

Article

A Fuzzy Graph Theory Approach to the Facility Location Problem: A Case Study in the Indian Banking System

Anushree Bhattacharya [†]  and Madhumangal Pal ^{*,†} 

Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102, India; rsmo_anumath96@mail.vidyasagar.ac.in

* Correspondence: mmpalvu@mail.vidyasagar.ac.in

† These authors contributed equally to this work.

Abstract: A fuzzy graph G is stated to have a set of trees as its tree cover if all the vertices of G are in their union. The maximum weight tree in the tree cover is assumed to be the cost of a tree cover for a fuzzy graph. For an integer $\beta > 0$, finding a set of trees to cover all the vertices of a graph with minimum cost and at most β number of spanning trees is known as the β -tree cover problem. Combining the tree-covering concept and facility location problem in a fuzzy environment for solving critical real-life problems in the recent era is a more fruitful approach. This issue strongly inspires us to develop a model with a practical algorithm. This paper provides an algorithm and complexity analysis to determine the number of rooted trees s covering the given fuzzy graph. In addition, a model is constructed with three optimization programming problems in the facility location problem and a tree covering fuzzy graphs. The model includes two types of the facility location problem, simultaneously addressing a variable covering radius and a fixed covering radius. A numerical example is provided to further describe the model, then, in the application part of the paper, the proposed model is applied to solve the real-life problem of maximizing demand saturation by minimizing the number of small denominations in the Indian banking system. This problem involves the data input of different indicators in the banking system along with details of the denominations of banknotes.

Keywords: fuzzy graph; tree covers; covering problem; tree covering number; fuzzy optimization**MSC:** 05C72

Citation: Bhattacharya, A.; Pal, M. A Fuzzy Graph Theory Approach to the Facility Location Problem: A Case Study in the Indian Banking System. *Mathematics* **2023**, *11*, 2992. <https://doi.org/10.3390/math11132992>

Academic Editor: Michael Voskoglou

Received: 15 June 2023

Revised: 1 July 2023

Accepted: 2 July 2023

Published: 4 July 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Most of the problems with uncertainties in our daily life can be modelled by fuzzy graphs. Fuzzy graph theory plays a salient role in making real connections between various objects in a system, with applications in network routing, wireless sensor networks, computer science, medical science, operations research, and more. In this introductory situation, the concept of fuzzy graphs provided by Rosenfield [1] is presented as an interesting fuzzy mathematical tool in graph theory.

A tree cover of a fuzzy graph $G = (V, \sigma, \mu)$, denoted by $\tau(G)$, is the set of the minimum number of fuzzy trees as induced subgraphs that cover all the vertices of G . For a crisp graph, the tree cover number is approximated as the maximum positive semi-definite nullity as a conjecture. Although this graph parameter was first introduced as an essential tool in 2011, there is very little information in the existing literature.

For minimizing the maximum cost of a tree cover, we are interested in finding a tree covering of a fuzzy graph with at most s number of trees which span the whole graph, as it is known that constant factor approximations to travelling salesperson tracks can be used to find the minimum number of spanning trees.

The actual purpose of the facility location problem is to decide which node to allocate to a set of facilities in order to provide full coverage to the service needed for several

demand nodes in a system in the most efficient way. In addition, a customer or demand point may decide on the location of their facility nodes to cover the demand needed by every customer in the fuzzy network. Nowadays, one of the crucial problems is the placement of servers or data objects in any communication network to ensure that the latency of access is optimized. However, the primary purpose of the problem is always the same; the aim is to maximize the profit, that is, to minimize the allocation costs of different facility nodes in the system.

Many facility location problems include possible locations for facility nodes to be built on or opened at already-built facilities. In this case, information about the building or opening cost for a facility in that location is supplied. A set of demand nodes or customers can be considered, with detailed information about the amount to be covered and the connection cost between the demand point and a certain facility point. On the other hand, many conditions or restrictions in a facility's location problem need to be satisfied. An objective function which needs to be optimized is always present in such problems. One objective function is generally based on the cost, which includes all types of costs, such as demand, ordering, transportation, etc., for all system facilities.

In certain facility location problems, a profit can be associated with the assignment, then the objective profit function is optimized. One critical way of characterizing the facility location problem is on the basis of whether the facilities in the system are of variable-covering radius or fixed-covering radius in terms of how many demand nodes are to be covered by a facility within its covering radius. This kind of facility location problem is difficult to solve; we handle such problems in this article.

Now, we have to go through the rich literature on covering problems, which is the main topic of work within this article. The tree covering problems modelled and solved in this article are closely associated with the works of Arkin, Hassin, and Levin [2], who have dealt with various covering problems, including coverage of vertices for a crisp graph and a subset of edges of a graph with a partition in walks, paths, or stars. There are a variety of studies on approximation algorithms to minimize the number of objectives for covering purposes (such as paths) by satisfying some restrictions on the cost of each covering object.

An efficient feasible solution includes k -tours for covering the vertices of a crisp graph in the k -traveling salesperson problem. The main objective in this model is to minimize the total tour length. It is possible to partition all vertices of a graph into several clusters, and the weight of a star or low-cost spanning trees in the cluster is considered the weight of the cluster in that graph. Rana et al. [3] worked on unweighted interval graphs with conditional covering problems (CCP). In addition, Bhattacharya and Pal have presented works on different types of covering problems involving fuzzy graphs [4–7].

Mordeson and Peng [8] discussed concepts around various operations on fuzzy graphs. Samanta and Pal [9] have provided results on the fuzzy colouring of fuzzy graphs and fuzzy tolerance graphs. Fuzzy planar graphs with m -polarity are discussed in the work of Ghorai and Pal in [10]. Cardinal and Hofer [11] have worked on various types of covering games. Chang and Zadeh handled fuzzy mappings and control techniques in [12]. Chaudhry introduced new heuristics for solving CCP in [13]. Chen and Chen [14] have provided a new method of finding similarity measures between two fuzzy numbers. Ghorai and Pal [15] have studied faces and dual m -polar fuzzy graphs. Jurji and Borut combined research on the facility location and covering problems in [16]. Fuzzy graphs were used in network optimization by Koczy [17]. In a similar way, Hakimi [18] worked on communication networks. The problem of emergency service facility location was handled by Toregas et al. [19]. Related problems on fuzzy graphs were studied and solved by Bhattacharya and Pal in [20].

Nayeem and Pal [21] solved the shortest path problem on a network. Ni [22] worked on a research area involving covering problems of edges having minimum weights. Pal et al. [23] detailed new ideas with respect to modern trends in fuzzy graph theory. Pramanik et al. [24] discussed interval-valued fuzzy planar graphs. A study on bipolar fuzzy graphs was revealed

by Rashmanlou et al. in [25]. The research area of Samanta et al. in [26] included vague graphs and their strengths.

The motivation for the present article arises from the “Doctor station location” problem. Consider the problem of a hospital that requires s doctors to be located in the campus area for total coverage. A single doctor can be assigned to a particular number of patients, and the doctors make their rounds in shifts during the morning. The purpose of the problem is to find suitable locations for doctor stations and determine how to cover the requirement of treating every patient in such a way that the time for completing the entire process is minimized. Similar types of real-life problems motivate us to use the concept of tree covering in a modelled graph.

The “doctor station location” problem is similar to the problem of covering a graph with at most k number of tours while maintaining the minimum value of the maximum length of a tour. A constant factor approximation for a minimum number of trees spans the whole graph of travelling salesperson tours. We are interested in finding a covering of the graph with at most k number of spanning trees and the minimum value of the maximum weight of those trees. When a doctor must return to their station to pick up necessary supplies without first visiting every patient, a variation occurs in the considered problem. At such a point, the problem of covering the graph is formulated with the help of stars in the graph. In addition, we can consider the possibility of the hospital already having been built with all required doctor stations, in which case we only need to find the best possible way of assigning patients to doctors. Thus, these problems are transformed into the problem of finding rooted trees that covering the graph.

The idea described above inspires us to deal with tree-covering problems; for a realistic level of imprecision, we include the flavour of fuzzy graphs in this article. In addition, we chose the banking system as a case study in the application part, keeping in mind the importance of a developed model in solving real-life problems involving a combination of the facility location problem and tree covering of fuzzy graphs. The need to address such circumstances and situations influence our construction of a model with three optimization programming problems.

The main contributions in the sense of covering problems and real-life implications are organized in this portion. There are two variations of the facility location problems, one in which the covering radius is fixed for all facility vertices and one in which the covering radius differs for the vertices of different facilities. Earlier works have addressed covering radius variations and solved related problems involving a particular type of covering radius. However, it is more practical to have the number of demand points be saturated by a particular facility vertex of a fuzzy network depending on the covering radius type.

The contributions of this paper can be summarized as follows:

- (i) This paper is the first to handle the two cases of a fixed covering radius and variable covering radius simultaneously in one model. The concept behind our proposed model is intended to cover all demand vertices. In this paper, we solve such complex facility location problems by minimizing the number of facility vertices, minimizing the total cost for all demand vertices, and maximizing the demand saturation of facility vertices. All fuzzy solutions for the model are determined using ‘LINGO’ mathematical software 18.0.44.
- (ii) We introduce new definitions of tree cover and tree covering numbers for a fuzzy graph, design an efficient algorithm to determine the number of rooted trees s needed to cover a fuzzy graph, and present the algorithm’s complexity analysis.
- (iii) In the application part, we analyze the Indian banking system using our developed model and algorithm. As an outcome of this real-life application, the conclusion is reached that the number of small denominators (up to INR 100) is minimized. The cost required to maintain the flow of denominations for the Indian banking system is thereby minimized. Lastly, the demand saturation of denominations at the bottom level is assessed in the interest of economic freedom.

This article includes the following sections, excluding the present introduction. In Section 2, necessary definitions and concepts are provided. The covering problem considered in this article is described in Section 3 with the mathematical formulation. Section 4 illustrates the mathematical formulation of the developed model. Section 5 illustrates the model with a suitable example. A real-life application involving emergency aircraft landing situations is analyzed using the proposed model in Section 6. Finally, our overall conclusions are provided in Section 7.

2. Preliminaries

Several basic definitions and necessary aspects are provided in this portion of the paper. These parameters are beneficial in developing the model for a fuzzy graph with a facility location concept.

Here, all the components of a triangular fuzzy number (TFN) are normalized, i.e., between 0 and 1. Different types of scaling are used for the normalization process in different situations. Additionally, the format of the TFN with the representation of its components as in Definition 1 is followed throughout the article.

Definition 1 ([4]). Let $\tilde{P} = (p_1, p_2, p_3)$ be a TFN that has the following function as its membership function:

$$\mu_{\tilde{P}}(x) = \begin{cases} 0 & \text{if } x \leq p_1 \\ \frac{x - p_1}{p_2 - p_1} & \text{if } p_1 < x \leq p_2 \\ 1 & \text{if } x = p_2 \\ \frac{p_3 - x}{p_3 - p_2} & \text{if } p_2 < x \leq p_3 \\ 0 & \text{if } x \geq p_3. \end{cases}$$

Then, the defuzzified form of \tilde{P} is denoted by $|\tilde{P}|$, defined as

$$|\tilde{P}| = \frac{1}{6} |(p_1 + 4p_2 + p_3)|.$$

Definition 2 ([8]). A fuzzy graph $G = (V, \sigma, \mu)$ is a non-empty set which carries two functions, $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$, for all $x, y \in V$, where σ is the vertex-membership function and μ is the edge-membership function satisfying $\mu(x, y) \leq \min\{\sigma(x), \sigma(y)\}$.

Definition 3 ([4]). A set of vertices $\check{V}_C(G)$ of a fuzzy graph $G = (V, \sigma, \mu)$ is said to be a fuzzy vertex cover of G if each vertex of $G = (V, \sigma, \mu)$ is incident to every edge of G within a fuzzy covering radius \check{R}^s .

Definition 4. Consider a fuzzy graph $G = (V, \sigma, \mu)$. A coverage radius \check{R}^s of G is defined as a predefined number which reflects a distance in G . If $\check{d}(u, v) \leq \check{R}^s$ for an arbitrary vertex u of G , then this vertex v is covered by u in fuzzy graph G .

Definition 5. Let G be a crisp graph and let $F_G = \{G_1, G_2, \dots, G_n\}$ be a collection of subgraphs of G where G_i is a tree for all $i = 1, 2, \dots, n$. If $G_i \in F_G$ exists for every edge $e \in E(G)$ such that $e \in E(G_i)$, then F_G is a tree cover of G .

Definition 6. The tree covering number for a crisp graph G is provided by

$$t_C(G) = \min\{|F_G| : F_G \text{ is a tree cover of } G\}.$$

Definition 7. Let $G = (V, \sigma, \mu)$ be a fuzzy graph and let $\tau(G) = \{T_1, T_2, \dots, T_n\}$ be a collection of subgraphs of G where T_i is a fuzzy tree for all $i = 1, 2, \dots, n$. If there exists $T_i \in \tau(G)$ for every

edge uv with $\sigma(uv) \neq 0$ in G such that $\sigma(uv) \neq 0$ in T_i , then $\tau(G)$ is a tree cover of fuzzy graph $G = (V, \sigma, \mu)$.

Definition 8. The tree covering number of a fuzzy graph $G = (V, \sigma, \mu)$ is provided by

$$T_C(G) = \min_{T_i \text{ in } \tau(G)} \left\{ \sum \frac{\sigma(u)\sigma(v)}{\mu(uv)} : u, v \in V(T_i), \mu(uv) \neq 0 \right\}.$$

2.1. Notations and Symbols

This part clarifies all the necessary notations to signify the quantities used in the entire paper in deducing the theories. The symbols and their meanings are provided in tabular form in Table 1.

Table 1. Notations and their meanings.

Symbol	Meaning
G	fuzzy graph.
V	non-empty vertex set for fuzzy graph G .
σ	vertex membership function.
μ	edge-membership function.
T_{C_i}	i -th tree cover in any fuzzy graph.
$M_\mu(G)$	spanning tree with minimum size with respect to edge-membership function for a fuzzy graph G .
F_i	i -th forest present in a fuzzy graph.
$d(T, x)$	distance of the tree T from any arbitrary vertex x .
A	considered fixed bin capacity of the proposed algorithm FUZZY-BIN-PACK.
$c(T)$	cost of the tree T .
$\tau(G)$	tree-cover set for a fuzzy graph.
$I = \{1, 2, \dots, m\}$	the set of possible choices of demand locations in a fuzzy graph.
$J = \{1, 2, \dots, n\}$	the collection of possible choices of facility locations.
y_j	fuzzy variable for indicating demand saturation for demand point j .
c_{ij}	the approximate number of days to saturate demand by a facility vertex i to a demand vertex j , which is a fuzzy number.
δ_{ij}	it is 1, if the cost is paid by demand node j to facility node i , 0 otherwise.
f_i	the fuzzy cost (fuzzy) paid in facility node i for transportation.
d_j	fuzzy cost paid for demand ordering by demand point j .
K	maximum number of possible vertices (locations) which will be allocated in a fuzzy graph G .
M	maximum limit of the total amount of facility supplied in a fuzzy system for the developed model.
\tilde{R}^s	fuzzy covering radius.
$T_C(G)$	tree covering number.
F	set of facility points of the fuzzy graph.
D	set of demand points of the fuzzy graph.
x_i	fuzzy variable denoting the point for placing facility.
F_n	total number of facility nodes in a fuzzy graph.
Z_c	total cost involved in the fuzzy system.
D_s	demand saturation of fuzzy system.
$w(u, v)$	weight associated to an arbitrary edge (u, v) of fuzzy graph G .
w_i or $w(i)$	weight associated to an arbitrary vertex i .
$c(u, v)$	fuzzy cost associated to an arbitrary edge (u, v) .
$c(u)$	fuzzy cost associated to an arbitrary vertex u .

2.2. Problem under Consideration

Consider a fuzzy graph $G = (V, \sigma, \mu)$, where V is the set of vertices. There are three types of vertices in the graph: facility vertices, demand vertices, and free vertices. The free vertices are neither demand vertices nor facility vertices. The graph is assumed to contain some demand and facility vertices, while the free vertices may or may not exist.

The facility vertices can store commodities (such as food, water, weapons, etc.) required for emergency or regular distribution. These vertices may be hospitals, military camps/offices, ration shops, or distribution centres. Demand vertices take such commodities for use, and generally represent people, organizations, etc. Free vertices are neither demand vertices nor facility vertices. Thus, new facility vertices may be initiated with respect to any free vertices. Because these vertices have no demand, free vertices cannot be considered demand vertices.

There are two possible cases. The first is that all the demand vertices are covered by at least one facility vertex (i.e., the required demand is fulfilled by any one of the facility vertices), regardless of the cost or distance to reach the facility. In the second case, certain demand vertices are not covered by facility vertices (as the demand vertices may be far away from facility vertices, or transport or other costs may be too high).

In both situations, we insert more facility vertices to reduce the cost, distance, etc. It is assumed that each facility vertex may cover all the demand vertices within a specific circular region (i.e., provide the commodities to the demand vertices). The radius of this circular region is called the covering radius. This radius may be the same for all facility vertices (i.e., fixed) or different for different ones. In this case, the covering radius is variable or dynamic.

Many parameters or situations may arise with respect to constructing new facility vertices. Here, we select the free vertices as facility points under different assumptions, objectives, and constraints.

The main assumptions are:

- (a) The building cost for a facility point at an arbitrary vertex (say, i) can be taken as c_i (a fuzzy number).
- (b) If a customer at vertex k has decided to take services from a facility point (at i), then a cost δ_{ik} (a combination of the establishment cost, ordering cost, and transportation cost) is imposed on the demand point.
- (c) There are two types of covering radius: the first is a variable covering radius in which each facility serves unlimited customers, i.e., the covering radius is flexible depending on the situation, while the second is a fixed covering radius in which each facility can serve a maximum (say, u) number of demand points.

The objectives of our model are as follows:

- (a) Minimize the total number of facilities.
- (b) Minimize the total cost for all demand points.
- (c) Maximize demand saturation by the facilities.

We use the following constraints or restrictions:

- (a) The total number of facility nodes must be less than or equal to the maximum number of locations allocated in the fuzzy graph.
- (b) The sum of the value of supplying a service by an arbitrary facility point i to a customer at demand point j is assumed to take a fixed value (say, M).
- (c) The total demand cost in a certain period (say, t_1) is less than or equal to the total servicing cost in the fuzzy network.
- (d) The servicing cost paid by an arbitrary demand point i is always a positive amount.
- (e) The upper limit of the covering demand cost is the product of K and c_{ij} , where K is the maximum number of possible vertices (locations) allocated in a fuzzy graph G and c_{ij} is the approximate number of days needed by a facility vertex i to saturate demand with respect to a demand vertex j , with j being a fuzzy number.
- (f) The flexibility of a facility point j in terms of providing facilities is less than or equal to the amount of the covering demand.
- (g) There should always be a non-zero day for ordering demand and providing facilities from a facility point.

3. Problem Description

In this portion, the considered covering problem in a facility location model is described in a broader sense. A simple connected fuzzy graph is taken as an example throughout this article. The covering radius for the facility points is taken as a fuzzy number. In a system with fuzziness, distances and other types of costs are taken as fuzzy numbers as well.

Our model assumes that the resulting fuzzy system has two types of facility nodes.

(i) Facility location problem with variable covering radius. For the variable covering radius case, the following problem is considered: let $N = \{1, 2, \dots, n\}$ be a possible set of locations, and let d_{ij} for $i, j = 1, 2, \dots, n$ be the distances between them. For possible choices of points to open a facility, for each location $j \in D$, a subset $F \subseteq N$ is supplied, while for opened facility points a subset $D \subseteq N$ is assigned to the demand points. Corresponding to every location $i \in F$, the non-negative cost f_i is assumed at location i . In addition, δ_{ij} is the cost of demand location j assigning an opened facility point at i per unit of demand.

It is assumed that these costs are symmetric and non-negative fuzzy numbers and that they satisfy the following triangular inequality:

$$\delta_{ij} = \delta_{ji} \text{ for all } i, j \in N \text{ and } \delta_{ij} + \delta_{jk} \geq \delta_{ik} \text{ for all } i, j, k \in N.$$

To minimize the total cost, finding a feasible assignment to an open facility point of every location in the set D is required.

(ii) Facility location problem with fixed covering radius. In this case, we consider the situation where every existing facility point produces a total demand less than or equal to a positive integer. It is interesting to investigate whether we can prove that the adaptive covering radius in our developed algorithm affects the capacity of the facility location problems for finding the tree cover of a fuzzy graph and solving facility location problems with a fixed covering radius.

When the optimal value of the total demand is given in the variable covering radius type, the trivial problem is finding the corresponding facility points for the fuzzy graph. In this case, anyone can assign each location $j \in D$ to the location i such that δ_{ij} provides the minimum value among all possibilities ensuring that $y_i = 1$.

There is sufficient capacity for the fixed covering radius to saturate the total demand occurring in a fuzzy system. Two variants of facility location problems exist in our consideration. The situations are selected depending on the demand condition for each vertex of the fuzzy graph. An arbitrary facility point is allocated for only one demand point, or the facility point is fractionally split among more than one demand point in the system.

First, we must determine the type of the problem, i.e., whether it is of variable or fixed covering radius type. Then, we need to find an arrangement to ensure that there is a route to a facility point by optimizing all parameters. For example, we can take the amount of demand at each demand point together with an integer value representing the upper bound of demand for a particular problem. A minimum cost assignment is determined if the optimum value for total demand is supplied. In every problem, the presumption is that corresponding weights (edge weights and vertex weights) for facility points are provided.

To solve the considered problem, one crucial task is to find a tree cover of a fuzzy graph with an optimized cost different from the problem. In the next portion, we design an algorithm for finding such a tree cover.

3.1. Algorithm to Find Rooted Trees Covering Fuzzy Graph G

The bin packing problem is an optimization problem in which items of different sizes must be packed into a finite number of bins or containers, each of a fixed capacity and depending on the covering radius, in a way that minimizes the number of bins used. Generally, a BIN-PACK Algorithm refers to an approximation algorithm used to solve the above bin packing problem.

In this portion of the paper, we modify an existing BIN-PACK algorithm designed for crisp graphs for use with fuzzy graphs. We briefly describe the FUZZY-BIN-PACK Algorithm used to find a tree cover for the fuzzy graph $G = (V, \sigma, \mu)$ with an optimized cost

of at most $4A$. It should be noted here that the set of vertices for locating the facility /demand points is treated as a collection of elements in our modified algorithm.

In Algorithm 1, we take a fuzzy graph $G = (V, \sigma, \mu)$ with vertex and edge costs as input. We provide a fixed capacity A for a fixed covering radius case of facility vertices (i.e., bin capacity) for the given graph. We aim to find a tree cover of G with an optimized cost of at most $4A$. In the first step, we compare the cost of any arbitrary vertex with A . If the cost is higher than A , we remove it from the vertex set of G . Next, we find a spanning tree with a minimum size in G , a transformed version of G when the roots in a given sample set S of vertices to be either facility or demand nodes are contracted to a single vertex. Then, we collect all uncontracting from the minimum spanning tree and obtain forests $\{F_i\}_i$ for $i = 1, 2, \dots, z$. Now, to decompose the edges of each forest, we can find trees and an individual component such that the cost of the obtained trees must belong to $[A, 2A)$ and the cost of the other portion must be less than the fixed bin capacity A . In the next step, if the distances from the obtained trees to an arbitrary vertex of G are less than or equal to A , we check whether the trees are subgraphs or not for all possible cases. If all trees are not subgraphs of G , then we can conclude that the considered bin capacity is too low in cost; in this case, the process is repeated after modifying the value. On the other hand, if every tree is a subgraph of G , a success statement is returned for the choice of this bin capacity in obtaining an optimized tree cover of G . Finally, we obtain a set of trees which are all subgraphs of G for which the number of roots of these trees is finite, i.e., a tree covering with optimized bin capacity for the facility and demand vertices of the given fuzzy graph with finite size.

Algorithm 1: FUZZY-BIN-PACK.

- Input:** A fuzzy graph $G = (V, \sigma, \mu)$ with associated costs to the vertices and edges. Also, a fixed bin capacity ' A ' and a sample set S of vertices for choosing as facility and demand vertex are given.
- Output:** A tree cover for the fuzzy graph $G = (V, \sigma, \mu)$ with the optimized cost at most $4A$.
- Step 1: If $c(u) > A$, then
 $V(G) = V(G) \setminus u$ for any arbitrary $u \in V(G)$.
- Step 2: For a fuzzy graph $G = (V, \sigma, \mu)$, find $M_\mu(G)$ (spanning tree with minimum size) of G which is a transformed version of G when the roots in S are contracted to a single vertex.
- Step 3: In S , un-contracting roots from $M_\mu(G)$; obtain a forest $\{F_i\}_i$ for $i = 1, 2, \dots, z$.
- Step 4: Decompose edges of each tree F_i into trees $\{T_j^i\}_j + N_i$ such that $c(T_j^i) \in [A, 2A)$, for every j and $c(N_i) < A$.
- Step 5: If $d(T_j^i, x) \leq A$ for any arbitrary vertex $x \in V(G)$.
 Check whether T_j^i is a sub-graph of G for all j .
- Step 6: If not all trees are sub-graph of G , then
 Return failure: " A is a too low cost".
- Step 7: If every tree is a sub-graph of G , then
 Return to success and go to step 8.
- Step 8: Get a set of trees where each tree consists of T_j^i , a sub-graph of G , the number of roots is s , and the tree N (if any) in leftover, which contains the fuzzy root graph G , denoted by r .
-

3.2. Complexity Analysis of the Proposed Algorithm

We next explore the complexity analysis of our proposed FUZZY-BIN-PACK algorithm. In this algorithm, we fix the bin capacity to cover the whole fuzzy graph with the help of trees by limiting the cost to most four times the bin capacity, as per our previous considerations. If there are n number of vertices with non-zero membership values in the

fuzzy graph, then the cost verification for the total number of vertices takes $O(n)$ times. In the next step, if there are m numbers of contracting roots in S to find a connected tree it takes $O(mn)$ times, as this is the most generalized possibility for finding trees to make a tree cover. For other situations, every tree needs to be checked to determine whether it is a subgraph of the considered fuzzy graph G by verifying the tree's distance from any arbitrary vertex of the main graph. For this purpose, the required time is $O(mn)$. Therefore, the total time required to perform the proposed algorithm is $O(n) \times O(mn) \times O(mn) = O(m^2n^3)$. Thus, the time complexity of our proposed FUZZY-BIN-PACK algorithm is $O(m^2n^3)$.

3.3. Advantages of Proposed Algorithm

In brief, the approximation algorithm known as BIN-PACK can be used to solve problems related to precise data. This algorithm helps to find the minimum number of total bins needed to assign each item to a bin when considering a finite number of items of different weights and a fixed capacity of bins in the system. This is useful in finding the minimum number of facility vertices needed to saturate all the demand in a system with crisp data and parameters, i.e., it is helpful for crisp graph scenarios under a graph-theoretical approach.

Practically, however, such crisp models have little relation to real-life problems which involve several imprecise data types. Suppose that the number of the bins is not fixed; in this scenario, the existing algorithm is not applicable, as it cannot find any parameter with fuzziness. More clearly, it is obvious that in any network the accessible amount of facilities and order of demand are uncertain, and depend on various circumstances within the system. Therefore, it is notable that fuzzy graphs are a more practically justified tool for modeling realistic problems. In these cases, the existing BIN-PACK algorithm fails to compute the solution due to the need to handle fuzzy parameters. These more complex situations motivate us to modify the BIN-PACK algorithm for fuzzy environments and develop our proposed FUZZY-BIN-PACK algorithm. A complexity analysis of the proposed modified algorithm is provided in Section 3.2.

4. Mathematical Formulation

In this article, we consider fuzzy systems modeled as fuzzy graphs. If any uncertain or vague parameters are present in any system, it can be described as a fuzzy environment. Vertices are chosen as facility or demand vertices based on the relevant information of a particular case. It is obvious that in real life the available facilities and possible demand must be imprecise. Thus, the vertices are fuzzy with regard to their respective membership functions. Here, x_i represents the number of fuzzy vertices that are facility vertices; on the other hand, y_j represents the amount of demand saturation by facility nodes. It is assumed in the model that the maximum possible number for allocating demand or facility nodes is K . The other notations and their proper meaning in the proposed model are provided in Table 1. Next, we move through the mathematical formulation of objective functions subject to all conditions in the model to find the solution of the problem described in Section 2.2.

The main objectives of the developed model are as follows:

- (i) To construct the first objective function, we multiply the number of facilities n by the corresponding fuzzy variable x_i representing potential locations of facility vertices. Meanwhile, several demand points m are multiplied by the fuzzy variable y_j to represent demand saturation of the demand vertices. The operational cost is incorporated in the calculation of the facility vertices through multiplication by the parameter δ_{ij} . In this way, the total number of facility and demand nodes provided by

$$\text{Min } F_n = \sum_{j \in J} \delta_{ij} y_j m + \sum_{i \in I} \delta_{ij} x_i n$$

is minimized.

- (ii) The second objective function related to total fuzzy cost is constructed by taking the difference between two quantities, the first representing the product of the minimum

number of days among c_{ij} needed to saturate demand in the fuzzy network based on the amount of fuzzy cost f_i paid by the facility node for transportation and the second representing the product of the fuzzy cost d_j paid for demand ordering, where the maximum possible number for allocating demand or facility nodes in the network is K . Therefore, we seek to minimize the total cost, which is represented by

$$\text{Min } Z_c = \sum_{i \in I} \min_{j \in J} c_{ij} f_i - K(\max_{j \in J} d_j).$$

- (iii) The last objective function seeks to maximize the total demand saturation of the fuzzy network. The demand saturation of the network depends on the tree covering number corresponding to the tree coverage of the network, the fuzzy transportation and ordering costs, the fuzzy weights associated with demand, and the number of facility points M for facilities available in the network. Thus, this objective function involves four expressions: the first for fuzzy cost, the second for tree covering number $T_C(G)$, the third corresponding to the fuzzy weights w_i , and the last for the available facilities. Therefore, we seek to maximize the demand saturation, as follows:

$$\text{Max } D_s = \sum_{i \in I} \sum_{j \in J} c_{ij} \delta_{ij} + \sum_{j \in J} d_j y_j T_C(G) - \sum_{i \in I} w_i (\sum_{j \in J} \delta_{ij}) - M \times f_i.$$

The above-described functions are optimized in our model subject to certain conditions that need to be satisfied. The constraints subjected to these objective functions are as follows:

- (a) The total number of facility nodes must be less than or equal to the maximum number of locations that can be allocated in the fuzzy graph

$$1 \leq \sum_{j \in J} y_j \leq K.$$

- (b) The sum of the value of supplying services from an arbitrary facility node i to a customer at demand point j is assumed to be a fixed value (say, M):

$$\sum_{j \in J} \delta_{ij} n \leq M \text{ for } i \in I.$$

- (c) The amount of total demand cost in a certain period (say, t_1) is less than or equal to the total servicing cost in the fuzzy network:

$$\sum_{j \in J} (\max d_j) c_{ij} \leq \sum_{j \in J} \sum_{i \in I} f_i y_j m.$$

- (d) The servicing cost paid by an arbitrary demand node i is always a positive amount:

$$0 \leq f_i \leq 1.$$

- (e) The upper limit of the covering demand cost is the product of K and c_{ij} :

$$T_C(G) \times (\max_{j \in J} d_j) \times f_i \leq K \times \sum_{j \in J} c_{ij}$$

for $i \in I$.

- (f) The flexibility amount when a facility node j provides facilities is less than or equal to the amount of the covering demand

$$\sum_{j \in J} \sum_{i \in I} c_{ij} y_j \leq \sum_{j \in J} d_j T_C(G)$$

- (g) There should always be a non-zero day for ordering demand and for a facility node to provide facilities:

$$0 \leq c_{ij} \leq 1.$$

5. Example Illustration

This section illustrates use of the proposed model for the fixed covering radius case of the fuzzy graph $G = (V, \sigma, \mu)$ in Figure 1 along with its vertices and corresponding edges. In this section, the parameters taken as fuzzy numbers in the description of the problem, such as the covering radius, distances, weights, and the costs corresponding to the vertices and edges of the fuzzy graph, are triangular fuzzy numbers. The proposed model can be illustrated using other types of fuzzy numbers as well.

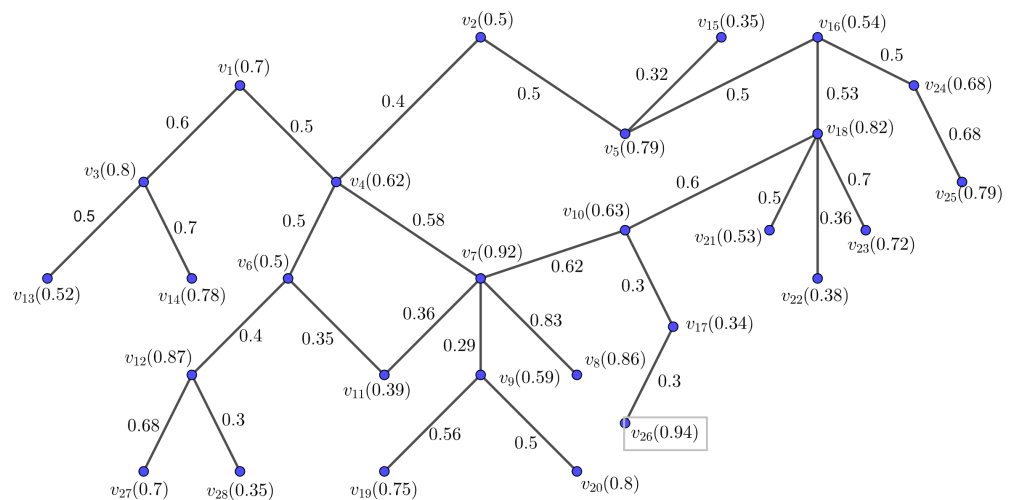


Figure 1. Considered fuzzy graph for illustration with the grey boxed value is the highest vertex-membership value.

In this example, the fuzzy graph $G = (V, \sigma, \mu)$ in Figure 1 with 26 vertices and 29 edges and their corresponding membership values, weights, and fuzzy costs is taken as an example. All of the weights and costs are triangular fuzzy numbers. We assume that all required data are supplied in Table 2.

To assign the edge membership values in the fuzzy graph, it is maintained that

$$\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\},$$

$$w(u, v) \leq \min\{w(u), w(v)\},$$

$$c(u, v) \leq \min\{c(u), c(v)\},$$

for any arbitrary vertices u and v in $V(G)$.

Using our developed algorithm, we can find a tree cover for the considered fuzzy graph as follows:

$$\tau(G) = \{T_1, T_2, T_3, T_4\}$$

where

$$T_1 = \{v_1, v_3, v_4, v_6, v_7, v_{12}, v_{13}, v_{14}, v_{27}, v_{28}\},$$

$$T_2 = \{v_8, v_9, v_{10}, v_{11}, v_{19}, v_{20}\},$$

$$T_3 = \{v_{17}, v_{18}, v_{21}, v_{22}, v_{23}, v_{26}\},$$

$$T_4 = \{v_2, v_5, v_{15}, v_{16}, v_{24}, v_{25}\}.$$

Table 2. Details of vertices for this illustration.

Vertices (v_i)	Membership Value ($\sigma(v_i)$)	Weights ($w(v_i)$)	Costs ($c(v_i)$)
v_1	0.7	(0.6, 0.8, 0.9)	(0.2, 0.5, 0.7)
v_2	0.5	(0.3, 0.4, 0.7)	(0.4, 0.5, 0.8)
v_3	0.8	(0.9, 0.97, 0.99)	(0.4, 0.52, 0.78)
v_4	0.62	(0.7, 0.83, 0.88)	(0.3, 0.48, 0.57)
v_5	0.79	(0.27, 0.35, 0.48)	(0.7, 0.82, 0.9)
v_6	0.5	(0.4, 0.57, 0.63)	(0.6, 0.73, 0.85)
v_7	0.92	(0.4, 0.6, 0.7)	(0.32, 0.46, 0.53)
v_8	0.86	(0.2, 0.37, 0.42)	(0.49, 0.58, 0.67)
v_9	0.59	(0.3, 0.42, 0.59)	(0.5, 0.63, 0.78)
v_{10}	0.63	(0.49, 0.56, 0.64)	(0.3, 0.5, 0.78)
v_{11}	0.39	(0.53, 0.68, 0.72)	(0.79, 0.82, 0.93)
v_{12}	0.87	(0.48, 0.53, 0.6)	(0.2, 0.37, 0.42)
v_{13}	0.52	(0.3, 0.42, 0.53)	(0.4, 0.5, 0.7)
v_{14}	0.78	(0.5, 0.7, 0.9)	(0.9, 0.95, 0.98)
v_{15}	0.35	(0.7, 0.72, 0.8)	(0.3, 0.34, 0.43)
v_{16}	0.54	(0.2, 0.32, 0.45)	(0.7, 0.81, 0.9)
v_{17}	0.34	(0.4, 0.57, 0.62)	(0.6, 0.72, 0.84)
v_{18}	0.82	(0.6, 0.8, 0.9)	(0.32, 0.46, 0.54)
v_{19}	0.75	(0.4, 0.53, 0.67)	(0.5, 0.62, 0.79)
v_{20}	0.8	(0.3, 0.42, 0.53)	(0.4, 0.5, 0.7)
v_{21}	0.53	(0.5, 0.62, 0.73)	(0.2, 0.35, 0.48)
v_{22}	0.38	(0.6, 0.72, 0.83)	(0.42, 0.57, 0.6)
v_{23}	0.72	(0.47, 0.52, 0.68)	(0.53, 0.68, 0.74)
v_{24}	0.68	(0.52, 0.68, 0.79)	(0.3, 0.4, 0.7)
v_{25}	0.79	(0.58, 0.62, 0.84)	(0.35, 0.48, 0.57)
v_{26}	0.94	(0.39, 0.46, 0.79)	(0.82, 0.87, 0.94)
v_{27}	0.7	(0.28, 0.37, 0.42)	(0.58, 0.62, 0.78)
v_{28}	0.35	(0.45, 0.51, 0.67)	(0.32, 0.48, 0.55)

The programming problem denoted as Problem 1 of the constructed model with three objective functions and associated conditions is provided as follows:

Problem 1. $Min F_n = \sum_{j \in J} \delta_{ij} y_j m + \sum_{i \in I} \delta_{ij} x_i n$
 $Min Z_c = \sum_{i \in I} \min_{j \in J} c_{ij} f_i - K(\max_{j \in J} d_j)$
 $Max D_s = \sum_{i \in I} \sum_{j \in J} c_{ij} \delta_{ij} + \sum_{j \in J} d_j y_j T_C(G) - \sum_{i \in I} w_i (\sum_{j \in J} \delta_{ij}) - M \times f_i$
 Subject to constraints
 $1 \leq \sum_{j \in J} y_j \leq K$
 $\sum_{j \in J} \delta_{ij} n \leq M$ for $i \in I$
 $\sum_{j \in J} (\max d_j) c_{ij} \leq \sum_{j \in J} \sum_{i \in I} f_i y_j m$
 $0 \leq f_i \leq 1$
 $T_C(G) \times (\max_{j \in J} d_j) \times f_i \leq K \times \sum_{j \in J} c_{ij}$
 for $i \in I$
 $\sum_{j \in J} \sum_{i \in I} c_{ij} y_j \leq \sum_{j \in J} d_j T_C(G)$
 $0 \leq c_{ij} \leq 1$

We assume the following for this example:

$\delta_{ij} = 1$ for the payment situation of demand point j with respect to facility point i .
 $min_{c_{ij}}$ is the approximate number of days needed by facility node i to saturate the demand of demand node $j=(0.58, 0.65, 0.79)$.

f_i is the fuzzy amount to be determined; it is the approximated cost paid at facility node i for transportation.

$K = \sigma(v_1) + \sigma(v_7) + \sigma_{10} + \sigma_{18} = 0.7 + 0.92 + 0.63 + 0.82 = 3.07$

$\max_{j \in J} d_j$ is the fuzzy cost paid for demand ordering by demand point $j = (0.79, 0.82, 0.95)$.

x_i is a fuzzy variable used to identify many facility nodes in a fuzzy network, which is a fuzzy number.

y_j is a fuzzy variable used to identify many demand nodes in a fuzzy system.

It is defined that $T_C(G) = \min\{T_{C_1}, T_{C_2}, T_{C_3}, T_{C_4}\}$.

Now, we have to evaluate the values $T_{C_1}, T_{C_2}, T_{C_3}, T_{C_4}$ corresponding to the trees T_1, T_2, T_3, T_4 , respectively, in the tree covering set of G .

$$T_{C_1} = \{\sigma(v_1) + \sigma(v_3) + \sigma(v_4) + \sigma(v_6) + \sigma(v_7) + \sigma(v_{11}) + \sigma(v_{12}) + \sigma(v_{13}) + \sigma(v_{14}) + \sigma(v_{27}) + \sigma(v_{28})\} = 7.15$$

$$T_{C_2} = \sigma(v_7) + \sigma(v_{11}) + \sigma(v_9) + \sigma(v_8) + \sigma(v_{10}) + \sigma(v_{19}) + \sigma(v_{20}) = 4.96$$

$$T_{C_3} = \sigma(v_{10}) + \sigma(v_{17}) + \sigma(v_{18}) + \sigma(v_{21}) + \sigma(v_{22}) + \sigma(v_{23}) + \sigma(v_{26}) = 4.36$$

$$T_{C_4} = \sigma(v_2) + \sigma(v_4) + \sigma(v_5) + \sigma(v_{15}) + \sigma(v_{16}) + \sigma(v_{10}) + \sigma(v_{24}) + \sigma(v_{25}) = 4.9$$

$T_C(G) = \min\{7.15, 4.96, 4.36, 4.9\} = 4.36$, which corresponds to the tree T_{C_3} . Thus, we have to calculate the total summed weight of the tree for the tree covering number of G .

$$\begin{aligned} \text{Therefore, } \sum_{i \in I} w_i &= w(v_{10}) + w(v_{17}) + w(v_{18}) + w(v_{21}) + w(v_{22}) + w(v_{23}) + w(v_{26}) = \\ &= (0.49, 0.56, 0.64) + (0.4, 0.57, 0.62) + (0.6, 0.8, 0.9) + (0.5, 0.62, 0.73) + (0.6, 0.72, 0.83) \\ &+ (0.47, 0.52, 0.63) + (0.39, 0.46, 0.79) = (3.45, 4.25, 5.14). \end{aligned}$$

In addition, the maximum fuzzy limit of the total amount of facilities supplied in the fuzzy system is $M = (8.92, 9.95, 10.98)$.

By solving the programming problems of the model with the help of the mathematical software 18.0.44 'LINGO', the variables to be determined are f_i, x_i, y_j, m and n . These are all fuzzy parameters, and are taken as triangular fuzzy numbers in this illustration. Then, Problem 1 is reduced to Problem 2 below.

Problem 2. $\text{Min } F_n = \sum_{j \in J} y_j m + \sum_{i \in I} x_i n$

$$\text{Min } Z_c = (0.58, 0.65, 0.79) f_i - 3.07 \times (0.79, 0.82, 0.95)$$

$$\text{Max } D_s = (0.58, 0.65, 0.79) m - (0.79, 0.82, 0.95) \times 4.36 y_j - \{(3.45, 4.25, 5.14) - (8.92 f_1, 9.95 f_2, 1.98 f_3)\}$$

Subject to conditions

$$1 \leq \sum_{j \in J} y_j \leq 3.07$$

$$(n_1, n_2, n_3) \leq (8.92, 9.95, 10.98)$$

$$(0.79, 0.82, 0.95) \times (0.58, 0.65, 0.79) \leq \sum_{j \in J} \sum_{i \in I} f_i y_j m$$

$$0 \leq f_i \leq 1$$

$$4.36 \times (0.79, 0.82, 0.95) \times f_i \leq 3.07 \times (0.58, 0.65, 0.79)$$

$$\sum_{j \in J} (0.58, 0.65, 0.79) y_j \leq 3.07 \times (0.79, 0.82, 0.95)$$

Using 'LINGO' mathematical software, we obtain the following solutions for these programming problems.

$f_i = (0.78, 0.84, 0.97), x_i = (2.84, 3.59, 4.01), y_j = (0.98, 1.29, 3.02), m = (3.12, 4.39, 5.74), n = (1.05, 2.71, 3.23)$. The minimum number of approximate choices for facility nodes is $\text{Min } F_n = (2.97, 3.28, 5.72)$, the minimized total fuzzy cost for the fuzzy network is $\text{Min } Z_c = (4.43, 5.12, 5.78)$, and the maximized demand saturation concerning weights and fuzzy costs is $\text{Max } D_s = (0.94, 2.37, 10.92)$.

6. Example Application

The recent COVID-19 pandemic has caused an abnormal situation for the global economy due to complicated circumstances such challenges within of the banking system and other financial institutions. In the Indian banking system. there is an ongoing revolution in the operations and services of banks and financial institutions due to the incorporation of new and updated technologies.

The Reserve Bank of India functions as a sunshade for the entire Indian banking industry, being responsible for the central banking system and regulatory processes. There

are two types of banks in the Indian banking system, namely, commercial banks and co-operative banks.

At present, different opportunities and challenges are faced by the Indian banking system. According to the Annual Report of the RBI, the main opportunities and challenges of the banking industry lie in continued development of the banking process to sustain the Indian banking system while reducing bankruptcies and other negative parameters.

Compared to the previous year, the level of consumer awareness is significantly higher. Today, consumers need internet banking, mobile banking, and ATM services with high frequency and reliability.

In the banking system, the optimal number of banknotes per denomination is not certain, and varies per year according to the annual report of the Reserve Bank of India. Any banknote denomination can be represented with the help of many other banknotes, a relation that is imprecise and varies over time. Thus, a great deal of vagueness and uncertainty exist in the banking system, and problems related this can be represented and solved with the help of fuzzy graphs. To construct a fuzzy graph related to this system, it is first necessary to choose proper vertices and edges; this selection process is described in the next portion.

6.1. Construction of Fuzzy Graph

In the last part of 2020–2021, the number of soiled banknotes needing to be disposed of was severely affected by the COVID-19 pandemic, a phenomenon that was expedited slowly in order to maintain a stable economic structure. Thus, there was a 32% decline in the disposal process of soiled banknotes in that financial year compared to the previous year.

Following the annual report of the Reserve Bank of India at the end of March 2021, here we consider the denominations of banknotes as vertices of a fuzzy graph. To assign vertex membership values, we first observed the denomination of banknotes from 2019 to 2021. Then, we took the average as the mean value of a triangular fuzzy number (TFN). The fuzzy graph uses these triangular fuzzy numbers as vertex membership values. On the other hand, if other banknotes can represent any banknote, then an edge exists between those vertices corresponding to the same banknotes. For example, there is an edge between the vertices corresponding to banknotes with values of INR 100 and 500. In addition, the vertex membership values are used to assign edge membership values in the fuzzy graph.

First, we must find the average percentage of parameters which affect the banking system, then turn this into a triangular fuzzy number. These TFNs are used to find similarity measures with the vertex membership values in order to determine the corresponding fuzzy weights for each vertex in the constructed fuzzy graph. The cost is taken as a fuzzy number representing the difference between denominations in the year 2021 and the denominations' averages corresponding to their vertices. Here, the facility points are assumed to be the smaller denominations and the demand points are to be taken as larger denominations. The edge membership values in the fuzzy graph must maintain the following:

$$\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}.$$

After constructing the fuzzy graph, we have to find a fuzzy tree cover and its tree covering number, then proceed to find the objectives and constraints of this particular part of the example. We aim to find the minimum number of smaller banknote denominations, minimize the fuzzy cost in the Indian banking system, and maximize demand saturation with respect to banknote circulation.

6.2. Case Study

At 21.4 percent of GDP for the end of December 2020, India's external debt remained lower than that of its emerging-market peers. The external vulnerability indicators at the end of March of 2021 are listed in Table 3, which reflect the approximate percentage unless indicated otherwise.

Table 3. Details of approximate percentage of parameters in the banking system.

Indicator	2013	2019	2020	End-March, 2021
Debt service ratio (debt service to current receipts)	5.9	6.4	6.5	9.0
Ratio of concessional debt to total debt	11.1	8.7	8.8	9.1
Ratio of short-term debt to reserves	33.1	426.3	22.4	17.7
Reserve cover of imports	7.0	9.6	12.0	18.6
External Debt to GDP ratio	22.4	19.9	20.6	21.4
Ratio of short-term debt to total debt	23.6	20.0	19.1	18.4

The number of frauds reported during the financial year 2020–2021 decreased by 15% in terms of number and 25% in terms of value. There was a decrease in the share of PSBs in terms of both value and number among the total number of frauds concerning private sector banks during the corresponding period.

Banknotes in Circulation

During 2020–2021, there were increases of 16.8% and 7.2% in the value and volume, respectively, in the circulation process of banknotes. The highest share in volume terms is for the INR 500 denomination, whereas the lowest is for INR 10 banknotes, as of 31 March 2021. Table 4 shows the details on banknotes in circulation to the end of March of 2021.

Table 4. Approximate number of banknotes in circulation (end March, 2021) as vertices of a fuzzy graph.

Denomination (Rs.) of Banknotes	2019	2020	2021	Average	v_i	$\sigma(v_i)$
2 and 5 Rs. notes	10.4	9.7	9.0	9.7	v_1	(9.5, 9.7, 10.2)
10 Rs. notes	28.7	26.2	23.6	26.16	v_2	(25.53, 26.16, 28.7)
20 Rs. notes	8.0	7.2	7.3	7.5	v_3	(6.73, 7.5, 8.25)
50 Rs. notes	7.9	7.4	7.0	7.43	v_4	(6.32, 7.43, 8.63)
100 Rs. notes	18.5	17.2	15.3	17	v_5	(16.38, 17, 18.72)
200 Rs. notes	3.7	4.6	4.7	4.33	v_6	(3.89, 4.33, 5.76)
500 Rs. notes	19.8	25.4	31.1	25.43	v_7	(23.81, 25.43, 26.95)
2000 Rs. notes	3.0	2.4	2.0	2.46	v_8	(2.37, 2.46, 3.54)

The fuzzy graph constructed using all these data is shown in Figure 2.

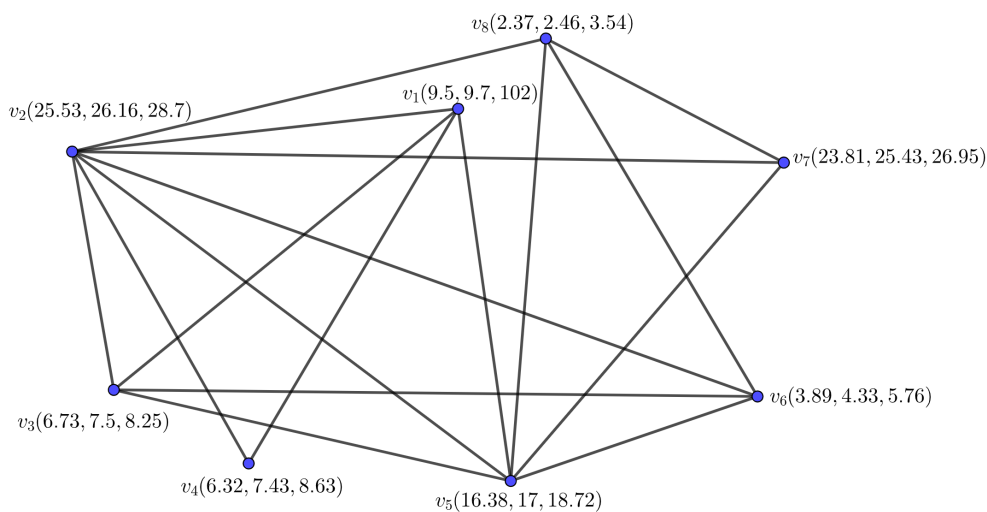


Figure 2. Fuzzy graph G for this application.

6.3. Formulation of the Problem

In this application area of our developed model, we have taken the Indian banking system as a fuzzy network, considering banknote denominations as items to be supplied from the facility to the demand vertices. The tree-like flow of denominations from the Reserve Bank of India to the common people helps in modeling such a system using the concept of tree cover in a fuzzy graph. The main problem is to minimize the number of small denominations without affecting the flow of demand in the lower layer related to the market level. For this purpose, we can use our proposed model with appropriate assumptions related to the Indian banking system to find all the solutions to the problem. The specific assumptions and considerations are described in the following portion.

Our objective for this application using the proposed model is to minimize the approximate number of smaller denominators (**Obj 1**), minimize the approximate total fuzzy cost in the Indian banking process concerning the corresponding weights mentioned earlier (**Obj 2**), and maximize demand saturation with respect to the economic need of the people through circulation of banknotes in the Indian economic system while accounting for external vulnerability indicators (**Obj 3**).

For this application, the data are as follows.

$\delta_{ij} = 1$ for the payment situation of the larger denomination j with respect to the smaller denomination i .

min_{ij} is the approximate number of days needed to saturate demand using a small denomination i with respect to a larger denomination $j = (8.78, 9.35, 10.59)$.

f_i is a fuzzy amount to be determined, representing the approximate cost of a smaller denomination i for transportation from one bank to another.

$$K = \sigma(v_3) + \sigma(v_4) + \sigma(v_1) + \sigma(v_8) + \sigma(v_5) = 7.5 + 7.43 + 9.7 + 2.46 + 17 = 44.09$$

$max_{j \in J} d_j$ is a fuzzy cost paid for demand ordering due to circulation of the larger denomination $j = (8.29, 9.32, 12.57)$.

x_i is a fuzzy variable used to identify a number of smaller denominations in the fuzzy banking system, and is a fuzzy number.

y_j is a fuzzy variable used to identify the number of larger denominations in the fuzzy banking network.

It is defined that $T_C(G) = \min\{T_{C_1}, T_{C_2}, T_{C_3}\}$, where

$$T_{C_1} = \{v_2, v_1, v_3, v_4, v_5\},$$

$$T_{C_2} = \{v_5, v_7, v_6, v_2\},$$

and

$$T_{C_3} = \{v_8, v_6, v_3, v_4\},$$

$T_C(G) = \min\{27.52, 42.36, 24.74\} = 24.74$, which corresponds to the tree T_{C_3} . Thus, we have to calculate the total summed weight of the tree for the tree covering number of G .

Therefore, $\sum_{i \in I} w_i = w(v_8) + w(v_6) + w(v_3) + w(v_4) = (3.49, 4.16, 5.34) + (2.34, 3.82, 4.32) + (5.26, 7.18, 8.39) + (4.35, 6.73, 9.81) = (15.44, 21.89, 27.86)$.

In addition, the maximum fuzzy limit of the total amount of facilities supplied in the fuzzy system is $M = (18.79, 19.65, 20.28)$.

By solving the programming problems of the model using 'LINGO' mathematical software, the variables to be determined are f_i, x_i, y_j, m , and n . In this illustration, all of these fuzzy parameters are taken as triangular fuzzy numbers.

6.4. Solution of Programming Problems

Then the problem Problem 1 is reduced to the problem, namely Problem 3 given by the following.

Problem 3. $Min F_n = \sum_{j \in J} y_j m + \sum_{i \in I} x_i n$
 $Min Z_c = (5.82, 6.35, 7.39)f_i - 26.23 \times (7.39, 8.42, 9.92)$

$$\text{Max } D_s = (5.82, 6.35, 7.39)m - (8.72, 8.92, 9.25) \times 24.69y_j - \{(12.45, 14.73, 15.48) - (18.79f_1, 19.65f_2, 20.28f_3)\}$$

Subject to conditions

$$1 \leq \sum_{j \in J} y_j \leq 33.947$$

$$(n_1, n_2, n_3) \leq (18.62, 19.25, 22.84)$$

$$(7.39, 8.42, 9.92) \times (5.82, 8.57, 9.91) \leq \sum_{j \in J} \sum_{i \in I} f_i y_j m$$

$$0 \leq f_i \leq 1$$

$$24.66 \times (7.39, 8.42, 9.92) \times f_i \leq 33.947 \times (5.82, 6.35, 7.39)$$

$$\sum_{j \in J} (5.82, 6.35, 7.39)y_j \leq 3.07 \times (7.39, 8.42, 9.92).$$

Using the mathematical software 'LINGO', we have the following solutions for these programming problems.

$f_i = (5.78, 6.84, 8.97)$, $x_i = (4.68, 5.79, 8.31)$, $y_j = (7.28, 9.89, 10.52)$, $m = (8.16, 8.53, 9.24)$, $n = (10.35, 12.92, 14.23)$. The minimum number of approximate choices for facility nodes is $\text{Min } F_n = (9.27, 9.86, 9.23)$, the minimum total fuzzy cost for the fuzzy network of the banking system is $\text{Min } Z_c = (14.38, 15.72, 16.98)$, and the maximum demand saturation concerning the weights and fuzzy costs is $\text{Max } D_s = (3.84, 4.37, 5.62)$.

6.5. Insightful Analysis

Based on the solutions obtained by the model in this application scenario, we can conclude the following:

- (i) The minimized number of total small denominations (i.e., banknotes of INR 2, 5, 10, 20, 50, and 100) provide a total coverage in the RBI banking system of (9.27, 9.86, 9.23), which is a fuzzy volume, i.e., in lakh; that is, a minimum total number of new banknotes of 9.27 lakh to 9.23 lakh per year in denominations of INR 2 to 100 need to be supplied for a smooth flow of denominations in the optimized sense, with the best results occurring for the case of 9.86 lakh.
- (ii) The minimized total fuzzy cost in the fuzzy system representing the banking process is (14.38, 15.72, 16.98) (in volume, i.e., INR notes in lakh) for smooth circulation of banknotes in the coming financial year to sustain the economic development of India. This objective function reflects the minimum denomination number needed to calculate cost for maintaining flow in the Indian banking system per year, and ranges from 14.38 lakh to 16.98 lakh, with 15.72 lakh being the best possibility.
- (iii) The maximized demand saturation of banknote circulation in the Indian economic system with respect to external vulnerability indicators can be deduced as (3.84, 4.37, 5.62), which is a fuzzy approximation. With the above-mentioned minimized objective function values in (i) and (ii), the demand for denominations (number of banknotes) at the population level is deduced as 3.84 lakh to 5.62 lakh. The best possible case is if the deduction in the number of denominations is 4.37 lakh.

7. Conclusions

In this article, we have considered the tree-covering of fuzzy graphs using tree-covering numbers. An efficient algorithm has been designed for evaluating tree cover for fuzzy graphs in two different situations involving the facility location problem: first, when the number of facility vertices in the fuzzy graph is variable, and second, when there is a fixed covering radius. As a fruitful solution to a realistic problem, we have modified the existing BIN-PACK algorithm to develop the FUZZY-BIN-PACK algorithm, which is very effective in fuzzy environments. A complexity analysis was carried out, showing its efficient performance. Most importantly, we have extended the common idea of tree cover to construct a model with a series of programming problems for solving complex versions

of the facility location problem in scenarios with fuzzy characteristics. The parameters obtained by our developed algorithm can help with constructing objective functions and relevant conditions to be maintained for developing the model for solving such facility location problems involving fuzzy graphs. The useful ‘LINGO’ mathematical software was used with the proposed model to separately find component-wise solutions and then combine them to obtain fuzzy solutions. The model was applied to a case study involving the circulation of banknotes in the Indian banking system, showing the usefulness of this model in efficiently solving real-life problems.

In our upcoming work, we intend to develop models by combining the concepts of graphoidal coverage and domination in fuzzy graphs. In addition, it might be interesting to construct an algorithm to evaluate graphoidal covering numbers for fuzzy graphs. Graphoidal coverage is an interesting topic in graph theory that can be extended to fuzzy graphs by considering the role of vertex and edge membership functions, and may be a better covering concept for path-related models. Incorporating the idea of domination can help to model real-world decisionmaking and choose the best possibilities for any facility vertices in a productive way. Further application to real-life problems related to sustainable development goals needs to be considered in future studies as well.

Author Contributions: A.B.: Conceptualization, Methodology, Software, Data Curation, Writing—Original draft preparation. M.P.: Supervision, Visualization, Investigation, Writing—Reviewing and Editing, Validation. All authors have read and agreed to the published version of the manuscript.

Funding: Financial support of the first author was provided by the Department of Science and Technology (DST), New Delhi, India, for research through the letter No. DST/INSPIRE/03/2019/001407 and Reg. No. IF190612, and is thankfully acknowledged.

Informed Consent Statement: Before submitting this work, all co-authors and the responsible authorities have declared their consent to submit this article tacitly or explicitly at the institute/organization where the work was developed.

Data Availability Statement: All of the data in the application part were taken from the Annual Report of the Reserve Bank of India, 2020–2021, published on 24 May 2021.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Rosenfield, A. Fuzzy graphs. In *Fuzzy Sets and Their Application*; Zadeh, L.A., Fu, K.S., Shimura, M., Eds.; Academic Press: New York, NY, USA, 1975; pp. 77–95.
2. Arkin, E.; Hassin, R.; Levin, A. Approximations for minimum and min-max vehicle routing problems. *J. Algorithms* **2006**, *59*, 1–18. [[CrossRef](#)]
3. Rana, A.; Pal, A.; Pal, M. The conditional covering problem on unweighted interval graphs with non-uniform coverage radius. *Math. Comput. Sci.* **2012**, *6*, 33–41. [[CrossRef](#)]
4. Bhattacharya, A.; Pal, M. Vertex covering problems of fuzzy graphs and their application in CCTV installation. *Neural Comput. Appl.* **2021**, *33*, 1–24. [[CrossRef](#)]
5. Bhattacharya, A.; Pal, M. Optimization in business strategy as a part of sustainable economic growth using clique covering of fuzzy graphs. *Soft Comput.* **2021**, *25*, 7095–7118. [[CrossRef](#)]
6. Bhattacharya, A.; Pal, M. Fuzzy covering problem of fuzzy graphs and its application to investigate the Indian economy in new normal. *J. Appl. Math. Comput.* **2022**, *68*, 479–510. [[CrossRef](#)]
7. Bhattacharya, A.; Pal, M. Fifth Sustainable Development Goal, Gender equality in India: Analysis by Mathematics of Uncertainty and Covering of Fuzzy Graphs. *Neural Comput. Appl.* **2021**, *33*, 15027–15057. [[CrossRef](#)]
8. Mordeson, J.N.; Peng, C.S. Operation on fuzzy graphs. *Inf. Sci.* **1994**, *79*, 159–170. [[CrossRef](#)]
9. Samanta, S.; Pramanik, T.; Pal, M. Fuzzy colouring of fuzzy graphs. *Afr. Math.* **2016**, *27*, 37–50. [[CrossRef](#)]
10. Ghorai, G.; Pal, M. A study on m-polar fuzzy planar graphs. *Int. J. Comput. Sci. Math.* **2016**, *7*, 283–292. [[CrossRef](#)]
11. Cardinal, J.; Hofer, M. Non-cooperative facility location and covering games. *Theor. Comput. Sci.* **2010**, *411*, 1855–1876. [[CrossRef](#)]
12. Chang, S.S.L.; Zadeh, L. On fuzzy mappings and control. *IEEE Trans. Syst. Man. Cybernet.* **1972**, *2*, 30–34. [[CrossRef](#)]
13. Chaudhry, S.S. New heuristics for the conditional covering problem. *Opsearch* **1993**, *30*, 42–47.
14. Chen, S.J.; Chen, S.M. A new method to measure the similarity between fuzzy numbers. *IEEE Int. Conf. Fuzzy Syst.* **2001**, *3*, 1123–1126.
15. Ghorai, G.; Pal, M. Faces and dual of m-polar fuzzy planar graphs. *J. Intell. Fuzzy Syst.* **2016**, *31*, 2043–2049. [[CrossRef](#)]

16. Mihelic, J.; Borut, R. Facility location and covering problems. In *Theoretical Computer Science*; Information Society: Ljubljana, Slovenia, 2004.
17. Koczy, L.T. Fuzzy graphs in the evaluation and optimization of networks. *Fuzzy Sets Syst.* **1992**, *46*, 307–319. [[CrossRef](#)]
18. Hakimi, S.L. Optimum Distribution of Switching Centers in a Communication Network and Some Related Graph Theoretic Problems. *Oper. Res.* **1965**, *13*, 462–475. [[CrossRef](#)]
19. Toregas, C.; ReVelle, C.; Swain, R.; Bergman, L. The location of emergency service facilities. *Oper. Res.* **1971**, *19*, 1363–1373. [[CrossRef](#)]
20. Bhattacharya, A.; Pal, M. Fuzzy tree covering number for fuzzy graphs with its real-life application in electricity distribution system. *Sadhana* **2022**, *47*, 1–10. [[CrossRef](#)]
21. Nayeem, S.M.A.; Pal, M. Shortest path problem on a network with imprecise edge weight. *Fuzzy Optim. Decis. Mak.* **2005**, *4*, 293–312. [[CrossRef](#)]
22. Ni, Y. Fuzzy minimum weight edge covering problem. *Appl. Math. Model.* **2008**, *32*, 1327–1337. [[CrossRef](#)]
23. Pal, M.; Samanta, S.; Ghorai, G. *Modern Trends in Fuzzy Graph Theory*; Springer: Berlin, Germany, 2020. [[CrossRef](#)]
24. Pramanik, T.; Samanta, S.; Pal, M. Interval-valued fuzzy planar graphs. *Int. J. Mach. Learn. Cybern.* **2016**, *7*, 653–664. [[CrossRef](#)]
25. Rashmanlou, H.; Samanta, S.; Pal, M.; Borzooei, R.A. A study on bipolar fuzzy graphs. *J. Intell. Fuzzy Syst.* **2015**, *28*, 571–580. [[CrossRef](#)]
26. Samanta, S.; Pal, M.; Rashmanlou, H.; Borzooei, R.A. Vague graphs and strengths. *J. Intell. Fuzzy Syst.* **2016**, *30*, 3675–3680. [[CrossRef](#)]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.