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Article

Novel Contributions to the System of Fractional Hamiltonian Equations

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Abstract: This work aims to analyze a new system of two fractional Hamiltonian equations. We propose an effective method for transforming the established model into a system of two distinct equations. Two functionals that are connected to the converted system of fractional Hamiltonian systems are introduced together with a new space, and it is demonstrated that these functionals are bounded below on this space. The hypotheses presented here differ from those provided in the literature.

Keywords: system; fractional Hamiltonian equations; bounded below

MSC: 35B38; 35A15; 26A33; 34C37

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1. Introduction

As a result of their numerous applications in the practice of mathematical modeling in mechanics, physics, biochemistry, control theory, economics, and biomechanics, fractional differential equation (FDE) theories have developed rapidly over the last two decades (see [1–9]). According to [10–14], some methods for solving fractional systems are obtained by extending procedures from differential equations theory. Important classes of these systems are Hamiltonian fractional systems, which now form a rich field of research in addition to their essential applications in a variety of domains. Several studies yielded intriguing results with respect to the multiplicity and existence of solutions for different fractional systems. Researchers have used various nonlinear analysis procedures in these studies, including the comparison method, topological degree theory, and many others. We discovered that critical point theory is an efficient tool for determining the presence of solutions to differential equations (cf. [15,16]).

Encouraged by the aforementioned classic work, the authors of [17] demonstrated that critical point theory is an efficient approach to determining the existence of solutions for the fractional boundary value problem.

$$\begin{cases} {}_{\sigma}\mathfrak{D}^{\alpha}_{T}({}_{0}\mathfrak{D}^{\alpha}_{\sigma}u(\sigma)) = \nabla V(\sigma, u(\sigma)), & \text{a.e. } \sigma \in [0,T], \\ u(0) = u(\sigma), \end{cases} \tag{1}$$

where α in $(\frac{1}{2}, 1)$, $V \in C^1([0,T] \times \mathbb{R}^N, \mathbb{R})$, $u \in \mathbb{R}^N$, and $\nabla V(\sigma, u)$ is the gradient of V at u. They also obtained the existence of at least one nontrivial solution. The author of [18] examined the following FHS:

$$\begin{cases} -{}_{\sigma}\mathfrak{D}^{\alpha}_{\infty}({}_{-\infty}\mathfrak{D}^{\alpha}_{\sigma}u(\sigma)) - \mathcal{A}(\sigma)u(\sigma) + \nabla V(\sigma, u(\sigma)) = 0, \\ u \in H^{\alpha}(\mathbb{R}, \mathbb{R}^N), \end{cases} \tag{2}$$

where the matrix $\mathcal{A}(\sigma) \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is positive, definite symmetric for all $\sigma \in \mathbb{R}$ and ${}_{\sigma}\mathfrak{D}^{\alpha}_{\infty}$ and ${}_{-\infty}\mathfrak{D}^{\alpha}_{\sigma}$ denote the right and the left Liouville–Weyl fractional derivatives with the order α.

The existence of solutions for the Hamiltonian system (1) was examined in several articles, including [19–30].

In [18], it is demonstrated using the Mountain Pass Theorem that Equation (2) has at least one nontrivial solution under some conditions on V and $\mathcal{A}$:

(Y_0) The matrix $\mathcal{A}(\sigma)$ is positive definite and symmetric $\forall \sigma \in \mathbb{R}$; moreover, $\exists\ l \in C(\mathbb{R}, (0, \infty)) : (\mathcal{A}(\sigma)y, y) \geq l(\sigma)|y|^2$, for all $(\sigma, y) \in \mathbb{R} \times \mathbb{R}^N$ and $l(\sigma) \to \infty$ as $|\sigma| \to \infty$;

($\mathcal{F}_1$) $|\nabla V(\sigma, y)| = o(|y|)$ as $|y| \to 0$ uniformly in $\sigma \in \mathbb{R}$;

($\mathcal{F}_2$) $\exists\ \overline{V} \in C(\mathbb{R}^N, \mathbb{R}) :\ |V(\sigma, y)| + |\nabla V(\sigma, y)| \leq |\overline{V}(y)|$ for all $(\sigma, y) \in \mathbb{R} \times \mathbb{R}^N$;

($\mathcal{F}_3$) $\exists\ \mu > 2 :\ \ 0 < \mu V(\sigma, y) \leq (\nabla V(\sigma, y), y)$ for all, $(\sigma, y) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$.

For $\alpha = 1$, problem (2) reads as

$$\ddot{u}(\sigma) - \mathcal{A}(\sigma)u(\sigma) + \nabla V(\sigma, u(\sigma)) = 0,$$

which is a basic second-order Hamiltonian system.

More recently, the authors of [31] examined the existence of solutions for the fractional Hamiltonian system (2) under some assumptions on $\mathcal{A}$ and V.

The present work aims to examine the system of fractional Hamiltonian equations of the form:

$$\begin{cases} -D^{\eta}_{\infty}\Big(D^{\eta}_{-\infty}a(\sigma)\Big) - \mathcal{S}(\sigma)a(\sigma) - \mathcal{K}(\sigma)b(\sigma) + \nabla\vartheta(\sigma, b(\sigma)) = 0, \\ -D^{\eta}_{\infty}\Big(D^{\eta}_{-\infty}b(\sigma)\Big) - \mathcal{S}(\sigma)b(\sigma) - \mathcal{K}(\sigma)a(\sigma) + \nabla\vartheta(\sigma, a(\sigma)) = 0. \end{cases}$$

We introduce a powerful technique for splitting the present model into a system of two different equations. We introduce a new space and two functionals that are related to the converted system of fractional Hamiltonian systems, and we prove that they are bounded below on this space. The theories presented here are distinct from those presented in the literature.

The remaining sections of this work are described in the following: The next section proceeds with a review of several fundamental concepts and terms used in fractional theory. Section 3 discusses the system of two fractional Hamiltonian equations. A new fractional space is introduced in Section 4. Proof of the primary findings is presented in Section 5. We complete our paper with a conclusion.

2. Preliminaries

This section begins with a review of several fundamental concepts and terms employed in fractional analyses.

Let Γ symbolize the basic Euler Gamma function in the fractional analysis.

Definition 1. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be an integrable function. The η-left-sided Liouville–Weyl fractional integral of ω is designated as follows:*

$$J^{\eta}_{-\infty}\omega(\sigma) := \frac{1}{\Gamma(\eta)} \int_{-\infty}^{\sigma} \frac{\omega(\varsigma)}{(\sigma - \varsigma)^{1-\eta}} d\varsigma, \quad \sigma \in \mathbb{R}\ \ \eta > 0.$$

Definition 2. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be an integrable function. The η-right-sided Liouville–Weyl fractional integral of ω is designated as follows:*

$$J^{\eta}_{\infty}\omega(\sigma) := \frac{1}{\Gamma(\eta)} \int_{\sigma}^{\infty} \frac{\omega(\varsigma)}{(\sigma - \varsigma)^{1-\eta}} d\varsigma, \quad \sigma \in \mathbb{R}\ \ \eta > 0.$$

Remark 1. *Letting*

$$\psi_\eta(\sigma) := \sigma^{\eta-1}/\Gamma(\eta),$$

then,

$$J^{\eta}_{0_+}\omega(\sigma) = (\psi_\eta \star \omega)(\sigma).$$

Definition 3. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a continuous function. The η-left-sided Liouville–Weyl fractional derivative of ω is given by*

$$D^{\eta}_{-\infty}\omega(\sigma) := \frac{1}{\Gamma(1-\eta)}\frac{d}{d\sigma}\int_{-\infty}^{\sigma}\frac{\omega(\varsigma)}{(\sigma-\varsigma)^{\eta}}d\varsigma, \quad \eta > 0.$$

Definition 4. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a continuous function. The η-right-sided Liouville–Weyl fractional derivative of ω is given by*

$$D^{\eta}_{\infty}\omega(\sigma) := \frac{1}{\Gamma(1-\eta)}\frac{d}{d\sigma}\int_{\sigma}^{\infty}\frac{\omega(\varsigma)}{(\sigma-\varsigma)^{\eta}}d\varsigma \quad \eta > 0.$$

Remark 2. *We have*

$$D^{\eta}_{-\infty}\omega(\sigma) = \frac{d}{d\sigma}J^{1-\eta}_{-\infty}\omega(\sigma), \quad \eta > 0$$

and

$$D^{\eta}_{\infty}\omega(\sigma) = -\frac{d}{d\sigma}J^{1-\eta}_{\infty}\omega(\sigma).$$

3. System of Two Fractional Hamiltonian Equations

The goal of the current research is to analyze a system of fractional Hamiltonian equations of the following form:

$$\begin{cases} -D^{\eta}_{\infty}\Big(D^{\eta}_{-\infty}a(\sigma)\Big) - \mathcal{S}(\sigma)a(\sigma) - \mathcal{K}(\sigma)b(\sigma) + \nabla\vartheta(\sigma, b(\sigma)) = 0, \\ -D^{\eta}_{\infty}\Big(D^{\eta}_{-\infty}b(\sigma)\Big) - \mathcal{S}(\sigma)b(\sigma) - \mathcal{K}(\sigma)a(\sigma) + \nabla\vartheta(\sigma, a(\sigma)) = 0. \end{cases} \tag{3}$$

We assume that

$$\nabla\vartheta(\sigma, \alpha u(\sigma) + \beta v(\sigma)) = \alpha\nabla\vartheta(\sigma, u(\sigma)) + \beta\nabla\vartheta(\sigma, v(\sigma)).$$

Using the procedure outlined in [32], consider the following transformation:

$$\mathrm{A} := a - b,$$

$$\mathrm{B} := a + b.$$

Lemma 1. *System (3) can be presented in the following form:*

$$-D^{\eta}_{\infty}\Big(D^{\eta}_{-\infty}B(\sigma)\Big) - \mathcal{S}(\sigma)B(\sigma) - \mathcal{K}(\sigma)A(\sigma) + \nabla\vartheta(\sigma, B(\sigma)) = 0, \tag{4}$$

$$-D^{\eta}_{\infty}\Big(D^{\eta}_{-\infty}A(\sigma)\Big) - \mathcal{S}(\sigma)A(\sigma) + \mathcal{K}(\sigma)A(\sigma) - \nabla\vartheta(\sigma, A(\sigma)) = 0. \tag{5}$$

Proof. We have,

$$a = \frac{\mathrm{A}+\mathrm{B}}{2},$$

$$b = \frac{\mathrm{A}-\mathrm{B}}{2}.$$

By substituting them into (3), we get

$$-D_{\infty}^{\eta}\Big(D_{-\infty}^{\eta}(\mathrm{B}+\mathrm{A})(\sigma)\Big)-\mathcal{S}(\sigma)(\mathrm{B}+\mathrm{A})(\sigma)-\mathcal{K}(\sigma)(\mathrm{B}-\mathrm{A})(\sigma)+\nabla\vartheta(\sigma,(\mathrm{B}-\mathrm{A})(\sigma))=0, \quad (6)$$

$$-D_{\infty}^{\eta}\Big(D_{-\infty}^{\eta}(\mathrm{B}-\mathrm{A})(\sigma)\Big)-\mathcal{S}(\sigma)(\mathrm{B}-\mathrm{A})(\sigma)-\mathcal{K}(\sigma)(\mathrm{B}+\mathrm{A})(\sigma)+\nabla\vartheta(\sigma,(\mathrm{B}+\mathrm{A})(\sigma))=0. \quad (7)$$

We obtain (4) by adding Equations (6) and (7) together and then subtracting (7) from (6). □

4. Fractional Space

Let us define the following semi-norm:

$$|\omega|_{J_{-\infty}^{\eta}} := \|D_{-\infty}^{\eta}\omega\|_{\mathcal{L}^2}, \quad \eta > 0.$$

Thus, the corresponding norm is given by

$$\|\omega\|_{J_{-\infty}^{\eta}} := \Big(\|\omega\|_{\mathcal{L}^2}^2 + |\omega|_{J_{-\infty}^{\eta}}^2\Big)^{1/2}.$$

The transform

$$\widehat{\omega}(\xi) = \int_{-\infty}^{\infty} e^{-i\sigma.\xi}\omega(\sigma)\,d\sigma$$

is the Fourier transform of $\omega(.)$. Furthermore, the norm $\|.\|_{\eta}$ is given by

$$\|\omega\|_{\eta} := (\|\omega\|_{\mathcal{L}^2}^2 + |\omega|_{\eta}^2)^{1/2},$$

where

$$|\omega|_{\eta} := \||\xi|^{\eta}\widehat{\omega}\|_{\mathcal{L}^2}, \quad 0 < \eta < 1$$

is the semi-norm. Thus,

$$J_{-\infty}^{\eta}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{J_{-\infty}^{\eta}}},$$

that is, $J_{-\infty}^{\eta}(\mathbb{R})$ is the completion of $C_0^{\infty}(\mathbb{R})$ with respect to the norm $\|\cdot\|_{J_{-\infty}^{\eta}}$.

With regard of the Fourier transform, we consider the following fractional Sobolev space:

$$H^{\eta}(\mathbb{R}) := \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{\eta}}.$$

The space $J_{-\infty}^{\eta}(\mathbb{R})$ is defined below:

$$J_{-\infty}^{\eta}(\mathbb{R}) := \Big\{\omega \in \mathcal{L}^2(\mathbb{R}) : \ |\xi|^{\alpha}\widehat{\omega} \in \mathcal{L}^2(\mathbb{R})\Big\}.$$

Specifically,

$$|\omega|_{J_{-\infty}^{\eta}} = \||\xi|^{\eta}\widehat{\omega}\|_{\mathcal{L}^2(\mathbb{R})}.$$

5. Proof of the Primary Findings

To begin, we will establish the fractional space and construct the variational foundation of the system of fractional Hamiltonian equations. To this purpose, we set

$$\begin{aligned}\mathcal{E} &:= X_{\mathcal{S},\mathcal{K}}^{\eta}\\ &= \Big\{\omega \in H^{\eta}(\mathbb{R},\mathbb{R}^n) : \int_{\mathbb{R}} |D_{-\infty}^{\eta}\omega(\sigma)|^2 + (\mathcal{S}(\sigma)\omega(\sigma),\omega(\sigma))\,d\sigma + (\mathcal{K}(\sigma)\omega(\sigma),\omega(\sigma))\,d\sigma < \infty\Big\}.\end{aligned}$$

Define the inner product

$$(\omega, \psi)_{\mathcal{E}} := \int_{\mathbb{R}} \Big[D^{\eta}_{-\infty}\omega(\sigma).D^{\eta}_{-\infty}\psi(\sigma) + (\mathcal{S}(\sigma)\omega(\sigma), \psi(\sigma)) + (\mathcal{K}(\sigma)\omega(\sigma), \psi(\sigma)) \Big] d\sigma.$$

The corresponding norm is

$$\|\omega\|^2_{\mathcal{E}} := (\omega, \omega)_{\mathcal{E}}.$$

Thus, the Hilbert space $\mathcal{E}$ is reflexive and separable.

Lemma 2. *Assume that the matrices $\mathcal{S}(\sigma)$, $\mathcal{K}(\sigma)$ are positive definite and symmetric for all $\sigma \in \mathbb{R}$, and there exist two functionals m_1, m_2 in $C(\mathbb{R}, (0, \infty))$ such that*

$$m_1(\sigma), m_2(\sigma) \to \infty \ \text{ as } \ |\sigma| \to \infty$$

and

$$(\mathcal{S}(\sigma)x, y) \geq m_1(\sigma)x.y, \ \ (\mathcal{K}(\sigma)x, y) \geq m_2(\sigma)x.y \ \text{ for all } \ \sigma \in \mathbb{R} \ \text{ and } \ x, y \in \mathbb{R}^N.$$

Then,

$$\|\omega\|^2_{\eta} \leq M^* \|\omega\|^2_{\mathcal{E}}, \ \text{ for some constant } \ M^*. \tag{8}$$

Proof. Since $m_1, m_2 \in C(\mathbb{R}, (0, \infty))$ and since m_1, m_2 are coercive, then $m_1^* := \min\limits_{\sigma \in \mathbb{R}} m_1(\sigma)$ and $m_2^* := \min\limits_{\sigma \in \mathbb{R}} m_2(\sigma)$ exist. So,

$$(\mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma)) \geq m_1(\sigma)|\omega(\sigma)|^2 \geq m_1^*|\omega(\sigma)|^2, \ \text{ for all } \ \sigma \in \mathbb{R},$$

and

$$(\mathcal{K}(\sigma)\omega(\sigma), \omega(\sigma)) \geq m_2(\sigma)|\omega(\sigma)|^2 \geq m_2^*|\omega(\sigma)|^2, \ \text{ for all } \ \sigma \in \mathbb{R}.$$

However,

$$\begin{aligned}
\|\omega\|^2_{\eta} &:= \|\omega\|^2_{\mathcal{L}^2} + |\omega|^2_{\eta} \\
&\leq \|\omega\|^2_{\mathcal{L}^2} + c_1 |\omega|^2_{J^{\eta}_{-\infty}}, \ \text{ for some constant } \ c_1 \\
&\leq \|\omega\|^2_{\mathcal{L}^2} + c_1 \left\| D^{\eta}_{-\infty}\omega \right\|^2_2 \\
&= \int_{\mathbb{R}} \Big(c_1 |D^{\eta}_{-\infty}\omega(\sigma)|^2 + |\omega(\sigma)|^2 \Big) d\sigma \\
&\leq \int_{\mathbb{R}} c_1 |D^{\eta}_{-\infty}\omega(\sigma)|^2 d\sigma + \frac{1}{m_1^*} \int_{\mathbb{R}} (\mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma) + \frac{1}{m_2^*} \int_{\mathbb{R}} (\mathcal{K}(\sigma)\omega(\sigma), \omega(\sigma)) d\sigma \\
&\leq M^* \left[\int_{\mathbb{R}} |D^{\eta}_{-\infty}\omega(\sigma)|^2 d\sigma + \int_{\mathbb{R}} (\mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma)) d\sigma + \int_{\mathbb{R}} (\mathcal{K}(\sigma)\omega(\sigma), \omega(\sigma)) d\sigma \right].
\end{aligned}$$

Thus,

$$\|\omega\|^2_{\eta} \leq M_1^* \|\omega\|^2_{\mathcal{E}}, \tag{9}$$

where $M_1^* := \max\left(c_1, \dfrac{1}{m_1^*}, \dfrac{1}{m_2^*} \right)$. □

Remark 3. *From Lemma 2, we deduce that $\mathcal{E}$ is continuously embedded in $H^{\alpha}(\mathbb{R}, \mathbb{R}^n)$.*

Lemma 3. *Under the assumption of Lemma 2, the embedding of $\mathcal{E}$ in $\mathcal{L}^2(\mathbb{R})$ is compact.*

Proof. Following [18] (Remark 2.2.) and from Lemma 2, we deduce the continuity of $\mathcal{E} \hookrightarrow \mathcal{L}^2(\mathbb{R})$. Now, let $(\omega_k) \in \mathcal{E}$ be a sequence such that $\omega_k \rightharpoonup \omega \in \mathcal{E}$. Let $\epsilon > 0$ and letting

$$\gamma := \sup_{k\in\mathbb{N}} \|\omega_k - \omega\|.$$

The Banach–Steinhauss Theorem establishes that $\gamma < \infty$. Since $\lim_{|\sigma|\to\infty} m_1(\sigma), m_2(\sigma) = \infty$, there exist two reals $T_1, T_2 > 0$ such that

$$\frac{1}{m_1(\sigma)} \leq \epsilon, \quad \text{for all} \quad |\sigma| \geq T_1,$$

and

$$\frac{1}{m_2(\sigma)} \leq \epsilon, \quad \text{for all} \quad |\sigma| \geq T_2.$$

Letting

$$T^* := \max\{T_1, T_2\} \quad \text{and} \quad m(\sigma) := \max\{m_1(\sigma), m_2(\sigma)\}, \quad \text{for all} \quad \sigma \in \mathbb{R},$$

we obtain

$$\begin{aligned} \int_{|t|\geq T^*} |\omega_k(\sigma) - \omega(\sigma)|^2 d\sigma &\leq \epsilon \int_{|t|\geq T^*} m(\sigma)|\omega_k(\sigma) - \omega(\sigma)|^2 d\sigma \\ &\leq \epsilon\|\omega_k - \omega\|^2 \leq \epsilon\gamma^2. \end{aligned} \tag{10}$$

As in [18], we conclude that $\omega_k \to \omega$ uniformly on $[-T^*, T^*]$. So, there is a $k_0 \in \mathbb{N}$ such that

$$\int_{|t|\leq T^*} |\omega_k(\sigma) - \omega(\sigma)|^2 d\sigma \leq \epsilon, \quad \text{for all} \quad k \geq k_0. \tag{11}$$

This yields from (10) and (11) that

$$\omega_k \to \omega \ \in \ \mathcal{L}^2(\mathbb{R}).$$

□

In order to prove our results using variational techniques, let us first define two variational functionals $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathcal{E}$ as follows:

$$\begin{aligned} \mathcal{F}_1(\omega) :=& \frac{1}{2}\int_{\mathbb{R}} |D^{\eta}_{-\infty}\omega(\sigma)|^2 d\sigma - \frac{1}{2}\int_{\mathbb{R}} \langle \mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma - \frac{1}{2}\int_{\mathbb{R}} \langle \mathcal{K}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma \\ &+ \int_{\mathbb{R}} \vartheta(\sigma, \omega(\sigma)) d\sigma, \\ :=& \frac{1}{2}\|\omega\|^2_{\mathcal{E}} - \int_{\mathbb{R}} \langle \mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma - \int_{\mathbb{R}} \langle \mathcal{K}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma + \int_{\mathbb{R}} \vartheta(\sigma, \omega(\sigma)) d\sigma, \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_2(\omega) :=& \frac{1}{2}\int_{\mathbb{R}} |D^{\eta}_{-\infty}\omega(\sigma)|^2 d\sigma - \frac{1}{2}\int_{\mathbb{R}} \langle \mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma + \frac{1}{2}\int_{\mathbb{R}} \langle \mathcal{K}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma \\ &- \int_{\mathbb{R}} \vartheta(\sigma, \omega(\sigma)) d\sigma, \\ :=& \frac{1}{2}\|\omega\|^2_{\mathcal{E}} - \int_{\mathbb{R}} \langle \mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma - \int_{\mathbb{R}} \vartheta(\sigma, \omega(\sigma)) d\sigma. \end{aligned}$$

These functionals are related to the fractional Hamiltonian system (4) and (5).

Theorem 1. *Assume that* $M_2^*(\|S\| + \|K\|) < \frac{1}{2}$, *with* $M_2^* := \max\left\{1, \frac{1}{m_1^*}, \frac{1}{m_2^*}\right\}$. *Then,*

$$\mathcal{F}_1(\omega) \to \infty \ \ as \ \ \|\omega\|_{\mathcal{E}} \to \infty.$$

Proof. We have

$$\int_{\mathbb{R}} |\omega(\sigma)|^2 d\sigma \leq \frac{1}{m_1^*} \int_{\mathbb{R}} \langle \mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma,$$
$$\int_{\mathbb{R}} |\omega(\sigma)|^2 d\sigma \leq \frac{1}{m_2^*} \int_{\mathbb{R}} \langle \mathcal{K}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma,$$

that is to say

$$\|\omega\|_{L^2}^2 \leq \frac{1}{m_1^*} \int_{\mathbb{R}} \langle \mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma,$$
$$\|\omega\|_{L^2}^2 \leq \frac{1}{m_2^*} \int_{\mathbb{R}} \langle \mathcal{K}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma.$$

Hence,

$$\|\omega\|_{L^2}^2 \leq \frac{1}{m_1^*} \int_{\mathbb{R}} \langle \mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma + \frac{1}{m_2^*} \int_{\mathbb{R}} \langle \mathcal{K}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma,$$

and hence

$$\|\omega\|_{L^2}^2 \leq \frac{1}{m_1^*} \int_{\mathbb{R}} \langle \mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma + \frac{1}{m_2^*} \int_{\mathbb{R}} \langle \mathcal{K}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma + \left\| D^{\eta}_{-\infty}, \omega \right\|_{L^2}.$$

Thus,

$$\|\omega\|_{L^2}^2 \leq M_2^* \|\omega\|_{\mathcal{E}}^2.$$

However,

$$\int_{\mathbb{R}} \langle \mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma \leq \|S\| \cdot \|\omega\|_{L^2}^2,$$
$$\int_{\mathbb{R}} \langle \mathcal{K}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma \leq \|K\| \cdot \|\omega\|_{L^2}^2$$

and thus,

$$\int_{\mathbb{R}} \langle \mathcal{S}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma + \int_{\mathbb{R}} \langle \mathcal{K}(\sigma)\omega(\sigma), \omega(\sigma)\rangle d\sigma \leq (\|S\| + \|K\|)\|\omega\|_{\mathcal{E}}^2$$
$$\leq M_2^*(\|S\| + \|K\|)\|\omega\|_{\mathcal{E}}^2$$

Since

$$\int_{\mathbb{R}} \vartheta(\sigma, \omega(\sigma)) d\sigma \geq 0,$$

we obtain

$$\mathcal{F}_1(\omega) \geq \frac{1}{2}\|\omega\|_{\mathcal{E}}^2 - M_2^*(\|S\| + \|K\|)\|\omega\|_{\mathcal{E}}^2,$$
$$\geq \left[\frac{1}{2} - M_2^*(\|S\| + \|K\|)\right]\|\omega\|_{\mathcal{E}}^2.$$

Consequently,

$$\mathcal{F}_1(\omega) \to \infty \ \ \text{as} \ \ \|\omega\|_{\mathcal{E}} \to \infty.$$

□

Theorem 2. *Assume that* $|\vartheta(\sigma, \omega(\sigma))| \leq \delta(\sigma)|x|^2$ *for all* $(\sigma, x) \in (\mathbb{R}, \mathbb{R}^N)$ *and* $M_2^* \|\delta\|_{L^2}^2 < \frac{1}{2}$. *Then,*

$$\mathcal{F}_2(\omega) \to \infty \ \ as \ \ \|\omega\|_{\mathcal{E}} \to \infty.$$

Proof. We have

$$\mathcal{F}_2(\omega) \geq \frac{1}{2}\|\omega\|_{\mathcal{E}}^2 - \int_{\mathbb{R}} \vartheta(\sigma, \omega(\sigma))d\sigma.$$

Since

$$\int_{\mathbb{R}} \vartheta(\sigma, \omega(\sigma))d\sigma \leq \|\delta\|_{L^2}^2 \|\omega\|_2^2,$$

we obtain

$$\int_{\mathbb{R}} \vartheta(\sigma, \omega(\sigma))d\sigma \leq M_2^* \|\delta\|_{L^2}^2 \|\omega\|_{\mathcal{E}}^2.$$

Thus,

$$\mathcal{F}_2(\omega) \geq \left(\frac{1}{2} - M_2^* \|\delta\|_{L^2}^2\right) \|\omega\|_{\mathcal{E}}^2.$$

Consequently,

$$\mathcal{F}_2(\omega) \to \infty \ \text{ as } \ \|\omega\|_{\mathcal{E}} \to \infty.$$

□

6. Conclusions

In physics, engineering, chemical science, economics, and bioengineering, fractional differential equations, including the fractional Hamiltonian, are used in the mathematical modeling of some processes. Numerous papers, including [20–24], have examined the existence of solutions for the Hamiltonian equations. The current research analyzed a system of two fractional Hamiltonian equations, which generalized the previous works. We investigated the solutions to a system of fractional Hamiltonian equations in this study. We have proposed an effective strategy for separating the current model into two distinct equation systems. We have introduced a new space and two functionals related to the converted system of fractional Hamiltonian systems, and demonstrate that they are below-bounded on this space. We demonstrated our findings using new hypotheses that differ from those presented in the literature.

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