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Stability of Nonlinear Implicit Differential Equations with Caputo–Katugampola Fractional Derivative

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Abstract: The purpose of this paper is to study nonlinear implicit differential equations with the Caputo–Katugampola fractional derivative. By using Gronwall inequality and Banach fixed-point theorem, the existence of the solution of the implicit equation is proved, and the relevant conclusions about the stability of Ulam–Hyers are obtained. Finally, the correctness of the conclusions is verified by an example.

Keywords: Caputo–Katugampola fractional derivative; Gronwall inequality; Banach fixed-point theorem; Ulam–Hyers stability; Ulam–Hyers–Rassias stability

MSC: 34A08; 34A09; 34D20; 34K37; 47H10

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1. Introduction

With the maturity of the development of fractional calculus theory, fractional differential equations have become research hot-spots for many mathematicians, and appear naturally in various fields such as fluid mechanics, fractals, environmental science, modeling and control theory, signal processing, bioengineering and biomedical science [1–4]. Due to the nonlocal properties of fractional derivatives, fractional differential equations can better describe complex processes and systems with genetic effects and memory. Their descriptions of complex phenomena have the advantages of clear physical meaning, fewer parameters and consistent experimental results [5–10], so they are useful tools in the mathematical modeling of complex mechanics and physical processes. Fractional differential equations are an important mathematical tool, in which the Caputo–Katugampola fractional derivative overcomes the shortcomings of traditional fractional derivative operators such as the Caputo derivative, which is a new research development at present. In addition, in the past 20 years, a large number of mathematicians have widely used Ulam stability to approximate the exact solution of the problem studied, which has effectively improved the level of scientific research.

Research on the stability of fractional differential equations has received extensive attention. J.Vanterler da C.Sousa and E.Capelas de Oliveira used Gronwall inequality to study the Ulam–Hyers and generalized Ulam–Hyers–Rassias stabilities of a class of fractional differential equations [11]; Sajedi Leila investigated the existence, uniqueness and different kinds of Ulam–Hyers stability of solutions of an impulsive coupled system of fractional differential equations by means of the Caputo–Katugampola fuzzy fractional derivative [12]; in 2022, Subramanian Muthaiah and Aljoudi Shorog obtained the existence and Hyers–Ulam stability of coupled differential equations that are related to Katugampola integrals [13]. For more details, see [14–19]. In fact, Ulam stability theory helps us to arrive at an efficient and reliable technique for approximating fractional differential equations, and when a given problem is stable, it is believed that there is an approximate solution to fractional differential equations. The study of Ulam–Hyers stability is widely used in algebra, functional analysis, calculus and dynamic systems [20–26]. The main methods include the successive approximation method, fixed-point theorem and the direct analysis method, among which the research on Ulam–Hyers stability and Ulam–Hyers–Rassias stability has become one of the central themes of mathematical analysis.

The Caputo–Katugampola fractional derivative is a very advanced theory of fractional calculus at present. It optimizes the shortcomings of Hadamard and Riemann–Liouville fractional derivatives. In [27,28], Katagampola unified the definition of Caputo and Caputo–Hadamard fractional derivatives, namely, the Caputo–Katugampola fractional derivative. The fractional derivative of Caputo–Katugampola not only includes the traditional Caputo derivative operator, but also adds a new integral form [29,30], and the form is relatively complex. In recent years, many outstanding scholars have studied and published a large number articles on the Caputo–Katugampola fractional derivative [31–37].

Tran Minh Duc, Ho Vu and Van Hoa Ngo [38] established the Ulam–Hyers–Mittag–Leffler stability, and presented the results of the global existence of fractional differential equations involving a generalized Caputo derivative with the case of the fractional-order derivative $\alpha \in (1,2)$ of the given problems

$$\begin{aligned} {}^{c}D^{\alpha,\rho}_{a+}y(t) &= f(t,y(t)), \forall t \in [a,b], \\ y(a) &= y_1, \\ y'(a) &= y_2. \end{aligned}$$

Benchohra M. and Lazreg E. J. [34] mainly studied the two types of Ulam-Hyers stability and Ulam-Hyers-Rassias stability of a class nonlinear implicit fractional differential equations by using generalized Gronwall inequality

$$\begin{aligned} {}^{c}D^{\alpha}y(t) &= f(t,y(t),{}^{c}D^{\alpha}y(t)), \forall t \in J, 0<\alpha\leq 1, \\ y(0) &= y_0, \end{aligned}$$

where $f : J \times R^d \times R^d \to R^d$ is a given function space, ${}^{c}D^{\alpha}$ is the Caputo fractional derivative.

In [31], the authors studied the following implicit Caputo fractional derivative and nonlocal fractional integral conditions by using Krasnoselskii's fixed-point theorem and Boyd–Wong nonlinear contraction

$$\begin{aligned} {}^{c}D^{q}_{0+}u(t) &= f(t,u(t),{}^{c}D^{q}_{0+}u(t)), t \in [0,T], \\ u(0) &= \eta, \\ u(T) &= {}^{RL}I^{p}_{0+}u(k), k \in (0,T), \end{aligned}$$

where $1 < q \leq 2, 0 < p \leq 1, {}^{c}D^{q}_{0+}u(t)$ is the Caputo fractional derivative of order q, ${}^{RL}I^{p}_{0+}u(k)$ is the Riemann–Liouville fractional integral of order p and $f : [0,T] \times R \times R \to R$ is a continuous function.

In this paper [35], Adjimi Naas, Maamar Benbachir and Mohamed S. Abdo used Schaefer's and Krasnoselskii's fixed-point theorems to study the existence and uniqueness of solutions to fractional differential equations of Riesz–Caputo operators with boundary value conditions

$$\begin{aligned} {}^{RC}_{0}D^{\vartheta}_{T}\aleph(t) + \Im(t,\aleph(t),{}^{RC}_{0}D^{\varsigma}_{T}\aleph(t)) &= 0, t \in J := [0,T], \\ \aleph(0) + \aleph(T) &= 0, \\ \mu\aleph'(0) + \sigma\aleph'(T) &= 0, \end{aligned}$$

where $1 < \vartheta \leq 2, 0 < \varsigma \leq 1,\ {}^{RC}_{0}D^{k}_{T}$ is the Riesz–Caputo fractional derivative of order $k \in \{\vartheta, \varsigma\}, \Im : J \times R \times R \to R$ is a continuous function and μ, σ are non-negative constants with $\mu > \sigma$.

In this article, we will study the existence, uniqueness, Ulam–Hyers stability and Ulam–Hyers–Rassias stability of solutions to the following fractional implicit fractional differential equation with a Caputo–Katugampola fractional derivative operator

$$
\begin{aligned}
{}^{c}D_{a+}^{\alpha,\rho}\varphi(t) &= f(t,\varphi(t),{}^{c}D_{a+}^{\alpha,\rho}\varphi(t)), &(1)\\
\varphi(a) &= \varphi_0, &(2)
\end{aligned}
$$

where $\varphi \in C_{\gamma}[a,b], t \in [a,b], \alpha \in (0,1], \rho > 0, \varphi_0$ is a constant, $f : [a,b] \times R^d \times R^d \to R^d$ is a nonlinear continuously vector-valued function and ${}^{c}D_{a+}^{\alpha,\rho}$ is the Caputo–Katugampola fractional derivative.

2. Preliminaries

In this section, we present some necessary definitions, lemmas and important theorems for obtaining the main results. Also, we introduce the concept of stability of Ulam–Hyers, Ulam–Hyers–Rassias and Banach fixed-point theorem. For more details, see [27,28,38–40]. The following function space plays a fundamental role in our discussion.

Let $[a,b] \subset R, 0 < a < b < \infty$ and $C([a,b],R^d) = \{\varphi : [a,b] \to R^d : \varphi$ is a continuously vector-valued function$\}$, then $C([a,b],R^d)$ is a Banach space equipped with the norm

$$\|\varphi\|_{C([a,b],R^d)} = \sup_{t\in[a,b]} |\varphi(t)|,$$

where $|\cdot|$ is the vector norm in R^d.

Let $C^n([a,b],R^d)$ be the space of the vector-valued function φ with an n-order continuous derivative, where $\varphi : [a,b] \to R^d$.

The weighted space $C_{\gamma}([a,b],R^d)$ of the vector-valued function φ is defined by

$$C_{\gamma}\Big([a,b],R^d\Big) = \left\{\varphi : [a,b] \to R^d : \left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma}\varphi(t) \in C\Big([a,b],R^d\Big)\right\}, \gamma \in (0,1].$$

Definition 1 ([41])**.** *Let $\varphi \in C([a,b],R^d), 0 < a < b < \infty, t \ge a, \alpha > 0, \rho > 0$ and $n = [\alpha]+1$, then the Caputo-Katugampola fractional derivatives are defined by*

$${}^{c}D_{a+}^{\alpha,\rho}\varphi(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_a^t \frac{s^{(\rho-1)(1-n)}}{(t^{\rho}-s^{\rho})^{\alpha-n+1}}\varphi^{(n)}(s)ds,$$

$${}^{c}D_{b-}^{\alpha,\rho}\varphi(t) = \frac{(-1)^n\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_t^b \frac{s^{(\rho-1)(1-n)}}{(s^{\rho}-t^{\rho})^{\alpha-n+1}}\varphi^{(n)}(s)ds.$$

Definition 2 ([41])**.** *Let $\varphi \in C([a,b],R^d), t \in (a,b], 0 < \alpha < 1, \rho > 0$, then the Katugampola fractional integrals are defined by*

$$I_{a+}^{\alpha,\rho}\varphi(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\alpha}}\varphi(s)ds,$$

$$I_{b-}^{\alpha,\rho}\varphi(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_t^b \frac{s^{\rho-1}}{(s^{\rho}-t^{\rho})^{1-\alpha}}\varphi(s)ds.$$

Definition 3 ([39])**.** *Let $\varphi \in C([a,b],R^d), t \in (a,b]$, then the Riemann–Liouville generalized fractional integral of φ is defined by*

$$\varphi_{\alpha,\rho}(t) = (I_{a+}^{\alpha,\rho}\varphi)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha-1}\varphi(s)ds.$$

Lemma 1 ([38])**.** *Let $n-1 < \alpha \le n \in N, \varphi \in C^n([a,b],R^d), t \in (a,b]$, then*

$$\left(I_{a+}^{\alpha,\rho c}D_{a+}^{\alpha,\rho}\varphi\right)(t)=\varphi(t)-\sum_{k=0}^{n-1}\frac{\varphi^{\rho,k}(a)}{k!}\left(\frac{t^\rho-a^\rho}{\rho}\right)^k,$$

where $\varphi^{\rho,k}(a)=\left[\left(t^{1-\rho}(\frac{d}{dt})\right)^k\varphi(t)\right]_{t=a}$*, n is the smallest integer greater than or equal to* α*.*

Lemma 2 (Gronwall inequality [38])**.** *Let* $p(t)$ *and* $q(t)$ *be integrable and non-negative functions. Let* $r(t)$ *be a continuous function that is non-negative and nondecreasing on* $[a,b]$*. If*

$$p(t)\le q(t)+r(t)\rho^{1-\alpha}\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}p(s)ds,\forall t\in[a,b],$$

then

$$p(t)\le q(t)+\int_a^t\sum_{k=1}^{\infty}\frac{\rho^{1-k\alpha}(r(t)\Gamma(\alpha))^k}{\Gamma(k\alpha)}s^{\rho-1}(t^\rho-s^\rho)^{k\alpha-1}q(s)ds,\forall t\in[a,b].$$

Furthermore, if the function $q(t)$ *is nondecreasing, then*

$$p(t)\le q(t)E_{\alpha,1}\left(r(t)\Gamma(\alpha)\left(\frac{t^\rho-a^\rho}{\rho}\right)^\alpha\right),\forall t\in[a,b].$$

Theorem 1 (Banach fixed-point theorem)**.** *Assume that* (X,d) *is a nonempty complete metric space. Furthermore, let the mapping* $T:X\to X$ *satisfy the inequality*

$$d(T(x),T(y))\le qd(x,y),$$

for a non-negative real number $q<1$ *and for any* $x,y\in X$*. Then, the operator T has a unique fixed point x.*

Definition 4. *For each* $\varepsilon>0$*, suppose that the function* $z\in C^1([a,b],R^d)$ *satisfies the inequality*

$$|^cD_{a+}^{\alpha,\rho}z(t)-f(t,z(t),^cD_{a+}^{\alpha,\rho}z(t))|\le\varepsilon,t\in[a,b]. \quad (3)$$

If there exist real numbers $C_f>0,\beta_f\ge0$ *and a solution* φ *of Equation (1), such that*

$$|z(t)-\varphi(t)|\le C_f\varepsilon E_{\alpha,1}\left(\beta_f\left(\frac{t^\rho-a^\rho}{\rho}\right)^\alpha\right),t\in[a,b],$$

then Equation (1) is Ulam–Hyers stable.

Definition 5. *Let function* $z\in C^1([a,b],R^d)$ *satisfy the inequality (3), and if there exists a function* $\psi_f\in C(R^+,R^+),\psi_f(0)=0$ *and a solution* φ *of Equation (1) such that*

$$|z(t)-\varphi(t)|\le\psi_f(\varepsilon)E_{\alpha,1}\left(\beta_f\left(\frac{t^\rho-a^\rho}{\rho}\right)^\alpha\right),t\in[a,b],\beta_f\ge0,$$

then Equation (1) is generalized Ulam–Hyers stable.

Definition 6. *For each* $\varepsilon>0$*, assume that the function* $\xi\in C([a,b],R^+)$ *and the function* $z\in C^1([a,b],R^d)$ *satisfy the inequality*

$$|^cD_{a+}^{\alpha,\rho}z(t)-f(t,z(t),^cD_{a+}^{\alpha,\rho}z(t))|\le\varepsilon\xi(t),t\in[a,b]. \quad (4)$$

If there exist real numbers $C_f>0,\beta_f\ge0$ *and a solution* φ *of Equation (1), such that*

$$|z(t)-\varphi(t)|\le C_f\varepsilon E_{\alpha,1}\left(\beta_f\left(\frac{t^\rho-a^\rho}{\rho}\right)^\alpha\right)\xi(t),t\in[a,b],$$

then Equation (1) is Ulam–Hyers–Rassias stable with respect to ξ.

Definition 7. *Assume the function $\xi \in C([a,b], R^+)$ and the function $z \in C^1([a,b], R^d)$ satisfy the inequality*

$$|^c D^{\alpha,\rho}_{a+} z(t) - f(t, z(t), ^c D^{\alpha,\rho}_{a+} z(t))| \le \xi(t), t \in [a,b]. \quad (5)$$

If there are real numbers $C_{f,\xi} > 0, \beta_f \ge 0$ and a solution φ of Equation (1) such that

$$|z(t) - \varphi(t)| \le C_{f,\xi} E_{\alpha,1}\left(\beta_f \left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha\right)\xi(t), t \in [a,b],$$

then Equation (1) is generalized Ulam–Hyers–Rassias stable with respect to ξ.

Remark 1. *A function $z \in C^1([a,b], R^d)$ is a solution of the inequality (2.1) if and only if there exists a function $g \in C([a,b], R^d)$ (which depends on φ) such that*

(i) $|g(t)| \le \varepsilon, \forall t \in [a,b], z_0 = \varphi_0$.

(ii) $^c D^{\alpha,\rho}_{a+} z(t) = f\left(t, z(t), ^c D^{\alpha,\rho}_{a+} z(t)\right) + g(t), t \in [a,b]$.

3. The Existence and Uniqueness of the Solution

Lemma 3. *Let a function $f : [a,b] \times R^d \times R^d \to R^d$ be a continuously vector-valued function. Then, the problems (1) and (2) are equivalent to the problem*

$$\varphi(t) = \varphi_0 + I^{\alpha,\rho}_{a+} g(t), \quad (6)$$

where $g \in C([a,b], R^d)$ satisfies the functional equation

$$g(t) = f(t, \varphi_0 + I^{\alpha,\rho}_{a+} g(t), g(t)).$$

Proof. If $^c D^{\alpha,\rho}_{a+} \varphi(t) = g(t)$, then $I^{\alpha,\rho}_{a+}(^c D^{\alpha,\rho}_{a+} \varphi(t)) = I^{\alpha,\rho}_{a+} g(t)$, so we obtain

$$\varphi(t) = \varphi_0 + I^{\alpha,\rho}_{a+} g(t).$$

□

Theorem 2 ([29]). *Assume*

(H1) $f : [a,b] \times R^d \times R^d \to R^d$ is a continuously vector-valued function;

(H2) There exist constants $K > 0$ and $0 < L < 1$, such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le K|u - \bar{u}| + L|v - \bar{v}|, u, \bar{u}, v, \bar{v} \in R^d, t \in [a,b].$$

If

$$\frac{k\rho^{-\alpha}(t^\rho - a^\rho)^\alpha}{(1-L)\Gamma(\alpha+1)} < 1,$$

then there exists a unique solution for Equations (1) and (2) .

Proof. Let $\Omega_T = \{\varphi : \varphi \in C([a,h], R^d), |\varphi(t) - \varphi_0| \le r\}, |g(t)| \le M$, where

$$h = \min\left\{b, \left(\left(r\frac{\rho^\alpha \Gamma(\alpha+1)}{M}\right)^{\frac{1}{\alpha}} + a^\rho\right)^{\frac{1}{\rho}}\right\}, g(t) = f(t, \varphi(t), g(t)).$$

Define the operator $T : \Omega_T \to \Omega_T$ by

$$(T\varphi)(t) = \varphi_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} g(s)ds.$$

For $\forall \varphi \in \Omega_T$,

$$\begin{aligned} |(T\varphi)(t) - \varphi_0| &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} |g(s)|ds \\ &\leq \frac{M\rho^{-\alpha}(t^\rho - a^\rho)^\alpha}{\Gamma(\alpha+1)} \leq r. \end{aligned}$$

Let $\varphi_n, \varphi \in \Omega_T$, such that $\varphi_n \to \varphi$ in $C([a,h], R^d)$ as $n \to \infty$. Since

$$|(T\varphi_n)(t) - (T\varphi)(t)| = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} |g_n(s) - g(s)|ds, \tag{7}$$

where $g_n, g \in C([a,h], R^d)$ satisfies the functional equation

$$g_n(t) = f(t, \varphi_n(t), g_n(t)), g(t) = f(t, \varphi(t), g(t)).$$

By $(H2), \forall t \in [a,b]$,

$$\begin{aligned} |g_n(t) - g(t)| &= |f(t, \varphi_n(t), g_n(t)) - f(t, \varphi(t), g(t))| \\ &\leq K|\varphi_n(t) - \varphi(t)| + L|g_n(t) - g(t)|. \end{aligned}$$

Then,

$$|g_n(t) - g(t)| \leq \frac{K}{1-L} |\varphi_n(t) - \varphi(t)|. \tag{8}$$

Thus, $\varphi_n \to \varphi, g_n \to g$ as $n \to \infty, (T\varphi_n)(t) \to (T\varphi)(t)$. On the other hand, T is continuous, $\varphi \in \Omega_T, T\varphi \in \Omega_T$, since $T\Omega_T \subseteq \Omega_T$.

By (7) and (8), we obtain

$$\begin{aligned} |(T\varphi_n)(t) - (T\varphi)(t)| &\leq \frac{K\rho^{1-\alpha}}{(1-L)\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} |\varphi_n(s) - \varphi(s)|ds \\ &\leq \frac{K}{(1-L)\Gamma(\alpha)} |\varphi_n(t) - \varphi(t)| \int_a^t s^{\rho-1}(\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} ds \\ &\leq \frac{K\rho^{-\alpha}(t^\rho - s^\rho)^\alpha}{(1-L)\Gamma(\alpha+1)} |\varphi_n(t) - \varphi(t)|. \end{aligned}$$

So we can obtain that Equations (1) and (2) have a unique fixed point as

$$\frac{K\rho^{-\alpha}(t^\rho - s^\rho)^\alpha}{(1-L)\Gamma(\alpha+1)} < 1,$$

then, there exists a unique solution for Equations (1) and (2). □

4. Ulam–Hyers Stability

Theorem 3. *Assume that the function (1) satisfies (H1) and (H2), then Equation (1) is Ulam–Hyers stable.*

Proof. Let $z \in C^1([a,b], R^d)$ be a solution of the inequation (3), i.e.,

$$|^c D_{a+}^{\alpha,\rho} z(t) - f(t, z(t), ^c D_{a+}^{\alpha,\rho} z(t))| \leq \varepsilon, t \in [a,b]. \tag{9}$$

Let us denote by $\varphi \in C^1([a,b], R^d)$ the unique solution of Equation (1)

$$\begin{aligned} {}^cD^{\alpha,\rho}_{a+}\varphi(t) &= f(t,\varphi(t), {}^c D^{\alpha,\rho}_{a+}\varphi(t)), \forall t \in [a,b], 0<\alpha\leq 1,\\ \varphi_0 &= z_0. \end{aligned}$$

By using Lemma 3, we have

$$\varphi(t) = z_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} g_\varphi(s)ds, t \in [a,b],$$

where $g_\varphi \in C([a,b], R^d)$ satisfies the functional equation

$$g_\varphi(t) = f(t, \varphi_0 + I^{\alpha,\rho}_{a+} g_\varphi(t), g_\varphi(t)).$$

Then, by integration of (9),

$$|z(t) - z_0 - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} g_z(s)ds| \leq \frac{\varepsilon\rho^{-\alpha}(t^\rho - a^\rho)^\alpha}{\Gamma(\alpha+1)} \tag{10}$$

where $g_z(t) \in C([a,b], R^d)$ satisfies the functional equation

$$g_z(t) = f(t, z_0 + I^{\alpha,\rho}_{a+} g_z(t), g_z(t)).$$

For $\forall t \in [a,b]$, we have

$$\begin{aligned} &|z(t) - \varphi(t)|\\ &= \left|z(t) - z_0 - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} g_\varphi(s)ds\right|\\ &= \left|z(t) - z_0 - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} g_z(s)ds + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}(g_z(s) - g_\varphi(s))ds\right|\\ &\leq \left|z(t) - z_0 - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} g_z(s)ds\right| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}|g_z(s) - g_\varphi(s)|ds, \end{aligned} \tag{11}$$

where $g_z(t) = f(t, z(t), g_z(t)), g_\varphi(t) = f(t, \varphi(t), g_\varphi(t))$.

By (H2), we have, for $\forall t \in [a,b]$,

$$\begin{aligned} |g_z(t) - g_\varphi(t)| &= |f(t, z(t), g_z(t)) - f(t, \varphi(t), g_\varphi(t))|\\ &\leq K|z(t) - \varphi(t)| + L|g_z(t) - g_\varphi(t)|. \end{aligned}$$

Furthermore,

$$|g_z(t) - g_\varphi(t)| \leq \frac{K}{1-L}|z(t) - \varphi(t)|. \tag{12}$$

Thus, by (10), (11) and (12) we obtain

$$|z(t) - \varphi(t)| \leq \frac{\varepsilon\rho^{-\alpha}(t^\rho - a^\rho)^\alpha}{\Gamma(\alpha+1)} + \frac{K\rho^{1-\alpha}}{(1-L)\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}|z(s) - \varphi(s)|ds.$$

Then by Lemma 2 (Gronwall inequality), we obtain

$$|z(t) - \varphi(t)| \leq \frac{\varepsilon\rho^{-\alpha}(t^\rho - a^\rho)^\alpha}{\Gamma(\alpha+1)} E_{\alpha,1}\left(\frac{K}{1-L}\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha\right), t \in [a,b],$$

where

$$C_f = \frac{\rho^{-\alpha}(t^\rho - a^\rho)^\alpha}{\Gamma(\alpha+1)}, \beta_f = \frac{K}{1-L}.$$

Thus, the function (1) is Ulam–Hyers stable. This completes the proof.

At this point, by putting $\psi_f(\varepsilon) = \frac{\varepsilon\rho^{-\alpha}(t^\rho - a^\rho)^\alpha}{\Gamma(\alpha+1)}, \psi_f(0) = 0, \beta_f = \frac{K}{1-L}$ yields that Equation (1) is generalized Ulam–Hyers stable. □

5. Ulam–Hyers–Rassias Stability

Theorem 4. *Assume that the function (1) satisfies simultaneously (H1), (H2) and (H3): the function $\xi \in C([a,b], R^+)$ is increasing and there exists $\lambda_\xi > 0$, for $\forall t \in [a,b]$, and we have*

$$I_{a+}^{\alpha,\rho}\xi(t) \leq \lambda_\xi \xi(t).$$

Then, Equation (1) is Ulam–Hyers–Rassias stable with respect to ξ.

Proof: Let $z \in C^1([a,b], R^d)$ be a solution of the inequation (4), i.e.,

$$|{}^c D_{a+}^{\alpha,\rho} z(t) - f(t, z(t), {}^c D_{a+}^{\alpha,\rho} z(t))| \leq \varepsilon\xi(t), t \in [a,b], \varepsilon > 0. \tag{13}$$

Let us denote by $\varphi \in C^1([a,b], R^d)$ the unique solution of Equation (1)

$$\begin{aligned} {}^c D_{a+}^{\alpha,\rho}\varphi(t) &= f(t, \varphi(t), {}^c D_{a+}^{\alpha,\rho}\varphi(t)), \forall t \in [a,b], 0 < \alpha \leq 1, \\ \varphi_0 &= z_0. \end{aligned}$$

By using Lemma 3, we have

$$\varphi(t) = z_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} g_\varphi(s) ds, t \in [a,b],$$

where $g_\varphi \in C([a,b], R^d)$ satisfies the functional equation

$$g_\varphi(t) = f(t, \varphi_0 + I_{a+}^{\alpha,\rho} g_\varphi(t), g_\varphi(t)).$$

But, by integration of Formula (13) and by $(H3)$, we obtain

$$\begin{aligned} |z(t) - z_0 - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} g_z(s) ds| &\leq \frac{\varepsilon\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}\xi(s) ds \\ &\leq \varepsilon\lambda_\xi\xi(t), \end{aligned} \tag{14}$$

where $g_z \in C([a,b], R^d)$ satisfies the functional equation

$$g_z(t) = f(t, z_0 + I_{a+}^{\alpha,\rho} g_z(t), g_z(t)).$$

On the other hand, we have, for $\forall t \in [a,b]$,

$$\begin{aligned} &|z(t) - \varphi(t)| \\ &= \left| z(t) - z_0 - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} g_\varphi(s) ds \right| \\ &= \left| z(t) - z_0 - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} g_z(s) ds + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}(g_z(s) - g_\varphi(s)) ds \right| \\ &\leq \left| z(t) - z_0 - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} g_z(s) ds \right| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} |g_z(s) - g_\varphi(s)| ds, \end{aligned} \tag{15}$$

where $g_z(t) = f(t, z(t), g_z(t)), g_\varphi(t) = f(t, \varphi(t), g_\varphi(t))$.

Therefore, by (H2), we obtain, for $\forall t \in [a,b]$,

$$|g_z(t) - g_\varphi(t)| = |f(t, z(t), g_z(t)) - f(t, \varphi(t), g_\varphi(t))|$$

$$\leq \quad K|z(t)-\varphi(t)| + L|g_z(t)-g_\varphi(t)|.$$

Then

$$|g_z(t)-g_\varphi(t)| \leq \frac{K}{1-L}|z(t)-\varphi(t)|. \tag{16}$$

Thus, by (14), (15) and (16) we obtain

$$|z(t)-\varphi(t)| \leq \varepsilon\lambda_\xi \xi(t) + \frac{K\rho^{1-\alpha}}{(1-L)\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}|z(s)-\varphi(s)|ds.$$

Then, Lemma 2 implies that, for $\forall t \in [a,b]$,

$$|z(t)-\varphi(t)| \leq \varepsilon\lambda_\xi \xi(t) E_{\alpha,1}\left(\frac{K}{1-L}\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha\right), t \in [a,b],$$

where

$$C_f = \lambda_\xi, \beta_f = \frac{K}{1-L}.$$

So, Equation (1) is Ulam–Hyers–Rassias stable. This completes the proof.

Putting $\varepsilon = 1, C_{f,\xi} = \lambda_\xi, \beta_f = \frac{K}{1-L}$ yields that Equation (1) is generalized Ulam–Hyers–Rassias stable.

6. Examples

Example 1. *We consider the following fractional Cauchy problem*

$$\begin{aligned} {}^cD_{0^+}^{\frac{1}{2},3}\varphi(t) &= \frac{3+|\varphi(t)|+|{}^cD_{0^+}^{\frac{1}{2},3}\varphi(t)|}{3e^{t+2}(1+|\varphi(t)|+|{}^cD_{0^+}^{\frac{1}{2},3}\varphi(t)|)}, \forall t \in [0,1], && (17) \\ \varphi(0) &= 1. && (18) \end{aligned}$$

Set

$$f(t,u,v) = \frac{3+|u|+|v|}{3e^{t+2}(1+|u|+|v|)}, t \in [0,1], u, v \in R^d.$$

Clearly, the function f is continuous.

For any $u, v, \bar{u}, \bar{v} \in R^d$ and $t \in [0,1]$,

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{3e^2}(|u-\bar{u}| + |v-\bar{v}|).$$

By (H2): $K = L = \frac{1}{3e^2}$, we obtain

$$\frac{K\rho^{-\alpha}(t^\rho - s^\rho)^\alpha}{(1-L)\Gamma(\alpha+1)} \leq \frac{2\sqrt{3}}{3(3e^2-1)\sqrt{\pi}} < 1.$$

It follows from Theorem 2 that the problem (17), (18) has a unique solution, and from Theorem 3, Equation (17) is Ulam–Hyers stable.

Example 2. *We consider the following fractional Cauchy problem*

$$\begin{aligned} {}^cD_{0^+}^{\frac{1}{2},2}\varphi(t) &= \frac{t}{50}(\cos\varphi(t) - \varphi(t)\sin t) + \frac{|{}^cD_{0^+}^{\frac{1}{2},2}\varphi(t)|}{50+|{}^cD_{0^+}^{\frac{1}{2},2}\varphi(t)|}, \forall t \in [0,1], && (19) \\ \varphi(0) &= 1. && (20) \end{aligned}$$

Set

$$f(t,u,v) = \frac{t}{50}(\cos u - u\sin t) + \frac{v}{50+v}, t \in [0,1], u, v \in R^d.$$

Clearly, the function f is continuous.

For any $u, v, \bar{u}, \bar{v} \in R^d$ and $t \in [0,1]$,

$$
\begin{aligned}
|f(t,u,v) - f(t,\bar{u},\bar{v})| &\leq \frac{1}{50}|\cos u - \cos \bar{u}| + \frac{1}{50}|\sin t||u-\bar{u}| + \frac{50|v-\bar{v}|}{(50+v)(50+\bar{v})} \\
&\leq \frac{1}{50}|u-\bar{u}| + \frac{1}{50}|u-\bar{u}| + \frac{1}{50}|v-\bar{v}| \\
&\leq \frac{1}{25}|u-\bar{u}| + \frac{1}{50}|v-\bar{v}|.
\end{aligned}
$$

By (H2): $K = \frac{1}{25}, L = \frac{1}{50}$, we obtain

$$\frac{K\rho^{-\alpha}(t^\rho - s^\rho)^\alpha}{(1-L)\Gamma(\alpha+1)} \leq \frac{2\sqrt{2}}{49\sqrt{\pi}} < 1.$$

Thus, according to Theorem 2, Equations (19) and (20) have a unique solution.

Let $\xi(t) = t^2$, we have

$$I_{a^+}^{\alpha,\rho}\xi(t) \leq \frac{\sqrt{2}}{\Gamma(\frac{3}{2})}t^2 := \lambda_\xi \xi(t).$$

Thus, condition $(H3)$ is satisfied with $\xi(t) = t^2$ and $\lambda_\xi = \frac{2\sqrt{2}}{\Gamma(\pi)}$, and it follows from Theorem 4 that Equation (19) is Ulam–Hyers–Rassias stable.

7. Conclusions

In this paper, we have analyzed the existence and uniqueness of solutions, the Ulam–Hyers stability and the Ulam–Hyers–Rassias stability for Caputo–Katugampola fractional implicit differential equations in terms of Banach fixed-point theorem and generalized Gronwall inequality. Finally, two examples are given to verify the correctness of the results.

With the wide application of fractional differential equations in many fields, more and more scholars began to study fractional operations, and a large number of definitions of fractional integrals and derivatives came into being. The (k, Ψ)-Hilfer fractional derivative operator is a new definition proposed recently [42–46]. Different types of fractional derivatives can be obtained when the parameter values are different. It is a more extensive and more complex fractional derivative definition, which will also be a future research direction for us.

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