*mathematics*

Article

# Characterizations of the Frame Bundle Admitting Metallic Structures on Almost Quadratic $\phi$-Manifolds

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**Abstract:** In this work, we have characterized the frame bundle $FM$ admitting metallic structures on almost quadratic $\phi$-manifolds $\phi^2 = p\phi + qI - q\eta \otimes \zeta$, where $p$ is an arbitrary constant and $q$ is a nonzero constant. The complete lifts of an almost quadratic $\phi$-structure to the metallic structure on $FM$ are constructed. We also prove the existence of a metallic structure on $FM$ with the aid of the $\tilde{J}$ tensor field, which we define. Results for the 2-Form and its derivative are then obtained. Additionally, we derive the expressions of the Nijenhuis tensor of a tensor field $\tilde{J}$ on $FM$. Finally, we construct an example of it to finish.

**Keywords:** metallic structure; frame bundle; partial differential equations; almost quadratic $\phi$-structure; 2-Form; diagonal lift; mathematical operators; nijenhuis tensor

**MSC:** 53C15; 58D17

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## 1. Introduction

Numerous types of $f$-structures on a differentiable manifold $M$ have been studied by Yano [1], Ishihara and Yano [2], Blair [3], Nakagawa [4] and others. Yano proposed the notion of an $f$-structure obeying $f^3 + f = 0$, $f$ is a tensor field of type (1,1), which is the generalization of an almost complex structure and an almost contact structure [5] and investigated some basic results of it. Later, Goldberg and Yano [6] and Goldberg and Perridis [7] defined a polynomial structure $P(J) = J^n + a_n J^{n-1} + ... + a_2 J + a_1 I$, where $a_1, a_2, \ldots, a_n$ are real numbers, $J$ is a tensor field of type (1,1) and $I$ is an identity tensor field of type (1,1) on $M$. Moreover, some important polynomial structures such as an $f(3,\varepsilon)$-structure [8], a general quadratic structure [9], an almost complex structure and an almost product structure [1], $\phi(4,\pm 2)$-structures [10] and an almost $r$-contact structure [11] are studied and the fundamental results are established in these papers.

Recently, the polynomial structure $J^2 = pJ + qI$, $p, q \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, of degree 2 is known as a metallic structure on $M$ [12–14]. For specific values of $p$ and $q$, metallic structures become prominent structures given below:

| $p$ | $q$ | Structure |
|---|---|---|
| 0 | 1 | an almost product structure [15] |
| 0 | −1 | an almost complex structure [16,17] |
| 1 | 1 | a golden structure [18,19] |
| 2 | 1 | a silver structure [20] |

Hretceanu and Crasmareanu [21] initiated the study of golden and metallic structures on a Riemannian manifold and interpreted the geometry of submanifolds admitting both structures on $M$. The various geometric properties of such structures in a metallic (and golden) Riemannian manifold and a metallic (and golden) warped product Riemannian manifold were studied in [22–26]. Debnath and Konar [27] defined a new type of structure named as an almost quadratic $\phi$-structure $(\phi, \zeta, \eta)$ on $M$ and studied some geometric properties of such structures. Next, Gonul et al. [28] established the relationship between an almost quadratic metric $\phi$-structure and a metallic structure on $M$. Most recently, Gok et. al. [29] defined a generalized structure namely $f_{(a,b)}(3,2,1)$-structures on manifolds and construct a framed $f_{(a,b)}(3,2,1)$-structures on $M$.

On the other hand, let $M$ be an $m$-dimensional differentiable manifold, $TM$ its tangent bundle and $FM$ its frame bundle. The notion of the mappings, namely vertical, complete and horzontal lifts from the manifold $M$ to its tangent bundle $TM$ were introduced by Sasaki [30], Yano and Ishihara [31] and Yano and Davis [32]. Kabayashi and Nomizu [33], Mok [34] and Okubo [35] have studied the complete lift of a vector field $\mathcal{A}$ to $FM$. The geometric structures such as an almost contact metric structure $(\phi, \zeta, \eta, g)$, and almost complex structures $J$ on $FM$ have been studied by Bonome et al. [16], who established the integrability and normality of such structures on $FM$.

In [36], Khan has introduced a tensor field $\tilde{J}$ on $FM$ and proved that $\tilde{J}$ is a metallic structure on $FM$. The integrability condition for the diagonal and horizontal lifts of the metallic structure $\tilde{J}$ on $FM$ is established. The geometric structures on $FM$ have been studied by Cordero et al. [37], Kowalski [38], Sekizawa [39], Kowalski and Sekizawa [40], Niedzialomski [41], Lachieze-Rey [42], Khan [43–45] and many more.

The main objective of this paper can be summarized as follows:

- We study the complete lifts of an almost quadratic $\phi$-structure to the metallic structure on $FM$.
- We establish the existence of a metallic structure on $FM$ in the tensor field $\tilde{J}$, which we define.
- We obtain results on the 2-Form and its derivative on $FM$.
- We derive the expressions of the Nijenhuis tensor of a tensor field $\tilde{J}$ on $FM$.
- We construct an example related to it.

Remark: $\Im_a^b(M)$ and $\Im_a^b(FM)$ are symbolized as the set of all $(a,b)$-type tensor fields in $M$ and $FM$ respectively [17].

## 2. Preliminaries

Let $F, \mathcal{A}, f$ and $\eta$ be a tensor field of type (1,1), a vector field, a function and a 1-form, respectively, on $M$. The horizontal, vertical and $\alpha$-vertical lifts of $F, \mathcal{A}, f$ and $\eta$ are represented by $F^H, \mathcal{A}^H, \mathcal{A}^{(\alpha)}, f^H, \eta^V$ and $\eta^{H_\alpha}$ on $FM$ and they are expressed in terms of partial differential equations as [16,17]

$$\mathcal{A}^H = \mathcal{A}^i \frac{\partial}{\partial \mathcal{A}^i} - \mathcal{A}^i \Gamma_{ik}^h \mathcal{A}_\alpha^k \frac{\partial}{\partial \mathcal{A}^h}, \tag{1}$$

$$\mathcal{A}^{(\alpha)} = \mathcal{A}^i \frac{\partial}{\partial \mathcal{A}_\alpha^i}, \tag{2}$$

$$\begin{aligned} F^H &= F_j^h \frac{\partial}{\partial \mathcal{A}^h} \otimes dx^j + \mathcal{A}_\alpha^k (\Gamma_{jk}^i F_i^h - \Gamma_{ik}^h F_j^i) \frac{\partial}{\partial \mathcal{A}_\alpha^h}, \\ &\otimes dx^j + \delta_\alpha^\beta F_j^h \frac{\partial}{\partial \mathcal{A}_\alpha^h} \otimes dX_\beta^j, \end{aligned} \tag{3}$$

$$\eta^V = \eta_i dx^i, \tag{4}$$

$$\eta^{H_\alpha} = \mathcal{A}_\alpha^j \Gamma_{ij}^h \eta_h dx^i + \eta_i dX_\alpha^i, \tag{5}$$

$$\mathcal{A}^H = \sum_{\alpha=1}^m (\mathcal{A}_\alpha^j \Gamma_{ij}^h \eta_h dx^i + \eta_i dX_\alpha^i), \tag{6}$$

where $\Gamma_{ij}^{h}, \mathcal{A}^{i}, F_{j}^{h}$ and $\eta_i$ are the local components of a linear connection $\nabla$, $\mathcal{A}$, $F$ and $\eta$, respectively on $M$.

**Proposition 1.** *$\forall \mathcal{A}, \mathcal{B} \in \Im_0^1(M)$, by using mathematical operators, we have the following*

$$\begin{array}{rcl} \mathcal{A}^H(f^V) & = & (\mathcal{A}(f))^V), \\ \mathcal{A}^{(\alpha)}(f^V) & = & 0, \\ F^H(\mathcal{A}^{(\alpha)}) & = & (F(\mathcal{A}))^{\alpha}, \\ F^H(\mathcal{A}^H) & = & (F(\mathcal{A}))^H, \\ \eta^V(\mathcal{A}^H) & = & (F(\mathcal{A}))^V, \\ \eta^V(\mathcal{A}^{(\alpha)}) & = & 0, \\ \eta^{H_\alpha}(\mathcal{A}^H) & = & 0, \\ \eta^{H_\alpha}(\mathcal{A}^{(\beta)}) & = & \delta_\alpha^\beta(\eta(\mathcal{A}))^V, \end{array} \tag{7}$$

*where $\alpha, \beta = 1, \ldots, m$ and $\delta_\beta^\alpha$ denotes the Kronecker delta.*

**Proposition 2.** *Let $\forall \mathcal{A}, \mathcal{B} \in \Im_0^1(M)$. Then, we have the following*

$$\begin{array}{rcl} [\mathcal{A}^{(\alpha)}, \mathcal{B}^{(\beta)}] & = & 0, \\ [\mathcal{A}^H, \mathcal{B}^{(\alpha)}] & = & (\nabla_X Y)^{(\alpha)}, \\ [\mathcal{A}^H, \mathcal{B}^H] & = & [\mathcal{A}, \mathcal{B}]^H - \gamma R(\mathcal{A}, \mathcal{B}), \end{array} \tag{8}$$

*where $R(\mathcal{A}, \mathcal{B}) = [\nabla_{\mathcal{A}}, \nabla_{\mathcal{B}}] - \nabla_{[\mathcal{A},\mathcal{B}]}$, $R$ is the curvature tensor of $\nabla$.*

Let $g$ be a Riemannian metric on a Riemannian manifold $M$ and $g^D$ its diagonal metric on $FM$, then

$$\begin{array}{rcl} g^D(\mathcal{A}^H, \mathcal{B}^H) & = & \{g(\mathcal{A}, \mathcal{B})\}^V, \\ g^D(\mathcal{A}^H, \mathcal{B}^{(\alpha)}) & = & 0, \\ g^D(\mathcal{A}^{(\alpha)}, \mathcal{B}^{(\beta)}) & = & \delta^{\alpha\beta}\{g(\mathcal{A}, \mathcal{B})\}^V, \forall \alpha, \beta = 1, \ldots, m \end{array} \tag{9}$$

and

$$\begin{array}{rcl} 2g^D(\tilde{\nabla}_{\tilde{\mathcal{A}}}\tilde{\mathcal{B}}, \tilde{\mathcal{C}}) & = & \tilde{\mathcal{A}}(g^D(\tilde{\mathcal{B}}, \tilde{\mathcal{C}})) + \tilde{\mathcal{B}}(g^D(\tilde{\mathcal{C}}, \tilde{\mathcal{A}})) - \tilde{\mathcal{C}}(g^D(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})) \\ & + & g^D([\tilde{\mathcal{A}}, \tilde{\mathcal{B}}], \tilde{\mathcal{C}}) + g^D([\tilde{\mathcal{C}}, \tilde{\mathcal{A}}], \tilde{\mathcal{B}}) + g^D(\tilde{\mathcal{A}}, [\tilde{\mathcal{C}}, \tilde{\mathcal{B}}]), \end{array} \tag{10}$$

$\forall \tilde{\mathcal{A}}, \tilde{\mathcal{B}} \in \Im_0^1(FM)$, where $\nabla$ and $\tilde{\nabla}$ represent the Levi-Civita connection of $(M, g)$ and $(FM, g^D)$, respectively.

**Proposition 3.** *$\forall \mathcal{A}, \mathcal{B} \in \Im_0^1(M)$ , by using mathematical operators, we have the following*

$$\begin{array}{rcl} \tilde{\nabla}_{\mathcal{A}^{(\alpha)}}\mathcal{B}^{(\beta)} & = & 0, \\ g^D(\tilde{\nabla}_{\mathcal{A}^{(\alpha)}}\mathcal{B}^H, \mathcal{C}^{(\beta)} & = & 0, \\ g^D(\tilde{\nabla}_{\mathcal{A}^{(\alpha)}}\mathcal{B}^H, \mathcal{C}^H) & = & -\dfrac{1}{2}g^D(\gamma R(\mathcal{C}, \mathcal{B}), \mathcal{A}^{(\alpha)}), \\ g^D(\tilde{\nabla}_{\mathcal{A}^H}\mathcal{B}^{(\alpha)}, \mathcal{C}^{(\beta)}) & = & \delta^{\alpha\beta}\{g(\nabla_{\mathcal{A}}\mathcal{B}, \mathcal{C})\}^V, \\ g^D(\tilde{\nabla}_{\mathcal{A}^H}\mathcal{B}^{(\alpha)}, \mathcal{C}^H) & = & -\dfrac{1}{2}g^D(\gamma R(\mathcal{C}, \mathcal{A}), \mathcal{B}^{(\alpha)}), \\ g^D(\tilde{\nabla}_{\mathcal{A}^H}\mathcal{B}^H, \mathcal{C}^{(\alpha)}) & = & -\dfrac{1}{2}g^D(\gamma R(\mathcal{A}, \mathcal{B}), \mathcal{C}^{(\alpha)}), \\ g^D(\tilde{\nabla}_{\mathcal{A}^H}\mathcal{B}^H, \mathcal{C}^H) & = & \{g(\nabla_{\mathcal{A}}\mathcal{B}, \mathcal{C})\}^V. \end{array} \tag{11}$$

*2.1. Metallic Structure*

If a (1, 1) tensor field $J$ obeying

$$J^2 = pJ + qI, \quad p, q \in \mathbb{N}, \tag{12}$$

where $\mathbb{N}$ is the set of natural numbers and $I$ is an identity operator, determines a polynomial structure on a manifold $M$, the structure is referred to as metallic. A metallic manifold is defined as $(M, J)$ when a manifold $M$ possesses a metallic structure (MS) $J$.

The Nijenhuis tensor $N_J$ of $J$ is expressed as

$$N_J(\mathcal{A}, \mathcal{B}) = [J\mathcal{A}, J\mathcal{B}] - J[J\mathcal{A}, \mathcal{B}] - J[\mathcal{A}, J\mathcal{B}] + J^2[\mathcal{A}, \mathcal{B}], \tag{13}$$

$\forall \mathcal{A}, \mathcal{B} \in \Im^1_0(M)$.

*2.2. Almost Quadratic $\phi$-Structure*

An $m(= 2n + 1)$-dimensional differentiable manifold $M$ with a non-null tensor field $\phi$ of type (1,1), a 1-form $\eta$ and a vector field $\zeta$ on $M$ satisfies

$$\phi^2 = p\phi + qI - q\eta \otimes \zeta, \quad p^2 + 4q \neq 0, \tag{14}$$

$$\eta(\zeta) = 1, \quad \eta \circ \phi = 0, \quad \phi(\zeta) = 0, \tag{15}$$

where $p$ is an arbitrary constant and $q \neq 0$. The structure $(\phi, \zeta, \eta)$ is called an almost quadratic $\phi$-structure on $M$ and the manifold $(M, \phi, \zeta, \eta)$ is called an almost quadratic $\phi$-manifold [27,28].

Furthermore,

$$g(\phi\mathcal{A}, \mathcal{B}) = g(\mathcal{A}, \phi\mathcal{B}) \tag{16}$$

and

$$g(\phi\mathcal{A}, \phi\mathcal{B}) = pg(\phi\mathcal{A}, \mathcal{B}) + qg(\mathcal{A}, \mathcal{B}) - q\eta(\mathcal{A})\eta(\mathcal{B})). \tag{17}$$

The structure $(\phi, \zeta, \eta, g)$ is referred to as an almost quadratic metric $\phi$-structure and $(M, \phi, \zeta, \eta, g)$ is called an almost quadratic metric $\phi$-manifold.

In addition, the 1-form $\eta$ is associated with $g$ such that

$$g(\mathcal{A}, \zeta) = \eta(\mathcal{A})$$

and the fundamental 2-Form $\Phi$ is given by [3]

$$\Phi(\mathcal{A}, \mathcal{B}) = g(\mathcal{A}, \phi\mathcal{B}). \tag{18}$$

The Nijenhuis tensor of $(\phi, \zeta, \eta)$ is denoted by $N_\phi$ and is given by

$$N_\phi(\mathcal{A}, \mathcal{B}) = [\phi\mathcal{A}, \phi\mathcal{B}] - \phi[\phi\mathcal{A}, \mathcal{B}] - \phi[\mathcal{A}, \phi\mathcal{B}] + \phi^2[\mathcal{A}, \mathcal{B}], \tag{19}$$

$\forall \mathcal{A}, \mathcal{B} \in \Im^1_0(M)$.

## 3. Proposed Theorems on *FM* Admitting Metallic Structures on Almost Quadratic $\phi$-Manifolds

In this section, we construct the complete lifts of an almost quadratic $\phi$-structure to the metallic structure on $FM$.

Next, we obtain the results on the 2-Form and its derivative on $FM$.

Boname et al. [16] proposed and gave the definition of $\tilde{J}$ on $FM$ as

$$\begin{aligned} \tilde{J} &= \phi^H + \sum_{\alpha=1}^{n} \eta^{H_\alpha} \otimes \zeta^{(\alpha+n)} - \sum_{\alpha=1}^{n} \eta^{H_{\alpha+n}} \otimes \zeta^{(\alpha)} \\ &+ \eta^V \otimes \zeta^{(2n+1)} - \eta^{H_{2n+1}} \otimes \zeta^H. \end{aligned} \tag{20}$$

Recently, Khan [36] proposed and gave the definition of the tensor field $\tilde{J}$ on $FM$ as

$$\begin{aligned}\tilde{J} &= \frac{p}{2}I - \left(\frac{2\sigma_p^q - p}{2}\right)[\phi^H + \sum_{\alpha=1}^{n}\eta^{H_\alpha}\otimes\zeta^{(\alpha+n)} \\ &- \sum_{\alpha=1}^{n}\eta^{H_{\alpha+n}}\otimes\zeta^{(\alpha)} + \eta^V\otimes\zeta^{(2n+1)} - \eta^{H_{2n+1}}\otimes\zeta^H],\end{aligned} \tag{21}$$

where $\eta = \eta_i dx^i$, $\eta^V = \eta_i dx^i$ and $\eta^{H_\alpha} = \mathcal{A}_\alpha^j\Gamma_{ij}^h\eta_h dx^i + \eta_i dx_\alpha^i$.

Motivated by the above definitions, let us introduce a tensor field $\tilde{J}$ of type (1,1) on $FM$ as

$$\begin{aligned}\tilde{J} &= \frac{p}{2}I - A[\phi^H + \sqrt{q}\{\sum_{\alpha=1}^{n}\eta^{H_\alpha}\otimes\zeta^{(\alpha+n)} \\ &- \sum_{\alpha=1}^{n}\eta^{H_{\alpha+n}}\otimes\zeta^{(\alpha)} + \eta^V\otimes\zeta^{(2n+1)} - \eta^{H_{2n+1}}\otimes\zeta^H\}],\end{aligned} \tag{22}$$

where $A = \frac{2\sigma_p^q - p}{2\sqrt{p\phi^H + q}}$, $\eta = \eta_i dx^i$,

$\eta^V = \eta_i dx^i$ and $\eta^{H_\alpha} = \mathcal{A}_\alpha^j\Gamma_{ij}^h\eta_h dx^i + \eta_i dx_\alpha^i$.

**Theorem 1.** *Let $\tilde{\mathcal{A}}$ be a vector field on FM. Then $\tilde{J}$ given by (22) is a metallic structure on FM.*

**Proof.** To prove that $\tilde{J}$ defined in (22) is a metallic structure, we have to prove that

$$\tilde{J}^2\tilde{\mathcal{A}} = p\tilde{J}(\tilde{\mathcal{A}}) + qI; p, q \in \mathbb{N}. \tag{23}$$

□

Taking the horizontal lift $\mathcal{A}^H$ and $\beta^{th}$-vertical lift $\mathcal{A}^{(\beta)}$ for each $\beta = 1, \ldots 2n+1$ on both sides of (22), we infer

$$\begin{aligned}\tilde{J}(\mathcal{A}^{(\beta)}) &= \frac{p}{2}\mathcal{A}^{(\beta)} - A[(\phi\mathcal{A})^{(\beta)} + \sqrt{q}\{\varepsilon(\beta)\zeta^{(\beta+\varepsilon(\beta)n)} \\ &- \delta_{2n+1}^{\beta}\eta(\mathcal{A})^V\xi^H\}],\end{aligned} \tag{24}$$

where

$$\varepsilon(\beta) = \begin{cases} 1, & \beta \le n, \\ -1, & n < \beta \le 2n, \\ 0, & \beta = 2n+1, \end{cases} \tag{25}$$

and

$$\tilde{J}(\mathcal{A}^H) = \frac{p}{2}\mathcal{A}^H - A[(\phi\mathcal{A})^H + \sqrt{q}\{\eta(\mathcal{A})^V\zeta^{(2n+1)}\}]. \tag{26}$$

In view of (22), we provide

$$\begin{aligned}\tilde{J}(\phi^H\tilde{\mathcal{A}}) &= \frac{p}{2}\phi^H\tilde{\mathcal{A}} - A[-\tilde{\mathcal{A}} + \sqrt{q}\{\sum_{\alpha=1}^{n}\eta^{H_\alpha}(\tilde{\mathcal{A}})\zeta^{(\alpha+n)} \\ &- \sum_{\alpha=1}^{n}\eta^{H_{\alpha+n}}(\tilde{\mathcal{A}})\zeta^{(\alpha)} + \eta^V(\tilde{\mathcal{A}})\zeta^{(2n+1)} - \eta^{H_{2n+1}}(\tilde{\mathcal{A}})\zeta^H\}],\end{aligned} \tag{27}$$

$$\tilde{J}(\zeta^{(\alpha)}) = \frac{p}{2}\zeta^{(\alpha)} - A\sqrt{q}(\zeta^{(\alpha+n)} - \zeta^H),$$

$$\tilde{J}(\zeta^H) = \frac{p}{2}\zeta^H - A\sqrt{q}\zeta^{(2n+1)},$$

and

$$
\begin{aligned}
\tilde{J}^2(\tilde{\mathcal{A}}) &= \frac{p}{2}J\tilde{\mathcal{A}} - A[\tilde{J}(\phi^H\tilde{\mathcal{A}}) + \sqrt{q}\{\sum_{\alpha=1}^{n}\eta^{H_\alpha}(\tilde{\mathcal{A}})\tilde{J}(\zeta^{(\alpha+n)}) \\
&- \sum_{\alpha=1}^{n}\eta^{H_{\alpha+n}}(\tilde{\mathcal{A}})\tilde{J}(\zeta^{(\alpha)}) + \eta^V(\tilde{\mathcal{A}})\tilde{J}(\zeta^{(2n+1)}) - \eta^{H_{2n+1}}(\tilde{\mathcal{A}})\tilde{J}(\zeta^H)\}], \\
\tilde{J}^2(\tilde{\mathcal{A}}) &= p\tilde{J}(\tilde{\mathcal{A}}) + q\tilde{\mathcal{A}}.
\end{aligned} \tag{28}
$$

**Definition 1.** *The 2-Form $\Omega$ of $\tilde{J}$ is given by*

$$
\Omega(\tilde{\mathcal{A}},\tilde{\mathcal{B}}) = g^D(\tilde{\mathcal{A}},\tilde{J}\tilde{\mathcal{B}}), \tag{29}
$$

$\forall\tilde{\mathcal{A}},\tilde{\mathcal{B}} \in \Im_0^1(FM)$.

**Theorem 2.** *The 2-Form $\Omega$ of $(g^D,\tilde{J})$ on $FM$ is given by*

$$
\begin{aligned}
(i)\ \Omega(\mathcal{A}^H,\mathcal{B}^H) &= \frac{p}{2}g(\mathcal{A},\mathcal{B})^V - A\Phi(\mathcal{A},\mathcal{B})^V, \\
(ii)\ \Omega(\mathcal{A}^H,\mathcal{B}^{(\beta)}) &= A\sqrt{q}\delta_{2n+1}^{\beta}\eta(\mathcal{A})^V\eta(\mathcal{B})^V, \\
(iii)\ \Omega(\mathcal{A}^{(\beta)},\mathcal{B}^{(\mu)}) &= \frac{p}{2}\delta_\mu^\beta(g(\mathcal{A},\mathcal{B}))^V - A[\delta_\mu^\beta\Phi(\mathcal{A},\mathcal{B})^V \\
&+ \sqrt{q}\varepsilon(\mu)\delta_{\mu+\varepsilon(\mu)n}^{\beta+\varepsilon(\beta)n}\eta(\mathcal{A})^V\eta(\mathcal{B})^V],
\end{aligned}
$$

*where $\alpha,\beta,\mu = 1,\dots,2n+1$ and $\forall\mathcal{A},\mathcal{B} \in \Im_0^1(M)$.*

**Proof.** Using (9) and (29), we infer

$$
\begin{aligned}
(i)\ \Omega(\mathcal{A}^H,\mathcal{B}^H) &= g^D(\mathcal{A}^H,\frac{p}{2}\mathcal{B}^H - A[(\phi\mathcal{B})^H + \sqrt{q}\eta(\mathcal{B})^V\zeta^{(2n+1)}]), \\
&= \frac{p}{2}g(\mathcal{A},\mathcal{B})^V - A\Phi(\mathcal{A},\mathcal{B})^V, \\
(ii)\ \Omega(\mathcal{A}^H,\mathcal{B}^{(\beta)}) &= g^D(\mathcal{A}^H,\frac{p}{2}\mathcal{B}^{(\beta)} - A[(\phi\mathcal{B})^{(\beta)} \\
&+ \sqrt{q}\{\varepsilon(\beta)\eta(\mathcal{B})^V\zeta^{(\beta+\varepsilon(\beta)n)} - \delta_{2n+1}^{\beta}\eta(\mathcal{B})^V\zeta^H]\}). \\
&= A\sqrt{q}\delta_{2n+1}^{\beta}\eta(\mathcal{A})^V\eta(\mathcal{B})^V, \\
(iii)\ \Omega(\mathcal{A}^{(\beta)},\mathcal{B}^{(\mu)}) &= g^D(\mathcal{A}^{(\beta)},\frac{p}{2}\mathcal{B}^{(\mu)} - A[(\phi\mathcal{B})^{(\mu)} \\
&+ \sqrt{q}\{\varepsilon(\beta)\eta(\mathcal{B})^V\zeta^{(\mu+\varepsilon(\mu)n)} - \delta_{2n+1}^{\mu}\eta(\mathcal{B})^V\zeta^H]\}) \\
&= \frac{p}{2}\delta_\mu^\beta(g(\mathcal{A},\mathcal{B}))^V - A[\delta_\mu^\beta\Phi(\mathcal{A},\mathcal{B})^V \\
&+ \sqrt{q}\varepsilon(\mu)\delta_{\mu+\varepsilon(\mu)n}^{\beta+\varepsilon(\beta)n}\eta(\mathcal{A})^V\eta(\mathcal{B})^V].
\end{aligned} \tag{30}
$$

□

**Theorem 3.** *The differential $d\Omega$ on $FM$ is expressed as*

$$\begin{aligned}
(i)\ d\Omega(\mathcal{A}^H,\mathcal{B}^H,\mathcal{C}^H) &= \frac{1}{3}\{\frac{p}{2}[(Xg(\mathcal{B},\mathcal{C}))^V - g([\mathcal{A},\mathcal{B}],\mathcal{C})^V - (Yg(\mathcal{B},\mathcal{C}))^V \\
&+ g([\mathcal{A},\mathcal{C}],\mathcal{B})^V + (Zg(\mathcal{A},\mathcal{B}))^V - g([\mathcal{B},\mathcal{C}],\mathcal{A})^V] \\
&- A[(\mathcal{A}(\Phi(\mathcal{B},\mathcal{C}))^V - (\mathcal{B}(\Phi(\mathcal{A},\mathcal{C}))^V \\
&+ (\mathcal{C}(\Phi(\mathcal{A},\mathcal{B}))^V - (\Phi([\mathcal{A},\mathcal{B}],\mathcal{C})^V) + (\Phi([\mathcal{A},\mathcal{C}],\mathcal{B})^V) \\
&- (\Phi([\mathcal{B},\mathcal{C}],\mathcal{A})^V) + \Omega(\gamma R(\mathcal{A},\mathcal{B}),\mathcal{C}^H) \\
&- \Omega(\gamma R(\mathcal{A},\mathcal{C}),\mathcal{B}^H) + \Omega(\gamma R(\mathcal{B},\mathcal{C}),\mathcal{A}^H)\}, \\
(ii)\ d\Omega(\mathcal{A}^H,\mathcal{B}^H,\mathcal{C}^{(\beta)}) &= \frac{1}{3}\{A\sqrt{q}[\delta^{\beta}_{2n+1}(\mathcal{A}\eta(\mathcal{C})\eta(\mathcal{B}))^V \\
&- \delta^{\beta}_{2n+1}(\mathcal{B}\eta(\mathcal{C})\eta(\mathcal{A}))^V \\
&- \delta^{\beta}_{2n+1}(\eta([\mathcal{A},\mathcal{B}])\eta(\mathcal{C}))^V + \Omega(\gamma R(\mathcal{A},\mathcal{B}),\mathcal{C}^{(\beta)}) \\
&+ \delta^{\beta}_{2n+1}(\eta(\nabla_X Z)\eta(\mathcal{B}))^V \\
&- \delta^{\beta}_{2n+1}(\eta(\nabla_Y Z)\eta(\mathcal{A}))^V]\}, \\
(iii)\ d\Omega(\mathcal{A}^H,\mathcal{B}^{(\beta)},\mathcal{C}^{(\mu)}) &= \frac{1}{3}\{\frac{p}{2}\delta^{\beta}_{\alpha}(\nabla_X g)(\mathcal{B},\mathcal{C})^V - A\delta^{\beta}_{\alpha}(\nabla_{\mathcal{A}}\Phi)(\mathcal{B},\mathcal{C})^V \\
&+ \sqrt{q}\varepsilon(\alpha)\delta^{\beta}_{\alpha+\sqrt{q}\varepsilon(\alpha)n}\eta(\mathcal{B})^V(\nabla_{\mathcal{A}}\eta)\mathcal{C})^V + \eta(\mathcal{C})^V(\nabla_{\mathcal{A}}\eta)\mathcal{B})^V\}, \\
(iv)\ d\Omega(\mathcal{A}^{(\alpha)},\mathcal{B}^{(\beta)},\mathcal{C}^{(\mu)}) &= 0,
\end{aligned}$$

$\forall \mathcal{A},\mathcal{B},\mathcal{C} \in \Im^1_0(M)$.

**Proof.** The differential $d\Omega$ is given by

$$\begin{aligned}
3d\Omega(\tilde{\mathcal{A}},\tilde{\mathcal{B}},\tilde{\mathcal{C}}) &= \{\tilde{\mathcal{A}}(\Omega(\tilde{\mathcal{B}},\tilde{\mathcal{C}})) - \tilde{\mathcal{B}}(\Omega(\tilde{\mathcal{A}},\tilde{\mathcal{C}})) + \tilde{\mathcal{C}}(\Omega(\tilde{\mathcal{A}},\tilde{\mathcal{B}})) \\
&- \Omega([\tilde{\mathcal{A}},\tilde{\mathcal{B}}],\tilde{\mathcal{C}}) + \Omega([\tilde{\mathcal{A}},\tilde{\mathcal{C}}],\tilde{\mathcal{B}}) - \Omega([\tilde{\mathcal{B}},\tilde{\mathcal{C}}],\tilde{\mathcal{A}})\},
\end{aligned}$$

$\forall \tilde{\mathcal{A}},\tilde{\mathcal{B}},\tilde{\mathcal{C}} \in \Im^1_0(FM)$.

$$\begin{aligned}
(i)\ 3d\Omega(\mathcal{A}^H,\mathcal{B}^H,\mathcal{C}^H) &= \frac{p}{2}[\mathcal{A}^H(g(\mathcal{B},\mathcal{C})^V) - \mathcal{B}^H(g(\mathcal{A},\mathcal{C})^V) \\
&+ \mathcal{C}^H(g(\mathcal{A},\mathcal{B})^V)] - A[\mathcal{A}^H(\Phi(\mathcal{B},\mathcal{C})^V) \\
&- \mathcal{B}^H(\Phi(\mathcal{A},\mathcal{C})^V) + \mathcal{C}^H(\Phi(\mathcal{A},\mathcal{B})^V)] \\
&- \frac{p}{2}g([\mathcal{A},\mathcal{B}],\mathcal{C})^V + A(\Phi([\mathcal{A},\mathcal{B}],\mathcal{C})^V) \\
&+ \Omega(\gamma R(\mathcal{A},\mathcal{B}),\mathcal{C}^H) + \frac{p}{2}g([\mathcal{A},\mathcal{C}],\mathcal{B})^V \\
&+ A(\Phi([\mathcal{A},\mathcal{C}],\mathcal{B})^V) - \Omega(\gamma R(\mathcal{A},\mathcal{C}),\mathcal{B}^H) \\
&- \frac{p}{2}g([\mathcal{B},\mathcal{C}],\mathcal{A})^V + A(\Phi([\mathcal{B},\mathcal{C}],\mathcal{A})^V) \\
&+ \Omega(\gamma R(\mathcal{B},\mathcal{C}),\mathcal{A}^H) \\
&= \frac{p}{2}[(Xg(\mathcal{B},\mathcal{C}))^V - g([\mathcal{A},\mathcal{B}],\mathcal{C})^V - (Yg(\mathcal{B},\mathcal{C}))^V \\
&+ g([\mathcal{A},\mathcal{C}],\mathcal{B})^V + (Zg(\mathcal{A},\mathcal{B}))^V - g([\mathcal{B},\mathcal{C}],\mathcal{A})^V] \\
&- A[(\mathcal{A}(\Phi(\mathcal{B},\mathcal{C}))^V - (\mathcal{B}(\Phi(\mathcal{A},\mathcal{C}))^V \\
&+ (\mathcal{C}(\Phi(\mathcal{A},\mathcal{B}))^V - (\Phi([\mathcal{A},\mathcal{B}],\mathcal{C})^V) + (\Phi([\mathcal{A},\mathcal{C}],\mathcal{B})^V) \\
&- (\Phi([\mathcal{B},\mathcal{C}],\mathcal{A})^V) + \Omega(\gamma R(\mathcal{A},\mathcal{B}),\mathcal{C}^H) \\
&- \Omega(\gamma R(\mathcal{A},\mathcal{C}),\mathcal{B}^H) + \Omega(\gamma R(\mathcal{B},\mathcal{C}),\mathcal{A}^H),
\end{aligned}$$

$$
\begin{aligned}
(ii)\ \ 3d\Omega(\mathcal{A}^H,\mathcal{B}^H,\mathcal{C}^{(\beta)}) &= A\sqrt{q}[\mathcal{A}^H\delta^{\beta}_{2n+1}\eta(\mathcal{C})^V\eta(\mathcal{B})^V \\
&- \mathcal{B}^H\delta^{\beta}_{2n+1}\eta(\mathcal{C})^V\eta(\mathcal{A})^V \\
&+ \mathcal{C}^{(\beta)}\{\frac{p}{2}g(\mathcal{A},\mathcal{B})^V - \Phi(\mathcal{A},\mathcal{B})^V\} \\
&- \delta^{\beta}_{2n+1}(\eta([\mathcal{A},\mathcal{B}])\eta(\mathcal{C}))^V + \Omega(\gamma R(\mathcal{A},\mathcal{B}),\mathcal{C}^{(\beta)}) \\
&+ \delta^{\beta}_{2n+1}(\eta(\nabla_X Z)\eta(\mathcal{B}))^V \\
&- \delta^{\beta}_{2n+1}(\eta(\nabla_Y Z)\eta(\mathcal{A}))^V] \\
&= A\sqrt{q}[\delta^{\beta}_{2n+1}(\mathcal{A}\eta(\mathcal{C})\eta(\mathcal{B}))^V \\
&- \delta^{\beta}_{2n+1}(\mathcal{B}\eta(\mathcal{C})\eta(\mathcal{A}))^V \\
&- \delta^{\beta}_{2n+1}(\eta([\mathcal{A},\mathcal{B}])\eta(\mathcal{C}))^V + \Omega(\gamma R(\mathcal{A},\mathcal{B}),\mathcal{C}^{(\beta)}) \\
&+ \delta^{\beta}_{2n+1}(\eta(\nabla_X Z)\eta(\mathcal{B}))^V \\
&- \delta^{\beta}_{2n+1}(\eta(\nabla_Y Z)\eta(\mathcal{A}))^V].
\end{aligned}
$$

Formulas $(iii)$ and $(iv)$ can be easily obtained. □

## 4. Behavior of the Nijehuis Tensor on *FM*

The Nijenhuis tensor of $\tilde{J}$ is expressed by

$$N(\tilde{\mathcal{A}},\tilde{\mathcal{B}}) = [\tilde{J}\tilde{\mathcal{A}},\tilde{J}\tilde{\mathcal{B}}] - \tilde{J}[\tilde{J}\tilde{\mathcal{A}},\tilde{\mathcal{B}}] - \tilde{J}[\tilde{\mathcal{A}},\tilde{J}\tilde{\mathcal{B}}] + \tilde{J}^2[\tilde{\mathcal{A}},\tilde{\mathcal{B}}].$$

**Theorem 4.** *$\forall\tilde{\mathcal{A}},\tilde{\mathcal{B}} \in \Im^1_0(FM)$, then*

$$
\begin{aligned}
(i)\ \ N(\mathcal{A}^H,\mathcal{B}^H) &= \frac{pA}{2}\{(\nabla_{\phi\mathcal{B}}\mathcal{A})^{(\beta)} - (\nabla_{\phi\mathcal{A}}\mathcal{B})^{(\beta)}\} \\
&+ A^2[\phi\mathcal{A},\phi\mathcal{B}]^H - A\tilde{J}[\phi\mathcal{A},\mathcal{B}]^H \\
&- A\tilde{J}[\mathcal{A},\phi\mathcal{B}]^H + \tilde{J}^2[\mathcal{A},\mathcal{B}]^H \\
&+ A^2(\eta(\mathcal{B})^V((\nabla_{\phi\mathcal{A}}\zeta)^{(2n+1)} - (\nabla_{\phi\mathcal{B}}\zeta)^{(2n+1)}) \\
&+ A^2((\nabla_{\phi\mathcal{A}}\zeta)^{(2n+1)} + (\phi\nabla_{\mathcal{A}}\zeta)^{(2n+1)})(\eta(\mathcal{B})^V \\
&- A^2((\nabla_{\phi\mathcal{B}}\zeta)^{(2n+1)} + (\phi\nabla_{\mathcal{B}}\zeta)^{(2n+1)})(\eta(\mathcal{A})^V \\
&+ A^2(\eta(\nabla_{\mathcal{B}}\zeta)^V\eta(\mathcal{A})^V - \eta(\nabla_{\mathcal{A}}\zeta)^V\eta(\mathcal{B})^V)\zeta^H \\
&+ \frac{pA}{2}\{(\nabla_{\mathcal{B}}\mathcal{A})^{(2n+1)} - (\nabla_{\mathcal{A}}\mathcal{B})^{(2n+1)}\} \\
&- A^2\gamma R(\phi\mathcal{A},\phi\mathcal{B}) + A\tilde{J}\gamma R(\phi\mathcal{A},\mathcal{B}) \\
&+ A\tilde{J}\gamma R(\phi\mathcal{A},\mathcal{B}) - \tilde{J}^2\gamma R(\mathcal{A},\mathcal{B}),
\end{aligned}
$$

$$
\begin{aligned}
(ii)\ \ N(\mathcal{A}^{(\alpha)},\mathcal{B}^{(\beta)}) &= \sqrt{q}\{A^2[(\delta^{\beta}_{2n+1}\eta(\mathcal{B})^V(\nabla_\zeta(\phi\mathcal{A}))^\alpha \\
&+ \varepsilon(\alpha)\eta(\mathcal{A})^V\eta(\mathcal{B})^V\delta^{\beta}_{2n+1}(\nabla_\zeta\zeta)^{(\alpha+\varepsilon(\alpha)n)} \\
&- \delta^{\beta}_{2n+1}\eta(\mathcal{A})^V(\nabla_\zeta(\phi\mathcal{B}))^\alpha \\
&- \varepsilon(\beta)\eta(\mathcal{A})^V\eta(\mathcal{B})^V\delta^{\alpha}_{2n+1}(\nabla_\zeta\zeta)^{(\beta+\varepsilon(\beta)n)} \\
&+ \delta^{\alpha}_{2n+1}\delta^{\beta}_{2n+1}([\zeta,\zeta]^H-\gamma R(\zeta,\zeta))] \\
&- \frac{pA}{2}(\nabla_\zeta\mathcal{B})^{(\beta)} - \frac{p}{2}\delta^{\beta}_{2n+1}\eta(\mathcal{B})^V(\nabla_{\mathcal{A}}\zeta)^{(\alpha)} \\
&- \frac{pA}{2}\mathcal{A}^{(\alpha)}\delta^{\alpha}_{2n+1}\eta(\mathcal{A})^V \\
&+ A^2X^{(\alpha)}\delta^{\alpha}_{2n+1}\eta(\mathcal{A})^V((\phi\nabla_\zeta\mathcal{B})^{(\beta)} \\
&+ \varepsilon(\beta)\eta(\nabla_\zeta\mathcal{B})^V\zeta^{(\beta+\varepsilon(\beta)n)} - \delta^{\beta}_{2n+1}\eta(\nabla_\zeta\mathcal{B})^V\zeta^H) \\
&- \frac{pA}{2}\mathcal{B}^{(\beta)}\delta^{\alpha}_{2n+1}\eta(\mathcal{B})^V \\
&+ A^2Y^{(\alpha)}\delta^{\alpha}_{2n+1}\eta(\mathcal{B})^V((\phi\nabla_\zeta\mathcal{A})^{(\beta)} \\
&+ \varepsilon(\beta)\eta(\nabla_\zeta\mathcal{A})^V\zeta^{(\beta+\varepsilon(\beta)n)} - \delta^{\beta}_{2n+1}\eta(\nabla_\zeta\mathcal{A})^V\zeta^H)\},
\end{aligned}
$$

$$
\begin{aligned}
(iii)\ \ N(\mathcal{A}^H,\mathcal{B}^{(\beta)}) &= -\frac{pA}{2}\sqrt{q}\delta^{\beta}_{2n+1}\eta(\mathcal{B})^V(\nabla_\zeta\mathcal{A})^{(\beta)} - \frac{pA}{2}(\nabla_{\phi\mathcal{A}}\mathcal{B})^{(\beta)} \\
&+ A^2(\nabla_{\phi\mathcal{A}}\phi\mathcal{B})^{(\beta)} + A^2\sqrt{q}\{\varepsilon(\beta)\eta(\mathcal{B})^V(\nabla_{\phi\mathcal{A}}\zeta)^{(\beta+\varepsilon(\beta)n)} \\
&- \delta^{\beta}_{2n+1}\eta(\mathcal{B})^V([\phi\mathcal{A},\zeta]-\gamma R(\phi\mathcal{A},\zeta)) \\
&+ \delta^{\beta}_{2n+1}\eta(\mathcal{A})^V\eta(\mathcal{B})^V(\nabla_\zeta\zeta)^{(2n+1)} - \phi\nabla_{\phi\mathcal{A}}\mathcal{B})^{(\beta)} \\
&+ \varepsilon(\beta)\eta(\nabla_{\phi\mathcal{A}}\mathcal{B})^V\zeta^{(\beta+\varepsilon(\beta)n)} - \delta^{\beta}_{2n+1}\eta(\nabla_{\phi\mathcal{A}}\mathcal{B})^V\zeta^H)\} \\
&+ \frac{pA}{2}(\nabla_{\phi\mathcal{A}}\mathcal{B})^V - pA((\phi\nabla_XY)^{(\beta)} \\
&+ pA((\phi\nabla_{\mathcal{A}}\phi\mathcal{B})^{(\beta)} \\
&+ \sqrt{q}\{\varepsilon(\beta)\eta(\nabla_XY)^V\zeta^{(\beta+\varepsilon(\beta)n)} - \delta^{\beta}_{2n+1}\eta(\nabla_XY)^V\zeta^H) \\
&+ +\varepsilon(\beta)\eta(\nabla_{\mathcal{A}}\phi\mathcal{B})^V\zeta^{(\beta+\varepsilon(\beta)n)} \\
&- \delta^{\beta}_{2n+1}\eta(\nabla_{\mathcal{A}}\phi\mathcal{B})^V\zeta^H) + \varepsilon(\beta)\eta(\mathcal{B})^V(\phi\nabla_{\mathcal{A}}\zeta)^{(\beta+\varepsilon(\beta)n)} \\
&+ \varepsilon^2(\beta)\eta(\mathcal{B})^V\eta(\phi\nabla_{\mathcal{A}}\zeta)^V\zeta^{(\beta+\varepsilon(\beta)n)} - \delta^{\beta}_{2n+1}\varepsilon(\beta)\eta(\mathcal{B})^V\eta(\phi\nabla_{\mathcal{A}}\zeta)^V\zeta^H) \\
&- \delta^{\beta}_{2n+1}\eta(\mathcal{B})^V((\phi[\mathcal{A},\zeta])^H + \eta[\mathcal{A},\zeta]^V\zeta^{(2n+1)}, -\gamma\tilde{J}R(\mathcal{A},\zeta)))\}
\end{aligned}
$$

*where* $\alpha,\beta = 1,\ldots,2n+1$.

**Proof.** Using (22) and Theorem (1), Theorem (4) is proven. □

## 5. Example

Let $\{e_i, \phi e_i, \zeta\}$ be a basis in $(M, \phi, \zeta, \eta, g)$ where $i$ denotes 1 to $n$. The coderivative $\delta\Omega$ with basis $\{e_i^H, (\phi e_i)^H, \zeta^H, e_i^{(\alpha)}, (\phi e_i)^{(\alpha)}, \zeta^{(\alpha)}\}$ can be expressed as [16]

$$
\begin{aligned}
\delta\Omega(\tilde{A}) &= -\sum_{i=1}^{n}\{(\tilde{\nabla}_{e_i^H}\Omega)(e_i^H, \tilde{A}) + (\tilde{\nabla}_{(\phi e_i)^H}\Omega)((\phi e_i)^H, \tilde{A})\} \\
&+ \sum_{j=1}^{n}(\tilde{\nabla}_{\zeta^{(j)}}\Omega)(\zeta^{(j)}, \tilde{A}) - (\tilde{\nabla}_{\zeta^{(2n+1)}}\Omega)(\zeta^{(2n+1)}, \tilde{A}) \\
&- (\tilde{\nabla}_{\zeta^H}\Omega)(\zeta^H, \tilde{A}) - \sum_{\alpha=1}^{2n+1}\sum_{i=1}^{n}\{\tilde{\nabla}_{e_i^{(\alpha)}}\Omega)(e_i^{(\alpha)}, \tilde{A}) \\
&+ (\tilde{\nabla}_{(\phi e_i)^{(\alpha)}}F)((\phi e_i)^{(\alpha)}, \tilde{A})\}.
\end{aligned} \tag{31}
$$

Taking $\tilde{A} = \mathcal{A}^{(\beta)}$ in (31), using (11) and (29), we acquire

$$
\begin{aligned}
\delta\Omega(\mathcal{A}^{(\beta)}) &= -\sum_{i=1}^{n}\{g^D(\nabla_{e_i^H}e_i^H, \tilde{J}\mathcal{A}^{(\beta)}) + g^D(\nabla_{(\phi E_i)^H}(\phi e_i)^H, \tilde{J}\mathcal{A}^{(\beta)})\} \\
&- g^D(\nabla_{\zeta^H}\zeta^H, \tilde{J}\mathcal{A}^{(\beta)}) \\
&= -\sum_{i=1}^{n}\{-g^D(\gamma R(e_i, e_i), \frac{p}{2}\mathcal{A}^{(\beta)}) - A[-g^D(\gamma R(\phi e_i, e_i), \mathcal{A}^{(\beta)}) \\
&- \sqrt{q}\delta_{2n+1}^{\beta}\eta(\mathcal{A})^V g(\nabla_{e_i}\zeta, e_i)^V - \sqrt{q}\delta_{2n+1}^{\beta}\eta(\mathcal{A})^V g(\nabla_{\phi e_i}\zeta, \phi e_i)^V\} \\
&+ \sqrt{q}\delta_{2n+1}^{\beta} g(\nabla_{\zeta^H}\zeta^H, \mathcal{A}^V)] \\
&= \frac{p}{2}\sum_{i=1}^{n}\{g^D(\gamma R(e_i, e_i), \mathcal{A}^{(\beta)}) - A[-g^D(\gamma R(e_i, \phi e_i), \mathcal{A}^{(\beta)}) \\
&+ \sqrt{q}\delta_{2n+1}^{\beta}\{\eta(\mathcal{A})^V(\delta\eta)^V, (\nabla_\zeta\eta)\mathcal{A}^V\}],
\end{aligned}
$$

where

$$\delta\eta = -\sum_{i=1}^{n}\{(\nabla_{e_i}\eta)\zeta_i + (\nabla_{\phi e_i}\eta)\phi\zeta_i\}$$

and

$$(\nabla_\zeta\eta)\mathcal{A} = g(\mathcal{A}, \nabla_\zeta\zeta).$$

Taking $\tilde{A} = \mathcal{A}^H$ in (31), using (11) and (29), we acquire

$$\begin{aligned}
\delta\Omega(\mathcal{A}^H) &= -\sum_{i=1}^{n}\{g^D(\nabla_{e_i^H}e_i^H, \tilde{J}\mathcal{A}^H) + g^D(\nabla_{(\phi E_i)^H}(\phi e_i)^H, \tilde{J}\mathcal{A}^H\} \\
&- g^D(\nabla_{\zeta^{(2n+1)}}\zeta^{(2n+1)}, \tilde{J}\mathcal{A}^H) - g^D(\nabla_{\zeta^H}\zeta^H, \tilde{J}\mathcal{A}^H) \\
&- \sum_{\alpha=1}^{2n+1}\sum_{i=1}^{n}(g^D(\nabla_{e_i^{(\alpha)}}e_i^{(\alpha)}, \tilde{J}\mathcal{A}^H) + g^D(\nabla_{(\phi E_i)^{(\alpha)}}(\phi e_i)^{(\alpha)}, \tilde{J}\mathcal{A}^H). \\
&= -\frac{p}{2}\sum_{i=1}^{n}[(g(\nabla_{e_i}e_i, \mathcal{A}))^V + (g(\nabla_{\phi e_i}\phi e_i, \mathcal{A}))^V + (g(\nabla_{\zeta}\zeta, \mathcal{A}))^V] \\
&- A[-\sum_{i=1}^{n}(-g((\nabla_{e_i}\phi)e_i, \mathcal{A})^V - g((\nabla_{\phi e_i}\phi)\phi e_i, \mathcal{A})^V) \\
&+ g((\nabla_{\zeta}\phi)\zeta, \mathcal{A})^V]. \\
&= -\frac{p}{2}\sum_{i=1}^{n}[(g(\nabla_{e_i}e_i, \mathcal{A}))^V + (g(\nabla_{\phi e_i}\phi e_i, \mathcal{A}))^V + (g(\nabla_{\zeta}\zeta, \mathcal{A}))^V] \\
&- A(\delta\Phi(\mathcal{A}))^V,
\end{aligned}$$

where

$$\delta\Phi(\mathcal{A}) = -\sum_{i=1}^{n}(\nabla_{e_i}\Phi)(e_i, \mathcal{A}) + (\nabla_{\phi e_i}\Phi)(\phi e_i, \mathcal{A})) - (\nabla_{\zeta}\Phi)(\zeta, \mathcal{A}).$$

and

$$(\nabla_{\mathcal{A}}\Phi)(\mathcal{B}, \mathcal{C}) = -g((\nabla_{\mathcal{A}}\phi)(\mathcal{B}, \mathcal{C}).$$

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