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Stabilization and Chaos Control of an Economic Model via a Time-Delayed Feedback Scheme

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Abstract: This paper addresses the problem of chaos control in an economic mathematical dynamical model. By regarding the control variables as the bifurcation parameters, the stability of equilibria and the existence of Hopf bifurcations of the relevance feedback system are investigated, and the criterion of controllability for the chaotic system is obtained based on a time-delayed feedback control technique. Furthermore, numerical simulations are provided to demonstrate the feasibility of our methods and results.

Keywords: economic model; chaos; time delay; stability; Hopf bifurcation

MSC: 37N40; 91B55; 93B52

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1. Introduction

Nowadays, mathematical models and analysis have been extensively used in the field of economic research [1–4], which can help better understand and predict the dynamical behavior of economical operating processes. Generally, the economical operating process can be described by nonlinear systems [5], which gives rise to the complexity and diversity of dynamical characteristics. Dynamical properties such as stability, vibration, deterministic chaos, and the stochastic process usually provide us with useful information in understanding various physical and biological phenomena [6]. However, more often than not, the unstable scenarios and irregular behaviors exhibited by economic systems are unfavorable for the study of state prediction. As we know, when it comes to studying dynamical economic models, "All stable processes, we shall predict. All unstable processes, we shall control" [7]. Thus, how to suppress or to stabilize an unstable economic behavior and convert it into other stable processes has become the core issue of the forecast and control study for economical systems.

Dynamical chaos happens quite often in nonlinear dynamical models but is usually undesirable in practice, which severely affects the reliability of prediction. Hence, in applied economic systems, the need to develop an effective solution to control chaotic vibrations has become increasingly pressing. At the present stage, two methods are mainly used for the control of chaos: by applying a specially designed external state feedback controller to chaotic systems [8–10], or by delayed feedback control method [11,12]. Both methods do not require an a priori analytical knowledge of systematical dynamics and are applicable to the experiment. In particular, for the latter method without using any external force, it does not require any computational analyses and can be particularly convenient for an experimental application. Therefore, the delayed feedback control method has been widely used in the study of control and suppression for chaos [11–13].

Models that describe features and predictive control functions in the field of economics have proven to be valuable applications. To understand and reliably predict the behavior of economic management systems, Shapovalov [13] proposed a differential equation model that describes the behavior of a mid-size firm that takes the following form [14,15]

$$\begin{cases} \dot{x} = -\sigma x + \delta y, \\ \dot{y} = \mu x + \mu y - \beta xz, \\ \dot{z} = -\gamma z + \alpha xy, \end{cases} \tag{1}$$

where α, β, σ, δ, μ, and γ are positive parameters, and the variables x, y, and z denote the growth of three main factors of production: the loan amount x, the fixed capital y, and the number of employees z.

By the coordinate transformation $(x, y, z) \to \left(\frac{\mu}{\sqrt{\alpha\beta}} x, \frac{\mu\sigma}{\delta\sqrt{\alpha\beta}} y, \frac{\mu\sigma}{\delta\beta} z \right)$, $t \to \frac{t}{\mu}$, system (1) can be reduced to a Lorenz-like system

$$\begin{cases} \dot{x} = -cx + cy, \\ \dot{y} = rx + y - xz, \\ \dot{z} = -bz + xy, \end{cases} \tag{2}$$

where $c = \frac{\sigma}{\mu}, r = \frac{\delta}{\sigma}, b = \frac{\gamma}{\mu}$. Although system (2) is slightly different from the classical Lorenz system in the coefficient of y in the second equation, which is 1 here, while in the Lorenz system this coefficient is -1, chaos would occur in system (2) as in the Lorenz system [16].

In addition, when the contained parameters satisfy the relations $\sigma^2/(\sigma - \delta) = \mu$ and $\delta < \sigma < \mu$, system (1) can be reduced to the well-known Chen system [17]

$$\begin{cases} \dot{x} = -dx + dy, \\ \dot{y} = (c - d)x + cy - xz, \\ \dot{z} = -bz + xy, \end{cases} \tag{3}$$

with $b = \gamma$, $c = \frac{\sigma^2}{\sigma - \delta} = \mu$, $d = \sigma$, $d < c$, using coordinate substitutions

$$(x, y, z) \to \left(\frac{1}{\sqrt{\alpha\beta}} x, \frac{\sigma}{\delta\sqrt{\alpha\beta}} y, \frac{\sigma}{\delta\beta} z \right).$$

As noted above, system (1) can be converted to the Lorenz-like system or the Chen system under the above conditions (cf. [15,18]). This indicates that the dynamical characteristics of system (1) can be complicated chaos, which was reflected in a series of recent studies. The quantitative characteristics of global attractor of this model, such as dimension and entropy, have been studied in Ref. [18]. It shows that when $c = 18.3$, $r = 51$ and $b = 5.7$, system (2) is chaotic (see Figure 1). Using the time-delayed feedback control approach in the Shapovalov model, the appearance of chaotic behaviors was inhibited.While the control method appears feasible in theory, additional research is needed to design and implement an effective chaos control strategy for the Shapovalov model.

The aim of this article is to show how to control and suppress the occurrence of the chaotic attractor in system (2), and to control chaotic state to the equilibrium or periodic orbits. To this end, following the idea of Refs. [19–21], we shall apply the delay feedback control approach by adding a time-delayed force $K[y - y(t - \tau)]$ to the second equation of system (2). That is,

$$\begin{cases} \dot{x} = -cx + cy, \\ \dot{y} = rx + y - xz + K[y - y(t - \tau)], \\ \dot{z} = -bz + xy, \end{cases} \tag{4}$$

where $K \in R$ and $\tau \in (0, +\infty)$ are predetermined controllable parameters. Clearly, the delayed feedback control system (4) has the same equilibria to the corresponding system (2), and when $\tau = 0$, system (4) becomes (2).

In particular, the work in Refs. [20,21] characterizes the existence of Hopf bifurcations and the stability of the equilibria for the control system with time delay τ. It should be pointed that those in Refs. [20,21] are concerned with how the equilibria lose their original stability when time delays pass through the critical values. The starting point for these studies is based on the premise that the homogeneous states of the controlled system are stable when $\tau = 0$. In this paper, we will attempt to control the chaotic behavior of a possible appearance in economic model (2). Therefore, considering that the control object should be aimed at those unstable processes rather than stable processes, we start with the assumption that the homogeneous states of the controlled system are unstable when $\tau = 0$, to investigate how the equilibrium point changes from an unstable state to a stable state with the presence of the delay, which is different from the existing studies [20,21]. This is not only the starting point of this paper and the end result, but also the core of this paper.

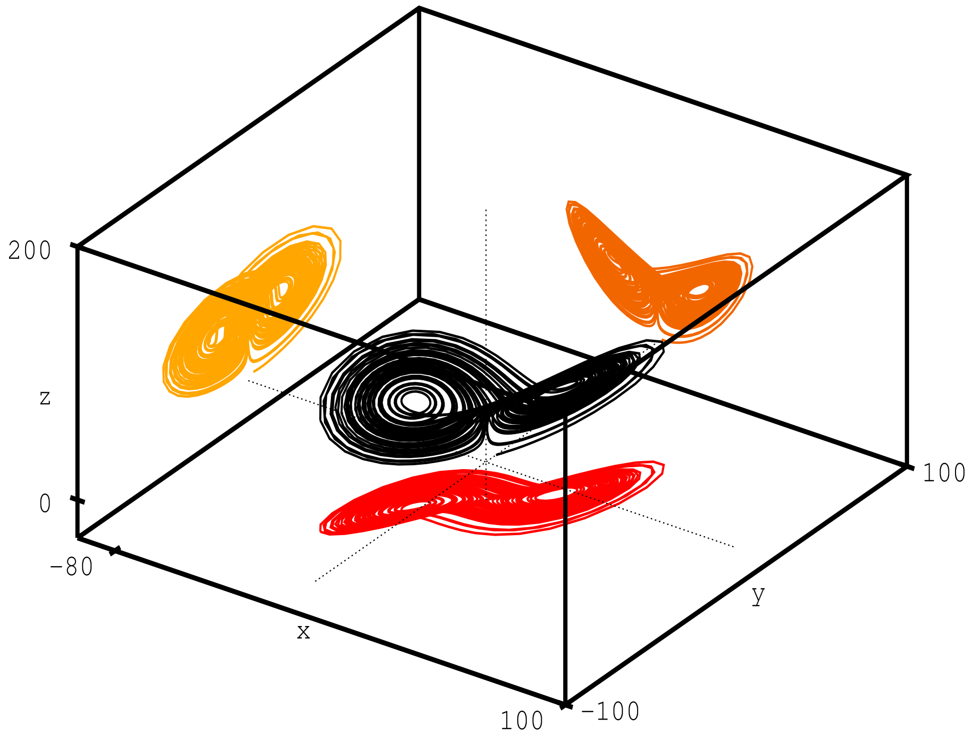

Figure 1. The chaotic attractor (uncolored trajectory) of system (2) with $c = 18.3$, $r = 51$, $b = 5.7$, and its projections (colored trajectories) along the coordinate axes.

The paper is organized as follows. In Section 2, we discuss the stability of equilibria and the existence of Hopf bifurcations of the feedback system and the criterion of the chaotic system by applying the time-delayed feedback control technique. In Section 3, numerical simulations are illustrated to demonstrate the feasibility of our methods and results. Section 4 provides a brief conclusion.

2. Stability and Bifurcation Analysis

In this section, we investigate the effect of control parameters K and τ on the dynamic stability of system (4), and present the chaos control strategy.

It easy to see that system (2) always has three equilibria: $E_0 = (0,0,0)$, $E_- = (-x_0, -y_0, z_0)$ and $E_+ = (x_0, y_0, z_0)$, where $x_0 = y_0 = \sqrt{b(r+1)}$ and $z_0 = r + 1$. We first cite two lemmas [15] on the stability of the equilibrium points of system (2).

Lemma 1. *The equilibrium state E_0 of system (2) is unstable for all parameter values.*

Lemma 2. *If one of the relations*

$$\left[\begin{array}{l} r > \frac{c(3-(c+b))}{b-(c+1)}, \;\; b > c+1, \\ r < \frac{c(3-(c+b))}{b-(c+1)}, \;\; 3-c < b < c+1, \end{array}\right. \tag{5}$$

holds for system (2)*, then the equilibria E_- and E_+ of system* (2) *are stable. If both relations* (5) *are not satisfied, then the equilibria E_- and E_+ of system* (2) *are unstable.*

Now we consider how to eliminate or suppress the possible chaotic motion in system (2). Firstly, this requests that the equilibria of system (2) are unstable. From Lemmas 1 and 2, this requires that both conditions described in (5) are not satisfied. On the other hand, considering the economic significance of system (2), it is worth nothing that the system exhibits symmetric invariance about the z-axis. In what follows, for simplicity we only discuss the stability of E_+ of system (4).

By linearizing system (4) at $E_+ = (x_0, y_0, z_0)$, the associated Jacobi matrix $J(E_+)$ takes the following form

$$J(E_+) = \begin{pmatrix} -c & c & 0 \\ r - z_0 & 1 + K - Ke^{-\lambda\tau} & -x_0 \\ y_0 & x_0 & -b \end{pmatrix}, \tag{6}$$

By substituting $x_0 = y_0 = \sqrt{b(r+1)}$ and $z_0 = r+1$ into (6), the characteristic equation corresponding to $J(E_*)$ will be

$$\begin{aligned} \lambda^3 + (b+c-1-K)\lambda^2 + (br+bc-bK-cK)\lambda + 2bc(r+1) - bcK \\ + \Big[\lambda^2 + (b+c)\lambda + bc\Big]Ke^{-\lambda\tau} = 0. \end{aligned} \tag{7}$$

Let

$$\begin{aligned} m_2 &= b+c-1-K, & m_1 &= br+bc-(b+c)K, & m_0 &= 2bc(r+1)-bcK, \\ n_2 &= K, & n_1 &= (b+c)K, & n_0 &= bcK. \end{aligned}$$

Then Equation (7) reduces to

$$\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau} = 0. \tag{8}$$

Therefore, the stability problem of equilibrium E_+ of system (4) is transformed into the distribution problem of the roots of the transcendental Equation (8). In the discussion that follows, we also need the following result, which was proved by Ruan and Wei in Ref. [22].

Lemma 3. *For the exponential polynomial*

$$\begin{aligned} & p\Big(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}\Big) = \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ & = \Big[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\Big]e^{-\lambda\tau_1} + \cdots + \Big[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\Big]e^{-\lambda\tau_m}, \end{aligned}$$

as $(\tau_1, \tau_2, \cdots, \tau_m)$ vary, the sum of orders of the zeros of $p\big(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}\big)$ on the open right half plane can change only if a zero appears on or crosses the imaginary axis.

Obviously, $\lambda = i\omega$ $(\omega > 0)$ is a root of (8) if and only if

$$-\omega^3 i - m_2\omega^2 + m_1\omega i + m_0 + (-n_2\omega^2 + n_1\omega i + n_0)e^{-\omega\tau i} = 0.$$

Separating the real and imaginary parts, one can get that

$$\left\{\begin{array}{l} -\omega^3 + m_1\omega + n_1\omega\cos\omega\tau + (n_2\omega^2 - n_0)\sin\omega\tau = 0, \\ -m_2\omega^2 + m_0 + (n_0 - n_2\omega^2)\cos\omega\tau + n_1\omega\sin\omega\tau = 0. \end{array}\right. \tag{9}$$

It follows from (9) that

$$\omega^6 + p_0\omega^4 + q_0\omega^2 + r_0 = 0, \tag{10}$$

where

$$p_0 = m_2^2 - 2m_1 - n_2^2,\ \ q_0 = m_1^2 - 2m_0m_2 + 2n_0n_2 - n_1^2,\ \ r_0 = m_0^2 - n_0^2.$$

Let $z = \omega^2$, and Equation (10) can be written as

$$h(z) := z^3 + p_0z^2 + q_0z + r_0 = 0. \tag{11}$$

Since $\lim_{t\to+\infty} = +\infty$ and $h(0) = r_0$, then (11) has at least one positive real root when $r_0 = m_0^2 - n_0^2 < 0$.

In Refs. [20,23], the distribution of roots of (11) is discussed in detail. We present here only the relevant results, providing a concise overview for further study without delving into excessive details.

Lemma 4. *For the polynomial Equation* (11)

(i) *If $r_0 < 0$, then* (11) *has at least one positive root;*
(ii) *If $r_0 \geq 0$ and $\Delta = p_0^2 - 3q_0 \leq 0$, then* (11) *has no positive roots;*
(iii) *If $r_0 \geq 0$ and $\Delta = p_0^2 - 3q_0 > 0$, then* (11) *has positive roots if and only if $z_1^* = \frac{-p_0+\sqrt{\Delta}}{3} > 0$ and $h(z_1^*) \leq 0$.*

Now, suppose that Equation (11) has positive roots. Without loss of generality, we assume that it has three positive roots, defined by z_1, z_2, and z_3. Consequently, Equation (10) then has three positive roots: $\omega_1 = \sqrt{z_1}$, $\omega_2 = \sqrt{z_2}$ and $\omega_3 = \sqrt{z_3}$.

From (9), we can derive

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left\{\arccos\left[\frac{n_1\omega_k^2(\omega_k^2 - m_1) - (m_2\omega_k^2 - m_0)(n_2\omega_k^2 - n_0)}{(n_1\omega_k)^2 + (n_2\omega_k^2 - n_0)^2}\right] + 2j\pi\right\}, \tag{12}$$

where $k = 1,2,3;\ j = 0,1,2,\ldots,$ then $\pm\omega_k i$ is a pair of purely imaginary roots of (8) when $\tau = \tau_k^{(j)}$.

Define

$$\tau_0 = \tau_{k_o}^{(0)} = \min_{k\in\{1,2,3\}} \tau_k^{(0)}. \tag{13}$$

Note that when $\tau = 0$, Equation (8) reduces to

$$\lambda^3 + (m_2 + n_2)\lambda^2 + (m_1 + n_1)\lambda + m_0 + n_0 = 0. \tag{14}$$

The above analysis proves that (8) has no purely imaginary roots for any $\tau \geq 0$ when (11) has no positive roots. Thus, applying Lemmas 3 and 4 to Equation (8), we can obtain the following result.

Lemma 5. *For the transcendental Equation* (8)

(i) *If $r_0 \geq 0$ and $\Delta = p_0^2 - 3q_0 \leq 0$, then all roots with positive real parts of* (8) *have the same sum to those of the polynomial Equation* (14) *for all $\tau \geq 0$;*
(ii) *If either $r_0 < 0$ or $r_0 \geq 0$, $\Delta = p_0^2 - 3q_0 > 0$, $z_1^* > 0$ and $h(z_1^*) \leq 0$, then all roots with positive parts of* (8) *have the same sum to those of the polynomial Equation* (14) *for $\tau \in [0, \tau_0)$.*

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (8) near $\tau = \tau_k^{(j)}$ satisfying $\alpha(\tau_k^{(j)}) = 0$, $\omega(\tau_k^{(j)}) = \omega_k$. Noticing that λ is a continuously differentiable function of τ, substituting $\lambda(\tau)$ into the left hand side of (8) and taking derivative with respect to τ, we can obtain

$$(3\lambda^2 + 2m_2\lambda + m_1)\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} + (2n_2\lambda + n_1)e^{-\lambda\tau}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} - (n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau}\left(\tau\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} + \lambda\right) = 0.$$

This gives

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{(3\lambda^2+2m_2\lambda+m_1)+(2n_2\lambda+n_1)e^{-\lambda\tau}-(n_2\lambda^2+n_1\lambda+n_0)\tau e^{-\lambda\tau}}{(n_2\lambda^2+n_1\lambda+n_0)\lambda e^{-\lambda\tau}}$$
$$= \frac{(3\lambda^2+2m_2\lambda+m_1)e^{\lambda\tau}}{(n_2\lambda^2+n_1\lambda+n_0)\lambda} + \frac{2n_2\lambda+n_1}{(n_2\lambda^2+n_1\lambda+n_0)\lambda} - \frac{\tau}{\lambda}.$$

Thus

$$\begin{aligned}
&\operatorname{sign}\left[\frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau}\right]_{\tau=\tau_k^{(j)}}\\
=&\operatorname{sign}\left[\mathrm{Re}\left(\frac{(3\lambda^2+2m_2\lambda+m_1)e^{\lambda\tau}}{(n_2\lambda^2+n_1\lambda+n_0)\lambda} + \frac{2n_2\lambda+n_1}{(n_2\lambda^2+n_1\lambda+n_0)\lambda} - \frac{\tau}{\lambda}\right)^{-1}\right]_{\tau=\tau_k^{(j)}}\\
=&\operatorname{sign}\mathrm{Re}\left[\frac{(m_1-3\omega_k^2+2m_2\omega_k i)\left(\cos\omega_k\tau_k^{(j)}+i\sin\omega_k\tau_k^{(j)}\right)}{-n_1\omega_k^2+i(-n_2\omega_k^2+n_0)\omega_k} + \frac{n_1+i2n_2\omega_k}{-n_1\omega_k^2+i(-n_2\omega_k^2+n_0)\omega_k}\right]\\
=&\operatorname{sign}\frac{1}{R_0}\left[\left\{(m_1-3\omega_k^2)\cos\omega_k\tau_k^{(j)}-2m_2\omega_k\sin\omega_k\tau_k^{(j)}\right\}(-n_1\omega_k^2)+\left\{(m_1-3\omega_k^2)\sin\omega_k\tau_k^{(j)}\right.\right.\\
&\left.\left.+2m_2\omega_k\cos\omega_k\tau_k^{(j)}\right\}(-n_2\omega_k^2+n_0)\omega_k - n_1^2\omega_k^2+2n_2\omega_k^2(-n_2\omega_k^2+n_0)\right]\\
=&\operatorname{sign}\frac{1}{R_0}\left[(3\omega_k^2-m_1)\omega_k\left\{n_1\omega_k\cos\omega_k\tau_k^{(j)}+(n_2\omega_k^2-n_0)\sin\omega_k\tau_k^{(j)}\right\}\right.\\
&\left.+2m_2\omega_k^2\left\{(n_0-n_2\omega_k^2)\cos\omega_k\tau_k^{(j)}+n_1\omega_k\sin\omega_k\tau_k^{(j)}\right\}-n_1^2\omega_k^2+2n_2\omega_k^2(-n_2\omega_k^2+n_0)\right]\\
=&\operatorname{sign}\frac{1}{R_0}\left[(3\omega_k^2-m_1)\omega_k\{\omega_k^3-m_1\omega_k\}+2m_2\omega_k^2\{m_2\omega_k^2-m_0\}\right.\\
&\left.-n_1^2\omega_k^2+2n_2\omega_k^2(-n_2\omega_k^2+n_0)\right]\\
=&\operatorname{sign}\frac{1}{R_0}\left[(3\omega_k^2-m_1)(\omega_k^2-m_1)+2m_2(m_2\omega_k^2-m_0)-n_1^2\omega_k^2+2n_2\omega_k^2(-n_2\omega_k^2+n_0)\right]\\
=&\operatorname{sign}\frac{1}{R_0}\left[3\omega_k^4+2(m_2^2-2m_1-n_2^2)\omega_k^2+(m_1^2-2m_0m_2+2n_0n_2-n_1^2)\right]\\
=&\operatorname{sign}\frac{1}{R_0}\left[3\omega_k^4+2p_0\omega_k^2+q_0\right]\\
=&\operatorname{sign}\frac{\omega_k^2}{R_0}\left[h'(\omega_k^2)\right]\\
=&\operatorname{sign}\left[h'(\omega_k^2)\right]\\
=&\operatorname{sign}\left[h'(z_k)\right],
\end{aligned}$$

where $R_0 = n_1^2\omega_k^4+\left(n_0-n_2\omega_k^2\right)^2$, $z_k = \omega_k^2$, and ω_k are the three positive roots of Equation (10). We observe that if $h'(z_k)\neq 0$, then the following transversality conditions

$$\left.\frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau}\right|_{\tau=\tau_k^{(j)}} \neq 0$$

are satisfied.

We recall that, under both relations (5) is violated, the equilibrium E_+ of system (4) with $\tau=0$ is unstable, and the polynomial Equation (14) then has at least one root with a positive real part. In view of Lemma 5, the multiplicity of roots with positive real parts of Equation (8) can change only if a root appears on or crosses the imaginary axis as the time delay τ varies. Based on the above analyses, and with the Hopf bifurcation theorem for functional differential equations, in summary we obtain the following result.

Theorem 1. *Let $\tau_k^{(j)}$ and τ_0 be defined by (12) and (13), respectively. For the delayed feedback control system (4)*

(i) If $r_0 \geq 0$ and $\Delta = p_0^2 - 3q_0 \leq 0$, then the equilibrium E_+ of system (4) is unstable for all $\tau \geq 0$;

(ii) If either $r_0 < 0$ or $r_0 \geq 0$, $\Delta = p_0^2 - 3q_0 > 0$, $z_1^* > 0$ and $h(z_1^*) \leq 0$, then the equilibrium E_+ of system (4) is unstable for $\tau \in [0, \tau_0)$;

(iii) If the conditions of (ii) are satisfied, and $h'(z_k) \neq 0$, then system (4) undergoes a series of Hopf bifurcations at the equilibrium E_+ when $\tau = \tau_k^{(j)}$.

3. Numerical Simulations

In this section, we numerically validate the previous analytical findings. As an example, we take the fixed values of $c = 18.3$, $r = 51$ and $b = 5.7$ in the delayed feedback control system (4), and consider K and τ as the controlling parameters. Then, system (4) takes the form:

$$\begin{cases} \dot{x} = -18.3x + 18.3y, \\ \dot{y} = 5.1x + y - xz + K[y - y(t-\tau)], \\ \dot{z} = -5.7z + xy, \end{cases} \tag{15}$$

The system (15) has three equilibria $E_0 = (0,0,0)$, $E_- = (-x_0, -y_0, z_0)$ and $E_+ = (x_0, y_0, z_0)$, where $x_0 = y_0 = \sqrt{296.4}$ and $z_0 = 52$. Clearly, when $K = 0$ or $\tau = 0$, system (15) is chaotic (cf. Figure 1).

From the discussions in the previous section, we get the corresponding characteristic equation of system (15) at E_+

$$\lambda^3 + (23-K)\lambda^2 + (395.01 - 24K)\lambda + 10848.24 - 104.31K + (\lambda^2 + 24\lambda + 104.31)Ke^{-\lambda\tau} = 0. \tag{16}$$

In the case of $\tau = 0$, (16) reduces into

$$\lambda^3 + 23\lambda^2 + 395.01\lambda + 10848.24 = 0. \tag{17}$$

By a direct calculation we get the three roots of Equation (16):

$$\lambda_1 \approx -24.75, \; \lambda_\pm \approx 0.87 \pm 20.92i.$$

In particular, we obtain the following expressions for the expected parameters

$$\begin{aligned} p_0 &= 2K - 261.02, \\ q_0 &= -342986.14 + 7534.18K, \\ r_0 &= 2263159.83(52 - K), \\ \Delta &= p_0^2 - 3q_0 = 4K^2 - 23646.62K + 1097089.86. \end{aligned}$$

From Theorem 1, we know that when $K > 52$, the characteristic (16) always has roots with positive real parts for all $\tau > 0$. Thus, for the purpose of controlling chaos, we consider $K < 52$, and we also recognize that $\Delta > 0$ if $K < 0$. So we take $K = -2$.

In this case, we have

$$h(z) = z^3 - 265.02z^2 - 358054.66z + 122210630.76. \tag{18}$$

It follows from (12) and (18) that

$$
\begin{aligned}
&z_1 \approx 407.0160, && z_2 \approx 481.5423, && z_3 \approx -623.5383,\\
&\omega_1 \approx 20.1746, && \omega_2 \approx 21.9441. &&\\
&\tau_1^{(j)} \approx 0.0714 + \frac{2j\pi}{\omega_1}, && h'(z_1) \approx -76719, &&\\
&\tau_2^{(j)} \approx 0.0757 + \frac{2j\pi}{\omega_2}, && h'(z_2) \approx 82358, &&\\
&\Delta \approx 1144399.58, && z_1^* \approx 444.93, && h(z_1^*) \approx -1.4832 \times 10^6.
\end{aligned}
$$

In particular, we get $\tau_0 = \min\{\tau_1^{(0)}, \tau_2^{(0)}\} = \tau_1^{(0)}$ and Hopf bifurcation values

$$\tau_1^{(0)} \approx 0.0714 < \tau_2^{(0)} \approx 0.0757 < \tau_2^{(1)} \approx 0.3620 < \tau_1^{(1)} \approx 0.3828 < \cdots.$$

Based on these calculations and Theorem 1, we therefore know that the steady states $E_\pm$ of system (15) are unstable when $\tau \in [0, \tau_1^{(0)})$. This property is illustrated by the numerical simulations in Figures 2 and 3. When $\tau_1^{(0)} < \tau < \tau_2^{(0)}$, the steady states $E_\pm$ are asymptotically stable (cf. Figure 4).

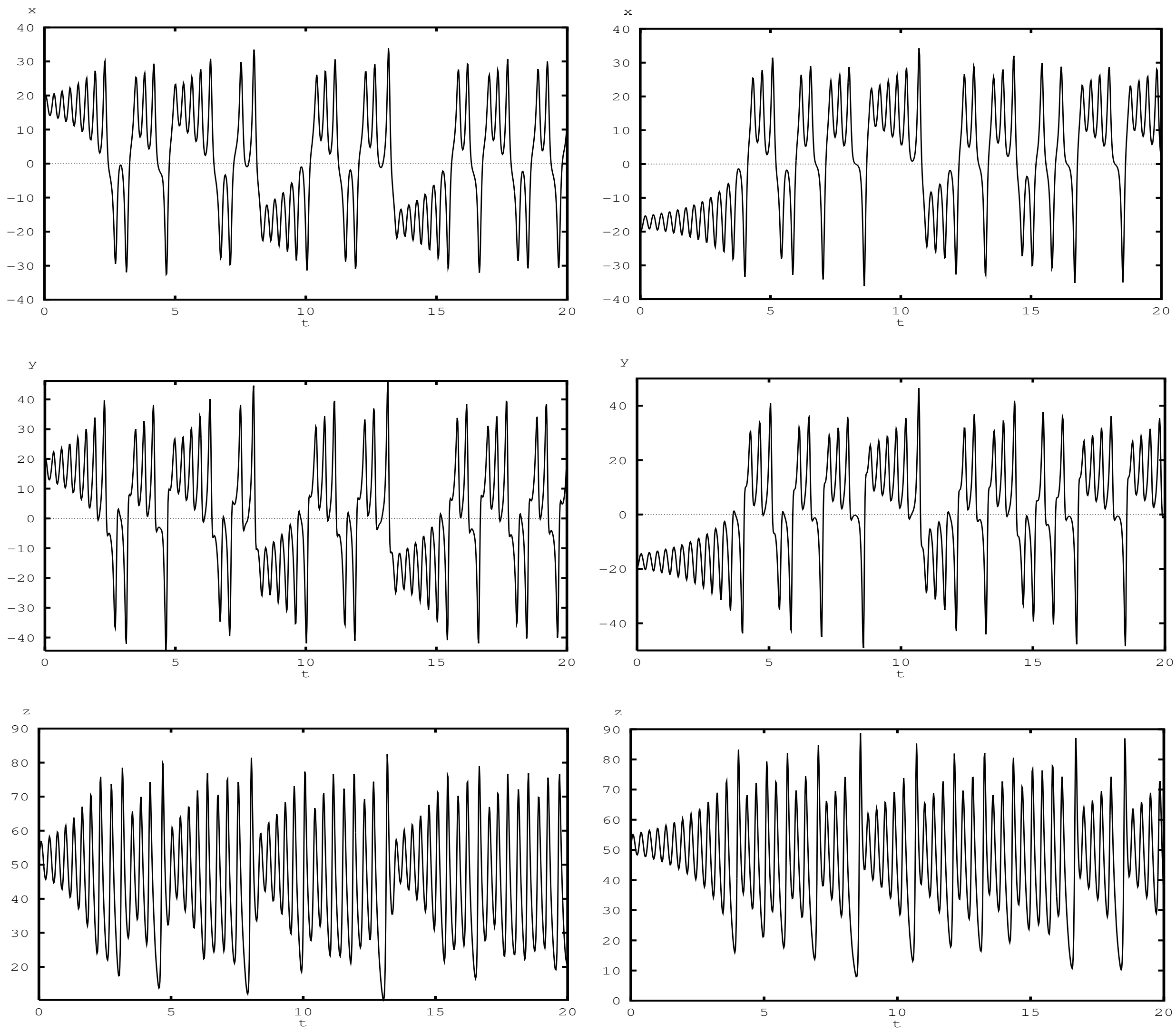

Figure 2. Instability of the steady states E_+ ((**left**) column) and E_- ((**right**) column) of system (15). Here $K = -2$, $\tau = 0.03$, initial value $(x(0), y(t), z(0)) = (20, 20, 50)$, $t \in [-0.03, 0]$.

When $\tau = \tau_1^{(j)}$ or $\tau = \tau_2^{(j)}$, system (15) undergoes a sequence of Hopf bifurcations near the equilibria $E_\pm$. Moreover, properties of the bifurcated periodic solutions, such as stability and direction, can be clearly demonstrated by applying the normal form theory and the center manifold reduction for functional differential equations. We will not cover them in the present paper. To illustrate and test the existence of the stable bifurcating periodic solutions that might appear in system (15), we only give a tentative computation. Let $\tau = 0.22 > \tau_2^{(0)}$ so that the steady states $E_\pm$ of system (15) are unstable. Meanwhile, a family of periodic orbits bifurcate from $E_\pm$ might emerge. This property is illustrated by the numerical simulations in Figure 5.

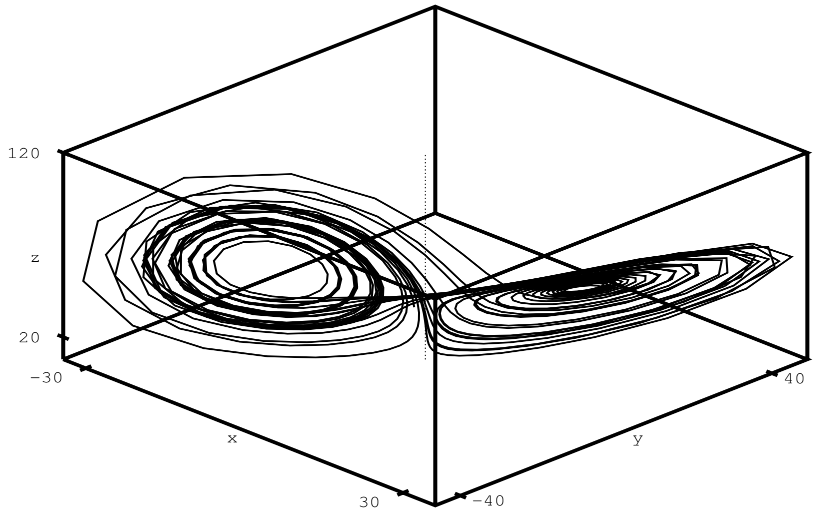

Figure 3. Chaos still exists in system (15). Here $K = -2$, $\tau = 0.03$, initial value $(x(0), y(t), z(0)) = (20, 20, 50)$, $t \in [-0.03, 0]$.

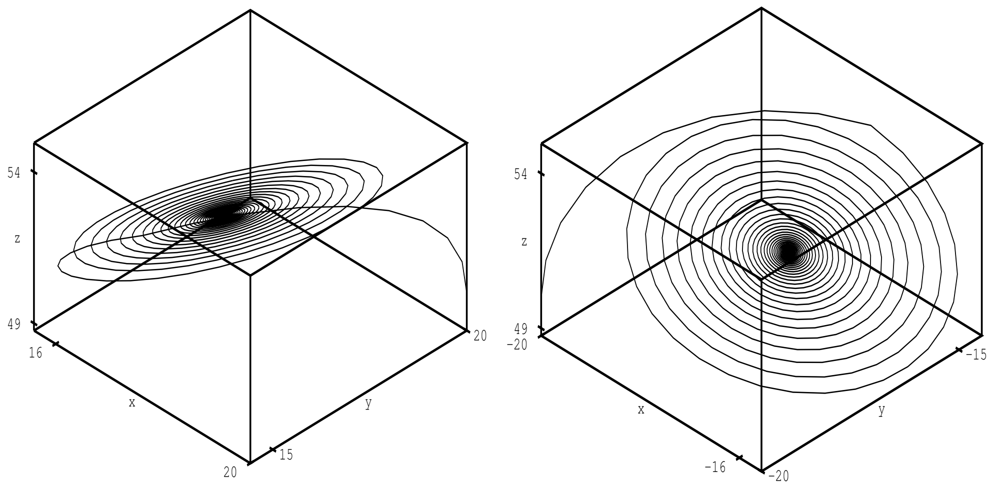

Figure 4. The local asymptotic stability of the steady states E_+ (**left**) and E_- (**right**) of system (15). Here $K = -2$, $\tau = 0.073$, initial value $(x(0), y(t), z(0)) = (20, 20, 50)$, $t \in [-0.073, 0]$.

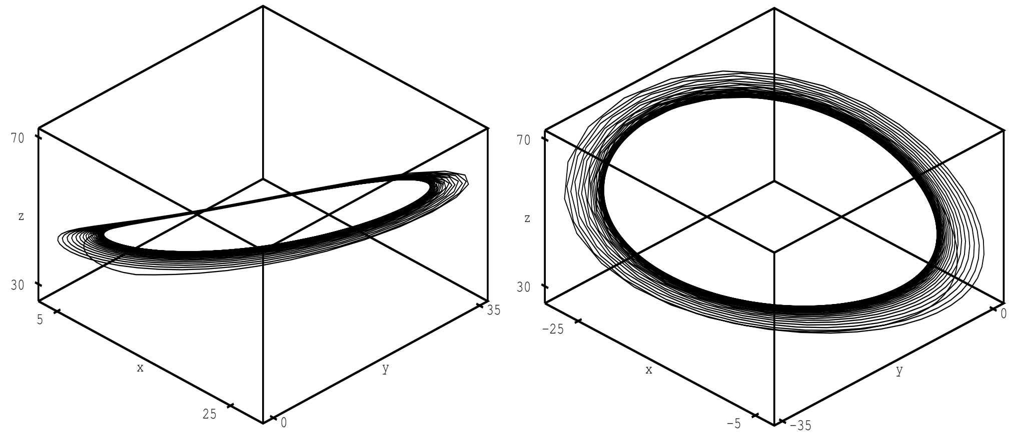

Figure 5. Instability of $E_{\pm}$ and stable bifurcating periodic solutions of the system (15) from $E_{\pm}$. Here $K = -2$, $\tau = 0.22$, initial value $(x(0), y(t), z(0)) = (20, 20, 50)$, $t \in [-0.22, 0]$.

The above calculated results indicate that when the steady states are stable or the bifurcating periodic solutions are asymptotically stable, chaos will disappear, which means that the original chaotic attractor of system (15) can be effectively controlled when the proper control parameters are selected. At this point, a specific control scheme for controlling chaos is now complete, and the operational processing goes as follows.

First, note that when $K = 0$ or $\tau = 0$, system (15) becomes

$$\begin{cases} \dot{x} = -18.3x + 18.3y, \\ \dot{y} = 5.1x + y - xz, \\ \dot{z} = -5.7z + xy, \end{cases} \tag{19}$$

and it is chaotic (cf. Figure 1). In order to achieve the purpose of controlling chaos, we add the delayed feedback item to system (19) at a certain moment, such as at $t = 8$. Then, the dynamical character of system (19) can change. Figure 6 shows the state response curves of x, y, and z. It is interesting to observe that after adding the feedback controller, system (19) will eventually converge to the steady states or the periodic solutions, depending on the different control parameters.

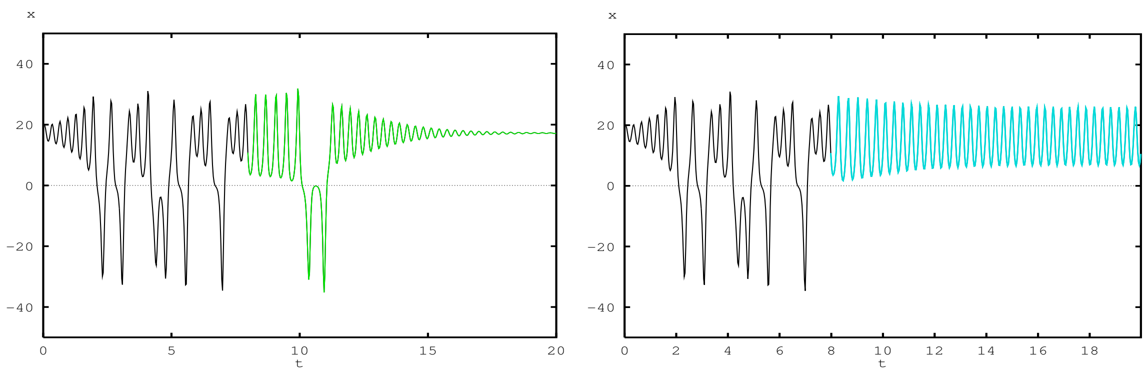

Figure 6. *Cont.*

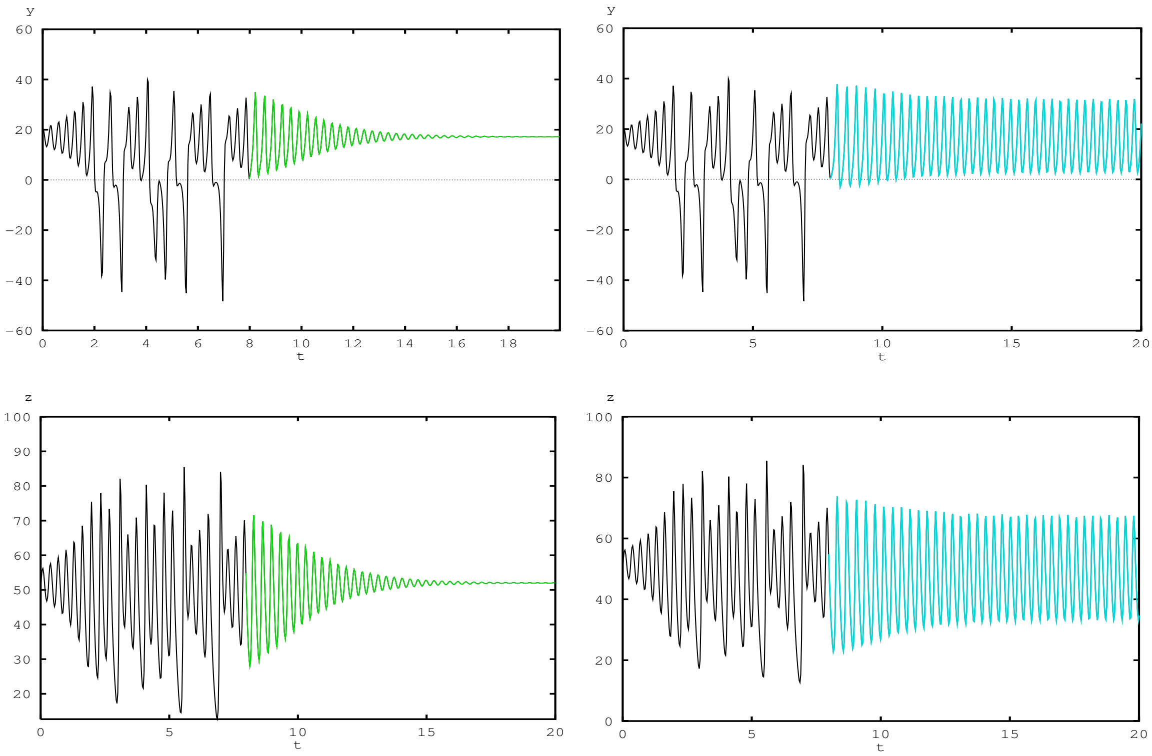

Figure 6. Response curves of the components of system (19) when adding the delayed feedback item $-2[y - y(t - \tau)]$ to its second equation. The figures in the (**left**) and (**right**) columns correspond to $\tau = 0.073$ and $\tau = 0.22$, respectively. Here, the initial value $(x(0), y(0), z(0)) = (20, 20, 50)$.

4. Conclusions

In this paper, a time-delayed feedback control approach was used to control the chaotic behavior of possible appearance in the Shapovalov model (2). By adding a time-delayed force to the second equation of system (2), we investigated the effect of time delay on the stability of equilibria $E_\pm = (\pm x_0, \pm y_0, z_0)$ of system (2). Consequently, we have proposed a simple but effective scheme that can be used to control and suppress the occurrence of the chaotic attractor in system (2). Both theoretical analysis and numerical simulations demonstrate that the chaos can be changed into equilibrium states or stable periodic orbits by using this scheme. This way, the unpredictable chaotic behaviors may be suppressed, which improves the dynamic and static performance of the economic system. This is significant for the study on state prediction of economic processes.

It should also be noted that the Shapovalov model (2) and the corresponding control model (4) are two completely different systems. Hence, their dynamic characteristics may not be the same. Further details about this sort of information are beyond the scope of this article. However, note that when adding time-delayed term $K[y - y(t - \tau)]$ to the second equation of system (2) only, the other two equations of (2) remain unchanged, and it then becomes (4). From a modeling standpoint, the initial value conditions are changing with the introduction of the time-delayed feedback item in Shapovalov model (2). That is, in the study of dynamics of (2), we are of the opinion that the growth of the variable $y(t)$ with time t not only depends on the current state $y(0)$, but also depends on the previous states $y(-\tau)$. In fact, the factor affecting every indicator of a firm is various, and it is indisputable that the volume of the upfront investment of fixed capital is one of the main factors. Given that the y in model (2) stands for fixed capital of a mid-size firm, it is believed that the delay functional differential Equation (4) is a more accurate mathematical description for the firm than the ordinary differential Equation (2). The analysis and simulation results in this paper also show that the proposed chaos control scheme based on model (4) is reasonable and feasible.

In the present paper we only add a time-delayed term to the second equation of (2) for the sake of simplicity. As a result, we have obtained a control system with delay that is capable of controlling chaos. It is also worth pointing out that, in theory by adding time-delayed force to any or all of the equations in (2), we can get different delayed feedback control systems, which can be used to realize the purpose of chaos control. It should also be noted that we used a standard linearization method to analyze the stability of the equilibrium point of the delayed feedback system (4). The obtained result about the dynamical behavior of the equilibrium point is of a local character rather than a global one. It would be difficult to give the exact scope of parameter values in which the equilibrium point is locally asymptotically stable, as this requires the global stability analysis, so we will address this important problem in future work.

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