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Invertibility and Fredholm Property of Fock Toeplitz Operators

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Abstract: We characterize some necessary and sufficient conditions of invertible Toeplitz operators acting on the Fock space. In particular, we study the Fredholm properties of Toeplitz operators with BMO^1 symbols, where their Berezin transforms are bounded functions of vanishing oscillation. We show the Fredholm index of the Toeplitz operator via the winding of its Berezin transform along a sufficiently large circle and provide a characterization of invertible Toeplitz operators with non-negative symbols, possibly unbounded, such that the Berezin transforms of the symbols are bounded and of vanishing oscillation.

Keywords: Toeplitz operator; Berezin transform; bounded mean oscillation; Fock space; invertible operator; Fredholm operator

MSC: 47B35; 30H20; 30H20; 47A53

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1. Introduction and Notations

Let dA be the Lebesgue area measure on the complex plane $\mathbb{C}$, and let $L^2(d\lambda) := L^2(\mathbb{C}, d\lambda)$ be the space of square-integrable functions with respect to the Gaussian measure

$$d\lambda(z) = \frac{1}{2\pi} e^{-\frac{|z|^2}{2}} dA(z).$$

It is easy to show that $d\lambda$ is a probability measure. Fock space F^2 denotes all functions on $\mathbb{C}$, which are also in $L^2(d\lambda)$, and F^2 is a closed subspace of $L^2(d\lambda)$. Let $L^\infty = L^\infty(\mathbb{C})$ be the space of the measurable functions f on $\mathbb{C}$ such that

$$\|f\|_\infty = esssup\{|f(z)| : z \in \mathbb{C}\} < +\infty.$$

Fock space F^2 is a Hilbert space under the inner product

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} d\lambda(z).$$

For $z \in \mathbb{C}$, $K_z(w) = K(w,z) = e^{\frac{\bar{z}w}{2}}$ is the reproducing kernel of F^2 and let $k_z = \frac{K_z}{\|K_z\|_2}$ be the normalized reproducing kernel, where $||\cdot||_2$ denotes the norm of F^2.

Let $0 < p < \infty$, and a measurable function f on $\mathbb{C}$ is said to satisfy condition (I_p) if

$$\int_{\mathbb{C}} |K(z,w)||f(w)|^p d\lambda(w) < \infty \quad (1)$$

for all $z \in \mathbb{C}$. It is easy to show that the above condition is equivalent to

$$\int_{\mathbb{C}} |K(z,w)|^2 |f(w)|^p d\lambda(w) < +\infty \quad (2)$$

for all $z \in \mathbb{C}$. Let P denote the integral operator given by

$$(Pf)(z) = \int_{\mathbb{C}} f(w)K(z,w)d\lambda(w), \tag{3}$$

where f satisfies condition (I_1). Note that P is the orthogonal projection from $L^2(d\lambda)$ onto F^2. If f satisfies condition (I_1) and M_f is the multiplication operator by f, let the Toeplitz operator $T_f = PM_fP$ on $L^2(d\lambda)$ and the Hankel operator $H_f = (I-P)M_fP$ on $L^2(d\lambda)$. The Toeplitz operator T_f is well-defined on a dense subset of F^2 when f satisfies condition (I_2). In fact, if

$$g(w) = \sum_{k=1}^{N} c_k K(w, w_k)$$

is any finite linear combination of the kernel functions in F^2, using condition (I_2) and the Cauchy–Schwarz inequality implies that $T_f(g)$ is well-defined. It is easy to check that the set of all finite linear combinations of kernel functions is dense in F^2. The operator T_f is bounded from $L^2(d\lambda)$ to F^2 for a special f that satisfies condition I_1 (see [1]). In particular, T_f is bounded whenever $f \in L^\infty$.

Each operator S on F^2 induces a function $\widetilde{S}$ on $\mathbb{C}$, namely

$$\widetilde{S}(z) = \langle Sk_z, k_z \rangle, \quad z \in \mathbb{C}.$$

We call $\widetilde{S}$ the Berezin transform of S. If f satisfies condition (I_p), the Berezin transform of f is defined by

$$\widetilde{f}(z) = \int_{\mathbb{C}} f(w)|k_z(w)|^2 d\lambda(w).$$

We have $\widetilde{f} = \widetilde{T_f}$ by the definition of T_f when f satisfies condition (I_2).

For the Fock space case, Wang and Zhao showed that the bounded Toeplitz operator T_f is invertible, implying that $\widetilde{|f|}$ is bounded below $\mathbb{C}$ (see [2], Theorem 4.2). The invertibility of Toeplitz operators T_f is completely determined by $\widetilde{f}$ and is bounded below when f is bounded and non-negative (see [2], Theorem 1.1). Furthermore, let $0 < f \in L^\infty$ be a Lipschitz function that satisfies the Lipschitz constant $L > 0$ if $\inf_{z\in\mathbb{C}} |f(z)| > 4L$, then T_f is invertible (see [2], Theorem 4.3). In the setting of Bergman space, more information about the invertibility of Toeplitz operators can be found in [3]. We focus on the invertibility and Fredholm properties of the Toeplitz operator, which are quite different from the approach and content in the literature [4].

The organization of the paper is as follows. Section 2 provides some sufficient and necessary conditions for the invertibility of the Toeplitz operator. In Section 3, we characterize the Fredholm Toeplitz operators with BMO^1 symbols and the invertible Toeplitz operators with non-negative symbols, which satisfy certain conditions. Additionally, we show the Fredholm index of the Toeplitz operator.

2. Invertibility of Toeplitz Operators with BMO^1 Symbols

In recent years, many scholars have studied Toeplitz operators with either a bounded symbol or a symbol in BMO^1. For example, the boundedness and the compactness of Toeplitz operators are determined by the behavior of their Berezin transforms (see [5–7]). The characterization of the compactness of Toeplitz operators with bounded symbols on the Bergman space in terms of the Berezin transform was first obtained by Axler and Zheng [8] and later generalized to BMO^1 symbols by Zorborska [9]. In this section, we demonstrate some necessary and sufficient conditions for Toeplitz operators with BMO^1 symbols to be invertible. We first provide the concept of BMO^p.

Let $p \geq 1$, and we define the bounded mean oscillation $f \in BMO^p$ when f satisfies condition (I_p) and

$$||f||_{BMO^p} = \sup_{z\in\mathbb{C}} ||f \circ \tau_z - \widetilde{f}(z)||_{L^p(d\lambda)} < +\infty,$$

where $\tau_z(w) = z - w$. For $z \in \mathbb{C}$, denote $MO_{p,f}(z) = ||f \circ \tau_z - \widetilde{f}(z)||_{L^p(d\lambda)}$. Similarly, we denote the vanishing mean oscillation VMO^p when $f \in BMO^p$ and

$$\lim_{z\to\infty} ||f \circ \tau_z - \widetilde{f}(z)||_{L^p(d\lambda)} = 0.$$

It is easy to see that

$$L^\infty \subset BMO^p \subset L^p(d\lambda), \ \ p \geq 1.$$

$$BMO^q \subset BMO^p \subset BMO^1, \ \ 1 \leq p < q,$$

and each of these inclusions is strict.

By using Theorem 8.4 and Proposition 8.3 in [10], if $f \in BMO^2$, the Hankel operators H_f and $H_{\overline{f}}$ are bounded. That is, there exists a positive constant M independent of f such that $||H_{\overline{f}}|| \leq M||f||_{BMO^2}$ and $||H_f|| \leq M||f||_{BMO^2}$. Furthermore, for any given $z \in \mathbb{C}$,

$$MO_{2,f}(z) \leq ||H_f k_z||_{L^2(d\lambda)} + ||H_{\overline{f}} k_z||_{L^2(d\lambda)} \leq ||H_f|| + ||H_{\overline{f}}||.$$

Hence,

$$\max\{||H_f||, ||H_{\overline{f}}||\} \geq \frac{1}{2}\Big(||H_f|| + ||H_{\overline{f}}||\Big) \geq \frac{1}{2}||f||_{BMO^2}, \tag{4}$$

and

$$\max\{\widetilde{H_f^* H_f}(z), \widetilde{H_{\overline{f}}^* H_{\overline{f}}}(z)\} \geq \frac{1}{4}(MO_{2,f}(z))^2. \tag{5}$$

For more information on BMO^p, refer to [6,10].

From the results of [5], T_f is bounded on F^2 when $f \in L^\infty$. Furthermore, T_f is compact if and only if its Berezin transform vanishes at infinity. Other scenarios of unbounded symbols are also considered such as BMO^p symbols for some $p \geq 1$.

Theorem 1 ([6])**.** *Let $f \in BMO^1$. Then, T_f is bounded on F^2 if and only if $\widetilde{f}$ is bounded, and T_f is compact if and if only if $\widetilde{f}$ vanishes at infinity.*

Fock space is an analytic function space defined on a complex plane using a Gaussian measure. This is completely different from the definition of Bergman space, although some conclusions and methods in this paper are similar to Bergman space. Next, we provide some necessary and sufficient conditions for Toeplitz operators with BMO^1 symbols to be invertible. We first introduce a lemma, the proof of which can be found in [11].

Lemma 1. *For bounded operator T on Fock space F^2:*

(a) There exists a constant $c > 0$ such that $||Th||_2 \geq c||h||_2$ for all $h \in F^2$. Then, $\widetilde{T^*T}(z) \geq c^2$ for any $z \in \mathbb{C}$;

(b) If operator T is invertible and positive, we obtain $\widetilde{T}(z) \geq ||T^{-1}||^{-1}$ for any $z \in \mathbb{C}$.

Lemma 2. *Let $f \in BMO^q$, $q > p \geq 1$, with $\widetilde{f}$ bounded on $\mathbb{C}$, and let $M_q = \sup_{z\in\mathbb{C}} \widetilde{|f|^q}(z)$. Then $\widetilde{|f|}$ is bounded below if the Berezin transform $\widetilde{|f|^p}$ is bounded below.*

Proof. Since $\widetilde{f}$ is bounded by the definition of BMO^q, $M_q < \infty$. By using Hölder's inequality, we have

$$\begin{aligned}\left(|\widetilde{f}|(z)\right)^p &= \left(\int_{\mathbb{C}} |f(w)||k_z(w)|^2 d\lambda(w)\right)^p \\ &\leq \int_{\mathbb{C}} |f(w)|^p |k_z(w)|^2 d\lambda(w) \\ &= \int_{\mathbb{C}} |f(w)|^{\frac{q-p}{q-1}+\frac{q(p-1)}{q-1}} |k_z(w)|^2 d\lambda(w) \\ &\leq \left(\int_{\mathbb{C}} |f(w)||k_z(w)|^2 d\lambda(w)\right)^{\frac{q-p}{q-1}} \left(\int_{\mathbb{C}} |f(w)|^q |k_z(w)|^2 d\lambda(w)\right)^{\frac{p-1}{q-1}} \\ &\leq (M_q)^{\frac{p-1}{q-1}} \left(|\widetilde{f}|(z)\right)^{\frac{q-p}{q-1}}.\end{aligned}$$

That is,

$$\left(|\widetilde{f}|(z)\right)^p \leq \widetilde{|f|^p}(z) \leq (M_q)^{\frac{p-1}{q-1}} \left(|\widetilde{f}|(z)\right)^{\frac{q-p}{q-1}}. \tag{6}$$

□

The Toeplitz and Hankel operators are closely related. We provide necessary and sufficient conditions for Toeplitz operators with unbounded symbols to be invertible.

Theorem 2. *Let $f \in BMO^q$ for some $q > 2$. Let T_f be a bounded operator on F^2 and $M_q = \sup_{z\in\mathbb{C}} \widetilde{|f|^q}(z)$. Then:*

(a) If T_f is invertible on F^2,

$$|\widetilde{f}|(z) \geq \frac{1}{(M_q)^{\frac{1}{q-2}}} \left(||T_f^{-1}||^{-2} + \frac{1}{4}(MO_{2,f}(z))^2\right)^{\frac{q-1}{q-2}}, \quad z \in \mathbb{C}.$$

(b) If $|\widetilde{f}|$ is bounded below on $\mathbb{C}$ and $T_{|f|^2}$ is invertible with $||T_{|f|^2}^{-1}||^{-1} > \max\{||H_f||^2, ||H_{\overline{f}}||^2\}$, then T_f is invertible. Moreover,

$$|\widetilde{f}|(z) \geq \frac{1}{2^{\frac{q-1}{q-2}}(M_q)^{\frac{1}{q-2}}} \left(||T_{|f|^2}^{-1}||^{-1} + \frac{1}{4}||f||^2_{BMO^2}\right)^{\frac{q-1}{q-2}}, \quad z \in \mathbb{C}. \tag{7}$$

Proof. Since T_f is bounded, $\widetilde{f}$ is bounded, and since $f \in BMO^q$, M^q is finite. Let $p = 2$ in (6). Then, we have

$$\left(|\widetilde{f}|(z)\right)^2 \leq \widetilde{|f|^2}(z) \leq (M_q)^{\frac{1}{q-1}} \left(|\widetilde{f}|(z)\right)^{\frac{q-2}{q-1}}, \tag{8}$$

and we also have that $|\widetilde{f}|$ and $\widetilde{|f|^2}$ are both bounded. From Theorem 1, we have that $T_{|f|}$ and $T_{|f|^2}$ are both bounded operators.

(a) Since T_f is invertible, $T_f^* T_f$ and $T_f T_f^*$ are positive invertible operators, and by using Lemma 1 (b), we know that

$$||T_f^{-1}||^{-2} = ||(T_f^{-1})^*||^{-2} \leq \min\{\widetilde{T_f^* T_f}(z), \widetilde{T_f T_f^*}(z)\}, \ \forall z \in \mathbb{C}.$$

Since

$$T_f^* T_f = T_{|f|^2} - H_f^* H_f \ \textit{and} \ T_f T_f^* = T_{|f|^2} - H_{\overline{f}}^* H_{\overline{f}},$$

$H_f^* H_f$ and $H_{\overline{f}}^* H_{\overline{f}}$ are positive operators. This means that their Berezin transforms are non-negative. Hence,

$$\widetilde{|f|^2}(z) = \widetilde{T_{|f|^2}}(z) \geq \max\{\widetilde{H_f^* H_f}(z), \widetilde{H_{\overline{f}}^* H_{\overline{f}}}(z)\} + ||T_f^{-1}||^{-2}.$$

By using (8) and (5), this completes the proof.

(b) Since $T_{|f|^2}$ is invertible and $\widetilde{|f|^2}$ is bounded below, we can utilize

$$T_f^* T_f = T_{|f|^2} - H_f^* H_f \ \ and \ \ T_f T_f^* = T_{|f|^2} - H_{\overline{f}}^* H_{\overline{f}}.$$

By multiplying both sides of the above equation by $T_{|f|^2}^{-1}$, we obtain

$$T_{|f|^2}^{-1} H_f^* H_f = I - T_{|f|^2}^{-1} T_f^* T_f \ \ and \ \ T_{|f|^2}^{-1} H_{\overline{f}}^* H_{\overline{f}} = I - T_{|f|^2}^{-1} T_f T_f^*.$$

Since $||H_f^* H_f|| = ||H_f||^2$ and $||H_{\overline{f}}^* H_{\overline{f}}|| = ||H_{\overline{f}}||^2$, under the assumption that $||T_{|f|^2}^{-1}||^{-1} > \max\{||H_f||^2, ||H_{\overline{f}}||^2\}$, we have

$$||I - T_{|f|^2}^{-1} T_f^* T_f|| < 1 \ \ and \ \ ||I - T_{|f|^2}^{-1} T_f T_f^*|| < 1.$$

Then, $T_{|f|^2}^{-1} T_f^* T_f$ and $T_{|f|^2}^{-1} T_f T_f^*$ are invertible and $T_f^* T_f$ and $T_f T_f^*$ are also invertible. This implies that T_f and T_f^* are both invertible.

Since $T_{|f|^2}$ is positively invertible, again, by using Lemma 1 (b), we have $\widetilde{|f|^2}(z) \geq ||T_{|f|^2}^{-1}||^{-1}$ for all $z \in \mathbb{C}$. By using (4); the assumption condition, $||T_{|f|^2}^{-1}||^{-1} > \frac{1}{4}||f||_{BMO^2}^2$; and (8), it is easy to show that (7) holds. □

By using Theorem 1.1 in [2] and a proof similar to that in Theorem 2, we can obtain the following corollary.

Corollary 1. *Let $f \in L^\infty$. Then:*

(a) *If T_f is invertible on F^2,*

$$|\widetilde{f}|^2(z) \geq \max\{\widetilde{H_f^* H_f}(z), \widetilde{H_{\overline{f}}^* H_{\overline{f}}}(z)\} + ||T_f^{-1}||^{-2}, \quad z \in \mathbb{C}.$$

Furthermore, we obtain

$$\widetilde{|f|}(z) > ||f||_\infty^{-1} \Big(||T_f^{-1}||^{-2} + \max\{\widetilde{H_f^* H_f}(z), \widetilde{H_{\overline{f}}^* H_{\overline{f}}}(z)\} \Big)$$

for all $z \in \mathbb{C}$.

(b) *If $|\widetilde{f}|$ is bounded below, $T_{|f|^2}$ is invertible on F^2, and if $||T_{|f|^2}^{-1}||^{-1} > \max\{||H_f||^2, ||H_{\overline{f}}||^2\}$, T_f is invertible on F^2.*

Zorboska ([11], Theorem 3.6) provided a sufficient condition for Toeplitz operators with symbols BMO^2 to be invertible. For Fock space, we provide a similar result, and as its proof is similar to [11], we omit it.

Theorem 3. *Let $f \in BMO^2$. If T_f is a bounded operator on F^2, $\widetilde{|f|^2}$ is bounded on $\mathbb{C}$. Furthermore, if there exist constants $\varepsilon > 0$, $\delta > 0$, and $C_2 > 0$ satisfying $1 > \delta > \frac{\sqrt{C_2^2 - 1}}{C_2^2}$ and $1 > \varepsilon > C_2\sqrt{1-\delta^2}$ for all $z \in \mathbb{C}$ such that*

$$|f(z)| \geq \varepsilon \Big(||\widetilde{|f|^2}||_\infty \Big)^{\frac{1}{2}} \ \ and \ \ |\widetilde{f}(z)| \geq \varepsilon \Big(||\widetilde{|f|^2}||_\infty \Big)^{\frac{1}{2}},$$

and we have that T_f *is invertible and* $||T_f^{-1}|| \leq \left((\varepsilon^2 - C_2^2(1-\delta^2)) || \widetilde{|f|^2} ||_\infty \right)^{-\frac{1}{2}}$.

3. Fredholmness of Toeplitz Operators with BMO^1 Symbols

In this section, we consider Fredholm Toeplitz operators with unbounded symbols. On F^2, Berger and Coburn [12] studied Fredholm Toeplitz operators with bounded symbols of vanishing oscillation at infinity. In the setting of weighted Fock spaces F^p_α $(p > 1)$, Stroethoff [13] provided a more elementary proof of Berger and Coburn's result. Al-Qabani and Virtanen [14] considered the essential spectra of Toeplitz operators when symbols satisfy certain conditions. They generalized the results of Berger and Coburn [12] and Stroethoff [13] on the essential spectra of Toeplitz operators. That is, they showed that for $f \in L^\infty \bigcap VMO^1$,

$$\sigma_{ess}(T_f) = \bigcap_{R>0} cl\widetilde{f}(\mathbb{C} \backslash R\mathbb{D}),$$

where clX is the closure of the set X and $R\mathbb{D} = \{Rz : z \in \mathbb{D},\ R \in (0,\infty)\}$. In ([14], Theorem 13), Al-Qabani and Virtanen completely characterized the Fredholm properties of a Toeplitz operator T_f on $F^p_\alpha(1 < p < \infty)$. Later, Hu and Virtanen [15] studied the Fredholm–Toeplitz operators on generalized Fock spaces of the n-dimensional complex space, considering symbols that satisfy certain conditions. In the context of Bergman space, Zorboska [11] considered Fredholm–Toeplitz operators with the symbol $f \in BMO^1$ when $\widetilde{f}$ is a bounded symbol of vanishing oscillation. We use this result to characterize the Fredholm–Toeplitz operators with symbols in BMO^1. For more information on the Fredholm operators, refer to [14].

A function $f : \mathbb{C} \to \mathbb{C}$ is called bounded away from zero when there exist numbers $R > 0$ and $\delta > 0$ such that for $|z| \geq R$,

$$|f(z)| \geq \delta.$$

Recall that bounded oscillation on a complex plane is defined by

$$BO = \{f \in C(\mathbb{C}) : ||f||_{BO} := \sup_{z\in\mathbb{C}} \sup_{w\in B(z,1)} |f(z) - f(w)| < +\infty\}.$$

We say that f has vanishing oscillation VO when $f \in BO$ and

$$\lim_{z\to\infty} \sup_{w\in B(z,1)} |f(z) - f(w)| = 0.$$

It is well-known from [6] that $\widetilde{f}$ is Lipschitz whenever $f \in BMO^1$ and $\widetilde{f}$ is bounded. Before we can state the theorem, we introduce the following lemma.

Lemma 3 ([16])**.** *Suppose that f is a continuous function on $\mathbb{C}$. Then, the following statements are equivalent:*

(a) $f \in VO$;

(b) $\lim_{z\to\infty} ||f \circ \tau_z - \widetilde{f}(z)||_{L^2(d\lambda)} = 0$;

(c) $f \in VMO^2$ *and* $f - \widetilde{f} \in C_0(\mathbb{C})$, *where* $C_0(\mathbb{C})$ *is the space of complex continuous functions vanishing at infinity.*

We are now ready to characterize the Fredholm–Toeplitz operators with symbols in BMO^1. Fredholm–Toeplitz operators with unbounded symbols on Bergman space can be found in [17]. Let the Toeplitz algebra $\mathfrak{T}$ be the norm closure of the algebra generated by Toeplitz operators with bounded symbols acting on F^2.

Theorem 4 ([5], Theorem 1.1)**.** *Let A be a bounded operator on F^2. Then, A is compact if and if only if $A \in \mathfrak{T}$ and $\widetilde{A}$ vanishes at infinity.*

Theorem 5. *Let $f \in BMO^1$, and let $\widetilde{f} \in L^\infty \cap VO$. Then:*
(a) T_f *is a Fredholm operator if the Berezin transform $\widetilde{f}$ is bounded away from zero;*
(b) $T_f \in \mathfrak{T}$.

Proof. (a) Since $f \in BMO^1$ and $\widetilde{f}$ is bounded, then according to Theorem 1, T_f is bounded. We write

$$T_f = T_{\widetilde{f}} + T_{f-\widetilde{f}},$$

and since $T_{\widetilde{f}}$ is bounded, we have that $T_{f-\widetilde{f}}$ is also bounded. We also have that $f - \widetilde{f} \in BMO^1$ and $\widetilde{f} - \widetilde{\widetilde{f}}$ is bounded. According to Theorem 4, $T_{f-\widetilde{f}}$ is compact if $\widetilde{f}(z) - \widetilde{\widetilde{f}}(z) \to 0$ when $z \to \infty$. Since $\widetilde{f}$ is a Lipschitz function, by using Theorem 3, we immediately obtain $\widetilde{f} \in VO$ if and only if $\widetilde{f} \in VMO^2$ and $\widetilde{f} - \widetilde{\widetilde{f}} \in C_0(\mathbb{C})$. Hence, $T_{f-\widetilde{f}}$ is compact when $\widetilde{f} \in VO$, and we have that T_f is a Fredholm operator if $T_{\widetilde{f}}$ is a Fredholm operator. By using Theorem 13 in [14], we have that $T_{\widetilde{f}}$ is a Fredholm operator if the Berezin transform $\widetilde{f}$ is bounded away from zero.

(b) By using the proof of (a), $T_{\widetilde{f}}$ and $T_{f-\widetilde{f}}$ are compact. By using Theorem 4, we have $T_f \in \mathfrak{T}$. □

Next, we introduce the index formula of the Toeplitz operators with symbols BMO^1.

Theorem 6. *Let $f \in BMO^1$ and $\widetilde{f} \in L^\infty \cap VO$. If the Berezin transform $\widetilde{f}$ is bounded away from zero,*

$$ind\ T_f = -wind(\widetilde{f}|_{|z|=R}),$$

where $R > 0$ is chosen such that $|\widetilde{f}(z)| > m > 0$ for $|z| \geq R$, and $wind\ \widetilde{f}|_{|z|=R}$ is the winding of the curve $\widetilde{f}$ around the origin.

Proof. Since $f \in BMO^1$ and $\widetilde{f} \in L^\infty \cap VO$, if $\widetilde{f}$ is bounded away from zero, by using Theorem 5, T_f is a Fredholm operator on F^2. Moreover, since $\widetilde{f} \in L^\infty \cap VO$, if $\widetilde{f}$ is bounded away from zero, by using Theorem 20 in [14],

$$ind\ T_{\widetilde{f}} = -wind(\widetilde{f}|_{|z|=R}).$$

From Theorem 5, we have that $T_{f-\widetilde{f}}$ is compact. Hence,

$$ind\ T_f = ind\ T_{\widetilde{f}} = -wind(\widetilde{f}|_{|z|=R}).$$

□

Finally, by using the previous theorem, we can obtain a description of invertible Toeplitz operators with non-negative symbols, possibly unbounded, such that the Berezin transform of symbols is bounded and of vanishing oscillation.

Corollary 2. *Let $0 < f \in L^1(\mathbb{C}, d\lambda)$ and $\widetilde{f} \in L^\infty \cap VO$. Then, T_f is invertible if the Berezin transform $\widetilde{f}$ is bounded away from zero.*

Proof. Since $f > 0$ and $f \in L^\infty$, then $\widetilde{f} > 0$ and $f \in BMO^1$, and T_f is a positive bounded operator on F^2.

If T_f is invertible, by using Theorem 5, we have that the Berezin transform $\widetilde{f}$ is bounded away from zero.

Conversely, we assume that $\widetilde{f}$ is bounded away from zero. Next, we show that $\text{Ker}T_f = \{0\}$. Let $E \subset \mathbb{C}$ be a family of positive Lebesgue measures such that $f(z) > 0$. If $g \in F^2$ is in $\text{Ker}T_f$, we have

$$0 = \langle T_f g, g \rangle = \int_{\mathbb{C}} f(w)|g(w)|^2 d\lambda(w) \geq \int_E f(w)|g(w)|^2 d\lambda(w) \geq 0,$$

and so $g = 0$ on some family $E_1 \subset E$ of positive Lebesgue measures. Since g is analytic on $\mathbb{C}$, this means that $g = 0$. Since $\widetilde{f} \in L^\infty \cap VO$, T_f must be a Fredholm operator according to Theorem 5. We know that T_f is also a self-adjoint operator with a trivial kernel so T_f is invertible. □

4. Conclusions

In this paper, we study some necessary and sufficient conditions of invertible Toeplitz operators with unbounded symbols acting on Fock space. Unfortunately, we do not provide sufficient necessary conditions for the invertibility of the Toeplitz operator with unbounded symbols. Our next goal is to explore sufficient and necessary conditions for the invertibility of the Toeplitz operator with unbounded symbols.

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