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(ω, ρ)-BVP Solutions of Impulsive Differential Equations of Fractional Order on Banach Spaces

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Abstract: The paper focuses on exploring the existence and uniqueness of a specific solution to a class of Caputo impulsive fractional differential equations with boundary value conditions on Banach space, referred to as (ω, ρ)-BVP solution. The proof of the main results of this study involves the application of the Banach contraction mapping principle and Schaefer's fixed point theorem. Furthermore, we provide the necessary conditions for the convexity of the set of solutions of the analyzed impulsive fractional differential boundary value problem. To enhance the comprehension and practical application of our findings, we conclude the paper by presenting two illustrative examples that demonstrate the applicability of the obtained results.

Keywords: (ω, ρ)-BVP solutions; boundary value problem; impulsive fractional equations

MSC: 34A37; 34C25; 34C27

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1. Introduction and Preliminaries

The advancement of modern technology and its continuous development have led to a growing interest in systems characterized by discontinuous trajectories, such as impulsive automatic control systems and impulsive computing systems. These systems have garnered substantial significance and are presently experiencing rapid growth, being applied to a wide range of technical problems, biological phenomena characterized by thresholds, models of bursting rhythms in medicine and biology, optimal control models in economics, pharmacokinetics, and frequency-modulated systems [1]. These processes experience short-term perturbations that have negligible durations compared to the overall process duration (see [1–6]). Consequently, there is a strong rationale to investigate the qualitative characteristics of the solutions of these impulsive systems given the considerable interest in understanding their behavior and properties.

The concept of (ω, c)-periodic functions, defined as $y(\cdot + \omega) = cy(\cdot)$, where $c \in \mathbb{C}$, was introduced and explored by E. Alvarez et al. in their work [7,8], naturally arising in the investigation of Mathieu's equations $y'' + ay = 2q\cos(2t)y$. In a related study by M. Agaoglou et al. [9], the authors investigated the existence and uniqueness of (ω, c)-periodic solutions for semilinear equations of the form $u' = Au + f(t, u)$ in complex Banach spaces. Expanding upon this notion, M. Fečkan, K. Liu, and J. R. Wang [10] extended the concept to $(\omega, \mathbb{T})$-periodic solutions for the aforementioned class of semilinear equations, where $\mathbb{T}$ represents a linear isomorphism on a Banach space X.

In the work by L. Ren and J. R. Wang [11], a necessary and sufficient condition for the existence of the (ω, c)-periodic solutions to a specific type of impulsive fractional differential equations is provided. Furthermore, the paper discusses the existence and uniqueness of (ω, c)-periodic solution to semilinear problems. The existence and uniqueness of solution for impulsive fractional differential equations with Caputo derivatives in Banach spaces were examined in the study conducted in [12]. Considering existence and uniqueness of solution of impulsive and regular fractional differential equations, we recommend the recent results in [13–18]. In the paper by M. Fečkan et al. [10], results regarding the existence and uniqueness of the $(\omega, \mathbb{T})$-periodic solution for impulsive linear and semilinear problems are established.

Building upon the previous investigations into (ω, c) and $(\omega, \mathbb{T})$-periodic solution for linear and semilinear problems involving ordinary and fractional order derivatives, we extend our focus to encompass the (ω, ρ)-BVP solution of impulsive fractional differential equations with boundary value conditions. This research serves as a generalization of previous studies (see [3,9–28]).

The primary goal of this paper is to showcase groundbreaking findings within the context of (ω, ρ)-BVP solutions of impulsive fractional differential equations. Specifically, the paper focuses on examining the scenario where the linear isomorphism ρ operates on the Banach space X, in contrast to previous results that primarily explored (ω, c)-periodicity, where $c \in \mathbb{C}$. The main focus of the authors of this paper lies in exploring the existence and uniqueness of the (ω, ρ)-BVP solution for specific classes of impulsive fractional differential equations, considering the boundary value conditions within the context of a Banach space X. The paper strives to present these pioneering results, shedding light on the distinct aspects and advancements achieved in this particular case.

The organization of this paper can be described briefly as follows. At the beginning, we recall some preliminary results and definitions from fractional derivatives, impulsive fractional differential equations, and we provide the definition of (ω, c)-BVP functions. In the main part of this paper, under certain conditions, we show several results on the existence and uniqueness of the (ω, ρ)-BVP solution of impulsive Caputo fractional differential equations with boundary conditions.

Preliminaries

Here, by $(X, \|\cdot\|)$, a complex Banach space is denoted. Abbreviation $\mathcal{C}(K : X)$, where K is a non-empty compact subset of $\mathbb{R}$, stands for the space of continuous functions $K \mapsto X$. This space is a Banach space, endowed with the sup-norm. The space of X-valued piecewise continuous functions on $[0, \omega]$ is given by

$$\begin{aligned}\mathcal{PC}([0,\omega] : X) \equiv \big\{ & u : [0,\omega] \to X : u \in \mathcal{C}\big((t_i, t_{i+1}] : X\big), \\ & u(t_i^-) = u(t_i) \text{ and } u(t_i^+) \text{ exist for any } i \in \{0, \cdots, m-1\}\big\},\end{aligned}$$

where $t_0 = 0 < t_1 < t_2 < \ldots < t_{m-1} < t_m = \omega$ and the symbols $u(t_i^-)$ and $u(t_i^+)$ denote the left and the right limits of the function $u(t)$ at the point $t = t_i$, $i \in \{0, \cdots, m-1\}$, respectively. Let us recall that $\mathcal{PC}([0, \omega] : X)$ is a Banach space endowed with the sup-norm.

The Gamma function $\Gamma(z)$ is defined as $\Gamma(z) = \int_0^\infty t^{z-1} e^{-z}\, dt$, $\Re z > 0$. Note that $\Gamma(n) = (n-1)!$, for n positive integer. We let u be a given function defined on the closed interval $[t_0, t_1]$. The Caputo fractional derivative of the function u is defined as

$${}^{c}D^{\alpha}_{t_0} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds,$$

where $n = [\alpha] + 1$.

We let $\rho : X \to X$ be linear isomorphism, and the set of all piecewise continuous and (ω, ρ)-BVP functions be denoted by $\Phi_{\omega,\rho}$, i.e.,

$$\Phi_{\omega,\rho} = \big\{ u \,:\, u \in \mathcal{PC}([0,\omega] : X) \text{ and } u(\omega) = \rho u(0) \big\}.$$

We continue the investigations in [10,11] by studying the (ω, ρ)-BVP solutions of the following impulsive fractional problem:

$$\begin{cases} {}^cD^{\alpha}_{t_0}u(t) = f(t, u(t)), \quad \alpha \in (0,1),\ t \neq t_k,\ t \in [0,\omega], \\ \Delta u(t_k) = I_k\big(u(t_k)\big), \quad k = 1,2,\ldots,m, \\ u(\omega) = \rho u(0), \end{cases} \tag{1}$$

where ${}^cD^{\alpha}_{t_0}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$ with the lower time at t_0, $f : [0,\omega] \times X \to X$ and $I_k : X \to X$ are continuous linear mappings, $\rho : X \to X$ is a linear isomorphism and $0 < t_1 < t_2 < \ldots < t_{m-1} < \omega$.

In this paper, we consider the following conditions:

(C1) There is a constant $L > 0$ such that

$$\|f(t,u) - f(t,v)\| \leq L\|u - v\| \quad \text{for all } t \in [0,\omega] \text{ and } u,\ v \in X.$$

(C2) There are constants $B > 0$ and $P > 0$ such that

$$\|f(t,u)\| \leq B\|u\| + P, \quad \text{for all } t \in [0,\omega] \text{ and } u \in X.$$

(C3) The operator $\rho : X \to X$ is a linear isomorphism and $(\rho - E)$ is injective, where E is the identity operator on X.
(C4) There is a constant $M > 0$ such that $\|(\rho - E)^{-1}\| \leq M$.
(C5) There is a finite real number $C_1 > 0$ such that

$$\|I_k(u)\| \leq C_1\|u\|, \quad k = 1,2,\ldots,m \quad \text{and for all } u \in X.$$

(C6) The operator $I_k : X \to X$ is continuous and there exists a constant $C_I \in [0, 1/m)$ such that

$$\big\|I_k(u) - I_k(v)\big\| \leq C_I\|u - v\|, \quad k = 1,2,\ldots,m, \quad \text{and for all } u,v \in X.$$

2. (ω, ρ)-BVP Solutions to Semilinear Impulsive Fractional Differential Boundary Value Problem

We consider the (ω, ρ)-BVP solutions of the impulsive fractional differential problem (1). We set $t_0 = 0$ and $t_m = \omega$.

If function $u \in \Psi$ is such that the equation ${}^cD^{\alpha}_0u(t) = f(t, u(t))$ is satisfied almost everywhere on $[0,\omega]$, and the conditions $\Delta u(t_k) = I_k\big(u(t_k)\big)$, $k = 1,2,\ldots,m$ and $u(\omega) = \rho u(0)$ hold, then the function u is said to be a solution of Equation (1).

The following holds true:

Proposition 1. *Let* (C3) *hold. Then, solution* $u \in \Psi = \mathcal{PC}([0,\omega] : X)$ *of Equation* (1) *is given by*

$$\begin{aligned} u(t) = (\rho - E)^{-1}\Bigg(&\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}f(s,u(s))\,ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{\omega}(\omega - s)^{\alpha-1}f(s,u(s))\,ds + \sum_{0<t_k<\omega}I_k\big(u(t_k)\big)\Bigg) \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}f(s,u(s))\,ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t - s)^{\alpha-1}f(s,u(s))\,ds + \sum_{0<t_k<t}I_k\big(u(t_k)\big),\ t \in (t_k, t_{k+1}], k = 0,1\ldots,m-1. \end{aligned}$$

Proof. By ([12] [Lemma 3.1]), solution $u \in \Psi$ of Equation (1) satisfies

$$u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, u(s))\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, u(s))\, ds + \sum_{0<t_k<t} I_k\big(u(t_k)\big),$$

$$t \in (t_k, t_{k+1}],\ k = 0, 1, \ldots, m-1.$$

We have

$$u(\omega) = u(0) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<\omega} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, u(s))\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{\omega} (\omega-s)^{\alpha-1} f(s, u(s))\, ds + \sum_{0<t_k<\omega} I_k\big(u(t_k)\big).$$

Using the boundary value condition, we have

$$u(0) = (\rho - E)^{-1}\left(\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<\omega} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, u(s))\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{\omega} (\omega-s)^{\alpha-1} f(s, u(s))\, ds + \sum_{0<t_k<\omega} I_k\big(u(t_k)\big)\right).$$

Hence,

$$\begin{aligned} u(t) = (\rho - E)^{-1}\Bigg(&\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<\omega} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, u(s))\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{\omega} (\omega-s)^{\alpha-1} f(s, u(s))\, ds + \sum_{0<t_k<\omega} I_k\big(u(t_k)\big)\Bigg) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, u(s))\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, u(s))\, ds \\ &+ \sum_{0<t_k<t} I_k\big(u(t_k)\big), \quad t \in (t_k, t_{k+1}], \quad k = 0, 1 \ldots, m-1. \end{aligned}$$

□

Theorem 1. *Suppose that* (C1) *and* (C3)–(C6) *hold. If* $0 < K < 1$, *where*

$$K = \frac{(M+1)\big(L(m+1)\omega^{\alpha} + \Gamma(\alpha+1) C_I m\big)}{\Gamma(\alpha+1)},$$

then the impulsive fractional differential boundary value problem (1) *has a unique* (ω, ρ)*-BVP solution* $u \in \Phi_{\omega,\rho}$*. Additionally,*

$$\|u\| \leq \frac{(M+1)(m+1)\omega^{\alpha}\|f\|_0}{\Gamma(\alpha+1)-(M+1)\big(L(m+1)\omega^{\alpha}+mC_1\Gamma(\alpha+1)\big)},$$

where $\|f\|_0 = \sup\limits_{s\in[0,\omega]} \|f(s,0)\|$.

Proof. Define operator $R : \Psi \to \Psi$ by

$$(Ru)(t) = (\rho - E)^{-1}\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int\limits_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,u(s))\,ds$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_{t_k}^{\omega}(\omega-s)^{\alpha-1}f(s,u(s))\,ds + \sum_{0<t_k<\omega} I_k\big(u(t_k)\big)\Bigg)$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int\limits_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,u(s))\,ds$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_{t_k}^{t}(t-s)^{\alpha-1}f(s,u(s))\,ds + \sum_{0<t_k<t} I_k\big(u(t_k)\big),\ k=1,2,\ldots,m.$$

The fixed points of R clearly determine the solutions of the (ω,ρ)-boundary value problem (1). Additionally, it is evident that $R(\Psi)\subseteq\Psi$. For any given u and v belonging to Ψ, we can observe the following:

$$\|(Ru)(t)-(Rv)(t)\| = \Bigg\|(\rho - E)^{-1}\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int\limits_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,u(s))\,ds$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_{t_k}^{\omega}(\omega-s)^{\alpha-1}f(s,u(s))\,ds + \sum_{0<t_k<\omega} I_k\big(u(t_k)\big)\Bigg)$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int\limits_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,u(s))\,ds$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_{t_k}^{t}(t-s)^{\alpha-1}f(s,u(s))\,ds + \sum_{0<t_k<t} I_k\big(u(t_k)\big)$$

$$-(\rho - E)^{-1}\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int\limits_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,v(s))\,ds$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_{t_k}^{\omega}(\omega-s)^{\alpha-1}f(s,v(s))\,ds + \sum_{0<t_k<\omega} I_k(v(t_k))\Bigg)$$

$$-\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int\limits_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,v(s))\,ds$$

$$- \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, v(s))\, ds - \sum_{0<t_k<t} I_k(v(t_k)) \Bigg\|$$

$$\leq \left\|(\rho - E)^{-1}\right\| \Bigg(\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<\omega} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, u(s)) - f(s, v(s))\|\, ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{\omega} (\omega - s)^{\alpha-1} \|f(s, u(s)) - f(s, v(s))\|\, ds + \sum_{0<t_k<\omega} \left\| I_k\big(u(t_k)\big) - I_k(v(t_k)) \right\| \Bigg)$$

$$+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, u(s)) - f(s, v(s))\|\, ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \|f(s, u(s)) - f(s, v(s))\|\, ds + \sum_{0<t_k<t} \left\| I_k\big(u(t_k)\big) - I_k(v(t_k)) \right\|$$

$$\leq M \Bigg(\frac{L}{\Gamma(\alpha)} \sum_{0<t_k<\omega} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|u(s) - v(s)\|\, ds$$

$$+ \frac{L}{\Gamma(\alpha)} \int_{t_k}^{\omega} (\omega - s)^{\alpha} \|u(s) - v(s)\|\, ds + C_I m \|u - v\| \Bigg)$$

$$+ \frac{L}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|u(s) - v(s)\|\, ds$$

$$+ \frac{L}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \|u(s) - v(s)\|\, ds + C_I m \|u - v\|$$

$$\leq \frac{(M+1)\big(L(m+1)\omega^{\alpha} + \Gamma(\alpha+1) C_I m\big)}{\Gamma(\alpha+1)} \|u - v\| = K \|u - v\|.$$

Given that $0 < K < 1$, it follows that operator R is a contraction. As a result, there exists a unique fixed point for the operator R, satisfying the condition $u(0) = \rho u(\omega)$. Consequently, Equation (1) possesses a unique (ω, ρ)-BVP solution $u \in \Phi_{\omega,\rho}$. Additionally,

$$\|u(t)\| \leq \left\|(\rho - E)^{-1}\right\| \Bigg(\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<\omega} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, u(s)) - f(s, 0)\|\, ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{\omega} (\omega - s)^{\alpha-1} \|f(s, u(s)) - f(s, 0)\|\, ds \Bigg)$$

$$+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, u(s)) - f(s, 0)\|\, ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \|f(s, u(s)) - f(s, 0)\|\, ds$$

$$+ \left\|(\rho - E)^{-1}\right\| \Bigg(\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<\omega} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, 0)\|\, ds$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{\omega} (\omega - s)^{\alpha-1} \|f(s,0)\| \, ds \Bigg) \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s,0)\| \, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \|f(s,0)\| \, ds \\
&+ \left\|(\rho - E)^{-1}\right\| \cdot \sum_{0<t_k<\omega} \left\|I_k\big(u(t_k)\big)\right\| + \sum_{0<t_k<t} \left\|I_k\big(u(t_k)\big)\right\| \\
&\le \left\|(\rho - E)^{-1}\right\| \Bigg(\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<\omega} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} L\|u\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{\omega} (\omega - s)^{\alpha-1} L\|u\| \, ds + \sum_{0<t_k<\omega} \left\|I_k\big(u(t_k)\big)\right\| \Bigg) \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} L\|u\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} L\|u\| \, ds + \sum_{0<t_k<t} \left\|I_k\big(u(t_k)\big)\right\| \\
&+ \left\|(\rho - E)^{-1}\right\| \left(\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<\omega} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f\|_0 \, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{\omega} (\omega - s)^{\alpha-1} \|f\|_0 \, ds \right) \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f\|_0 \, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \|f\|_0 \, ds \\
&\le \frac{ML\|u\|}{\Gamma(\alpha+1)} \big(m\omega^\alpha + \omega^\alpha\big) + MmC_1\|u\| + \frac{L\|u\|}{\Gamma(\alpha+1)} \big(m\omega^\alpha + \omega^\alpha\big) + mC_1\|u\| \\
&+ \frac{M\|f\|_0}{\Gamma(\alpha+1)} \big(m\omega^\alpha + \omega^\alpha\big) + \frac{\|f\|_0}{\Gamma(\alpha+1)} \big(m\omega^\alpha + \omega^\alpha\big).
\end{aligned}
$$

Hence,

$$
\|u\| \le \frac{(M+1)(m+1)\omega^\alpha \|f\|_0}{\Gamma(\alpha+1) - (M+1)\big(L(m+1)\omega^\alpha + mC_1\Gamma(\alpha+1)\big)}.
$$

□

Theorem 2. *Suppose that* (C2)–(C5) *hold. Then, the impulsive fractional differential boundary value problem* (1) *has at least one* (ω, ρ)*-BVP solution* $u \in \Phi_{\omega,\rho}$.

Proof. Let $B_r = \{u \in \Psi \,:\, \|u\| \le r\}$, where

$$
\frac{(M+1)P(m+1)\omega^\alpha}{\Gamma(\alpha+1) - (M+1)\big(B(m+1)\omega^\alpha + C_1 m\Gamma(\alpha+1)\big)} \le r.
$$

We consider again the operator R

$$(Ru)(t) = (\rho - E)^{-1}\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}f(s,u(s))\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{\omega}(\omega - s)^{\alpha-1}f(s,u(s))\,ds + \sum_{0<t_k<\omega}I_k\big(u(t_k)\big)\Bigg) + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}f(s,u(s))\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t - s)^{\alpha-1}f(s,u(s))\,ds + \sum_{0<t_k<t}I_k\big(u(t_k)\big),\ k = 1,2\ldots,m$$

defined on B_r.

Step 1. We show that operator $R : \Psi \to \Psi$ is bounded. It is sufficient to show that for any $r > 0$, there exists a constant $K > 0$, such that for each $u \in B_r$, we have $\|Ru\| \leq K$. We let (u_n) be a sequence on a bounded subset $\mathcal{B} \subseteq B_r$. Then, by (C2) and (C5), we obtain

$$\begin{aligned}
\|(Ru_n)(t)\| &\leq \|(\rho - E)^{-1}\|\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}\|f(s,u_n(s))\|\,ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{\omega}(\omega - s)^{\alpha-1}\|f(s,u_n(s))\|\,ds + \sum_{0<t_k<\omega}\|I_k(u_n(t_k))\|\Bigg) \\
&+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}\|f(s,u_n(s))\|\,ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t - s)^{\alpha-1}\|f(s,u_n(s))\|\,ds + \sum_{0<t_k<t}\|I_k(u_n(t_k))\| \\
&\leq M\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}(B\|u_n\| + P)\,ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{\omega}(\omega - s)^{\alpha-1}(B\|u_n\| + P)\,ds + C_1 m\|u_n\|\Bigg) \\
&+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}(B\|u_n\| + P)\,ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t - s)^{\alpha-1}(B\|u_n\| + P)\,ds + C_1 m\|u_n\| \\
&\leq \frac{(M+1)\big((m+1)\omega^{\alpha}(Br + P) + C_1 mr\Gamma(\alpha+1)\big)}{\Gamma(\alpha+1)} = K.
\end{aligned}$$

Hence, (Ru_n) is uniformly bounded on B_r, which implies $R(\mathcal{B})$ is bounded in B_r.

Step 2. We prove that operator $R : \Psi \to \Psi$ is equicontinuous. We let (u_n) be a sequence on a bounded subset $\mathcal{B} \subseteq B_r$. We let $t_1, t_2 \in [0, \omega]$ and $t_1 < t_2$. Therefore,

$$\begin{aligned}
\|(Ru_n)(t_2) - (Ru_n)(t_1)\| = \Bigg\| &(\rho - E)^{-1}\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1} f(s, u_n(s))\,ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{\omega}(\omega - s)^{\alpha-1} f(s, u_n(s))\,ds + \sum_{0<t_k<\omega} I_k(u_n(t_k))\Bigg) \\
&+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t_2}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1} f(s, u_n(s))\,ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t_2}(t_2 - s)^{\alpha-1} f(s, u_n(s))\,ds + \sum_{0<t_k<t_2} I_k(u_n(t_k)) \\
&- (\rho - E)^{-1}\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1} f(s, u_n(s))\,ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{\omega}(\omega - s)^{\alpha-1} f(s, u_n(s))\,ds + \sum_{0<t_k<\omega} I_k(u_n(t_k))\Bigg) \\
&- \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t_1}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1} f(s, u_n(s))\,ds \\
&- \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t_1}(t_1 - s)^{\alpha-1} f(s, u_n(s))\,ds - \sum_{0<t_k<t_1} I_k(u_n(t_k))\Bigg\| \\
= \Bigg\| &\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t_2}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1} f(s, u_n(s))\,ds \\
&- \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t_1}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1} f(s, u_n(s))\,ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t_2}(t_2 - s)^{\alpha-1} f(s, u_n(s))\,ds - \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t_1}(t_1 - s)^{\alpha-1} f(s, u_n(s))\,ds \\
&+ \Bigg(\sum_{0<t_k<t_2} I_k(u_n(t_k)) - \sum_{0<t_k<t_1} I_k(u_n(t_k))\Bigg)\Bigg\| \\
= \Bigg\| &\frac{1}{\Gamma(\alpha)}\int_{0}^{t_1}(t_2 - s)^{\alpha-1} f(s, u_n(s))\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2 - s)^{\alpha-1} f(s, u_n(s))\,ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t_k}(t_2 - s)^{\alpha-1} f(s, u_n(s))\,ds - \frac{1}{\Gamma(\alpha)}\int_{0}^{t_1}(t_1 - s)^{\alpha-1} f(s, u_n(s))\,ds \\
&- \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_k}(t_1 - s)^{\alpha-1} f(s, u_n(s))\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t_2}(t_2 - s)^{\alpha-1} f(s, u_n(s))\,ds
\end{aligned}$$

$$
- \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_1} (t_1 - s)^{\alpha-1} f(s, u_n(s))\, ds + \sum_{t_1 < t_k < t_2} I_k(u_n(t_k)) \Bigg\|
$$

$$
\leq \frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right) \| f(s, u_n(s)) \|\, ds
$$

$$
+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \| f(s, u_n(s)) \|\, ds + \sum_{t_1 < t_k < t_2} \| I_k(u_n(t_k)) \|.
$$

Putting $t_2 \to t_1$, we determine that the right-hand side of the last inequality tends to 0, so (Ru_n) is equicontinuous.

Step 3. Operator $R : \Psi \to \Psi$ is a compact operator. Indeed, we let $\mathcal{B} \subseteq \Psi$. By the Arzela–Ascoli theorem, since R is bounded and equicontinuous, we can conclude that $R(\mathcal{B})$ is a relatively compact subset of Ψ. Therefore, $R : \Psi \to \Psi$ is a compact operator.

Step 4. Set $\mathcal{F}(R) = \{u \in \Psi : u = K \cdot Ru, \quad \text{for some } K \in [0, 1]\}$ is bounded. Now, it is clear that the fixed points of R are solutions of Equation (1). Since R is continuous, we have to prove that set

$$
\mathcal{F}(R) = \{u \in \Psi : u = K \cdot Ru, \quad \text{for some } K \in (0, 1)\}
$$

is bounded. We let $u \in \mathcal{F}(R)$. Then, $u = K \cdot Ru$, for some $K \in (0, 1)$. Now,

$$
\|u(t)\| = K\|Ru(t)\| \leq \|(\rho - E)^{-1}\| \left(\frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < \omega} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, u(s))\|\, ds \right.
$$

$$
\left. + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{\omega} (\omega - s)^{\alpha-1} \|f(s, u(s))\|\, ds + \sum_{0 < t_k < \omega} \| I_k(u(t_k)) \| \right)
$$

$$
+ \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, u(s))\|\, ds
$$

$$
+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \|f(s, u(s))\|\, ds + \sum_{0 < t_k < t} \| I_k(u(t_k)) \|
$$

$$
\leq \frac{(M+1)\big((m+1)(Br + P)\omega^{\alpha} + C_1 m r \Gamma(\alpha + 1)\big)}{\Gamma(\alpha + 1)},
$$

so $\mathcal{F}(R)$ is bounded in Ψ.

Now, using Schaefer's fixed point theorem, we conclude that R has a fixed point, and by the above, this point is a solution of Equation (1). □

Theorem 3. *Suppose that* (C2)–(C5) *hold. Then, the set of the* (ω, ρ)*-BVP solutions to the impulsive fractional differential boundary value problem* (1) *is convex.*

Proof. Using Theorem 2, we determine that the impulsive fractional differential Equation (1) has a solution in Ψ. We put $K = 1$. Then, the set of solutions is given by $\mathcal{F}(R) = \{u \in \Psi : u = Ru\}$. For every $u_1, u_2 \in \mathcal{F}(R)$, $0 \leq \lambda \leq 1$, $t \in [0, \omega]$, we have

$$
\lambda u_1(t) + (1 - \lambda) u_2(t) = \lambda (Ru_1)(t) + (1 - \lambda)(Ru_2)(t)
$$

$$= \lambda\Bigg((\rho - E)^{-1}\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}f(s,u_1(s))\,ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{\omega}(\omega - s)^{\alpha-1}f(s,u_1(s))\,ds + \sum_{0<t_k<\omega}I_k(u_1(t_k))\Bigg)$$

$$+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}f(s,u_1(s))\,ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t - s)^{\alpha-1}f(s,u_1(s))\,ds + \sum_{0<t_k<t}I_k(u_1(t_k))\Bigg)$$

$$+ (1-\lambda)\Bigg((\rho - E)^{-1}\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}f(s,u_2(s))\,ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{\omega}(\omega - s)^{\alpha-1}f(s,u_2(s))\,ds + \sum_{0<t_k<\omega}I_k(u_2(t_k))\Bigg)$$

$$+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}f(s,u_2(s))\,ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t - s)^{\alpha-1}f(s,u_2(s))\,ds + \sum_{0<t_k<t}I_k(u_2(t_k))\Bigg)$$

$$= (\rho - E)^{-1}\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}\big(\lambda f(s,u_1(s)) + (1-\lambda)f(s,u_2(s))\big)\,ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{\omega}(\omega - s)^{\alpha-1}\big(\lambda f(s,u_1(s)) + (1-\lambda)f(s,u_2(s))\big)\,ds$$

$$+ \sum_{0<t_k<\omega}\big(\lambda I_k(u_1(t_k)) + (1-\lambda)I_k(u_2(t_k))\big)\Bigg)$$

$$+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}\big(\lambda f(s,u_1(s)) + (1-\lambda)f(s,u_2(s))\big)\,ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t - s)^{\alpha-1}\big(\lambda f(s,u_1(s)) + (1-\lambda)f(u_2(s))\big)\,ds$$

$$+ \sum_{0<t_k<t}\big(\lambda I_k(u_1(t_k)) + (1-\lambda)I_k(u_2(t_k))\big)$$

$$= (\rho - E)^{-1}\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\omega}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}f\big(s,(\lambda u_1 + (1-\lambda)u_2)(s)\big)\,ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{\omega}(\omega - s)^{\alpha-1}f\big(s,(\lambda u_1 + (1-\lambda)u_2)(s)\big)\,ds$$

$$+ \sum_{0<t_k<\omega} I_k\big((\lambda u_1 + (1-\lambda)u_2)(t_k)\big)\bigg)$$

$$+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, (\lambda u_1 + (1-\lambda)u_2)(s))\, ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, (\lambda u_1 + (1-\lambda)u_2)(s))\, ds$$

$$+ \sum_{0<t_k<t} I_k((\lambda u_1 + (1-\lambda)u_2)(t_k)).$$

Hence,

$$\big(\lambda u_1 + (1-\lambda)u_2\big)(t) = \big(R\big(\lambda u_1 + (1-\lambda)u_2\big)\big)(t),$$

so $\lambda u_1 + (1-\lambda)u_2 \in \mathcal{F}(R)$, implying $\mathcal{F}(R)$ is a convex set, meaning that the set of (ω, ρ)-BVP solutions of Equation (1) is a convex set. □

We end this section with two illustrative examples:

Example 1. *We consider the following impulsive fractional differential boundary value problem:*

$$\begin{cases} {}^cD_0^{\frac{1}{2}} u(t) = a \sin 2t \cos u(t), \quad t \neq t_k,\ t \in [0, \infty), \\ \Delta u(t_k) = \dfrac{1}{8m}\big(u(t_k)\big), \quad k = 1, 2, \ldots, m, \\ u(\pi) = \rho u(0), \end{cases} \tag{2}$$

where $a \in \mathbb{R}$, $t_k = \frac{k\pi}{2}$, $k = 1, 2, \ldots, m$. Therefore, $f(t, u(t)) = a \sin 2t \cos u(t)$ and $I_k(u(t_k)) = \frac{1}{8m}\big(u(t_k)\big)$. We set $\omega = \pi$ and $\rho(x) = cx$, where $c \in \mathbb{C}\backslash\{0\}$. For any $t \in [0, \pi]$, $u, v \in \mathbb{R}$ we have $|f(t,u) - f(t,v)| \leq |a|\,|u - v|$, so (C1) is satisfied for $L = |a|$. Conditions (C3) and (C4) are trivially satisfied for $M = 1$. Since $|I_k(u(t_k))| \leq \frac{1}{8m}|u|$, (C5) is fulfilled for $C_1 = \frac{1}{8m}$ and since $|I_k(u(t_k)) - I_k(v(t_k))| \leq \frac{1}{8m}|u - v|$, (C6) holds for $C_I = \frac{1}{8m}$. Moreover, we let a and m be such that $\frac{3}{8}\Gamma\big(\frac{3}{2}\big) - 2|a|(m+1)\sqrt{\pi} > 0$. Then, using Theorem 1, the impulsive fractional differential boundary value problem (2) has a unique $(\pi, c\cdot)$-BVP solution $u \in \Phi_{\pi,c\cdot}$. Additionally, it holds that

$$|u| \leq \frac{2|a|(m+1)\sqrt{\pi}}{\Gamma\big(\frac{3}{2}\big) - 2\Big(|a|(m+1)\sqrt{\pi} + \frac{1}{8}\Gamma\big(\frac{3}{2}\big)\Big)}.$$

Example 2. *We consider the following impulsive fractional differential boundary value problem:*

$$\begin{cases} {}^cD_0^{\frac{2}{3}} u(t) = a \sin u(t), \quad t \neq t_k,\ t \in [0, \infty), \\ \Delta u(t_k) = \cos k\pi, \quad k = 1, 2, \ldots, m, \\ u(2\pi) = \rho u(0), \end{cases} \tag{3}$$

where $a \in \mathbb{R}$, $t_k = \frac{k\pi}{2}$, $k = 1, 2, \ldots, m$. Now, $f(t, u(t)) = a \sin u(t)$ and $I_k(u(t_k)) = \cos k\pi$. We set $\omega = 2\pi$ and $\rho(x) = \exp(x)$. It is clear that (C2) holds for $B = |a|$ and $P > 0$. Also, it is obvious that (C3) and (C4) are satisfied with $M = 1$. Additionally, for $C_1 = 1$, condition (C5) is fulfilled. Hence, the conditions of Theorem 2 are satisfied; then, the impulsive fractional differential boundary value problem (3) has at least one $(2\pi, \exp(\cdot))$-BVP solution $u \in \Phi_{2\pi,\exp(\cdot)}$. Furthermore, by Theorem 3, the set of solutions of Equation (3) is a convex set.

3. Conclusions

This paper presents established results concerning the existence of (ω, ρ)-BVP solutions for a specific class of Caputo impulsive fractional differential equations with boundary value conditions on Banach spaces. The main focus of the investigation involves providing sufficient conditions for the existence and uniqueness of the (ω, ρ)-periodic solution of Equation (1) using the Banach contraction mapping principle. Additionally, the paper presents sufficient conditions for the existence of (ω, ρ)-BVP solutions of Equation (1) using Schaefer's fixed point theorem.

As a future research endeavor, the authors plan to further explore the existence and uniqueness of the (ω, ρ)-BVP solution for other types of abstract fractional differential equations, including various types of fractional derivatives.

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