\alpha_1-1}}{\Gamma_q(\alpha1)(1-l_1l_2)}\Bigg[\int_0^1 l_1\Bigg(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}g_1(\tau,qs)d_q\tau+\sum_{j=1}^{n}b_{2j}g_1(\eta_j,qs)\Bigg)h_1(s)d_qs\\
&+\int_0^1\Bigg(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}g_2(\tau,qs)d_q\tau+\sum_{j=1}^{n}b_{1j}g_2(\eta_j,qs)\Bigg)h_2(s)d_qs\Bigg]\\
=&\int_0^1 K_1(t,qs)h_1(s)d_qs+\int_0^1 H_1(t,qs)h_2(s)d_qs,
\end{aligned}
$$

where $K_1(t,qs)$ and $H_1(t,qs)$ are introduced by (7). Similarly, we also have

$$
\begin{aligned}
v(t)=&\int_0^1 g_2(t,qs)h_2(s)d_qs\\
&+\frac{t^{\alpha_2-1}}{\Gamma_q(\alpha_2)(1-l_1l_2)}\Bigg[\int_0^1 l_2\Bigg(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}g_2(\tau,qs)d_q\tau+\sum_{j=1}^{n}b_{1j}g_2(\eta_j,qs)\Bigg)h_2(s)d_qs\\
&+\int_0^1\Bigg(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}g_1(\tau,qs)d_q\tau+\sum_{j=1}^{n}b_{2j}g_1(\eta_j,qs)\Bigg)h_1(s)d_qs\Bigg]\\
=&\int_0^1 K_2(t,qs)h_2(s)d_qs+\int_0^1 H_2(t,qs)h_1(s)d_qs,
\end{aligned}
$$

where $K_2(t,qs)$ and $H_2(t,qs)$ are also given by (8).

This completes the proof of the lemma. □

Let

$$
\begin{aligned}
K_{\tilde{i}1}(t,qs) =& D_q^{\gamma_{\tilde{i}}} K_1(t,qs) \\
=& D_q^{\gamma_{\tilde{i}}} g_1(t,qs) + \frac{l_1 t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})(1-l_1l_2)} \\
& \cdot \left[\sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{2i}-1)} g_1(\tau,qs) d\tau + \sum_{j=1}^{n} b_{2j} g_1(\eta_j,qs) \right], \\
H_{\tilde{i}1}(t,qs) =& D_q^{\gamma_{\tilde{i}}} H_1(t,qs) \\
=& \frac{t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})(1-l_1l_2)} \\
& \cdot \left[\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{1i}-1)} g_2(\tau,qs) d_q\tau + \sum_{j=1}^{n} b_{1j} g_2(\eta_j,qs) \right],
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
K_{\tilde{i}2}(t,qs) =& D_q^{\gamma_{\tilde{i}}} K_2(t,qs) \\
=& D_q^{\gamma_{\tilde{i}}} g_2(t,qs) + \frac{l_2 t^{\alpha_2-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_2-\gamma_{\tilde{i}})(1-l_1l_2)} \\
& \cdot \left[\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{1i}-1)} g_2(\tau,qs) d\tau + \sum_{j=1}^{n} b_{1j} g_2(\eta_j,qs) \right], \\
H_{\tilde{i}2}(t,qs) =& D_q^{\gamma_{\tilde{i}}} H_2(t,qs) \\
=& \frac{t^{\alpha_2-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_2-\gamma_{\tilde{i}})(1-l_1l_2)} \\
& \cdot \left[\sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{2i}-1)} g_1(\tau,qs) d_q\tau + \sum_{j=1}^{n} b_{2j} g_1(\eta_j,qs) \right],
\end{aligned} \tag{19}
$$

$$
D_q^{\gamma_{\tilde{i}}} g_k(t,qs) = \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})} \begin{cases} t^{\alpha_k-\gamma_{\tilde{i}}-1} - (t-qs)^{(\alpha_k-\gamma_{\tilde{i}}-1)}, & 0 \leqslant qs \leqslant t \leqslant 1, \\ t^{\alpha_k-\gamma_{\tilde{i}}-1}, & 0 \leqslant t \leqslant qs \leqslant 1, \end{cases} \tag{20}
$$

where $\tilde{i} = 1,2$ and $k = 1,2$.

Lemma 6. *Assume that* (F_1) *holds. Then the functions* $g_k(t,qs)$ *defined by (9) and* $D_q^{\gamma_{\tilde{i}}} g_k(t,qs)$ *defined by (20) for* $\tilde{i} = 1,2, k = 1,2$ *have the following properties:*

(1) $0 \leqslant \frac{1}{\Gamma_q(\alpha_k)}[1-(1-qs)^{(\alpha_k-1)}]t^{\alpha_k-1} \leqslant g_k(t,qs) \leqslant \frac{1}{\Gamma_q(\alpha_k)} t^{\alpha_k-1}$;

(2) $0 \leqslant \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})}[1-(1-qs)^{(\alpha_k-\gamma_{\tilde{i}}-1)}]t^{\alpha_k-\gamma_{\tilde{i}}-1} \leqslant D_q^{\gamma_{\tilde{i}}} g_k(t,qs) \leqslant \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})} t^{\alpha_k-\gamma_{\tilde{i}}-1}$, *for* $t, qs \in [0,1]$.

Proof. (1) For $0 \leqslant qs \leqslant t \leqslant 1$, we can get

$$\begin{aligned} g_k(t,qs) &= \frac{1}{\Gamma_q(\alpha_k)}\left[t^{\alpha_k-1} - (t-qs)^{(\alpha_k-1)}\right] \\ &\geqslant \frac{1}{\Gamma_q(\alpha_k)}\left[t^{\alpha_k-1} - (t-qs\cdot t)^{(\alpha_k-1)}\right] \\ &= \frac{1}{\Gamma_q(\alpha_k)}[1-(1-qs)^{(\alpha_k-1)}]t^{\alpha_k-1} \geqslant 0, \\ g_k(t,qs) &= \frac{1}{\Gamma_q(\alpha_k)}\left[t^{\alpha_k-1} - (t-qs)^{(\alpha_k-1)}\right] \\ &\leqslant \frac{1}{\Gamma_q(\alpha_k)}t^{\alpha_k-1}. \end{aligned}$$

For $0 \leqslant t \leqslant qs \leqslant 1$, we have

$$\begin{aligned} g_k(t,qs) &= \frac{1}{\Gamma_q(\alpha_k)}t^{\alpha_k-1} \\ &\geqslant \frac{1}{\Gamma_q(\alpha_k)}t^{\alpha_k-1} - \frac{1}{\Gamma_q(\alpha_k)}t^{\alpha_k-1}\cdot(1-qs)^{(\alpha_k-1)} \\ &= \frac{1}{\Gamma_q(\alpha_k)}[1-(1-qs)^{(\alpha_k-1)}]t^{\alpha_k-1} \geqslant 0, \\ g_k(t,qs) &= \frac{1}{\Gamma_q(\alpha_k)}t^{\alpha_k-1}. \end{aligned}$$

(2) For $0 \leqslant qs \leqslant t \leqslant 1$, we have

$$\begin{aligned} D_q^{\gamma_{\tilde{i}}} g_k(t,qs) &= \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})}\left[t^{\alpha_k-\gamma_{\tilde{i}}-1} - (t-qs)^{(\alpha_k-\gamma_{\tilde{i}}-1)}\right] \\ &\geqslant \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})}\left[t^{\alpha_k-\gamma_{\tilde{i}}-1} - (t-qs\cdot t)^{(\alpha_k-\gamma_{\tilde{i}}-1)}\right] \\ &= \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})}[1-(1-qs)^{(\alpha_k-\gamma_{\tilde{i}}-1)}]t^{\alpha_k-\gamma_{\tilde{i}}-1} \geqslant 0, \\ D_q^{\gamma_{\tilde{i}}} g_k(t,qs) &= \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})}\left[t^{\alpha_k-\gamma_{\tilde{i}}-1} - (t-qs)^{(\alpha_k-\gamma_{\tilde{i}}-1)}\right] \\ &\leqslant \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})}t^{\alpha_k-\gamma_{\tilde{i}}-1}. \end{aligned}$$

For $0 \leqslant t \leqslant qs \leqslant 1$, we have

$$\begin{aligned} D_q^{\gamma_{\tilde{i}}} g_k(t,qs) &= \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})}t^{\alpha_k-\gamma_{\tilde{i}}-1} \\ &\geqslant \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})}t^{\alpha_k-\gamma_{\tilde{i}}-1} - \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})}t^{\alpha_k-\gamma_{\tilde{i}}-1}\cdot(1-qs)^{(\alpha_k-\gamma_{\tilde{i}}-1)} \\ &= \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})}[1-(1-qs)^{(\alpha_k-\gamma_{\tilde{i}}-1)}]t^{\alpha_k-\gamma_{\tilde{i}}-1} \geqslant 0, \\ D_q^{\gamma_{\tilde{i}}} g_k(t,qs) &= \frac{1}{\Gamma_q(\alpha_k-\gamma_{\tilde{i}})}t^{\alpha_k-\gamma_{\tilde{i}}-1}. \end{aligned}$$

This completes the proof of the lemma. □

For computational convenience, we introduce the following notations:

$$\varrho_1 = \frac{1}{\Gamma_q(\alpha_1)}\left[1 + \frac{l_1}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{2i}-1)}\tau^{\alpha_1-1}d_q\tau + \sum_{j=1}^{n} b_{2j}{\eta_j}^{\alpha_1-1}\right)\right], \tag{21}$$

$$\varrho_2 = \frac{1}{\Gamma_q(\alpha_2)}\left[1 + \frac{l_2}{\Gamma_q(\alpha_2)(1-l_1l_2)}\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{1i}-1)}\tau^{\alpha_2-1}d_q\tau + \sum_{j=1}^{n} b_{1j}{\eta_j}^{\alpha_2-1}\right)\right], \tag{22}$$

$$\varrho_3 = 1 + \frac{l_1}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{2i}-1)}\tau^{\alpha_1-1}d_q\tau + \sum_{j=1}^{n} b_{2j}{\eta_j}^{\alpha_1-1}\right), \tag{23}$$

$$\varrho_4 = 1 + \frac{l_2}{\Gamma_q(\alpha_2)(1-l_1l_2)}\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{1i}-1)}\tau^{\alpha_2-1}d_q\tau + \sum_{j=1}^{n} b_{1j}{\eta_j}^{\alpha_2-1}\right), \tag{24}$$

$$\rho_1 = \frac{1}{\Gamma_q(\alpha_1)\Gamma_q(\alpha_2)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{1i}-1)}\tau^{\alpha_2-1}d_q\tau + \sum_{j=1}^{n} b_{1j}{\eta_j}^{\alpha_2-1}\right], \tag{25}$$

$$\rho_2 = \frac{1}{\Gamma_q(\alpha_1)\Gamma_q(\alpha_2)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{2i}-1)}\tau^{\alpha 1-1}d_q\tau + \sum_{j=1}^{n} b_{2j}{\eta_j}^{\alpha_1-1}\right], \tag{26}$$

$$\rho_3 = \frac{1}{\Gamma_q(\alpha_2)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{1i}-1)}\tau^{\alpha_2-1}d_q\tau + \sum_{j=1}^{n} b_{1j}{\eta_j}^{\alpha_2-1}\right], \tag{27}$$

$$\rho_4 = \frac{1}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i - q\tau)^{(\beta_{2i}-1)}\tau^{\alpha 1-1}d_q\tau + \sum_{j=1}^{n} b_{2j}{\eta_j}^{\alpha_1-1}\right]. \tag{28}$$

Lemma 7. *Assume that* (F_1)–(F_3) *hold. Then for* $(t, qs) \in [0,1]\times[0,1]$, *the functions* $K_{\tilde{j}}(t,qs)$, $H_{\tilde{j}}(t,qs)$, $K_{\tilde{i}\tilde{j}}(t,qs)$ *and* $H_{\tilde{i}\tilde{j}}(t,qs)$ *for* $\tilde{i}=1,2, \tilde{j}=1,2$ *defined by (7), (8), (18) and (19) satisfy the following results:*

(1) $0 \le \varrho_1 t^{\alpha_1-1}\left[1-(1-qs)^{(\alpha_1-1)}\right] \leqslant K_1(t,qs) \leqslant \varrho_1 t^{\alpha_1-1}$,

$0 \le \varrho_2 t^{\alpha_2-1}\left[1-(1-qs)^{(\alpha_2-1)}\right] \leqslant K_2(t,qs) \leqslant \varrho_2 t^{\alpha_2-1}$,

$0 \le \frac{\varrho_3}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})} t^{\alpha_1-\gamma_{\tilde{i}}-1}\left[1-(1-qs)^{(\alpha_1-\gamma_{\tilde{i}}-1)}\right] \leqslant K_{\tilde{i}1}(t,qs) \leqslant \frac{\varrho_3}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})} t^{\alpha_1-\gamma_{\tilde{i}}-1}$,

$0 \le \frac{\varrho_4}{\Gamma_q(\alpha_2-\gamma_{\tilde{i}})} t^{\alpha_2-\gamma_{\tilde{i}}-1}\left[1-(1-qs)^{(\alpha_2-\gamma_{\tilde{i}}-1)}\right] \leqslant K_{\tilde{i}2}(t,qs) \leqslant \frac{\varrho_4}{\Gamma_q(\alpha_2-\gamma_{\tilde{i}})} t^{\alpha_2-\gamma_{\tilde{i}}-1}$,

(2) $0 \le \rho_1 t^{\alpha_1-1}\left[1-(1-qs)^{(\alpha_2-1)}\right] \leqslant H_1(t,qs) \leqslant \rho_1 t^{\alpha_1-1}$,

$0 \le \rho_2 t^{\alpha_2-1}\left[1-(1-qs)^{(\alpha_1-1)}\right] \leqslant H_2(t,qs) \leqslant \rho_2 t^{\alpha_2-1}$,

$0 \le \frac{\rho_3}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})} t^{\alpha_1-\gamma_{\tilde{i}}-1}\left[1-(1-qs)^{(\alpha_2-\gamma_{\tilde{i}}-1)}\right] \leqslant H_{\tilde{i}1}(t,qs) \leqslant \frac{\rho_3}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})} t^{\alpha_1-\gamma_{\tilde{i}}-1}$,

$0 \le \frac{\rho_4}{\Gamma_q(\alpha_2-\gamma_{\tilde{i}})} t^{\alpha_2-\gamma_{\tilde{i}}-1}\left[1-(1-qs)^{(\alpha_1-\gamma_{\tilde{i}}-1)}\right] \leqslant H_{\tilde{i}2}(t,qs) \leqslant \frac{\rho_4}{\Gamma_q(\alpha_2-\gamma_{\tilde{i}})} t^{\alpha_2-\gamma_{\tilde{i}}-1}$.

Proof. (1) Accordance with (F_3), Lemma 6 and the definition of $K_1(t,qs)$, we have

$$
\begin{aligned}
K_1(t,qs) &= g_1(t,qs) + \frac{l_1 t^{\alpha_1-1}}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}g_1(\tau,qs)d_q\tau + \sum_{j=1}^{n}b_{2j}g_1(\eta_j,qs)\right]\\
&\geqslant \frac{t^{\alpha_1-1}}{\Gamma_q(\alpha_1)}[1-(1-qs)^{(\alpha_1-1)}] + \frac{l_1 t^{\alpha_1-1}}{\Gamma_q(\alpha_1)(1-l_1l_2)}\\
&\quad\cdot\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\frac{\tau^{\alpha_1-1}}{\Gamma_q(\alpha_1)}[1-(1-qs)^{(\alpha_1-1)}]d_q\tau + \sum_{j=1}^{n}b_{2j}\frac{\eta_j^{\alpha_1-1}}{\Gamma_q(\alpha_1)}[1-(1-qs)^{(\alpha_1-1)}]\right]\\
&= \frac{t^{\alpha_1-1}}{\Gamma_q(\alpha_1)}[1-(1-qs)^{(\alpha_1-1)}]\\
&\quad\cdot\left[1+\frac{l_1}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\tau^{\alpha_1-1}d_q\tau + \sum_{j=1}^{n}b_{2j}\eta_j^{\alpha_1-1}\right)\right]\\
&=\varrho_1 t^{\alpha_1-1}\left[1-(1-qs)^{(\alpha_1-1)}\right] \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
K_1(t,qs) &= g_1(t,qs) + \frac{l_1 t^{\alpha_1-1}}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}g_1(\tau,qs)d_q\tau + \sum_{j=1}^{n}b_{2j}g_1(\eta_j,qs)\right]\\
&\leqslant \frac{t^{\alpha_1-1}}{\Gamma_q(\alpha_1)} + \frac{l_1 t^{\alpha_1-1}}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})\Gamma_q(\alpha_1)}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\tau^{\alpha_1-1}d_q\tau + \frac{1}{\Gamma_q(\alpha_1)}\sum_{j=1}^{n}b_{2j}\eta_j^{\alpha_1-1}\right]\\
&=\frac{t^{\alpha_1-1}}{\Gamma_q(\alpha_1)}\left[1+\frac{l_1}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\tau^{\alpha_1-1}d_q\tau + \sum_{j=1}^{n}b_{2j}\eta_j^{\alpha_1-1}\right)\right]\\
&=\varrho_1 t^{\alpha_1-1},
\end{aligned}
$$

where ϱ_1 is defined by (21).

$$
\begin{aligned}
K_{\tilde{i}1}(t,qs) &= D_q^{\gamma_{\tilde{i}}}g_1(t,qs) + \frac{l_1 t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}g_1(\tau,qs)d_q\tau + \sum_{j=1}^{n}b_{2j}g_1(\eta_j,qs)\right]\\
&\geqslant \frac{t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}[1-(1-qs)^{(\alpha_1-\gamma_{\tilde{i}}-1)}] + \frac{l_1 t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})(1-l_1l_2)}\\
&\quad\cdot\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\frac{\tau^{\alpha_1-1}}{\Gamma_q(\alpha_1)}[1-(1-qs)^{(\alpha_1-1)}]d_q\tau + \sum_{j=1}^{n}b_{2j}\frac{\eta_j^{\alpha_1-1}}{\Gamma_q(\alpha_1)}[1-(1-qs)^{(\alpha_1-1)}]\right]\\
&\geqslant \frac{t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}[1-(1-qs)^{(\alpha_1-\gamma_{\tilde{i}}-1)}]\\
&\quad\cdot\left[1+\frac{l_1}{\Gamma_q(\alpha_1-\gamma_1)(1-l_1l_2)}\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\tau^{\alpha_1-1}d_q\tau + \sum_{j=1}^{n}b_{2j}\eta_j^{\alpha_1-1}\right)\right]\\
&=\frac{\varrho_3}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}t^{\alpha_1-\gamma_{\tilde{i}}-1}\left[1-(1-qs)^{(\alpha_1-\gamma_{\tilde{i}}-1)}\right] \geq 0,
\end{aligned}
$$

$$\begin{aligned}
K_{\tilde{i}1}(t,qs) =& D_q^{\gamma_{\tilde{i}}} g_1(t,qs) + \frac{l_1 t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}g_1(\tau,qs)d_q\tau + \sum_{j=1}^{n} b_{2j}g_1(\eta_j,qs)\right] \\
\leqslant& \frac{t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})} + \frac{l_1 t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})(1-l_1l_2)} \\
&\cdot\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})\Gamma_q(\alpha_1)}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\tau^{\alpha_1-1}d_q\tau + \frac{1}{\Gamma_q(\alpha_1)}\sum_{j=1}^{n} b_{2j}{\eta_j}^{\alpha_1-1}\right] \\
=&\frac{t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}\left[1+\frac{l_1}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})(1-l_1l_2)}\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\tau^{\alpha_1-1}d_q\tau + \sum_{j=1}^{n} b_{2j}{\eta_j}^{\alpha_1-1}\right)\right] \\
=&\frac{\varrho_3}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}t^{\alpha_1-\gamma_{\tilde{i}}-1},
\end{aligned}$$

where $\tilde{i}=1,2$ and ϱ_3 is defined by (23). Similarly, we get

$$0 \leqslant \varrho_2 t^{\alpha_2-1}\left[1-(1-qs)^{(\alpha_2-1)}\right] \leqslant K_2(t,qs) \leqslant \varrho_2 t^{\alpha_2-1},$$

$$0 \leq \frac{\varrho_4}{\Gamma_q(\alpha_2-\gamma_{\tilde{i}})}t^{\alpha_2-\gamma_{\tilde{i}}-1}\left[1-(1-qs)^{(\alpha_2-\gamma_{\tilde{i}}-1)}\right] \leqslant K_{\tilde{i}2}(t,qs) \leqslant \frac{\varrho_4}{\Gamma_q(\alpha_2-\gamma_{\tilde{i}})}t^{\alpha_2-\gamma_{\tilde{i}}-1},$$

where ϱ_2 and ϱ_4 are defined by (22) and (24).

(2) According to (F_3), Lemma 6 and the definition of $H_1(t,qs)$, we can obtain

$$\begin{aligned}
H_1(t,qs) =& \frac{t^{\alpha_1-1}}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}g_2(\tau,qs)d_q\tau + \sum_{j=1}^{n} b_{1j}g_2(\eta_j,qs)\right] \\
\geqslant& \frac{t^{\alpha_1-1}}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\frac{\tau^{\alpha_2-1}}{\Gamma_q(\alpha_2)}[1-(1-qs)^{(\alpha_2-1)}]d_q\tau\right. \\
&\left.+\sum_{j=1}^{n} b_{1j}\frac{\eta_j^{\alpha_2-1}}{\Gamma_q(\alpha_2)}[1-(1-qs)^{(\alpha_2-1)}]\right] \\
=& \frac{t^{\alpha_1-1}}{\Gamma_q(\alpha_1)\Gamma_q(\alpha_2)(1-l_1l_2)}[1-(1-qs)^{(\alpha_2-1)}]\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\tau^{\alpha_2-1}d_q\tau + \sum_{j=1}^{n} b_{1j}\eta_j^{\alpha_2-1}\right] \\
=& \rho_1 t^{\alpha_1-1}\left[1-(1-qs)^{(\alpha_2-1)}\right] \geq 0,
\end{aligned}$$

$$\begin{aligned}
H_1(t,qs) =& \frac{t^{\alpha_1-1}}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}g_2(\tau,qs)d_q\tau + \sum_{j=1}^{n} b_{1j}g_2(\eta_j,qs)\right] \\
\leqslant& \frac{t^{\alpha_1-1}}{\Gamma_q(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})\Gamma_q(\alpha_2)}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\tau^{\alpha_2-1}d_q\tau + \frac{1}{\Gamma_q(\alpha_2)}\sum_{j=1}^{n} b_{1j}\eta_j^{\alpha_2-1}\right] \\
=& \frac{t^{\alpha_1-1}}{\Gamma_q(\alpha_1)\Gamma_q(\alpha_2)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\tau^{\alpha_2-1}d_q\tau + \sum_{j=1}^{n} b_{1j}\eta_j^{\alpha_2-1}\right] \\
=& \rho_1 t^{\alpha_1-1},
\end{aligned}$$

where ρ_1 is defined by (27).

$$
\begin{aligned}
H_{\tilde{i}1}(t,qs) &= \frac{t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}g_2(\tau,qs)d_q\tau+\sum_{j=1}^{n}b_{1j}g_2(\eta_j,qs)\right] \\
&\geqslant \frac{t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\frac{\tau^{\alpha_2-1}}{\Gamma_q(\alpha_2)}[1-(1-qs)^{(\alpha_2-1)}]d_q\tau\right. \\
&\quad\left.+\sum_{j=1}^{n}b_{1j}\frac{\eta_j^{\alpha_2-1}}{\Gamma_q(\alpha_2)}[1-(1-qs)^{(\alpha_2-1)}]\right] \\
&\geqslant \frac{t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})\Gamma_q(\alpha_2)(1-l_1l_2)}[1-(1-qs)^{(\alpha_2-\gamma_{\tilde{i}}-1)}] \\
&\quad\cdot\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\tau^{\alpha_2-1}d_q\tau+\sum_{j=1}^{n}b_{1j}\eta_j^{\alpha_2-1}\right] \\
&= \frac{\rho_3}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}t^{\alpha_1-\gamma_{\tilde{i}}-1}\left[1-(1-qs)^{(\alpha_2-\gamma_{\tilde{i}}-1)}\right]\geq 0,
\end{aligned}
$$

$$
\begin{aligned}
H_{\tilde{i}1}(t,qs) &= \frac{t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}g_2(\tau,qs)d_q\tau+\sum_{j=1}^{n}b_{1j}g_2(\eta_j,qs)\right] \\
&\leqslant \frac{t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})\Gamma_q(\alpha_2)}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\tau^{\alpha_2-1}d_q\tau+\frac{1}{\Gamma_q(\alpha_2)}\sum_{j=1}^{n}b_{1j}\eta_j^{\alpha_2-1}\right] \\
&= \frac{t^{\alpha_1-\gamma_{\tilde{i}}-1}}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})\Gamma_q(\alpha_2)(1-l_1l_2)}\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\tau^{\alpha_2-1}d_q\tau+\sum_{j=1}^{n}b_{1j}\eta_j^{\alpha_2-1}\right] \\
&= \frac{\rho_3}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}t^{\alpha_1-\gamma_{\tilde{i}}-1},
\end{aligned}
$$

where $\tilde{i}=1,2$ and ρ_3 is defined by (27). Analogously, we get

$$
\begin{aligned}
0\leq \rho_2 t^{\alpha_2-1}\left[1-(1-qs)^{(\alpha_1-1)}\right] &\leqslant H_2(t,qs)\leqslant \rho_2 t^{\alpha_2-1}, \\
0\leq \frac{\rho_4}{\Gamma_q(\alpha_2-\gamma_{\tilde{i}})}t^{\alpha_2-\gamma_{\tilde{i}}-1}\left[1-(1-qs)^{(\alpha_1-\gamma_{\tilde{i}}-1)}\right] &\leqslant H_{\tilde{i}2}(t,qs)\leqslant \frac{\rho_4}{\Gamma_q(\alpha_2-\gamma_{\tilde{i}})}t^{\alpha_2-\gamma_{\tilde{i}}-1},
\end{aligned}
$$

where ρ_2 and ρ_4 are defined by (26) and (28).

This completes the proof of the lemma. □

Lemma 8 ([38]). *$K_h=P_{h_1}\times P_{h_2}$, where $K=P\times P$ and $h(\tau)=(h_1(\tau),h_2(\tau))$.*

3. Existence Results of Monotone Iterative Non-Negative Solutions

Let $E=\{x|x,D_q^{\gamma_1}x(t),D_q^{\gamma_2}x(t)\in C[0,1]\}$ endowed with the norm

$$\|x\|=\max\left\{\max_{0\leqslant t\leqslant 1}|x(t)|,\max_{0\leqslant t\leqslant 1}D_q^{\gamma_1}|x(t)|,\max_{0\leqslant t\leqslant 1}D_q^{\gamma_2}|x(t)|\right\}.$$

Let $\|(x,y)\|=\max\{\|x\|,\|y\|\}$ for $(x,y)\in E\times E$, then $(E\times E,\|(x,y)\|)$ is a Banach space. Define a cone $P=\{x\in E\,|\,x,D_q^{\gamma_1}x(t),D_q^{\gamma_2}x(t)\geqslant\theta\}$. Let $K=P\times P$, it is obvious that K is a normal cone equipped with the following partial order:

$$(x_1,y_1)\preceq(x_2,y_2)\quad\Leftrightarrow\quad\begin{cases}x_1\leqslant x_2, & D_q^{\gamma_1}x_1\leqslant D_q^{\gamma_1}x_2,\ \ D_q^{\gamma_2}x_1\leqslant D_q^{\gamma_2}x_2,\\ y_1\leqslant y_2, & D_q^{\gamma_1}y_1\leqslant D_q^{\gamma_1}y_2,\ \ D_q^{\gamma_2}y_1\leqslant D_q^{\gamma_2}y_2.\end{cases}\qquad(29)$$

For all $(u,v) \in P \times P$, in view of Lemma 5, let $T : K \times K \to K$ be the operator defined by

$$T(u,v) = \begin{pmatrix} T_1(u,v) \\ T_2(u,v) \end{pmatrix},$$

where

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 K_1(t,qs) f_1(s,u(s),v(s),D_q^{\gamma_1}u(s),D_q^{\gamma_2}v(s))d_q s \\ &\quad + \int_0^1 H_1(t,qs) f_2(s,u(s),v(s),D_q^{\gamma_1}u(s),D_q^{\gamma_2}v(s))d_q s, \end{aligned}$$

$$\begin{aligned} T_2(u,v)(t) &= \int_0^1 K_2(t,qs) f_2(s,u(s),v(s),D_q^{\gamma_1}u(s),D_q^{\gamma_2}v(s))d_q s \\ &\quad + \int_0^1 H_2(t,qs) f_1(s,u(s),v(s),D_q^{\gamma_1}u(s),D_q^{\gamma_2}v(s))d_q s. \end{aligned}$$

Theorem 1. *Assume that*

(S_1) For $t \in [0,1]$, $f_j(t,x_1,y_1,x_2,y_2)$ is increasing in $x_i \in [0,\infty)$ $(i=1,2)$ and decreasing in $y_i \in [0,\infty)$ $(i=1,2)$ for $j=1,2$;

(S_2) $\forall\, r \in (0,1), \exists\, \varphi_1(r),\ \varphi_2(r) \in (r,1]$ such that

$$f_i(t,\, rx_1,\, r^{-1}y_1,\, rx_2,\, r^{-1}y_2) \geqslant \varphi_i(r) f_i(t,\, x_1,\, y_1,\, x_2,\, y_2) \ \ (i=1,2),$$
$$\varphi_0(r) = \min\{\varphi_1(r),\ \varphi_2(r)\}.$$

Then

(1) $T(h,h) \in K_h$, where $h(t) = (h_1(t),h_2(t)) = (t^{\alpha_1-1},t^{\alpha_2-1}), 0 \leqslant t \leqslant 1$;

(2) $T(ru,r^{-1}v) \geqslant \varphi_0(r)\, T(u,v)$;

(3) BVP (1) and (2) has a unique non-negative solutions (u^*,v^*) in K_h. For any initial $(x_{01},x_{02}),(y_{01},y_{02}) \in K_h$, there are two iterative sequences $\{(x_{n1},x_{n2})\},\{(y_{n1},y_{n2})\}$ satisfying that $(x_{n1},x_{n2}) \to (u^*,v^*)$, $(y_{n1},y_{n2}) \to (u^*,v^*)$, where

$$\begin{aligned} (x_{n1},x_{n2}) &= \Big(T_1(x_{(n-1)1},y_{(n-1)1}),T_2(x_{(n-1)2},y_{(n-1)2})\Big) \\ &= \left(\begin{aligned} &\int_0^1 K_1(t,qs) f_1(s,x_{(n-1)1}(s),y_{(n-1)1}(s),D_q^{\gamma_1}x_{(n-1)1}(s),D_q^{\gamma_2}y_{(n-1)1}(s))d_q s \\ &\quad + \int_0^1 H_1(t,qs) f_2(s,x_{(n-1)1}(s),y_{(n-1)1}(s),D_q^{\gamma_1}x_{(n-1)1}(s),D_q^{\gamma_2}y_{(n-1)1}(s))d_q s, \\ &\int_0^1 K_2(t,qs) f_2(s,x_{(n-1)1}(s),y_{(n-1)1}(s),D_q^{\gamma_1}x_{(n-1)1}(s),D_q^{\gamma_2}y_{(n-1)1}(s))d_q s \\ &\quad + \int_0^1 H_2(t,qs) f_1(s,x_{(n-1)1}(s),y_{(n-1)1}(s),D_q^{\gamma_1}x_{(n-1)1}(s),D_q^{\gamma_2}y_{(n-1)1}(s))d_q s \end{aligned}\right), \end{aligned}$$

$$\begin{aligned} (y_{n1},y_{n2}) &= \Big(T_1(y_{(n-1)1},x_{(n-1)1}),T_2(y_{(n-1)2},x_{(n-1)2})\Big) \\ &= \left(\begin{aligned} &\int_0^1 K_1(t,qs) f_1(s,y_{(n-1)1}(s),x_{(n-1)1}(s),D_q^{\gamma_1}y_{(n-1)1}(s),D_q^{\gamma_2}x_{(n-1)1}(s))d_q s \\ &\quad + \int_0^1 H_1(t,qs) f_2(s,y_{(n-1)1}(s),x_{(n-1)1}(s),D_q^{\gamma_1}y_{(n-1)1}(s),D_q^{\gamma_2}x_{(n-1)1}(s))d_q s, \\ &\int_0^1 K_2(t,qs) f_2(s,y_{(n-1)1}(s),x_{(n-1)1}(s),D_q^{\gamma_1}y_{(n-1)1}(s),D_q^{\gamma_2}x_{(n-1)1}(s))d_q s \\ &\quad + \int_0^1 H_2(t,qs) f_1(s,y_{(n-1)1}(s),x_{(n-1)1}(s),D_q^{\gamma_1}y_{(n-1)1}(s),D_q^{\gamma_2}x_{(n-1)1}(s))d_q s \end{aligned}\right), \end{aligned}$$

$n = 1,2,\ldots.$

Proof. By Lemma 7 we have

$$K_{\tilde{j}}(t,qs), K_{\tilde{i}\tilde{j}}(t,qs), H_{\tilde{j}}(t,qs), H_{\tilde{i}\tilde{j}}(t,qs) \geqslant 0, \quad \tilde{i}=1,2, \tilde{j}=1,2. \tag{30}$$

Regarding (30) and $(F4)$, we get $T_1, T_2 : P \times P \to P$, $T : K \times K \to K$. It is obvious that T is a mixed monotone operator, because for any $(u_1, v_1), (u_2, v_2) \in K$ with $(u_1, v_1) \preceq (u_2, v_2)$, considering $(S1)$, we acquire

$$T(u_1, v_1) \preceq T(u_2, v_1) \ \ for\ fixed\ v_1\ and\ \ T(u_1, v_1) \succeq T(u_1, v_2) \ \ for\ fixed\ u_1.$$

From Lemma 8, we obtain $K_h = P_{h_1} \times P_{h_2}$, where $h(t) = (h_1(t), h_2(t)) = (t^{\alpha_1 - 1}, t^{\alpha_2 - 1})$.

(1) In view of $h(t) = (h_1(t), h_2(t)) = (t^{\alpha_1 - 1}, t^{\alpha_2 - 1})$, we get

$$\begin{aligned} &h_1(t) = t^{\alpha_1 - 1} \geqslant 0, \ h_2(t) = t^{\alpha_2 - 1} \geqslant 0, \\ D_q^{\gamma_1} h_1(t) = D_q^{\gamma_1} t^{\alpha_1 - 1} = \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_1)} t^{\alpha_1 - \gamma_1 - 1} \geqslant 0,\ &D_q^{\gamma_1} h_2(t) = D_q^{\gamma_1} t^{\alpha_2 - 1} = \frac{\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_2 - \gamma_1)} t^{\alpha_2 - \gamma_1 - 1} \geqslant 0, \\ D_q^{\gamma_2} h_1(t) = D_q^{\gamma_2} t^{\alpha_1 - 1} = \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_2)} t^{\alpha_1 - \gamma_2 - 1} \geqslant 0,\ &D_q^{\gamma_2} h_2(t) = D_q^{\gamma_2} t^{\alpha_2 - 1} = \frac{\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_2 - \gamma_2)} t^{\alpha_2 - \gamma_2 - 1} \geqslant 0. \end{aligned} \tag{31}$$

Consequently, from (31), we can see that $h_1, h_2 \in P$ and $h \in K$. Indeed

$$\begin{aligned} T_1(h_1, h_1)(t) &= \int_0^1 K_1(t,qs) f_1(s, h_1(s), h_1(s), D_q^{\gamma_1} h_1(s), D_q^{\gamma_2} h_1(s)) d_q s \\ &\quad + \int_0^1 H_1(t,qs) f_2(s, h_1(s), h_1(s), D_q^{\gamma_1} h_1(s), D_q^{\gamma_2} h_1(s)) d_q s \\ &= \int_0^1 K_1(t,qs) f_1\left(s, s^{\alpha_1 - 1}, s^{\alpha_1 - 1}, \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_1)} s^{\alpha_1 - \gamma_1 - 1}, \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_2)} s^{\alpha_1 - \gamma_2 - 1}\right) d_q s \\ &\quad + \int_0^1 H_1(t,qs) f_2\left(s, s^{\alpha_1 - 1}, s^{\alpha_1 - 1}, \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_1)} s^{\alpha_1 - \gamma_1 - 1}, \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_2)} s^{\alpha_1 - \gamma_2 - 1}\right) d_q s \\ &\geqslant \int_0^1 \varrho_1 t^{\alpha_1 - 1} [1 - (1 - qs)^{(\alpha_1 - 1)}] f_1\left(s, 0, 1, 0, \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_2)}\right) d_q s \\ &\quad + \int_0^1 \rho_1 t^{\alpha_1 - 1} [1 - (1 - qs)^{(\alpha_2 - 1)}] f_2\left(s, 0, 1, 0, \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_2)}\right) d_q s, \end{aligned}$$

$$\begin{aligned} T_1(h_1, h_1)(t) &= \int_0^1 K_1(t,qs) f_1(s, h_1(s), h_1(s), D_q^{\gamma_1} h_1(s), D_q^{\gamma_2} h_1(s)) d_q s \\ &\quad + \int_0^1 H_1(t,qs) f_2(s, h_1(s), h_1(s), D_q^{\gamma_1} h_1(s), D_q^{\gamma_2} h_1(s)) d_q s \\ &= \int_0^1 K_1(t,qs) f_1\left(s, s^{\alpha_1 - 1}, s^{\alpha_1 - 1}, \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_1)} s^{\alpha_1 - \gamma_1 - 1}, \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_2)} s^{\alpha_1 - \gamma_2 - 1}\right) d_q s \\ &\quad + \int_0^1 H_1(t,qs) f_2\left(s, s^{\alpha_1 - 1}, s^{\alpha_1 - 1}, \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_1)} s^{\alpha_1 - \gamma_1 - 1}, \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_2)} s^{\alpha_1 - \gamma_2 - 1}\right) d_q s \\ &\leqslant \int_0^1 \varrho_1 t^{\alpha_1 - 1} f_1\left(s, 1, 0, \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_1)}, 0\right) d_q s + \int_0^1 \rho_1 t^{\alpha_1 - 1} f_2\left(s, 1, 0, \frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1 - \gamma_1)}, 0\right) d_q s, \end{aligned}$$

$$
\begin{aligned}
D_q^{\gamma_{\tilde{i}}} T_1(h_1,h_1)(t) &= \int_0^1 K_{\tilde{i}1}(t,qs) f_1(s,h_1(s),h_1(s),D_q^{\gamma_1}h_1(s),D_q^{\gamma_2}h_1(s))d_qs \\
&\quad + \int_0^1 H_{\tilde{i}1}(t,qs) f_2(s,h_1(s),h_1(s),D_q^{\gamma_1}h_1(s),D_q^{\gamma_2}h_1(s))d_qs \\
&= \int_0^1 K_{\tilde{i}1}(t,qs) f_1\left(s,s^{\alpha_1-1},s^{\alpha_1-1},\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_1)}s^{\alpha_1-\gamma_1-1},\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_2)}s^{\alpha_1-\gamma_2-1}\right)d_qs \\
&\quad + \int_0^1 H_{\tilde{i}1}(t,qs) f_2\left(s,s^{\alpha_1-1},s^{\alpha_1-1},\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_1)}s^{\alpha_1-\gamma_1-1},\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_2)}s^{\alpha_1-\gamma_2-1}\right)d_qs \\
&\geqslant \int_0^1 \frac{\varrho_3}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}t^{\alpha_1-\gamma_{\tilde{i}}-1}[1-(1-qs)^{(\alpha_1-\gamma_{\tilde{i}}-1)}]f_1\left(s,0,1,0,\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_2)}\right)d_qs \\
&\quad + \int_0^1 \frac{\rho_3}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}t^{\alpha_1-\gamma_{\tilde{i}}-1}[1-(1-qs)^{(\alpha_2-\gamma_{\tilde{i}}-1)}]f_2\left(s,0,1,0,\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_2)}\right)d_qs,
\end{aligned}
$$

$$
\begin{aligned}
D_q^{\gamma_{\tilde{i}}} T_1(h_1,h_1)(t) &= \int_0^1 K_{\tilde{i}1}(t,qs) f_1(s,h_1(s),h_1(s),D_q^{\gamma_1}h_1(s),D_q^{\gamma_2}h_1(s))d_qs \\
&\quad + \int_0^1 H_{\tilde{i}1}(t,qs) f_2(s,h_1(s),h_1(s),D_q^{\gamma_1}h_1(s),D_q^{\gamma_2}h_1(s))d_qs \\
&= \int_0^1 K_{\tilde{i}1}(t,qs) f_1\left(s,s^{\alpha_1-1},s^{\alpha_1-1},\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_1)}s^{\alpha_1-\gamma_1-1},\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_2)}s^{\alpha_1-\gamma_2-1}\right)d_qs \\
&\quad + \int_0^1 H_{\tilde{i}1}(t,qs) f_2\left(s,s^{\alpha_1-1},s^{\alpha_1-1},\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_1)}s^{\alpha_1-\gamma_1-1},\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_2)}s^{\alpha_1-\gamma_2-1}\right)d_qs \\
&\leqslant \int_0^1 \frac{\varrho_3}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}t^{\alpha_1-\gamma_{\tilde{i}}-1}f_1\left(s,1,0,\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_1)},0\right)d_qs \\
&\quad + \int_0^1 \frac{\rho_3}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}t^{\alpha_1-\gamma_{\tilde{i}}-1}f_2\left(s,1,0,\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_1)},0\right)d_qs.
\end{aligned}
$$

Let

$$
\begin{aligned}
a_{11} &= \int_0^1 \varrho_1[1-(1-qs)^{(\alpha_1-1)}]f_1\left(s,0,1,0,\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_2)}\right)d_qs \\
&\quad + \int_0^1 \rho_1[1-(1-qs)^{(\alpha_2-1)}]f_2\left(s,0,1,0,\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_2)}\right)d_qs, \\
a_{12} &= \int_0^1 \varrho_1 f_1\left(s,1,0,\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_1)},0\right)d_qs + \int_0^1 \rho_1 f_2\left(s,1,0,\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_1)},0\right)d_qs, \\
a_{11}^{'} &= \int_0^1 \frac{\varrho_3}{\Gamma_q(\alpha_1)}[1-(1-qs)^{(\alpha_1-\gamma-1)}]f_1\left(s,0,1,0,\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_2)}\right)d_qs \\
&\quad + \int_0^1 \frac{\rho_3}{\Gamma_q(\alpha_1)}[1-(1-qs)^{(\alpha_2-\gamma-1)}]f_2\left(s,0,1,0,\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_2)}\right)d_qs, \\
a_{12}^{'} &= \int_0^1 \frac{\varrho_3}{\Gamma_q(\alpha_1)}f_1\left(s,1,0,\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_1)},0\right)d_qs + \int_0^1 \frac{\rho_3}{\Gamma_q(\alpha_1)}f_2\left(s,1,0,\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_1)},0\right)d_qs,
\end{aligned}
$$

where $\gamma = \max\{\gamma_1,\gamma_2\}$. Then, we obtain

$$
\begin{aligned}
&T_1(h_1,h_1)(t) \geqslant a_{11}t^{\alpha_1-1} = a_{11}h_1(t), \\
&T_1(h_1,h_1)(t) \leqslant a_{12}t^{\alpha_1-1} = a_{12}h_1(t), \\
&D_q^{\gamma_{\tilde{i}}}T_1(h_1,h_1)(t) \geqslant a_{11}^{'}\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}t^{\alpha_1-\gamma_{\tilde{i}}-1} = a_{11}^{'}D_q^{\gamma_{\tilde{i}}}h_1(t), \\
&D_q^{\gamma_{\tilde{i}}}T_1(h_1,h_1)(t) \leqslant a_{12}^{'}\frac{\Gamma_q(\alpha_1)}{\Gamma_q(\alpha_1-\gamma_{\tilde{i}})}t^{\alpha_1-\gamma_{\tilde{i}}-1} = a_{12}^{'}D_q^{\gamma_{\tilde{i}}}h_1(t).
\end{aligned} \tag{32}
$$

Also, let

$$a_{21}=\int_0^1 \varrho_2[1-(1-qs)^{(\alpha_2-1)}]f_2\left(s,0,1,0,\frac{\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_2-\gamma_2)}\right)d_qs$$
$$+\int_0^1 \rho_2[1-(1-qs)^{(\alpha_1-1)}]f_1\left(s,0,1,0,\frac{\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_2-\gamma_2)}\right)d_qs,$$
$$a_{22}=\int_0^1 \varrho_2 f_2\left(s,1,0,\frac{\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_2-\gamma_1)},0\right)d_qs+\int_0^1 \rho_2 f_2\left(s,1,0,\frac{\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_2-\gamma_1)},0\right)d_qs,$$
$$a_{21}'=\int_0^1 \frac{\varrho_4}{\Gamma_q(\alpha_2)}[1-(1-qs)^{(\alpha_2-\gamma-1)}]f_2\left(s,0,1,0,\frac{\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_2-\gamma_2)}\right)d_qs$$
$$+\int_0^1 \frac{\rho_4}{\Gamma_q(\alpha_2)}[1-(1-qs)^{(\alpha_1-\gamma-1)}]f_1\left(s,0,1,0,\frac{\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_2-\gamma_2)}\right)d_qs,$$
$$a_{22}'=\int_0^1 \frac{\varrho_4}{\Gamma_q(\alpha_2)}f_2\left(s,1,0,\frac{\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_2-\gamma_1)},0\right)d_qs+\int_0^1 \frac{\rho_4}{\Gamma_q(\alpha_2)}f_1\left(s,1,0,\frac{\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_2-\gamma_1)},0\right)d_qs,$$

we have

$$\begin{aligned}
&T_2(h_2,h_2)(t)\geqslant a_{21}t^{\alpha_2-1}=a_{21}h_2(t),\\
&T_2(h_2,h_2)(t)\leqslant a_{22}t^{\alpha_2-1}=a_{22}h_2(t),\\
&D_q^{\gamma_{\tilde{i}}}T_2(h_2,h_2)(t)\geqslant a_{21}'\frac{\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_2-\gamma_{\tilde{i}})}t^{\alpha_2-\gamma_{\tilde{i}}-1}=a_{21}'D_q^{\gamma_{\tilde{i}}}h_2(t),\\
&D_q^{\gamma_{\tilde{i}}}T_2(h_2,h_2)(t)\leqslant a_{22}'\frac{\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_2-\gamma_{\tilde{i}})}t^{\alpha_2-\gamma_{\tilde{i}}-1}=a_{22}'D_q^{\gamma_{\tilde{i}}}h_2(t).
\end{aligned} \tag{33}$$

As a result, according to (32) and (33), we can get $T_1(h_1,h_1)\in P_{h_1}$, $T_2(h_2,h_2)\in P_{h_2}$. Further, it can be see that $T(h,h)\in K_h$, which satisfies (A1) in Lemma 1.

(2) For $u,v\in P$ and $t\in(0,1)$, it can be obtain that

$$\begin{aligned}
T_1(ru,r^{-1}v)&=\int_0^1 K_1(t,qs)f_1(s,ru,r^{-1}v,D_q^{\gamma_1}ru,D_q^{\gamma_2}r^{-1}v)d_qs\\
&\quad+\int_0^1 H_1(t,qs)f_2(s,ru,r^{-1}v,D_q^{\gamma_1}ru,D_q^{\gamma_2}r^{-1}v)d_qs\\
&\geqslant\int_0^1 K_1(t,qs)\varphi_1(r)f_1(s,u,v,D_q^{\gamma_1}u,D_q^{\gamma_2}v)d_qs\\
&\quad+\int_0^1 H_1(t,qs)\varphi_2(r)f_2(s,u,v,D_q^{\gamma_1}u,D_q^{\gamma_2}v)d_qs\\
&\geqslant\varphi_0(r)T_1(u,v),
\end{aligned}$$

$$\begin{aligned}
T_2(ru,r^{-1}v)&=\int_0^1 K_2(t,qs)f_2(s,ru,r^{-1}v,D_q^{\gamma_1}ru,D_q^{\gamma_2}r^{-1}v)d_qs\\
&\quad+\int_0^1 H_2(t,qs)f_1(s,ru,r^{-1}v,D_q^{\gamma_1}ru,D_q^{\gamma_2}r^{-1}v)d_qs\\
&\geqslant\int_0^1 K_2(t,qs)\varphi_2(r)f_2(s,u,v,D_q^{\gamma_1}u,D_q^{\gamma_2}v)d_qs\\
&\quad+\int_0^1 H_2(t,qs)\varphi_1(r)f_1(s,u,v,D_q^{\gamma_1}u,D_q^{\gamma_2}v)d_qs\\
&\geqslant\varphi_0(r)T_2(u,v),
\end{aligned}$$

$$
\begin{aligned}
D_q^{\gamma_{\tilde{i}}}T_1(ru,r^{-1}v)=&\int_0^1 K_{\tilde{i}1}(t,qs)f_1(s,ru,r^{-1}v,D_q^{\gamma_1}ru,D_q^{\gamma_2}r^{-1}v)d_qs\\
&+\int_0^1 H_{\tilde{i}1}(t,qs)f_2(s,ru,r^{-1}v,D_q^{\gamma_1}ru,D_q^{\gamma_2}r^{-1}v)d_qs\\
\geqslant&\int_0^1 K_{\tilde{i}1}(t,qs)\varphi_1(r)f_1(s,u,v,D_q^{\gamma_1}u,D_q^{\gamma_2}v)d_qs\\
&+\int_0^1 H_{\tilde{i}1}(t,qs)\varphi_2(r)f_2(s,u,v,D_q^{\gamma_1}u,D_q^{\gamma_2}v)d_qs\\
\geqslant&\varphi_0(r)D_q^{\gamma_{\tilde{i}}}T_1(u,v),\\
D_q^{\gamma_{\tilde{i}}}T_2(ru,r^{-1}v)=&\int_0^1 K_{\tilde{i}2}(t,qs)f_2(s,ru,r^{-1}v,D_q^{\gamma_1}ru,D_q^{\gamma_2}r^{-1}v)d_qs\\
&+\int_0^1 H_{\tilde{i}2}(t,qs)f_1(s,ru,r^{-1}v,D_q^{\gamma_1}ru,D_q^{\gamma_2}r^{-1}v)d_qs\\
\geqslant&\int_0^1 K_{\tilde{i}2}(t,qs)\varphi_2(r)f_2(s,u,v,D_q^{\gamma_1}u,D_q^{\gamma_2}v)d_qs\\
&+\int_0^1 H_{\tilde{i}2}(t,qs)\varphi_1(r)f_1(s,u,v,D_q^{\gamma_1}u,D_q^{\gamma_2}v)d_qs\\
\geqslant&\varphi_0(r)D_q^{\gamma_{\tilde{i}}}T_2(u,v).
\end{aligned}
$$

Thus, it is obvious that $T(ru,r^{-1}v)\geqslant\varphi_0(r)T(u,v)$, which satisfies (A2) in Lemma 1.

(3) From what has been discussed in (1) and (2), according to Lemma 1, we obtain that BVP (1) and (2) has a unique non-negative solutions (u^*,v^*) in K_h. For any initial (x_{01},x_{02}), $(y_{01},y_{02})\in K_h$, there are two iterative sequences $\{(x_{n1},x_{n2})\},\{(y_{n1},y_{n2})\}$ satisfying that $(x_{n1},x_{n2})\to(u^*,v^*)$, $(y_{n1},y_{n2})\to(u^*,v^*)$.

This completes the proof of the theorem. □

Remark 1. *Let (u,v) be the solution of the BVP (1) and (2). If $u\geq 0$, $v\geq 0$, then (u,v) be the non-negative solution of the BVP (1) and (2).*

Example 1. *For $t\in[0,1]$, consider the following fractional differential system:*

$$
\begin{cases}
D_{0.5}^{\frac{6}{5}}u(t)+(u(t))^{\frac{1}{6}}+(v(t))^{-\frac{1}{4}}+\left(D_q^{\frac{1}{2}}u(t)\right)^{\frac{1}{2}}+t\left(D_q^{\frac{1}{2}}v(t)\right)^{-\frac{1}{3}}=0,\\
D_{0.5}^{\frac{7}{5}}v(t)+\dfrac{u}{u+1}+\dfrac{1}{v+2}+D_q^{\frac{1}{2}}u(t)+\dfrac{1}{D_q^{\frac{1}{2}}v(t)}=0,
\end{cases}
\tag{34}
$$

with the coupled integral and discrete mixed boundary conditions:

$$
\begin{cases}
u(0)=v(0)=0,\\
{}^cD_{0.5}^{0.2}u(1)=\sum_{i=1}^{2}\lambda_{1i}I_{0.5}^{\beta_{1i}}v(\xi_i)+\sum_{j=1}^{2}b_{1j}v(\eta_j),\\
{}^cD_{0.5}^{0.4}v(1)=\sum_{i=1}^{2}\lambda_{2i}I_{0.5}^{\beta_{2i}}u(\xi_i)+\sum_{j=1}^{2}b_{2j}u(\eta_j).
\end{cases}
\tag{35}
$$

In this model, we set

$$
\begin{aligned}
&\lambda_{11}=0.25,\quad \lambda_{21}=0.2,\quad \beta_{11}=1.5,\quad \beta_{21}=1.4,\quad \xi_1=0.25,\quad b_{11}=0.33,\quad b_{21}=0.17,\quad \eta_1=0.33,\\
&\lambda_{12}=0.5,\quad \lambda_{22}=0.1,\quad \beta_{12}=2.5,\quad \beta_{22}=2.4,\quad \xi_2=0.75,\quad b_{12}=0.67,\quad b_{22}=0.83,\quad \eta_2=0.67,
\end{aligned}
$$

$$f_1(t,x_1,y_1,x_2,y_2) = {x_1}^{\frac{1}{6}} + {y_1}^{-\frac{1}{4}} + {x_2}^{\frac{1}{2}} + t{y_2}^{-\frac{1}{3}},$$
$$f_2(t,x_1,y_1,x_2,y_2) = \frac{x_1}{x_1+1} + \frac{1}{y_1+2} + x_2 + \frac{1}{y_2}.$$

It is not difficult to find that $f_i(t,x_1,y_1,x_2,y_2), (i=1,2)$ *are satisfy* (S_1) *in Theorem 1. Further, for* $\forall\, r \in (0,1)$, *we have*

$$\begin{aligned} f_1(t,rx_1,r^{-1}y_1,rx_2,r^{-1}y_2) &= (rx_1)^{\frac{1}{6}} + (r^{-1}y_1)^{-\frac{1}{4}} + (rx_2)^{\frac{1}{2}} + t(r^{-1}y_2)^{-\frac{1}{3}} \\ &= r^{\frac{1}{6}}{x_1}^{\frac{1}{6}} + r^{\frac{1}{4}}{y_1}^{-\frac{1}{4}} + r^{\frac{1}{2}}{x_2}^{\frac{1}{2}} + r^{\frac{1}{3}}t{y_2}^{-\frac{1}{3}} \\ &\geqslant r{x_1}^{\frac{1}{6}} + r{y_1}^{-\frac{1}{4}} + r{x_2}^{\frac{1}{2}} + rt{y_2}^{-\frac{1}{3}} \\ &= rf_1(t,x_1,y_1,x_2,y_2) \\ &= \varphi_1(r)f_1(t,x_1,y_1,x_2,y_2), \end{aligned}$$

$$\begin{aligned} f_2(t,rx_1,r^{-1}y_1,rx_2,r^{-1}y_2) &= \frac{rx_1}{rx_1+1} + \frac{1}{r^{-1}y_1+2} + rx_2 + \frac{1}{r^{-1}y_2} \\ &\geqslant \frac{rx_1}{x_1+1} + \frac{r}{y_1+2} + rx_2 + \frac{r}{y_2} \\ &= rf_2(t,x_1,y_1,x_2,y_2) \\ &= \varphi_2(r)f_2(t,x_1,y_1,x_2,y_2). \end{aligned}$$

So $\varphi_0(r) = \min\{\varphi_1(r),\varphi_2(r)\} = \min\{r,r\} = r$, *which satisfy* (S_2) *in Theorem 1. Then from Theorem 1, we can assert that BVP (34) and (35) has a unique non-negative solutions* (u^*,v^*) *in* $K_h = P_{h_1} \times P_{h_2}$, *where* $(h_1,h_2) = (t^{\frac{1}{5}}, t^{\frac{2}{5}})$.

4. Conclusions

The Q derivative has important applications in many fields, such as quantum physics, spectral analysis and dynamical systems, which make it as a powerful tool for solving physics problems mathematically. In the model studied in this paper, the equations and boundary conditions are universal, but it can be seen from Theorem 1 that utilizing the fixed point theorem for mixed monotone operators, it can be acquired that the conclusion of the existence and uniqueness of the solution only by two easily attainable constraints on the nonlinear term. One of the conditions is mixed monotonicity, and the other is to restrict its properties similar to the upper bound using another function. Compared with the monotone iterative method in [30], we prove the existence and uniqueness of non-negative solutions for more complex systems using more looser conditions.

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References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands; Boston, MA, USA, 2006.
2. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
3. Khan, H.; Gómez-Aguilar, J.F.; Abdeljawad, T.; Khan, A. Existence results and stability criteria for ABC-fuzzy-Volterra integro-differential equation. *Fractals* **2020**, *28*, 2040048. [CrossRef]
4. Shah, A.; Khan, R.A.; Khan, A.; Khan, H.; Gómez-Aguilar, J.F. Investigation of a system of nonlinear fractional order hybrid differential equations under usual boundary conditions for existence of solution. *Math. Meth. Appl. Sci.* **2020**, *44*, 1628–1638. [CrossRef]

5. Matar, M.M.; Abbas, M.I.; Alzabut, J.; Kaabar, M.K.A.; Etemad, S.; Rezapour, S. Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives. *Adv. Differ. Equ.* **2021**, *1*, 68. [CrossRef]
6. Subramanian, M.; Manigandan, M.; Tunc, C.; Gopal, T.N.; Alzabut, N. On system of nonlinear coupled differential equations and inclusions involving Caputo-type sequential derivatives of fractional order. *J. Taibah Univ. Sci.* **2022**, *16*, 1–23. [CrossRef]
7. Zhao, X.; Liu, Y.; Pang, H. Iterative positive solutions to a coupled fractional differential system with the multistrip and multipoint mixed boundary conditions. *Adv. Differ. Equ.* **2019**, *1*, 389. [CrossRef]
8. Du, X.; Meng, Y.; Pang, H. Iterative positive solutions to a coupled Hadamard-type fractional differential system on infinite domain with the multistrip and multipoint mixed boundary conditions. *J. Funct. Spaces* **2020**, *2020*, 6508075. [CrossRef]
9. Afshari, H.; Sajjadmanesh, M.; Baleanu, D. Existence and uniqueness of positive solutions for a new class of coupled system via fractional derivatives. *Adv. Differ. Equ.* **2020**, *1*, 111. [CrossRef]
10. Song, S.; Zhang, L.; Zhou, B.; Zhang, N. Existence-uniqueness of positive solutions to nonlinear impulsive fractional differential systems and optimal control. *Bound. Value Probl.* **2020**, *1*, 162. [CrossRef]
11. Wang, T.; Hao, Z. Existence and Uniqueness of Positive Solutions for Singular Nonlinear Fractional Differential Equation via Mixed Monotone Operator Method. *J. Funct. Spaces* **2020**, *2020*, 2354927. [CrossRef]
12. Sang, Y.; He, L. Existence of an approximate solution for a class of fractional multi-point boundary value problems with the derivative term. *Bound. Value Probl.* **2021**, *1*, 20. [CrossRef]
13. Harjani, J.; López, B.; Sadarangani, K. Positive solutions for a fractional boundary value problem via a mixed monotone operator. *Fixed Point Theory* **2021**, *22*, 189–203. [CrossRef]
14. Jackson, F.H. On q-Functions and a certain Difference Operator. *Trans. R. Soc. Edinb.* **1909**, *46*, 253–281. [CrossRef]
15. Jackson, F.H. q-difference equations. *Am. J. Math.* **1910**, *32*, 305–314. [CrossRef]
16. Al-Salam, W.A. Some fractional q-integrals and q-derivatives. *Proc. Edinb. Math. Soc.* **1966**, *15*, 135–140. [CrossRef]
17. Agarwal, R.P. Certain fractional q-integrals and q-derivatives. *Math. Proc. Camb. Philos. Soc.* **1969**, *66*, 365–370. [CrossRef]
18. Bekker, M.B.; Bohner, M.J.; Herega, A.N.; Voulov, H. Spectral analysis of a q-difference operator. *J. Phys. A-Math. Theor.* **2010**, *43*, 1189–1195. [CrossRef]
19. Field, C.M.; Joshi, N.; Nijhoff, F. q-Difference equations of KdV type and "Chazy-type" second-degree difference equations. *J. Phys. A-Math. Theor.* **2008**, *41*, 2444–2454. [CrossRef]
20. Mabrouk, H. q-heat operator and q-Poisson's operator. *Fract. Calc. Appl. Anal.* **2006**, *9*, 265–286.
21. Nemri, A.; Fitouhi, A. Polynomial expansions for solution of wave equation in quantum calculus. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **2011**, *14*, 345–359. [CrossRef]
22. Ernst, T. Some results for q-functions of many variables. *Rend. Semin. Mat. Univ. Padova* **2004**, *112*, 199–235.
23. Ernst, T. q-Bernoulli and q-Euler polynomials, An umbral approach. *Int. J. Differ. Equ.* **2006**, *1*, 31–80.
24. Rui, A. Nontrivial solutions for fractional q-difference boundary value problems. *Electron. J. Qual. Theory Differ. Equ.* **2010**, *70*, 1–10.
25. Yuan, Q.; Yang, W. Positive solutions of nonlinear boundary value problems for delayed fractional q-difference systems. *Adv. Differ. Equ.* **2014**, *2014*, 51. [CrossRef]
26. Jafari, H.; Haghbin, A.; Johnston, S.J.; Baleanu, D. A new algorithm for solving dynamic equations on a time scale. *J. Comput. Appl. Math.* **2017**, *312*, 167–173. [CrossRef]
27. Zhang L.; Sun, S. Existence and uniqueness of solutions for mixed fractional q-difference boundary value problems. *Bound. Value Probl.* **2019**, *2019*, 100. [CrossRef]
28. Samei, M.E. Existence of solutions for a system of singular sum fractional q-differential equations via quantum calculus. *Adv. Differ. Equ.* **2020**, *1*, 23. [CrossRef]
29. Samei, M.E.; Ahmadi, A.; Hajiseyedazizi, S.N.; Mishra, S.K.; Ram, B. The existence of nonnegative solutions for a nonlinear fractional q-differential problem via a different numerical approach. *J. Inequalities Appl.* **2021**, *1*, 75. [CrossRef]
30. Bai, C.; Yang, D. The iterative positive solution for a system of fractional q-difference equations with four-point boundary conditions. *Discret. Dyn. Nat. Soc.* **2020**, 3970903. [CrossRef]
31. Guo, F.; Kang, S. Positive solutions for a class of fractional boundary value problem with q-derivatives. *Mediterr. J. Math.* **2019**, *16*, 113. [CrossRef]
32. Yang, C. Positive solutions for a three-point boundary value problem of fractional q-difference equations. *Symmetry* **2018**, *10*, 358. [CrossRef]
33. Lakshmikantham, D.G. Coupled fixed points of nonlinear operators with applications. *Nonlinear Anal.* **1987**, *11*, 623–632.
34. Guo, D.J. Fixed points of mixed monotone operators with applications. *Appl. Anal.* **1988**, *31*, 215–224. [CrossRef]
35. Zhai, C.; Zhang, L. New fixed point theorems for mixed monotone operators and local existence-uniqueness of positive solutions for nonlinear boundary value problems. *J. Math. Anal. Appl.* **2011**, *382*, 594–614. [CrossRef]
36. Annaby, M.H.; Mansour, Z.S. q-fractional calculus and equations. *Lect. Notes Math.* **2012**, *2056*, 1–318.

37. Kac, V.; Cheung, P. *Quantum Calculus*; Mathematics Subject Classification; Springer: Berlin/Heidelberg, Germany, 2002.
38. Yang, C.; Zhai, C.; Zhang, L. Local uniqueness of positive solutions for a coupled system of fractional differential equations with integral boundary conditions. *Adv. Differ. Equ.* **2017**, *2017*, 282. [CrossRef]