mathematics

Article

A New Extension of C_J Metric Spaces—Partially Controlled J Metric Spaces

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Abstract: This article introduces the concept of partially controlled J metric spaces; in particular, the J metric space with self-distance is not necessarily zero, which is important in computer science. We prove the existence of a unique fixed point for linear and nonlinear contractions, provide some examples to prove the existence of this metric space, and present some important applications in fractional differential equations, i.e., "Riemann–Liouville derivatives".

Keywords: C_J metric spaces; partially controlled J metric spaces; fixed point; fractional differential equations

MSC: 54H25; 47H10

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Citation: Subhi Aiadi, S.; Mior Othman, W.A.; Wong, K.B.; Mlaiki, N. A New Extension of C_J Metric Spaces—Partially Controlled J Metric Spaces. *Mathematics* **2023**, *11*, 2973. https://doi.org/10.3390/math11132973

Academic Editors: Mircea Balaj, Vasile Berinde and Massimiliano Giuli

Received: 6 June 2023
Revised: 29 June 2023
Accepted: 30 June 2023
Published: 3 July 2023

1. Introduction

In 1922, Banach [1] introduced the theory of fixed points; this theory has been further developed through generalizations of linear and nonlinear contractions [2,3]. Generalizations have been made for metric spaces, such as the b metric space and its generalization [4], as well as the J metric space [5], and many more such as [6]. However, all of these extensions assume that the self-distance is zero.

In 1994, Matthews suggested the concept of "non-zero self-distance" to help researchers in the computer science field, where the self-distance in many applications is not necessarily zero. Matthews introduced the concept of partial metric spaces in two dimensions; in 2012, Sedgi [7] introduced the S metric space (the ordinary metric space in three dimensions), where the self-distance is zero. In 2014, Mlaiki introduced S_p (the partial S metric space) [8], where the self-distance is not necessarily zero, and is in three dimensions.

This fixed theory has many applications in economics, such as game theory, for finding equilibrium points and optimization problems [9]. It is very interesting and useful in the existence of orbits and the study of dynamical systems [9]. Moreover, in mathematics, it has significant applications, especially in solving nonlinear hybrid differential equations [10,11], and it is also used to solve some nontrivial equations [12], which has motivated researchers to work further on the fixed point theory.

In 2022, Souayah introduced J metric spaces [5], where the self-distance equals zero. However, the triangle inequality includes a constant c that is greater than zero and a limit of the supremum for certain sequences $\{\alpha_n\}$ that converge to certain values in the metric space, leading to important applications. In 2022, Aiadi introduced the controlled J metric space as a generalization of the J metric space [13]; they replaced the constant c in the J metric space with a function f. They proved the existence and the uniqueness of the fixed point for linear and nonlinear self-mapping contractions, and they put forward certain applications for solving the linear system.

In this paper, after checking the importance of "non-zero self-distance" for computer science, and how most applications in data science and computer science need self-distance to not necessarily be zero [14,15], we introduce the partially controlled J metric space, which is a generalization of the controlled J metric space, where the self-distance is not necessarily zero. When the self-distance is zero, it will be a special case of a "partial controlled J metric space"; we provide examples to prove the existence of the defined metric space and the uniqueness of self-mapping (linear and non-linear contractions). At the end of this paper, we present important applications in fractional differential equations, specifically the "Riemann–Liouville derivatives", which represent the most significant extensions of ordinary calculus, with other definitions being considered as special cases.

2. Preliminaries

First, we start by recalling some basic definitions of partial metric spaces.

Definition 1 ([16])**.** *Let $p : \Upsilon \times \Upsilon \to R^+$, where Υ is a nonempty set and it is denoted as a partial metric on Υ if for any $\Phi, \Psi, \Omega \in \Upsilon$ the following conditions hold true:*

(P1) $\Phi = \Psi$ *if and only if* $p(\Phi,\Phi) = p(\Psi,\Psi) = p(\Phi,\Psi)$;

(P2) $p(\Phi,\Phi) \leq p(\Phi,\Psi)$;

(P3) $p(\Phi,\Psi) = p(\Psi,\Phi)$;

(P4) $p(\Phi,\Psi) \leq p(\Phi,\Omega) + p(\Psi,\Omega) - p(\Omega,\Omega)$.

Definition 2 ([5])**.** *Consider a nonempty set Υ, and a function $J : \Upsilon^3 \to [0,+\infty)$. Let us define the set,*

$$S(J,\Upsilon,\phi) = \{\{\phi_n\} \subset \Upsilon : \lim_{n\to+\infty} J(\phi,\phi,\phi_n) = 0\}$$

for all $\phi \in \Upsilon$.

Definition 3 ([5])**.** *Let Υ be a nonempty set and $J : \Upsilon^3 \to [0,+\infty)$, which satisfies the conditions below:*

(i) $J(\Phi,\Psi,\Omega) = 0$ *implies* $\Phi = \Psi = \Omega$ *for any* $\Phi,\Psi,\Omega \in \Upsilon$.

(ii) *There are some* $K > 0$, *where for each* $(\Phi,\Psi,\Omega) \in \Upsilon^3$ *and* $\{\nu_n\} \in S(J,\delta,\nu)$

$$J(\Phi,\Psi,\Omega) \leq K \limsup_{n\to+\infty} \Big(J(\Phi,\Phi,\nu_n) + J(\Psi,\Psi,\nu_n) + J(\Omega,\Omega,\nu_n) \Big).$$

Then, (Υ, J) is defined as a J metric space. In addition, if $J(\Phi,\Phi,\Psi) = J(\Psi,\Psi,\Phi)$ for each $\Phi,\Psi \in \Upsilon$, the pair (Υ, J) is defined as a symmetric J metric space.

Definition 4 ([13])**.** *Let Υ be a set with at least one element and $C_J : \Upsilon^3 \to [0,+\infty)$ fulfill the following conditions:*

(i) $C_J(\Phi,\Psi,\Omega) = 0$ *implies* $\Phi = \Psi = \Omega$ *for all* $\Phi,\Psi,\Omega \in \Upsilon$.

(ii) *There exists a function* $\theta : \Upsilon^3 \to [0,+\infty)$, *and* $\alpha_n \in S(C_J, \Upsilon, \phi)$,

$$C_J(\Phi,\Psi,\Omega) \leq \theta(\Phi,\Psi,\Omega) \limsup_{n\to+\infty} \Big(C_J(\Phi,\Phi,\alpha_n) + C_J(\Psi,\Psi,\alpha_n) + C_J(\Omega,\Omega,\alpha_n) \Big).$$

Then, (Υ, C_J) is defined as C_J metric space. In addition, if

$$C_J(\Phi,\Phi,\Psi) = C_J(\Psi,\Psi,\Phi).$$

For each $\Phi,\Psi \in \Upsilon$, then (Υ, C_J) is defined as a symmetric C_J metric space.

3. Main Result

In this section, we generalize both metric spaces [partial and J metric spaces] to obtain the new extension defined below as partially controlled metric space.

Definition 5. *Let* Υ *be a nonempty set and a function* $P_J : \Upsilon^3 \to [0, +\infty)$*. Then the set is defined as follows:*

$$S(P_J, \Upsilon, \Phi) = \{\{\Phi_n\} \subset \Upsilon : \lim_{n\to+\infty} P_J(\Phi, \Phi, \Phi_n) = P_J(\Phi, \Phi, \Phi) = \lim_{n\to+\infty} P_J(\Phi_n, \Phi_n, \Phi_n)\}$$

for each $\Phi \in \Upsilon$*.*

Definition 6. *Let* Υ *be a nonempty set and a function* $P_J : \Upsilon^3 \to [0, +\infty)$ *is said to be a partially controlled J metric space if, for all* $\Phi, \Psi, \Omega \in \Upsilon$ *the following conditions hold*

(i) $\Phi = \Psi = \Omega$ *if* $P_J(\Phi, \Phi, \Phi) = P_J(\Psi, \Psi, \Psi) = P_J(\Omega, \Omega, \Omega) = P_J(\Psi, \Phi, \Omega)$*.*

(ii) $P_J(\Phi, \Phi, \Phi) \leq P_J(\Phi, \Phi, \Psi)$*.*

(iii) There exists a continuous function $\theta : \Upsilon^3 \to [0, +\infty)$*, and* $\mu_n \in S(P_J, Y, \Phi)$*, such that for all* $\Psi, \Phi, \Omega \in \Upsilon$ *we have*

$$P_J(\Phi, \Psi, \Omega) \leq \theta(\Phi, \Psi, \Omega). \limsup_{n\to+\infty} \Big(P_J(\Phi, \Phi, \mu_n) + P_J(\Psi, \Psi, \mu_n) + P_J(\Omega, \Omega, \mu_n) \Big).$$

Then, (Υ, P_J) *is defined as a* P_J *metric space. In addition, if*

$$P_J(\Phi, \Phi, \Psi) = P_J(\Psi, \Psi, \Phi),$$

for each $\Phi, \Psi \in \Upsilon$*, then* (Υ, P_J) *is defined as a symmetric* P_J *metric space.*

Remark 1. *This symmetry hypothesis does not necessarily mean that*

$$P_J(\Phi, \Psi, \Omega) = P_J(\Psi, \Omega, \Phi) = P_J(\Omega, \Psi, \Phi) = \cdots$$

We will start by presenting some properties in the topology of P_J metric spaces.

Definition 7.

(1) Let (Υ, P_J) *be a* P_J *metric space. A sequence* $\{\Phi_n\} \subset \Upsilon$ *converges to an element* $\Phi \in \Upsilon$ *if and only if*
$\lim_{n\to+\infty} P_J(\Phi_n, \Phi_n, \Phi_n) = P_J(\Phi, \Phi, \Phi) = \lim_{n\to+\infty} P_J(\Phi_n, \Phi_n, \Phi) = \lim_{n\to+\infty} P_J(\Phi, \Phi, \Phi_n)$

(2) Let (Υ, P_J) *be a* P_J *metric space. A sequence* $\{\Phi_n\} \subset \Upsilon$ *is called Cauchy if and only if* $\lim_{n,m\to+\infty} P_J(\Phi_n, \Phi_n, \Phi_m)$ *exists and it is finite.*

(3) A P_J *metric space is denoted as complete if each Cauchy sequence in* Υ *is convergent.*

Remark 2. *In a* P_J *space, the limit is not necessarily unique.*

In the following proposition, we show that the limit is unique if and only if $\theta(\Phi, \Psi, \Omega) \leq \frac{1}{3}$.

Proposition 1. *In a* P_J *space,* (Υ, P_J)*, if* $\{\Phi_n\}$ *is convergent, and* $\theta(\Phi, \Psi, \Omega) \leq \frac{1}{3}$ *for all* $\Phi, \Psi, \Omega \in \Upsilon$*, then it converges to only one element.*

Proof. Let us assume that $\{\Phi_n\}$ converges to Φ_1 and Φ_2. By using the definition of convergence,

$$P_J(\Phi_1, \Phi_1, \Phi_1) = \lim_{n\to+\infty} P_J(\Phi_n, \Phi_n, \Phi_n) = P_J(\Phi_2, \Phi_2, \Phi_2),$$

$$\begin{aligned}
P_J(\Phi_1,\Phi_1,\Phi_2) &\leq \theta(\Phi_1,\Phi_1,\Phi_2).\limsup_{n\to+\infty}\Big(2P_J(\Phi_1,\Phi_1,\Phi_n)+P_J(\Phi_2,\Phi_2,\Phi_n)\Big)\\
&\leq \frac{1}{3}\limsup_{n\to+\infty}\Big(2P_J(\Phi_1,\Phi_1,\Phi_n)+P_J(\Phi_2,\Phi_2,\Phi_n)\Big)\\
&= \frac{1}{3}(2P_J(\Phi_1,\Phi_1,\Phi_1)+P_J(\Phi_2,\Phi_2,\Phi_2))\\
&= P_J(\Phi_1,\Phi_1,\Phi_1).
\end{aligned}$$

Thus, $P_J(\Phi_1,\Phi_1,\Phi_2) \leq P_J(\Phi_1,\Phi_1,\Phi_1) = P_J(\Phi_2,\Phi_2,\Phi_2))$. On the other hand, by the definition of the metric spaces, we have $P_J(\Phi_1,\Phi_1,\Phi_1) \leq P_J(\Phi_1,\Phi_1,\Phi_2)$. Hence, $\Phi_1 = \Phi_2$ is desired. □

Definition 8. *Let (Υ,ψ) and (Λ,ϕ) be P_J metric spaces and Γ is said to be a continuous function $\Gamma : \Upsilon \to \Lambda$ at $\alpha \in \Upsilon$ if for every $\epsilon > 0$ there is a $\delta > 0$, such that for all $x \in \Upsilon, \psi(x,x,\alpha) < \delta$ implies $\phi(\Gamma(x),\Gamma(x),\Gamma(\alpha)) < \epsilon$.*

Definition 9. *Let (Υ, P_J) be a partially controlled J metric space and $\Phi_0 \in \Upsilon$, $\epsilon > 0$ is the P_J open ball, and P_J is the closed ball of radius λ with centered Φ_0 are*

$$B_{P_J}(\Phi_0,\epsilon) = \{\Phi \in \Upsilon :\mid P_J(\Phi_0,\Phi_0,\Phi) - P_J(\Phi_0,\Phi_0,\Phi_0) \mid < \epsilon\}$$
$$B_{P_J}[\Phi_0,\epsilon] = \{\Phi \in \Upsilon :\mid P_J(\Phi_0,\Phi_0,\Phi) - P_J(\Phi_0,\Phi_0,\Phi_0) \mid \leq \epsilon\}$$

Example 1. *Let $\Upsilon = [0,+\infty)$ and $P_J : \Upsilon^3 \to [0,+\infty)$ be defined by*

$$P_J(\Phi,\Psi,\Omega) = max\{\Phi,\Psi,\Omega\},$$

for all $\Phi,\Psi,\Omega \in [0,+\infty)$.
(1) $P_J(\Phi,\Phi,\Phi) = P_J(\Psi,\Psi,\Psi) = P_J(\Omega,\Omega,\Omega)$, which means that $max(\Phi,\Phi,\Phi) = max(\Psi,\Psi,\Psi) = max(\Omega,\Omega,\Omega)$, which implies that $\Phi = \Psi = \Omega$.
(2) Due to $\Phi,\Psi,\Omega \in [0,+\infty)$, then $P_J(\Phi,\Phi,\Phi) = \Phi \leq P_J(\Phi,\Phi,\Omega) = max(\Phi,\Phi,\Omega)$.
(3) Let

$$\theta(\Phi,\Psi,\Omega) = \begin{cases}\Phi+\Psi, & \text{if } \Phi,\Psi \text{ are even and } \Omega = 2n+1\\ \Psi, & \text{if } \Phi,\Psi \text{ are odd and } \Omega = 2n\\ 1, & \text{otherwise.}\end{cases}$$

$$P_J(\Phi,\Psi,\Omega) = max\{\Phi,\Psi,\Omega\}$$

$$max(\Phi,\Psi,\Omega) \leq \theta(\Phi,\Psi,\Omega)\limsup_{n\to+\infty}\Big(P_J(\Phi,\Phi,\Phi_n)+P_J(\Psi,\Psi,\Phi_n)+P_J(\Omega,\Omega,\Phi_n)\Big).$$

To show this, note that for $\Phi_n = \Phi + \frac{1}{n} \in S(P_J, Y, \Phi)$ and $\Phi,\Psi,\Omega \in \Upsilon$, and using the fact that $\theta(\Phi,\Psi,\Omega) \geq 1$, we deduce

$$\begin{aligned}
&\theta(\Phi,\Psi,\Omega)\limsup_{n\to+\infty}\Big(P_J(\Phi,\Phi,\Phi_n)+P_J(\Psi,\Psi,\Phi_n)+P_J(\Omega,\Omega,\Phi_n)\Big)\\
&\geq \limsup_{n\to+\infty}\Big(P_J(\Phi,\Phi,\Phi_n)+P_J(\Psi,\Psi,\Phi_n)+P_J(\Omega,\Omega,\Phi_n)\Big)\\
&= \Phi + max\{\Psi,\Phi\} + max\{\Omega,\Phi\}\\
&\geq max\{\Phi,\Psi,\Omega\}\\
&= P_J(\Phi,\Psi,\Omega).
\end{aligned}$$

Therefore, the third condition is satisfied as required. This example shows and proves the existence of the new metric space.

Theorem 1. *Let (Υ, P_J) be a partial complete symmetric P_J metric space and $\varphi : \Upsilon \to \Upsilon$ be a continuous map satisfying*

$$P_J(\varphi\Phi, \varphi\Psi, \varphi\Omega) \leq Q(P_J(\Phi, \Psi, \Omega)) \quad \text{for all } \Phi, \Psi, \Omega \in \Upsilon, \tag{1}$$

where $Q : [0, +\infty) \to [0, +\infty)$ is an increasing continuous function, such that

$$\lim_{n\to+\infty} Q^n(t) = 0 \ \text{ for each fixed } \ t > 0.$$

For every $\Phi \in \Upsilon$, let $M(P_J, \varphi, \Phi) = \sup\{P_J(\Phi, \Phi, \varphi^j\Phi) : j \in \mathbb{N} \cup \{0\}\}$. If there exists $\Phi_0 \in \Upsilon$, such that $M(P_j, \varphi, \Phi_0)$ is finite, then φ has a unique fixed point in Υ.

Proof. To prove the existence and the uniqueness of the fixed point, let us define

$$\{\Phi_n\}_{n\geq 0} \subset X$$

as

$$(\Phi_1 = \varphi\Phi_0), (\Phi_2 = \varphi\Phi_1), \ldots, (\Phi_n = \varphi^n\Phi_0). \quad n = 1, 2, \ldots. \tag{2}$$

Let us start by showing that $\{\Phi_n\}$ is a Cauchy sequence.

For every pair $n, m \in \mathbb{N}$ and $n < m$ by applying (2)

$$\begin{aligned}
P_J(\Phi_n, \Phi_n, \Phi_m) &= P_J(\varphi\Phi_{n-1}, \varphi\Phi_{n-1}, \varphi\Phi_{m-1}) \\
&\leq Q\big(P_J(\Phi_{n-1}, \Phi_{n-1}, \Phi_{m-1})\big) \\
&= Q\big(P_J(\varphi\Phi_{n-2}, \varphi\Phi_{n-2}, \varphi\Phi_{m-2})\big) \\
&\leq Q^2\big(P_J(\Phi_{n-2}, \Phi_{n-2}, \Phi_{m-2})\big) \\
&\leq \vdots \\
&\leq Q^n\big(P_J(\Phi_0, \Phi_0, \Phi_{m-n})\big).
\end{aligned}$$

Assume that $m = n + q$ is for some constant $q \in \mathbb{N}$ to obtain

$$P_J(\Phi_n, \Phi_n, \Phi_m) \leq Q^n\big(P_J(\Phi_0, \Phi_0, \Phi_q)\big) \tag{3}$$

and by using the definition of M and the properties of $M(P_J, \varphi, \Phi_0)$, we have

$$\begin{aligned}
P_J(\Phi_n, \Phi_n, \Phi_{n+q}) &= P_J(\varphi\Phi_{n-1}, \varphi\Phi_{n-1}, \varphi\Phi_{n+q-1}) \\
&\leq Q\big(P_J(\Phi_{n-1}, \Phi_{n-1}, \Phi_{n+q-1})\big) \\
&= Q(M(P_J, \varphi, \Phi_{n+q-1}) \\
&\leq \vdots \\
&\leq Q^n\big(M(P_J, \varphi, \Phi_0)
\end{aligned}$$

By applying the limit in (3) as $n \longrightarrow +\infty$ and $M(P_J, \varphi, \Phi_0) < +\infty$, we have

$$\lim_{n\to+\infty} P_J(\Phi_n, \Phi_n, \Phi_m) = 0. \tag{4}$$

So, that $\{\Phi_n\}$ is a Cauchy sequence in Υ.

Due to the completeness definition, $\{\Phi_n\}$ converges to a $\Phi \in \Upsilon$, which means

$$\begin{aligned}
\lim_{n\to+\infty} P_J(\Phi_n, \Phi_n, \Phi_n) = P_J(\Phi, \Phi, \Phi) \quad &= \lim_{n\to+\infty} P_J(\Phi_n, \Phi_n, \Phi) \\
&= \lim_{n\to+\infty} P_J(\Phi, \Phi, \Phi_n)
\end{aligned}$$

Secondly, we will attempt to prove the existence of the fixed point; that is

$$P_J(\Phi, \Phi, \Phi) = P_J(\varphi\Phi, \varphi\Phi, \varphi\Phi).$$

We will start with

$$\begin{aligned} P_J(\Phi, \Phi, \varphi\Phi) &\leq \theta(\Phi, \Phi, \varphi\Phi) \limsup_{n \to +\infty} (2P_J(\Phi, \Phi, \Phi_{n+1}) + P_J(\varphi\Phi, \varphi\Phi, \Phi_{n+1})) \\ &= \theta(\Phi, \Phi, \varphi\Phi) \limsup_{n \to +\infty} (2P_J(\Phi_{n+1}, \Phi_{n+1}, \Phi_{n+1}) + P_J(\varphi\Phi, \varphi\Phi, \varphi\Phi_n)) \\ &= \theta(\Phi, \Phi, \varphi\Phi) \limsup_{n \to +\infty} (2P_J(\varphi\Phi_n, \varphi\Phi_n, \varphi\Phi_n) + P_J(\varphi\Phi, \varphi\Phi, \varphi\Phi_n)) \\ &\leq 3\theta(\Phi, \Phi, \varphi\Phi) \limsup_{n \to +\infty} Q(P_J(\Phi_n, \Phi_n, \Phi_n) \\ &\leq \vdots \\ &\leq 3\theta(\Phi, \Phi, \varphi\Phi) \limsup_{n \to +\infty} Q^n\big(P_J(\Phi_0, \Phi_0, \Phi_0)\big). \end{aligned}$$

Since

$$\lim_{n \to +\infty} Q^n(t) = 0, \tag{5}$$

which implies that

$$P_J(\Phi, \Phi, \varphi\Phi) = 0. \tag{6}$$

Similarly, it can be shown that $P_J(\varphi\Phi, \varphi\Phi, \varphi\Phi) = 0$.

By applying the second condition of the P_J metric space

$$P_J(\Phi, \Phi, \varphi\Phi) \geq P_J(\Phi, \Phi, \Phi). \tag{7}$$

Which gives that

$$P_J(\Phi, \Phi, \Phi) = P_J(\varphi\Phi, \varphi\Phi, \varphi\Phi) = P_J(\Phi, \Phi, \Phi) = 0, \tag{8}$$

$$\Phi = \varphi\Phi \tag{9}$$

That is the definition of the fixed point.

The uniqueness of the fixed point is left to be proven. Assume that there are two fixed points, Φ_1 and Φ_2, we need to show that $\Phi_1 = \Phi_2$.

Because both are fixed points, then

$$P_J(\Phi_1, \Phi_1, \Phi_1) = P_J(\varphi\Phi_1, \varphi\Phi_1, \varphi\Phi_1), \tag{10}$$

and

$$P_J(\Phi_2, \Phi_2, \Phi_2) = P_J(\varphi\Phi_2, \varphi\Phi_2, \varphi\Phi_2).$$

Let us take

$$\begin{aligned} P_J(\Phi_1, \Phi_1, \Phi_2) = P_J(\varphi\Phi_1, \varphi\Phi_1, \varphi\Phi_2). \\ \leq Q(P_J(\Phi_1, \Phi_1, \Phi_2) \end{aligned}$$

Apply this property n times to obtain

$$P_J(\Phi_1, \Phi_1, \Phi_2) \leq Q^n(P_J(\varphi\Phi_1, \varphi\Phi_1, \varphi\Phi_2).$$

As

$$\lim_{n \to +\infty} Q^n(t) = 0 \to P_J(\Phi_1, \Phi_1, \Phi_2) = 0$$

and by using the second condition of the P_J metric space, then

$$P_J(\Phi_1, \Phi_1, \Phi_1) \leq P_J(\Phi_1, \Phi_1, \Phi_2) = 0.$$

Similarly, it can be shown that $P_J(\Phi_2, \Phi_2, \Phi_1) = 0$. From

$$P_J(\Phi_2, \Phi_2, \Phi_2) \leq P_J(\Phi_2, \Phi_2, \Phi_1) = 0.$$

we have

$$P_J(\Phi_2, \Phi_2, \Phi_2) = P_J(\Phi_1, \Phi_1, \Phi_1) = 0.$$

Thus, as desired, φ has a unique fixed point in Υ. □

Example 2. *Let* $\Upsilon = [0, +\infty)$ *and* $P_J : \Upsilon^3 \to [0, +\infty)$ *be defined by*

$$P_J(\Phi, \Psi, \Omega) = max\{\Phi, \Psi, \Omega\},$$

for all $\Phi, \Psi, \Omega \in [0, +\infty)$.

Let $Q(t) = \frac{t}{1+t}$. *Note that,*

$$Q^2(t) = \frac{\frac{t}{1+t}}{1 + \frac{t}{1+t}} = \frac{t}{1+2t}, Q^3(t) = \frac{\frac{t}{1+2t}}{1 + \frac{t}{1+2t}} = \frac{t}{1+3t}.$$

Now, by induction on n, we can easily deduce that

$$Q^n(t) = \frac{t}{1+nt}.$$

Hence,

$$\lim_{n \to +\infty} Q^n(t) = 0.$$

Next, let $\varphi(x) = \frac{x}{1+x}$.

(1) P_J *is a symmetric and complete partially controlled J metric space.*

(2) $P_J(\varphi\Phi, \varphi\Psi, \varphi\Omega) \leq Q(P_J(\Phi, \Psi, \Omega))$, *because*

$$\max\left\{\frac{\Phi}{1+\Phi}, \frac{\Psi}{1+\Psi}, \frac{\Omega}{1+\Omega}\right\} \leq \frac{\max\{\Phi, \Psi, \Omega\}}{1 + \max\{\Phi, \Psi, \Omega\}}, forall \Phi, \Psi, \Omega \geq 0.$$

This is an example that φ *has a unique fixed point in* Υ.

Theorem 2. *Let* (Υ, P_J) *be a complete symmetric* P_J *metric space and*
φ: $\Upsilon \to \Upsilon$ *be a mapping that satisfies*

$$P_J(\varphi\Phi, \varphi\Psi, \varphi\Omega) \leq \phi(\Phi, \Psi, \Omega) P_J(\Phi, \Psi, \Omega) \quad \textbf{for all} \Phi, \Psi, \Omega \in \Upsilon, \tag{11}$$

ϕ: $\Upsilon^3 \to (0, 1)$, *is a given mapping satisfying* $\phi(\varphi\Phi, \varphi\Psi, \varphi\Omega)) \leq \phi(\Phi, \Psi, \Omega)$, *and* $\varphi : \Upsilon \to \Upsilon$. *For every* $\Phi \in \Upsilon$, *let* $M(P_J, \varphi, \Phi) = \sup\{P_J(\Phi, \Phi, \varphi^j\Phi) : j \in \mathbb{N} \cup \{0\}\}$ *and*

$$M'(P_J, \varphi, \Phi) = \sup\left\{\phi(\varphi^i\Phi, \varphi^i\Phi, \varphi^j\Phi) : i, j \in \mathbb{N} \cup \{0\}\right\}.$$

If there exists $\Phi_0 \in \Upsilon$, *such that* $M(P_j, \varphi, \Phi_0)$ *is finite, and* $M'(P_j, \varphi, \Phi_0) < 1$, *then* ϕ *has a unique fixed point in* Υ.

Proof. We build a sequence $\{\Phi_n\}$ as follows $\{\Phi_n = \varphi^n\Phi_0\}$.

Let us start by proving that $\{\Phi_n\}$ is a Cauchy sequence. For any natural numbers, n, m, we assume that $n < m$ and we can assume that there exists $q \in N$, where $m = n + q$:

$$
\begin{aligned}
P_J(\Phi_n, \Phi_n, \Phi_m) &= P_J(\varphi\Phi_{n-1}, \varphi\Phi_{n-1}, \varphi\Phi_{m-1}) \\
&\leq \phi(\Phi_{n-1}, \Phi_{n-1}, \Phi_{m-1}) P_J(\Phi_{n-1}, \Phi_{n-1}, \Phi_{m-1}) \\
&\leq \phi^n(\Phi_0, \Phi_0, \Phi_q) P_J(\Phi_0, \Phi_0, \Phi_q).
\end{aligned}
$$

Since $P_J(\Phi_0, \Phi_0, \Phi_q) \leq M(P_J, \varphi, \Phi_0) < +\infty$ and $\phi(\Phi_0, \Phi_0, \Phi_q) \leq M'(P_J, \varphi, \Phi_0) < 1$, we have

$$
P_J(\Phi_n, \Phi_n, \Phi_m) \leq \big(M'(P_J, \varphi, \Phi_0)\big)^n M(P_J, \varphi, \Phi_0).
$$

Taking $n \to +\infty$ and noting that $\big(M'(P_J, \varphi, \Phi_0)\big)^n \to 0$, we have

$$
\lim_{n,m \to +\infty} P_J(\Phi_n, \Phi_n, \Phi_m) = 0. \tag{12}
$$

so $\{\Phi_n\}$ is a Cauchy sequence.

Then, by the completeness definition of Y, there is a $\Phi \in Y$, such that

$$
\lim_{n \to +\infty} P_J(\Phi_n, \Phi_n, \Phi_n) = P_J(\Phi, \Phi, \Phi) = \lim_{n \to +\infty} P_J(\Phi_n, \Phi_n, \Phi) = \lim_{n \to +\infty} P_J(\Phi, \Phi, \Phi_n). \tag{13}
$$

We will show that Φ is a fixed point of φ. From (13),

$$
\begin{aligned}
P_J(\Phi, \Phi, \varphi\Phi) \quad &= \quad \theta(\Phi, \Phi, \varphi\Phi). \limsup_{n \to +\infty} (2P_J(\Phi, \Phi, \Phi_{n+1}) + P_J(\varphi\Phi, \varphi\Phi, \Phi_{n+1})) \\
&= \quad \theta(\Phi, \Phi, \varphi\Phi). \limsup_{n \to +\infty} (2P_J(\Phi_{n+1}, \Phi_{n+1}, \Phi_{n+1}) + P_J(\varphi\Phi, \varphi\Phi, \varphi\Phi_n)) \\
&= \quad \theta(\Phi, \Phi, \varphi\Phi). \limsup_{n \to +\infty} (2P_J(\varphi\Phi_n, \varphi\Phi_n, \varphi\Phi_n) + P_J(\varphi\Phi, \varphi\Phi, \varphi\Phi_n)) \\
&\leq \quad \theta(\Phi, \Phi, \varphi\Phi). \limsup_{n \to +\infty} (2\phi(\Phi_n, \Phi_n, \Phi_n) P_J(\Phi_n, \Phi_n, \Phi_n) \\
&+ \quad \phi(\Phi, \Phi, \Phi_n) P_J(\Phi, \Phi, \Phi_n)) \\
&\leq \quad \theta(\Phi, \Phi, \varphi\Phi). \limsup_{n \to +\infty} (2P_J(\Phi_n, \Phi_n, \Phi_n) + P_J(\Phi, \Phi, \Phi_n)) \\
&= \quad 3\theta(\Phi, \Phi, \varphi\Phi). \limsup_{n \to +\infty} P_J(\Phi_n, \Phi_n, \Phi_n) \\
&= \quad 3\theta(\Phi, \Phi, \varphi\Phi). \limsup_{n \to +\infty} P_J(\varphi\Phi_{n-1}, \varphi\Phi_{n-1}, \varphi\Phi_{n-1}) \\
&\leq \quad 3\theta(\Phi, \Phi, \varphi\Phi). \limsup_{n \to +\infty} \phi(\Phi_{n-1}, \Phi_{n-1}, \Phi_{n-1}).P_J(\Phi_{n-1}, \Phi_{n-1}, \Phi_{n-1}) \\
&\leq \quad \cdots \\
&\vdots \\
&\leq \quad 3\theta(\Phi, \Phi, \varphi\Phi)\phi^{n-1}(\Phi_{n-1}, \Phi_{n-1}, \Phi_{n-1}).(P_J(\Phi_0, \Phi_0, \Phi_0)) \\
&\leq \quad 3\theta(\Phi, \Phi, \varphi\Phi)\big(M'(P_J, \varphi, \Phi_0)\big)^{n-1}.(P_J(\Phi_0.\Phi_0, \Phi_0))
\end{aligned}
$$

and by using the property $M'(P_J, \varphi, \Phi_0) < 1$.

$$
P_J(\Phi, \Phi, \varphi\Phi) = 0. \tag{14}
$$

Similarly, it can be shown that $P_J(\varphi\Phi, \varphi\Phi, \varphi\Phi) = 0$. Now, by using the property of the metric space

$$
0 = P_J(\Phi, \Phi, \varphi\Phi) \geq P_J(\Phi, \Phi, \Phi).
$$

and then

$$
P_J(\Phi, \Phi, \Phi) = P_J(\varphi\Phi, \varphi\Phi, \varphi\Phi) = 0.
$$

This takes us to the result
$$\Phi = \varphi\Phi$$

Finally, we need to prove the uniqueness of the fixed point. Assume that there are two fixed points, Φ_1 and Φ_2, we plan to show that $\Phi_1 = \Phi_2$.
Because both are fixed points, then
$$P_J(\Phi_1, \Phi_1, \Phi_1) = P_J(\varphi\Phi_1, \varphi\Phi_1, \varphi\Phi_1)), \tag{15}$$
and
$$P_J(\Phi_2, \Phi_2, \Phi_2) = P_J(\varphi\Phi_2, \varphi\Phi_2, \varphi\Phi_2)).$$
Let us start with
$$\begin{aligned} P_J(\Phi_1, \Phi_1, \Phi_2) &= P_J(\varphi\Phi_1, \varphi\Phi_1, \varphi\Phi_2) \\ &\leq \phi(\Phi_1, \Phi_1, \Phi_2)(P_J(\Phi_1, \Phi_1, \Phi_2)) \end{aligned}$$

Apply this property n times to obtain
$$P_J(\Phi_1, \Phi_1, \Phi_2) \leq \phi^n(\Phi_1, \Phi_1, \Phi_2)(P_J(\Phi_1, \Phi_1, \Phi_2)$$
as
$$\lim_{n \to +\infty} \phi^n(\Phi_1, \Phi_1, \Phi_2) = 0.$$
due to
$$\phi : \mathrm{Y}^3 \to (0, 1)$$
and by using the second condition of the P_J metric space, then
$$P_J(\Phi_1, \Phi_1, \Phi_1) \leq P_J(\Phi_1, \Phi_1, \Phi_2) = 0.$$
Similarly, it can be shown that $P_J(\Phi_2, \Phi_2, \Phi_1) = 0$. From
$$P_J(\Phi_2, \Phi_2, \Phi_2) \leq P_J(\Phi_2, \Phi_2, \Phi_1) = 0,$$
we have
$$P_J(\Phi_2, \Phi_2, \Phi_2) = P_J(\Phi_1, \Phi_1, \Phi_1).$$

Thus, φ has a unique fixed point in Y, as desired. □

Theorem 3. *Let* (Y, P_J) *be a complete symmetric* P_J*-metric space,* $\varphi : \mathrm{Y} \to \mathrm{Y}$ *is a continuous map, where:*
$$P_J(\varphi\Phi, \varphi\Psi, \varphi\Omega) \leq aP_J(\Phi, \Psi, \Omega) + bP_J(\Phi, \varphi\Phi, \varphi\Phi) + cP_J(\Psi, \varphi\Psi, \varphi\Psi) + dP_J(\Omega, \varphi\Omega, \varphi\Omega). \tag{16}$$
for each $\Phi, \Psi, \Omega \in \mathrm{Y}$ *and*
$$0 < a + b < 1 - c - d \tag{17}$$
$$0 < a < 1. \tag{18}$$

For every $\Phi \in \mathrm{Y}$*, let* $M(P_J, \varphi, \Phi) = \sup\{P_J(\Phi, \Phi, \varphi^j\Phi) : j \in \mathbb{N} \cup \{0\}\}$*. If there exists* $\Phi_0 \in \mathrm{Y}$*, such that* $M(P_j, \varphi, \Phi_0) < +\infty$*, then* Φ_0 *is a fixed point of* φ *in* Y*.*

Proof. Define $\{\Phi_n = \varphi^n\Phi_0\}$ be a sequence in Y. By (16), we have

$$\begin{aligned} P_J(\Phi_n, \Phi_{n+1}, \Phi_{n+1}) = P_J(\varphi\Phi_{n-1}, \varphi\Phi_n, \varphi\Phi_n) &\leq a P_J(\Phi_{n-1}, \Phi_n, \Phi_n) + b P_J(\Phi_{n-1}, \Phi_n, \Phi_n) \\ &\quad + c P_J(\Phi_n, \Phi_{n+1}, \Phi_{n+1}) + d P_J(\Phi_n, \Phi_{n+1}, \Phi_{n+1}) \\ &= (a+b) P_J(\Phi_{n-1}, \Phi_n, \Phi_n) + (c+d) P_J(\Phi_n, \Phi_{n+1}, \Phi_{n+1}). \end{aligned}$$

Then,

$$P_J(\Phi_n, \Phi_{n+1}, \Phi_{n+1}) \leq \frac{a+b}{1-c-d} P_J(\Phi_{n-1}, \Phi_n, \Phi_n).$$

By taking $k = \dfrac{a+b}{1-c-d}$, then by using (17) we will have $0 < k < 1$.

$$P_J(\Phi_n, \Phi_{n+1}, \Phi_{n+1}) \leq k^n P_J(\Phi_0, \Phi_1, \Phi_1),$$

which gives

$$\lim_{n\to+\infty} P_J(\Phi_n, \Phi_{n+1}, \Phi_{n+1}) = 0. \tag{19}$$

We denote $P_{J_n} = P_J(\Phi_n, \Phi_{n+1}, \Phi_{n+1})$. For each $n, m \in N$, and $n < m$, there is a $q \in N$, such that $m = n + q$. We have

$$\begin{aligned} P_J(\Phi_n, \Phi_n, \Phi_m) &= P_J(\Phi_n, \Phi_n, \Phi_{n+q}) = P_J(\varphi\Phi_{n-1}, \varphi\Phi_{n-1}, \varphi\Phi_{n+q-1}) \\ &\leq a P_J(\Phi_{n-1}, \Phi_{n-1}, \Phi_{n+p-1}) + b P_J(\Phi_{n-1}, \Phi_n, \Phi_n) + c P_J(\Phi_{n-1}, \Phi_n, \Phi_n) \\ &+ d P_J(\Phi_{n+q-1}, \Phi_{n+q}, \Phi_{n+q}) \\ &= a P_J(\Phi_{n-1}, \Phi_{n-1}, \Phi_{n+q-1}) + (b+c) P_{J_{n-1}} + d P_{J_{n+q-1}} \\ &\leq a[a P_J(\Phi_{n-2}, \Phi_{n-2}, \Phi_{n+p-2}) + (b+c) P_{J_{n-2}} + d\Phi P_{J_{n+q-2}}] + (b+c) P_{J_{n-1}} \\ &+ d P_{J_{n+q-1}} \\ &= a^2 P_J(\Phi_{n-2}, \Phi_{n-2}, \Phi_{n+q-2}) + a(b+c) P_{J_{n-2}} + ad\Phi P_{J_{n+q-2}} + (b+c) P_{J_{n-1}} \\ &+ d P_{J_{n+q-1}} \\ &\vdots \\ &\leq a^n P_J(\Phi_0, \Phi_0, \Phi_q) + (b+c) \sum_{k=1}^{n} a^{k-1} P_{J_{n-k}} + d \sum_{k=1}^{n} a^{k-1} P_{J_{n+q-k}}. \end{aligned} \tag{20}$$

Since $a^n P_j(\Phi_0, \Phi_0, \Phi_q) \leq a^n M(P_J, \varphi, \Phi_0)$, $a < 1$ and $M(P_J, \varphi, \Phi_0) < +\infty$,

$$\lim_{n\to+\infty} a^n P_j(\Phi_0, \Phi_0, \Phi_q) = 0.$$

As in the first paragraph, we have

$$P_{J_{n-k}} = P_J(\Phi_{n-k}, \Phi_{n-k+1}, \Phi_{n-k+1}) \leq l^{n-k} P_J(\Phi_0, \Phi_1, \Phi_1),$$

where $l = \frac{a+b}{1-c-d} < 1$. So,

$$\sum_{k=1}^{n} a^{k-1} P_{J_{n-k}} \leq P_J(\Phi_0, \Phi_1, \Phi_1) \sum_{k=1}^{n} a^{k-1} l^{n-k}.$$

There are two cases, if $a \leq l$, then

$$\begin{aligned} \sum_{k=1}^{n} a^{k-1} P_{J_{n-k}} \leq P_J(\Phi_0, \Phi_1, \Phi_1) \sum_{k=1}^{n} a^{k-1} l^{n-k} &\leq P_J(\Phi_0, \Phi_1, \Phi_1) \sum_{k=1}^{n} l^{n-1} \\ &\leq P_J(\Phi_0, \Phi_1, \Phi_1) n l^{n-1}. \end{aligned}$$

Since $0 < l < 1$, $\lim_{n\to+\infty} nl^{n-1} = 0$. So,

$$\lim_{n\to+\infty}(b+c)\sum_{k=1}^{n} a^{k-1}P_{J_{n-k}} = 0.$$

If $a > l$, then

$$\begin{aligned}\sum_{k=1}^{n} a^{k-1}P_{J_{n-k}} &\le P_J(\Phi_0,\Phi_1,\Phi_1)\sum_{k=1}^{n} a^{k-1}l^{n-k} \le P_J(\Phi_0,\Phi_1,\Phi_1)\sum_{k=1}^{n} a^{n-1} \\ &\le P_J(\Phi_0,\Phi_1,\Phi_1)na^{n-1}.\end{aligned}$$

Since $0 < a < 1$, $\lim_{n\to+\infty} na^{n-1} = 0$. So,

$$\lim_{n\to+\infty}(b+c)\sum_{k=1}^{n} a^{k-1}P_{J_{n-k}} = 0.$$

Similarly, it can be shown that (recall that $m = n + q$),

$$\lim_{n,m\to+\infty} d\sum_{k=1}^{n} a^{k-1}P_{J_{n+q-k}} = 0.$$

We obtain

$$\lim_{n,m\to+\infty} P_J(\Phi_n,\Phi_n,\Phi_m) = 0.$$

Then, $\{\Phi_n\}$ is a Cauchy sequence in Y by using the definition of a Cauchy sequence, which is finite and exists.

By the completeness definition, there is $\Phi \in Y$

$$\lim_{n\to+\infty} P_J(\Phi_n,\Phi_n,\Phi_n) = P_J(\Phi,\Phi,\Phi) = \lim_{n\to+\infty} P_J(\Phi_n,\Phi_n,\Phi) = \lim_{n\to+\infty} P_J(\Phi,\Phi,\Phi_n).$$

Now, since φ is continuous,

$$\Phi = \lim_{n\to+\infty}\Phi_{n+1} = \lim_{n\to+\infty}\varphi\Phi_n = \varphi\lim_{n\to+\infty}\Phi_n = \varphi\Phi.$$

Thus, Φ is a fixed point of φ.

Let $\Phi_1, \Phi_2 \in Y$ be two fixed points of φ, where $\varphi\Phi_1 = \Phi_1$, $\varphi\Phi_2 = \Phi_2$. By (2.16), if $P_J(\Phi_1,\Phi_1,\Phi_1) \neq 0$, then

$$\begin{aligned}P_J(\Phi_1,\Phi_1,\Phi_1) &= P_J(\varphi\Phi_1,\varphi\Phi_1,\varphi\Phi_1) \\ &\le aP_J(\Phi_1,\Phi_1,\Phi_1) + (b+c)P_J(\Phi_1,\varphi\Phi_1,\varphi\Phi_1) + dP_J(\Phi_1,\varphi\Phi_1,\varphi\Phi_1) \\ &= aP_J(\Phi_1,\Phi_1,\Phi_1) + (b+c)P_J(\Phi_1,\Phi_1,\Phi_1) + dP_J(\Phi_1,\Phi_1,\Phi_1) \\ &= (a+b+c+d)P_J(\Phi_1,\Phi_1,\Phi_1) \\ &< P_J(\Phi_1,\Phi_1,\Phi_1),\end{aligned}$$

a contradiction. So, $P_J(\Phi_1,\Phi_1,\Phi_1) = 0$. Similarly, $P_J(\Phi_2,\Phi_2,\Phi_2) = 0$. This means that

$$P_J(\Phi_1,\Phi_1,\Phi_1) = P_J(\Phi_2,\Phi_2,\Phi_2) = 0.$$

and

$$\Phi_1 = \Phi_2.$$

Finally, we could say that φ has a unique fixed point. □

4. Application of Theorem 1 to Polynomial Equations

In this section, in Example 3, we prove the existence and uniqueness of a solution to polynomial equations

Example 3. *For any natural number $n \geq 2$, consider that equation*

$$(x+2)^n = (n^3 3^n + 2)x(x+2)^n + n^3 3^n x, \tag{21}$$

has a unique solution in the interval $(0,1]$.

Proof. Define the mapping $T : (0,1] \to (0,1]$ by

$$Tx = \frac{(x+2)^n}{(n^3 3^n + 2)(x+2)^n + n^3 3^n}, \text{ for some } n \in \mathbb{N}. \tag{22}$$

Note that x is a fixed point of T if and only if x is a solution of (21). Hence, we will show that T has a unique fixed point in $(0,1]$, by using Theorem 1.
Consider the P_J metric space $P_J : (0,1]^3 \to [0,+\infty)$ that is defined in Example 1.

$$P_J(x,y,z) = \max\{x,y,z\}, \tag{23}$$

Then, $((0,1], P_J)$ is a complete P_J metric space.
Next, we show that

$$P_J(Tx,Ty,Tz) \leq Q(P_J(x,y,z)), \text{ for } x,y,z \in (0,1], \text{ and } Q(x) = \frac{n3^{n-1}x}{n^3 3^n + 2} \in (0,1).$$

First of all, note that Q is increasing on $(0,1]$ and since $\frac{n3^{n-1}}{n^3 3^n+2} < 1$ for all $n \geq 2$, we can easily deduce that for all $x \in (0,1]$ we have $Q(x) < x$. Thus, since $Q(x) \in (0,1)$, we deduce that for all $x \in (0,1]$, we have $\lim_{m \to +\infty} Q^m(x) = 0$.
Next, we have

$$\begin{aligned}
P_J(Tx,Ty,Tz) &= \max\{\frac{(x+2)^n}{(n^3 3^n + 2)(x+2)^n + n^3 3^n}, \frac{(y+2)^n}{(n^3 3^n + 2)(y+2)^n + n^3 3^n}, \\
&\quad \frac{(z+2)^n}{(n^3 3^n + 2)(z+2)^n + n^3 3^n}\} \\
&\leq \max\{\frac{(x+2)^n}{(n^3 3^n + 2)(x+2)^n}, \frac{(y+2)^n}{(n^3 3^n + 2)(y+2)^n}, \frac{(z+2)^n}{(n^3 3^n + 2)(z+2)^n}\}. \\
&= \max\{\frac{1}{n^3 3^n + 2}, \frac{1}{n^3 3^n + 2}, \frac{1}{n^3 3^n + 2}\}. \\
&\leq \max\{\frac{n3^{n-1}x}{n^3 3^n + 2}, \frac{n3^{n-1}y}{n^3 3^n + 2}, \frac{n3^{n-1}z}{n^3 3^n + 2}\}. \\
&= Q(P_J(x,y,z)).
\end{aligned}$$

Thus, all the assumptions of Theorem 1 are satisfied. So, T has a unique fixed point in $(0,1]$. Hence, Equation (21) has a unique solution in the interval $(0,1]$. □

Example 4.

$$(x+2)^3(731x-1) + 729x = 0, \tag{24}$$

has a unique solution in the interval $(0,1]$.

Proof. Note that the equation $(x+2)^3(731x-1)+729x = 0$ is equivalent to Equation (21), with $n = 3$. Hence, the result follows from Example 3 by taking $n = 3$. □

5. Application to Fractional Differential Equation

In this section, we provide an example of a fractional differential equation, which serves as an application of our new partially controlled J metric space.

$$(\mathcal{P}) : \left\{ \begin{array}{lcl} D^{\lambda}\chi(\tau) & = & g(\tau, \chi(\tau)) = F\chi(t) \text{ if } \tau \in I_0 = (0, \zeta] \\ \chi(0) & = & \chi(\zeta) = r \end{array} \right\}$$

where $\zeta > 0$ and $g : I \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions, $I = [0, \zeta]$ and $D^{\lambda}\chi$ indicate a Riemann–Liouville fractional derivative of χ with $\lambda \in (0, 1)$.

Let $C_{1-\lambda}(I, \mathbb{R}) = \{g \in C((0, \zeta], \mathbb{R}) : \tau^{1-\lambda} g \in C(I, \mathbb{R})\}$. We introduce the weighted norm

$$||g||^* = \max_{\tau \in [0,\zeta]} \tau^{1-\lambda} |g(\tau)|.$$

Theorem 4. *Let $\lambda \in (0, 1)$, $g \in C(I \times I, \mathbb{R})$ increasing and $0 < \alpha < 1$. Other than that, we make the following assumption;*

$$|g(u_1(\tau), v_1(\tau)) - g(u_2(\tau), v_2(\tau))| \leq \frac{\Gamma(2\lambda)}{T^{2\lambda-1}} \alpha |v_1 - v_2|$$

Then $(\mathcal{P})$ has a solution that is unique.

Proof. Problem $(\mathcal{P})$ is equivalent to problem $\mathcal{M}\chi(\tau) = \chi(\tau)$, where

$$\mathcal{M}\chi(t) = r\tau^{\lambda-1} + \frac{1}{\Gamma(\lambda)} \int_0^{\tau} (\tau - s)^{\lambda-1} F\chi(s) ds.$$

Indeed, demonstrating that $\mathcal{M}$ has a fixed point suffices to establish that the problem $\mathcal{P}$ has a unique solution. Let us take and assume that $\mathcal{M}\chi(\tau) = \chi(\tau)$ and by applying D^{λ} to both sides, we have $D^{\lambda}\chi(\tau) = F\chi(\tau)$. As a result, we must ensure that the hypothesis in Theorem 3 is satisfied, where $\beta = \gamma = \delta = 0$.

Let us start first with proofing that $(A = C_{1-\lambda}(I, \mathbb{R}), P_J)$ is a complete P_J metric space if we choose:

$$P_J(\chi, \Psi, \Omega) = \max_{[0,T]} \tau^{1-\lambda} \Big(|\chi(\tau) - \Psi(\tau)| + |\chi(\tau) - \Omega(\tau)| \Big), \chi, \Psi, \Omega \in C_{1-\lambda}(I, \mathbb{R}).$$

Moreover, let $f(\chi, \Psi, \Omega) = \max_{[0,T]}\{2, |\chi(\tau)|, |\Psi(\tau)|, |\Omega(\tau)|\}$, for all $\chi, \Psi, \Omega \in C_{1-\lambda}(I, \mathbb{R})$ let $\chi, \Psi, \Omega \in \Upsilon$, if $P_J(\chi, \Psi, \Omega) = 0$, then $|\chi(\tau) - \Psi(\tau)| + |\chi(\tau) - \Omega(\tau)| =$ for all $\tau \in [0, T]$ which provides us with $\chi = \Psi = \Omega$. On the other hand, let (π_n) be a convergent of the sequence, such that $\lim_{n\to+\infty} P_J(\Omega, \Omega, \pi_n)$, which implies that $\lim_{n\to+\infty} |\pi_n(\tau) - \Omega(\tau)| = 0$, we have

$$\begin{aligned}
P_J(\chi, \Psi, \Omega) &= \max_{[0,\zeta]} \tau^{1-\lambda} \Big(|\chi(\tau) - \Psi(\tau)| + |\Phi(\tau) - \Omega(\tau)| \Big) \\
&= \max_{[0,\zeta]} \tau^{1-\lambda} \Big(|\chi(\tau) - \pi_n(\tau) + \pi_n(\tau) - \Psi(\tau)| + |\chi(\tau) - \pi_n(\tau) + \pi_n(\tau) - \Omega(\tau)| \Big) \\
&\leq \max_{[0,\zeta]} \tau^{1-\lambda} \Big(2|\chi(\tau) - \pi_n(\tau)| + |\pi_n(\tau) - \Psi(\tau)| + |\pi_n(\tau) - \Omega(\tau)| \Big) \\
&\leq 2 \limsup_{n\to+\infty} \max_{[0,\zeta]} \tau^{1-\lambda} \Big(|\chi(\tau) - \pi_n(\tau)| + |\pi_n(\tau) - \Psi(\tau)| + |\pi_n(\tau) - \Omega(\tau)| \Big) \\
&\leq f(\chi, \Psi, \Omega) \limsup_{n\to+\infty} \max_{[0,\zeta]} \tau^{1-\lambda} \Big(P_J(\chi, \chi, \pi_n) + P_J(\Psi, \Psi, \pi_n) + P_J(\Omega, \Omega, \pi_n) \Big).
\end{aligned}$$

Therefore, $(A = C_{1-\lambda}(I, \mathbb{R}), P_J)$ is a P_J metric space.

Because g increases, so does the mapping $\mathcal{M}$. We need to prove that $\mathcal{M}$ is a contraction map. Let $\chi, \Psi, \Omega \in C_{1-\lambda}(P_J, \mathbb{R})$, $0 < \lambda < 1$.

$$
\begin{aligned}
P_J(\mathcal{M}\chi, \mathcal{M}\Psi, \mathcal{M}\Omega) &= \max_{[0,\zeta]} \tau^{1-\lambda}\Big(|\mathcal{M}\chi(\tau) - \mathcal{M}\Psi(\tau)| + |\mathcal{M}\chi(\tau) - \mathcal{M}\Omega(\tau)|\Big) \\
&\leq \frac{1}{\Gamma(\lambda)} \max_{\tau\in[0,T]} \tau^{1-\lambda} \int_0^\tau (\tau - s)^{\lambda-1}\Big(|g(\tau,\chi(s)) - g(\tau,\Psi(s))| \\
&+ \quad |g(\tau,\chi(s)) - g(\tau,\Omega(s))|\Big) ds.
\end{aligned}
$$

As a result of the theorem's hypothesis, we have:

$$
\begin{aligned}
P_J(\mathcal{M}\chi, \mathcal{M}\Psi, \mathcal{M}\Omega) &\leq \frac{1}{\Gamma(\lambda)} \max_{\tau\in[0,\zeta]} t^{1-\lambda} \int_0^\tau (\tau - s)^{\lambda-1} \frac{\Gamma(2\lambda)}{T^{2\lambda-1}}\Big[\alpha|\chi(s) - \Psi(s)| \\
&+ \alpha|\chi(s) - \Omega(s)|\Big] ds \\
&\leq \frac{1}{\Gamma(\lambda)} \max_{t\in[0,\zeta]} \tau^{1-\lambda} \int_0^\tau (\tau - s)^{\lambda-1} \frac{\Gamma(2\lambda)}{T^{2\lambda-1}}\Big[\alpha||\chi - \Psi||^* s^{\lambda-1} \\
&+ \alpha||\chi - \Omega||^* s^{\lambda-1}\Big] ds \\
&\leq \frac{1}{\Gamma(\lambda)} \max_{\tau\in[0,\zeta]} \tau^{1-\lambda}\alpha(||\chi - \Psi||^* + ||\chi - \Omega||^*) \frac{\Gamma(2\lambda)}{\zeta^{2\lambda-1}} \int_0^\tau (\tau - s)^{\lambda-1} s^{\lambda-1} ds.
\end{aligned}
$$

From the Riemann–Liouville fractional integral, we have

$$
\int_0^\tau (\tau - s)^{\lambda-1} s^{\lambda-1} ds = \frac{\Gamma(\lambda)}{\Gamma(2\lambda)} \tau^{2\lambda-1}.
$$

Therefore, we have

$$
P_J(\mathcal{M}\chi, \mathcal{M}\Psi, \mathcal{M}\Omega) \leq \alpha P_J(\chi, \Psi, \Omega).
$$

As a result of theorem 3, we can conclude that $\mathcal{M}$ has a unique fixed point, which brings us to the conclusion that $(\mathcal{P})$ has a unique solution, as desired. □

6. Conclusions

In this article, we introduce a new type of metric space called the $P_J$metric space, which serves as a generalization of the controlled J metric space and the J metric space. We provide examples to prove the existence of this metric space, and we prove the existence and uniqueness of the fixed points of self-mapping linear and nonlinear contraction. Moreover, we provide applications of our work to fractional differential equations.

Due to the importance of the partial metric space and its application in computer science, our plan is to cooperate with computer science researchers and concentrate on the application; for more details, see [14,15]. Finally, we would like to draw the researchers' attention to a few questions that we intend to address in upcoming research.

Question:

What will happen if $P_J(\Phi, \Phi, \Phi) \leq P_J(\Phi, \Psi, \Omega)$ is not necessarily held, which is a metric similar to this space?

Could this metric-like space be a generalization to the P_J metric and could we prove the existence and the uniqueness of the given contractions?

Author Contributions: S.S.A.: writing—original draft, methodology; W.A.M.O.: conceptualization, supervision, writing—original draft; K.B.W.: conceptualization, supervision, writing—original draft; N.M.: conceptualization, supervision, writing—original draft. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: S. S. Aiadi and N. Mlaiki thank Prince Sultan University for paying the publication fees for this work through TAS LAB. Moreover, we thank the reviewers for their valuable input, especially with Example 2, which significantly contributed to the enhancement of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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