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Article

On Meromorphic Parabolic Starlike Functions with Fixed Point Involving the q-Hypergeometric Function and Fixed Second Coefficients

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Abstract: This article defines a new class of meromorphic parabolic starlike functions in the punctured unit disc $D^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ that includes fixed second coefficients of class $\mathcal{A}^d_{s,c}(\psi, \tau, \nu, \eta)$ and the q- hypergeometric functions. For the function belonging to the class $\mathcal{A}^d_{s,c}(\psi, \tau, \nu, \eta)$, some properties are obtained, including the coefficient inequalities, closure theorems, and the radius of convexity.

Keywords: meromorphic parabolic functions; univalent; starlike function; q-hypergeometric function

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1. Introduction

Let η be a fixed point in the unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$. Using $\mathcal{H}(D)$, denote the class of functions that are regular and

$$\mathcal{A}(\eta) = \{f \in \mathcal{H}(D) : f(\eta) = f'(\eta) - 1 = 0\}$$

Using $\mathcal{S}_\eta = \{f \in \mathcal{A}(\eta), denotethefollowing : f$ is univalent in D $\}$, the subclass of $A(\eta)$ consisting of the functions of the form

$$\zeta(z) = z - \eta + \sum_{l=1}^{\infty} a_l (z-\eta)^l.$$

Let $\varkappa$ denote the class of meromorphic functions $\zeta(z)$ of the form

$$\zeta(z) = \frac{1}{z} + \sum_{l=1}^{\infty} a_l z^l \tag{1}$$

defined on the punctured unit disc $D^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Using $\varkappa_\eta$, denote the subclass of $\varkappa$ consisting of the form's functions

$$\zeta(z) = \frac{1}{z-\eta} + \sum_{l=1}^{\infty} a_l (z-\eta)^l, \quad a_l \geq 0; z \neq \eta, \quad z \in D. \tag{2}$$

If a function $\zeta(z)$ of the form (2) belongs to the class of meromorphic starlike of order σ $(0 \geq \sigma < 1)$, it is indicated by $\varkappa^*_\eta(\sigma)$, if

$$-\Re\left(\frac{(z-\eta)\zeta'(z)}{\zeta(z)}\right) > \sigma, \quad z \neq \eta, \quad z \in D, \tag{3}$$

and belongs to a class of meromorphic convex of order $\sigma(0 \leq \sigma < 1)$, which is indicated by $\varkappa_{\eta}^{k}(\sigma)$, if

$$-\Re\left(1 + \frac{(z-\eta)\zeta''(z)}{\zeta'(z)}\right) > \sigma, \quad z \neq \eta, \quad z \in D. \tag{4}$$

For functions $\zeta(z)$, given by (2) and $g(z) = \frac{1}{z-\eta} + \sum_{l=1}^{\infty} b_l (z-\eta)^l, (b_l \geq 0)$, we define the Hadamard product or convolution of $\zeta(z)$ and $g(z)$ by

$$(\zeta * g)(z) = \frac{1}{z-\eta} + \sum_{l=1}^{\infty} a_l b_l (z-\eta)^l = (g * \zeta)(z). \tag{5}$$

Define the following operator [1].

$$q_{\mu,\xi}(z) = \frac{1}{z-\eta} + \sum_{l=1}^{\infty} \left(\frac{\mu}{l+1+\mu}\right)^{\xi} (z-\eta)^l, \quad (\mu > 0, \quad \xi \geq 0). \tag{6}$$

Cho [2], Ghanim, and Darus [3] studied the above function when $\eta = 0$.

Corresponding to the function $q_{\mu,\xi}(z)$ and using the Hadamard product for $\zeta(z) \in \varkappa_{\eta}$, we define a new linear operator $\mathcal{J}(\mu, \xi, \eta)$ on $\varkappa_{\eta}(\sigma)$ by

$$\mathcal{J}_{\mu,\xi}\zeta(z) = (\zeta(z) * q_{\mu,\xi}(z)) = \frac{1}{z-\eta} + \sum_{l=1}^{\infty} \left(\frac{\mu}{l+1+\mu}\right)^{\xi} |a_l| (z-\eta)^l. \tag{7}$$

When $\eta = 0$, it reduces to Ghanim and Darus [4].

A generalized q-Taylars formula for fractional q-calculus was introduced more recently by Purohit and Raina [5], who also derived a few q-generating functions for q-hypergeometric functions.

As with the aforementioned functions, we attempt to derive a generalized differential operator on meromorphic functions in $D^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ in this paper and study some of their characteristics.

For complex parameters $\gamma_1, \ldots, \gamma_d$ and $\beta_1, \ldots, \beta_s (\beta_t \neq 0, -1, \ldots; t = 1, 2, \ldots, s)$ the q-hypergeometric function ${}_d\Phi_s(z)$ is defined by

$${}_d\Phi_s(\gamma_1, \ldots, \gamma_d; \beta_1, \ldots, \beta_s; q, z) = \sum_{l=0}^{\infty} \frac{(\gamma_1, q)_l \ldots (\gamma_d, q)_l}{(q,q)_l (\beta_1, q)_l \ldots (\beta_s, q)_l} \times [(-1)^l q^{\binom{l}{2}}]^{1+s-d} z^l, \tag{8}$$

with $\binom{l}{2} = l(l-1)/2$ where $q \neq 0$ when $d > s+1 (d, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in D^*)$. The q-shifted factorial is defined for $\gamma, q \in \mathbb{C}$ as a product of l factors by

$$(\gamma; q)_l = \begin{cases} (1-\gamma)(1-\gamma q) \ldots \left(1 - \gamma q^{l-1}\right) & (l \in \mathbb{N}) \\ 1 & (l = 0) \end{cases} \tag{9}$$

and in terms of basic analogue of the gamma function

$$(q^{\gamma}, q)_l = \frac{\Gamma_q(\gamma + l)(1-q)^l}{\Gamma_q(\gamma)} \qquad l > 0. \tag{10}$$

It is important to note that $\lim_{q \to -1}\left((q^{\gamma}; q)_l / (1-q)^l\right) = (\gamma)_l = \gamma(\gamma+1)\ldots(\gamma+l-1)$ is the familiar Pochhammer symbol and

$${}_d\Phi_s(\gamma_1, \ldots, \gamma_d; \beta_1, \ldots, \beta_s; z) = \sum_{l=0}^{\infty} \frac{(\gamma_1)_l \ldots (\gamma_d)_l}{(\beta_1)_l \ldots (\beta_s)_l} \frac{z^l}{l!}. \tag{11}$$

Now, for $z \in D, 0 < |q| < 1$, and $d = s + 1$, the basic hypergeometric function defined in (8) takes the form

$$ {}_d\Phi_s(\gamma_1, \ldots, \gamma_d; \beta_1, \ldots, \beta_s, ; q, z) = \sum_{l=0}^{\infty} \frac{(\gamma_1, q)_l \ldots (\gamma_d, q)_l}{(q, q)_l (\beta_1, q)_l \ldots (\beta_s, q)_l} z^l, \quad (12) $$

which converges absolutely in the open disc D.

According to the recently introduced function ${}_d\Phi_s(\gamma_1, \ldots, \gamma_d; \beta_1, \ldots, \beta_s; q, z)$ for meromorphic functions $\zeta \in \varkappa$ consisting of functions of the form (1), Al-dweby and Darus [6] developed the q-analogue of the Liu-Srivastava operator, as follows:

$$ \begin{aligned} {}_dY_s(\gamma_1, \ldots, \gamma_d; \beta_1, \ldots, \beta_s, ; q, z) * \zeta(z) &= \frac{1}{z}\, {}_d\Phi_s(\gamma_1, \ldots, \gamma_d; \beta_1, \ldots, \beta_s, ; q, z) * \zeta(z) \\ &= \frac{1}{z} + \sum_{l=1}^{\infty} \frac{\prod_{i=1}^{d} (\gamma_i, q)_{l+1}}{(q, q)_{l+1} \prod_{i=1}^{s} (\beta_i, q)_{l+1}} a_l z^l, \end{aligned} \quad (13) $$

where $\prod_{n=1}^{m} (\gamma_n, q)_{l+1} = (\gamma_1, q)_{l+1} (\gamma_2, q)_{l+1} \ldots (\gamma_m, q)_{l+1}$, where $z \in D^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$, and

$$ \begin{aligned} {}_dY_s(\gamma_1, \ldots, \gamma_d; \beta_1, \ldots, \beta_s, ; q, z) &= \frac{1}{z}\, {}_d\Phi_s(\gamma_1, \ldots, \gamma_d; \beta_1, \ldots, \beta_s, ; q, z) \\ &= \frac{1}{z} + \sum_{l=1}^{\infty} \frac{\prod_{i=1}^{d} (\gamma_i, q)_{l+1}}{(q, q)_{l+1} \prod_{i=1}^{s} (\beta_i, q)_{l+1}} z^l. \end{aligned} \quad (14) $$

Murugusundaramoorthy and Janani [7] defined the following linear operator for functions $\zeta \in \varkappa_\eta$ and for real parameters $\gamma_1, \ldots, \gamma_d$ and $\beta_1, \ldots, \beta_s (\beta_t \neq 0, -1, \ldots; t = 1, 2, \ldots, s)$:

$$ {}_dY_s(\gamma_1, \ldots, \gamma_d; \beta_1, \ldots, \beta_s, ; q, z - \eta) : \varkappa_\eta \to \varkappa_\eta, \quad (15) $$

$$ \begin{aligned} {}_dY_s(\gamma_1, \ldots, \gamma_d; \beta_1, \ldots, \beta_s, ; q, z - \eta) &= \frac{1}{z - \eta}\, {}_d\Phi_s(\gamma_1, \ldots, \gamma_d; \beta_1, \ldots, \beta_s, ; q, z - \eta) \\ &= \frac{1}{z - \eta} + \sum_{l=1}^{\infty} \frac{\prod_{i=1}^{d} (\gamma_i, q)_{l+1}}{(q, q)_{l+1} \prod_{i=1}^{s} (\beta_i, q)_{l+1}} (z - \eta)^l. \end{aligned} \quad (16) $$

Corresponding to the functions ${}_dY_s(\gamma_1, \ldots, \gamma_d; \beta_1, \ldots, \beta_s, ; q, z - \eta)$ and $q_{\mu,\xi}(z)$ given in (6) and using the Hadamard product for $\zeta(z) \in \varkappa_\eta$, we define a new linear operator $\mathcal{J}_\xi^\mu(\gamma_1, \gamma_2, \ldots \gamma_d; \beta_1, \beta_2, \ldots \beta_s; q)$ on $\varkappa_\eta$ by

$$ \mathcal{J}_\xi^\mu(\gamma_1, \gamma_2, \ldots \gamma_d; \beta_1, \beta_2, \ldots \beta_s; q) \zeta(z) $$

$$ = \left(\zeta(z) *_d Y_s(\gamma_1, \gamma_2, \ldots \gamma_d; \beta_1, \beta_2, \ldots \beta_s; q, z - \eta) * q_{\lambda,\xi}(z)\right) \quad (17) $$

$$ = \frac{1}{z - \eta} + \sum_{l=1}^{\infty} \frac{\prod_{i=1}^{d} (\gamma_i, q)_{l+1}}{(q, q)_{l+1} \prod_{i=1}^{s} (\beta_i, q)_{l+1}} \left(\frac{\mu}{l + 1 + \mu}\right)^\xi a_l (z - \eta)^l. $$

$$ = \frac{1}{z - \eta} + \sum_{l=1}^{\infty} \Omega_s^d(l) a_l (z - \eta)^l, \quad (18) $$

where

$$ \Omega_s^d(l) = \frac{\prod_{i=1}^{d} (\gamma_i, q)_{l+1}}{(q, q)_{l+1} \prod_{i=1}^{s} (\beta_i, q)_{l+1}} \left(\frac{\mu}{l + 1 + \mu}\right)^\xi. \quad (19) $$

For convenience, we will denote

$$\mathcal{J}_{\xi}^{\mu}(\gamma_1,\gamma_2,\ldots\gamma_d;\beta_1,\beta_2,\ldots\beta_s;q)\zeta(z)=\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z). \tag{20}$$

In (17), for $\xi=0$, the operator was investigated by Murugusundaramoorthy and Janani [7].

Recent studies on the meromorphic functions with generalized hypergeometric functions and with q-hypergeometric functions include those by Cho and Kim [8], Dziok and Srivastava [9,10], Ghanim [11], Ghanim et al. [12,13], Liu and Srivastava [14,15], Aldweby and Darus [6], Murugusundaramoorthy and Janani [7]. We define the following new subclass of functions in $\varkappa_\eta$ using the generalized operator $\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z)$. In response to earlier work on meromorphic functions by function theorists (see [15–22]).

For $0\le\nu<1$ and $0\le\psi\le1$, we let $\mathcal{A}_s^d(\psi,\tau,\nu,\eta)$ indicate a subclass of $\varkappa_\eta$ that consists of functions of the form (2) that satisfy the requirement that

$$-\Re\left(\frac{(z-\eta)(\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z))'+\psi(z-\eta)^2(\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z))''}{(1-\psi)\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z)+\psi(z-\eta)(\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z))'}\right)$$

$$>\tau\left|\frac{(z-\eta)(\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z))'+\psi(z-\eta)^2(\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z))''}{(1-\psi)\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z)+\psi(z-\eta)(\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z))'}+1\right|+\nu, \tag{21}$$

where (17) is used to give $\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z)$.

Additionally, we can state this condition by

$$-\Re\left(\frac{(z-\eta)F'(z)}{F(z)}\right)>\tau\left|\frac{(z-\eta)F'(z)}{F(z)}+1\right|+\nu, \tag{22}$$

where

$$F(z)=(1-\psi)\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z)+\psi(z-\eta)(\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z))'$$

$$=\frac{1-2\psi}{z-\eta}+\sum_{l=1}^{\infty}(l\psi-\psi+1)\Omega_s^d(l)a_l(z-\eta)^l,\quad a_l\ge0 \tag{23}$$

where $\Omega_s^d(l)$ defind by (18).

It is interesting to note that we can define a number of new subclasses of $\varkappa_\eta$ by specializing the parameters ψ,τ and d,s. In the examples that follow, we demonstrate two significant subclasses.

Example 1. *For $\psi=0$, we let $\mathcal{A}_s^d(0,\tau,\nu,\eta)=\mathcal{A}_s^d(\tau,\nu,\eta)$ indicate a subclass of $\varkappa_\eta$ that consists of functions of the form (2), which satisfy the requirement that*

$$-\Re\left(\frac{(z-\eta)(\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z))'}{\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z)}\right)>\tau\left|\frac{(z-\eta)(\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z))'}{\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z)}+1\right|+\nu, \tag{24}$$

where $\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z)$ is given by (17).

Example 2. *Example 2. For $\psi=0,\tau=0$, we let $\mathcal{A}_s^d(0,0,\nu,\eta)=\mathcal{A}_s^d(v,\eta)$ indicate a subclass of $\varkappa_\eta$ that consists of functions of form (2) that satisfy the requirement that*

$$-\Re\left(\frac{(z-\eta)(\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z))'}{\mathcal{J}_{\xi}^{\mu}(\gamma_d,\beta_s,q)\zeta(z)}\right)>\nu, \tag{25}$$

where $\mathcal{J}_{\xi}^{\mu}(\gamma_d, \beta_s, q)\zeta(z)$ *is given by (17).*

We begin by recalling the following lemma due to Challab, Darus and Ghanim [1].

Lemma 1 ([1])**.** *The function* $\zeta(z)$ *defined by (2) is in the class in* $\mathcal{A}_s^d(\psi,\tau,\nu,\eta)$ *if, and only if,*

$$\sum_{l=1}^{\infty}[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega_s^d(l)a_l \leq (1-2\psi)(1-\nu). \tag{26}$$

The result is sharp.

In view of Lemma 1, we can see that the function $\zeta(z)$, defined by (2) in the class $\mathcal{A}_s^d(\psi,\tau,\nu,\eta)$, satisfies the ceofficient inequality

$$a_1 \leq \frac{(1-2\psi)(1-\nu)}{[(1+v+2\tau)]\Omega_s^d(1)}, \tag{27}$$

where

$$\Omega_s^d(1) = \frac{\prod_{i=1}^{d}(\gamma_i,q)_2}{(q,q)_2\prod_{i=1}^{s}(\beta_i,q)_2}\left(\frac{\mu}{\mu+2}\right)^{\xi}.$$

Hence we may take

$$a_1 = \frac{(1-2\psi)(1-\nu)c}{[((1+v+2\tau))]\Omega_s^d(1)} \quad 0 \leq c \leq 1. \tag{28}$$

Making use of (27), we now introduce the following class of functions:

Let $\mathcal{A}_{s,c}^d(\psi,\tau,\nu,\eta)$ denote the subclass of $\mathcal{A}_s^d(\psi,\tau,\nu,\eta)$, consisting of a function of the form

$$\zeta(z) = \frac{1}{z-\eta} + \frac{(1-2\psi)(1-\nu)c}{[(1+\nu+2\tau)]\Omega_s^d(1)}(z-\eta) + \sum_{l=2}^{\infty}a_l(z-\eta)^l, \tag{29}$$

where

$$a_{l\geq 0} \quad and \quad 0 \leq c \leq 1.$$

In this paper, we obtain the coefficient inequalities for the class $\mathcal{A}_{s,c}^d(\psi,\tau,\nu,\eta)$ and closure theorems. Further, the radius of convexity are obtained for the class $\mathcal{A}_{s,c}^d(\psi,\tau,\nu,\eta)$.

2. Coefficients' Inequalities

Theorem 1. *Let the function* $\zeta(z)$ *be defined (28). Then,* $\zeta(z)$ *is in the class* $\mathcal{A}_{s,c}^d(\psi,\tau,\nu,\eta)$ *if, and only if,*

$$\sum_{l=2}^{\infty}[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega_s^d(l)a_l \leq (1-2\psi)(1-\nu)(1-c). \tag{30}$$

The result is sharp.

Proof. Putting

$$a_1 = \frac{(1-2\psi)(1-\nu)c}{[(1+\tau)+(v+\tau)]\Omega_s^d(1)} \quad 0 \leq c \leq 1, \tag{31}$$

Using (25) and simplification, we arrive at the result, which is sharp for the function

$$\zeta(z) = \frac{1}{z-\eta} + \frac{(1-2\psi)(1-\nu)c}{[(1+\tau)+(\nu+\tau)]\Omega_s^d(1)}(z-\eta) + \frac{(1-2\psi)(1-\nu)(1-c)}{[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega_s^d(l)}(z-\eta)^l(l \geq 2). \tag{32}$$

□

Corollary 1. *Let the function $\zeta(z)$ defined by (27) be in the class $\mathcal{A}^{d}_{s,c}(\psi,\tau,\nu,\eta)$. Then,*

$$a_l \leq \frac{(1-2\psi)(1-\nu)(1-c)}{[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega^{d}_{s}(l)} \quad (l \geq 2). \tag{33}$$

The result for the function $\zeta(z)$ given by (31) is sharp.

Corollary 2. *If $0 \leq c_1 \leq c_2 \leq 1$*

$$\mathcal{A}^{d}_{s,c_2}(\psi,\tau,\nu,\eta) \subseteq \mathcal{A}^{d}_{s,c_1}(\psi,\tau,\nu,\eta).$$

3. Closure Theorems

Using Theorem 1, we can prove the following theorems:

Theorem 2. *Let the function*

$$\zeta_j(z) = \frac{1}{z-\eta} + \frac{(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega^{d}_{s}(1)}(z-\eta) + \sum_{l=2}^{\infty} a_{l,j}(z-\eta)^l \quad (a_{l,j} \geq 0), \tag{34}$$

be in the class $\mathcal{A}^{d}_{s,c}(\psi,\tau,\nu,\eta)$. for every $j = 1,2,\ldots,s$. Then, the function

$$g(z) = \frac{1}{z-\eta} + \frac{(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega^{d}_{s}(1)}(z-\eta) + \sum_{l=2}^{\infty} b_l(z-\eta)^l (b_l \geq 0), \tag{35}$$

is also in the same class $\mathcal{A}^{d}_{s,c}(\psi,\tau,\nu,\eta)$, where

$$b_l = \frac{1}{m}\sum_{j=1}^{m} a_{l,j} (l = 1,2,\ldots). \tag{36}$$

Proof. Since $\zeta_j(z) \in \mathcal{A}^{d}_{s,c}(\psi,\tau,\nu,\eta)$, it follows from Theorem 1 that

$$\sum_{l=2}^{\infty} [l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega^{d}_{s}(l)a_{l,j} \leq (1-2\psi)(1-\nu)(1-c), \tag{37}$$

for every $j = 1,2,\ldots,m$. Hence,

$$\begin{aligned} &\sum_{l=2}^{\infty} [l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega^{d}_{s}(l)b_l \\ &= \sum_{l=2}^{\infty} [l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega^{d}_{s}(l)(\frac{1}{m}\sum_{j=1}^{m} a_{l,j}) \\ &= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{l=2}^{\infty} [l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega^{d}_{s}(l)a_{l,j}\right) \\ &\leq (1-2\psi)(1-\nu)(1-c). \end{aligned} \tag{38}$$

From Theorem 1, it follows that $g(z) \in \mathcal{A}^{d}_{s,c}(\psi,\tau,\nu,\eta)$. This completes the proof. □

Theorem 3. *The class $\mathcal{A}^{d}_{s,c}(\psi,\tau,\nu,\eta)$ is closed under convex linear combination*

Proof. Let $\zeta_j(z)(j = 1,2)$ be defined by (33)

$$h(z) = \phi\zeta_1(z) + (1-\phi)\zeta_2(z) \qquad (0 \leq \phi \leq 1), \tag{39}$$

It is sufficient to prove that the function h(z) is also in the class $\mathcal{A}_{s,c}^{d}(\psi,\tau,\nu,\eta)$.
Since

$$h(z)=\frac{1}{z-\eta}+\frac{(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega_s^d(1)}(z-\eta)+\sum_{l=2}^{\infty}[\phi a_{l,1}+(1-\phi)a_{l,2}](z-\eta)^l. \tag{40}$$

Then, we have Theorem 1, that

$$\sum_{l=2}^{\infty}[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega_s^d(l)[\phi a_{l,1}+(1-\phi)a_{l,2}]$$

$$\leq \phi(1-2\psi)(1-\nu)(1-c)+(1-\phi)(1-2\psi)(1-\nu)(1-c)$$

$$=(1-2\psi)(1-\nu)(1-c).$$

Therefore, $h(z)\in\mathcal{A}_{s,c}^{d}(\psi,\tau,\nu,\eta)$. □

Theorem 4. *Let*

$$\zeta(z)=\frac{1}{z-\eta}+\frac{(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega_s^d(1)}(z-\eta), \tag{41}$$

and

$$\zeta_l(z)=\frac{1}{z-\eta}+\frac{(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega_s^d(1)}(z-\eta)+\frac{(1-2\psi)(1-\nu)(1-c)}{[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega_s^d(l)}(z-\eta)^l \quad (l\geq 2). \tag{42}$$

Then, $\zeta(z)$ is in the class $\mathcal{A}_{s,c}^{d}(\psi,\tau,\nu,\eta)$, if, and only if, it can be expressed in the form

$$\zeta(z)=\sum_{l=1}^{\infty}\lambda_l\zeta_l(z), \tag{43}$$

where $\lambda_l\geq 0$ and $\sum_{l=1}^{\infty}\lambda_l=1$.

Proof. Let

$$\zeta(z)=\sum_{l=1}^{\infty}\lambda_l\zeta_l(z), \tag{44}$$

$$=\frac{1}{z-\eta}+\frac{(1-2\psi)(1-\nu)c}{[1+\nu+2\tau)]\Omega_s^d(1)}(z-\eta)+\sum_{l=2}^{\infty}\frac{(1-2\psi)(1-\nu)(1-c)}{[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega_s^d(l)}\lambda_l(z-\eta)^l.$$

Since

$$\sum_{l=2}^{\infty}\frac{(1-2\psi)(1-\nu)(1-c)\lambda_l}{[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega_s^d(l)}.\frac{[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega_s^d(l)}{(1-2\psi)(1-\nu)} \tag{45}$$

$$=(1-c)\sum_{l=2}^{\infty}\lambda_l=(1-c)(1-\lambda_1)\leq(1-c).$$

Hence, using Theorem 1, we have $\zeta(z)\in\mathcal{A}_{s,c}^{d}(\psi,\tau,\nu,\eta)$.
Conversely, we assume that $\zeta(z)$, defined by (28), is in the class $\mathcal{A}_{s,c}^{d}(\psi,\tau,\nu,\eta)$.
Then by applying (32), we can obtain

$$a_l\leq\frac{(1-2\psi)(1-\nu)(1-c)}{[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega_s^d(l)} \quad (l\geq 2). \tag{46}$$

Setting

$$\lambda_l=\frac{[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega_s^d(l)}{(1-2\psi)(1-\nu)(1-c)}a_l \quad (l\geq 2). \tag{47}$$

$$\lambda_1 = 1 - \sum_{l=2}^{\infty} \lambda_l. \tag{48}$$

we have (42). The proof of Theorem 4 is now complete. □

4. Radius of Convexity

Theorem 5. *Let the function $\zeta(z)$ be defined by (28) in the class $\mathcal{A}^d_{s,c}(\psi,\tau,\nu,\eta)$. Then, $\zeta(z)$ is meromorphically convex of order $\delta(0 \le \delta < 1)$ in $0 < |z-\eta| < r_1 = r_1(\psi,\tau,\nu,\eta,c,\delta)$ where $r_1(\psi,\tau,\nu,\eta,c,\delta)$, which has the highest value*

$$\frac{(1+\delta)(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega^d_s(1)} r^2 + \frac{(l(l+2-\delta)(1-2\psi)(1-\nu)(1-c)}{[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega^d_s(l_o)} r^{l+1} \le (1-\delta), \tag{49}$$

for $l \ge 2$. The result is sharp for the function

$$\zeta_l(z) = \frac{1}{z-\eta} + \frac{(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega^d_s(1)}(z-\eta) + \frac{(1-2\psi)(1-\nu)(1-c)}{[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega^d_s(l)}(z-\eta)^l, \tag{50}$$

for some l.

Proof. It is sufficient to show that

$$\left|\frac{(z-\eta)\zeta''(z)}{\zeta'(z)} + 2\right| \le 1-\delta \quad (0 \le \delta < 1)\text{for}0 < |z-\eta| < r_1(\psi,\tau,\nu,\eta,c,\delta).$$

Note that

$$\left|\frac{(z-\eta)\zeta''(z)}{\zeta'(z)} + 2\right| = \left|\frac{\frac{2(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega^d_s(1)}(z-\eta)^2 + \sum_{l=2}^{\infty} l(l+1)a_l(z-\eta)^{l+1}}{-1+\frac{(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega^d_s(1)}(z-\eta)^2 + \sum_{l=2}^{\infty} la_l(z-\eta)^{l+1}}\right| \le 1-\delta$$

$$= \frac{\frac{2(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega^d_s(1)}|z-\eta|^2 + \sum_{l=2}^{\infty} l(l+1)a_l|z-\eta|^{l+1}}{1-\left(\frac{(1-2\psi)(1-\nu)c}{[(1+\tau)+(\nu+\tau)]\Omega^d_s(1)}|z-\eta|^2 + \sum_{l=2}^{\infty} la_l|z-\eta|^{l+1}\right)} \le 1-\delta$$

$$\frac{\frac{2(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega^d_s(1)}r^2 + \sum_{l=2}^{\infty} l(l+1)a_l r^{l+1}}{1-\frac{(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega^d_s(1)}r^2 - \sum_{l=2}^{\infty} la_l r^{l+1}} \le 1-\delta, \tag{51}$$

for $0 < |z-\eta| < r$ if, and only if,

$$\frac{2(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega^d_s(1)}r^2 + \sum_{l=2}^{\infty} l(l+1)a_l r^{l+1} \le (1-\delta)\left(1-\frac{(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega^d_s(1)}r^2 - \sum_{l=2}^{\infty} la_l r^{l+1}\right)$$

$$\frac{(3-\delta)(1-2\psi)(1-\nu)c}{[1+\nu+2\tau]\Omega^d_s(1)}r^2 + \sum_{l=2}^{\infty} l(l+2-\delta)a_l r^{l+1} \le (1-\delta). \tag{52}$$

Since $\zeta(z)$ is in the class $\mathcal{A}^d_{s,c}(\psi,\tau,\nu,\eta)$, from (32), we may take

$$a_l = \frac{(1-2\psi)(1-\nu)(1-c)\lambda_l}{[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega^d_s(l)} \quad (l \ge 2), \tag{53}$$

where $\lambda_l \ge 0 \quad (l \ge 2)$ and

$$\sum_{l=1}^{\infty} \lambda_l \le 1. \tag{54}$$

We select the positive integer $l_0 = l_0(r)$ for each fixed r, where $\frac{l(l+2-\delta)r^{l+1}}{[l(1+\tau)+(\nu+\tau)](1+l\psi-\psi)\Omega_s^d(l)}$ is maximal. Then it follows that

$$\sum_{l=2}^{\infty} l(l+2-\delta)a_l r^{l+1} \leq \frac{(l_o(l_o+2-\delta)(1-2\psi)(1-\nu)(1-c)}{[l_o(1+\tau)+(\nu+\tau)](1+l_o\psi-\psi)\Omega_s^d(l_o)} r^{l_o+1} \quad (l \geq 2). \tag{55}$$

Then $\zeta(z)$ is convex of order δ in $0 < |z-\eta| < r_1(\psi,\tau,\nu,\eta,c,\delta)$ provided that

$$\frac{(3-\delta)(1-2\psi)(1-\nu)c}{[(1+\tau)+(\nu+\tau)]\Omega_s^d(1)} r^2 + \frac{(l_o(l_o+2-\delta)(1-2\psi)(1-\nu)(1-c)}{[l_o(1+\tau)+(\nu+\tau)](1+l_o\psi-\psi)\Omega_s^d(l_o)} r^{l_o+1} \leq (1-\delta). \tag{56}$$

We find the value $r_o = r_o(\psi,\tau,\nu,\eta,c,\delta)$ and the corresponding integer $l_o(r_o)$so that

$$\frac{(3-\delta)(1-2\psi)(1-\nu)c}{[(1+\tau)+(\nu+\tau)]\Omega_s^d(1)} r_o^2 + \frac{(l_o(l_o+2-\delta)(1-2\psi)(1-\nu)(1-c)}{[l_o(1+\tau)+(\nu+\tau)](1+l_o\psi-\psi)\Omega_s^d(l_o)} r_o^{l_o+1} = (1-\delta). \tag{57}$$

Then, this value r_o is the radius of meromorphically convex of order δ for functions belonging to the class $\mathcal{A}_{s,c}^d(\psi,\tau,\nu,\eta)$. □

5. Conclusions

The fixed second coefficients of class $\mathcal{A}_{s,c}^d(\psi,\tau,\nu,\eta)$ and the q- hypergeometric functions are included in the new class of meromorphic parabolic starlike functions defined in this article. Some features are obtained for the function in the class $\mathcal{A}_{s,c}^d(\psi,\tau,\nu,\eta)$, including the radius of convexity, closure theorems, and coefficient inequalities.

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