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L-Quasi (Pseudo)-Metric in L-Fuzzy Set Theory

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Abstract: The aim of this paper is to focus on the metrization question in L-fuzzy sets. Firstly, we put forward an L-quasi (pseudo)-metric on the completely distributive lattice L^X by comparing some existing lattice-valued metrics with the classical metric and show a series of its related properties. Secondly, we present two topologies: ψ_p and ζ_p, generated by an L-quasi-metric p with different spherical mappings, and prove $\psi_p = \zeta'_p$ if p is further an L-pseudo-metric on L^X. Thirdly, we characterize an equivalent form of L-pseudo-metric in terms of a class of mapping clusters and acquire several satisfactory results. Finally, based on this kind of L-metric, we assert that, on L^X, a Yang–Shi metric topology is $Q - C_I$, but an Erceg metric topology is not always so.

Keywords: L-quasi (pseudo)-metric; co-prime element; irreducible element; way below; R-neighborhood; T_1-space; $Q - C_I$.

MSC: 54A40

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1. Introduction

As we know, C.L. Chang [1] firstly introduced the fuzzy set theory of Zadeh [2] into topology in 1968, which declared the birth of $[0,1]$-topology. Soon after that, J.A. Goguen [3] further generalized the L-fuzzy set to $[0,1]$-topology, and his related theory has now been recognized as L-topology. From then on, L-topology formed another important, branch of topology and many creative results and original thoughts were presented (see [4–36], etc.).

However, how to reasonably generalize the classical metric to L-topology has been a great challenge for a long time. So far, there has been a lot of research work on this aspect, including at least three well-known L-fuzzy metrics, with which the academic community has gradually become familiar. In addition, there was an even more interesting L-fuzzy metric recently discovered, which is parallel to the mentioned three L-fuzzy metrics. To explain the four L-fuzzy metrics, we list them below one by one.

The first is the Erceg metric, presented in 1979 by M.A. Erceg [4]. Due to the complexity of its definition given by M.A. Erceg, it is very inconvenient and difficult to conduct in-depth research on this metric. In 1993, Peng Yuwei [5] provided a pointwise expression for the Erceg metric. Based on Peng's result, later on, this metric was further simplified by P. Chen and F.G. Shi (see [6,7]) as below.

(I) An *Erceg pseudo-metric* on L^X is a mapping $p : M \times M \to [0, +\infty)$, satisfying the following properties:

(A1) if $a \geq b$, then $p(a,b) = 0$;

(A2) $p(a,c) \leq p(a,b) + p(b,c)$;

(B1) $p(a,b) = \bigvee_{c \ll b} p(a,c)$;

(A3) $\forall a, b \in M$, $\exists x \not\leq a'$ s.t. $p(b, x) < r \Leftrightarrow \exists y \not\leq b'$ s.t. $p(a, y) < r$.

An Erceg pseudo-metric p is called an *Erceg metric* if it further satisfies the following property:

(A4) if $p(a, b) = 0$, then $a \geq b$,

where " $\ll$ " is the way below relation in Domain Theory and L^X is a completely distributive lattice [37,38].

The second is the Yang–Shi metric (or Shi $p.q.$ metric), proposed in 1988 by L.C. Yang [8]. After that, this kind of metric was studied in depth by F.G. Shi and P. Chen (see [6,7,9–11,39] etc.), and was ultimately defined [11] as follows.

(II) A *Yang–Shi pseudo-metric* (resp., *Yang–Shi metric*) on L^X is a mapping $p : M \times M \to [0, +\infty)$, satisfying (A1)–(A3) (resp., (A1)–(A4)) and the following property:

(B2) $p(a, b) = \bigwedge\limits_{c \ll a} p(c, b)$.

The third is the Deng metric, supplied in 1982 by Z.K. Deng [12], which was only limited to the special lattice I^X originally $(I = [0, 1])$. Recently, it was extended to L^X by P. Chen [13] as follows:

(III) A *Deng pseudo-metric (resp., Deng metric)* on L^X is a mapping $p : M \times M \to [0, +\infty)$, satisfying (A1)–(A3) (resp., (A1)–(A4)) and the following property:

(B3) $p(a, b) = \bigwedge\limits_{b \ll c} p(a, c)$.

In short, the above three L-fuzzy metrics are defined by using the same (A1)–(A4) but different (B1), (B2) and (B3). Inspired by this, we conclude that there is another new L-fuzzy metric [9], as below.

(IV) A *Chen pseudo-metric* (resp., *Chen metric*) on L^X is a mapping $p : M \times M \to [0, +\infty)$, satisfying (A1)–(A3) (resp., (A1)-(A4)) and the following property:

(B4) $p(a, b) = \bigvee\limits_{a \ll c} p(c, b)$.

Concerning the above four L-fuzzy metrics **(I)–(IV)**, we [9] have investigated the relationships between them on I^X and acquired the following conclusion.

Let the following be true: $C = \{\, p \mid p \text{ is a Chen metric}\}$; $E = \{\, p \mid p \text{ is an Erceg metric}\}$; $D = \{\, p \mid p \text{ is a Deng metric}\}$; $Y = \{\, p \mid p \text{ is a Yang–Shi metric}\}$. Then, $D = C \cap Y \cap E$.

In summary, although many scholars have engaged in the research of metrics in L-fuzzy sets, it is a pity that, at the same time, such an important issue has been ignored. Since the term fuzzy metric is a generalization of the classical metric, are there so few generalized L-fuzzy-metrics on L^X? Therefore, this naturally leads to the following problem: what should the most essential axiomatic system about L-fuzzy metrics consist of on earth? To inquire into these problems, we first of all compare these existing fuzzy metrics on L^X with the classical metric, which is defined as follows.

Definition 1 ([40])**.** *A pseudo-metric on a non-empty set X is a function d: $X \times X \longrightarrow [0, +\infty)$, satisfying the following properties:*

(1) if $x = y$, then $d(x, y) = 0$;

(2) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$;

(3) $d(x, y) = d(y, x)$ for all $x, y \in X$.

The function d is called a metric on X if d still satisfies the following property:

(4) if $d(x, y) = 0$, then $x = y$.

It is easy to check that (A1), (A2), (A3) and (A4) in **(I)–(IV)** are the generalizations of (1), (2), (3) and (4) in Definition 1, respectively. However, no axioms correspond to (B1), (B2), (B3) or (B4). Therefore, we guess that (B1), (B2), (B3) and (B4) in these fuzzy metrics on L^X are inessential for many purposes, especially their induced topologies. In this article, we affirm this guess, for this put forward a lattice-valued metric on L^X, and show some related properties.

2. Preliminary Information

Throughout this paper, L is a completely distributive lattice with an order reversing involution " $'$ " [37,38]. X is a nonempty set. L^X is the set of all L-fuzzy sets of X [3]. L^X inherits the structure of lattice L with an order reversing involution in a natural way, by defining $\vee, \wedge, '$ pointwise. The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$, respectively.

Let $e \in L$-$\{\underline{0}\}$; e is called a co-prime if, for any $p, q \in L$, $e \leq p \vee q$ implies $e \leq p$ or $e \leq q$. The set of all nonzero co-prime elements in L is denoted by $M(L)$. We define $M(L^X) = \{x_\lambda \mid x \in X, \lambda \in M(L)\}$, where x_λ is an L-fuzzy point [38]. Conveniently, we omit L^X from the notation, namely, we write $M(L^X)$ simply as M. Therefore, M is the set of all nonzero co-prime elements in L^X. Similarly, L-fuzzy set a is called an irreducible element if, for any $x, y \in L^X$, $x \wedge y \leq a$ implies $x \leq a$ or $y \leq a$. The set of all nonzero irreducible elements on L^X is denoted as J.

Let $a, b \in L^X$ and a is much lower than b, denoted by $a \ll b$, if, for every directed subset $D \subseteq L^X$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$. Let $a \in L^X$ and $B \subset L^X$. If $a \leq \sup B$ (resp., $a = \sup B$), then B is called a cover (resp., proper cover) of a. Let $B, C \subset L^X$. If, for any $x \in B$, there exists some $y \in C$ such that $x \leq y$, then B is called a refinement C. If B is a proper cover of a and B refines each cover of a, then B is called a minimal set of a. Let $\mathcal{T}(a)$ be all minimal sets of a. Clearly, the union of the elements of any subfamily of $\mathcal{T}(a)$ is still a minimal set of a. Therefore, each L-fuzzy set a must correspond to a greatest minimal set, denoted by $\beta(a)$ [38]. Let $\beta^*(a) = \beta(a) \cap M$. Then, x_λ belongs to $\beta^*(a)$ if and only if x_λ is much lower than a. Let $a \in L^X$ and $A \subset L^X$. Similarly, if A satisfies the following properties: (1) $\inf A = a$; (2) if $B \subset L^X$ and $\inf B \leq a$, then, for any $x \in A$, there exists some $y \in B$ such that $y \leq x$; then, A is claimed as a maximum set of a. Let $\mathcal{W}(a)$ be all maximum sets of a. Obviously, the union of the elements of any subfamily of $\mathcal{W}(a)$ is still a maximum set of a. Thus, if there exists a maximum set of a, then there must exist a greatest maximal set of a, denoted as $\alpha(a)$ [38]. In addition, we stipulate $\vee\emptyset = \underline{0}$ and $\wedge\emptyset = \underline{1}$. Other unexplained terminologies, notations and further details can be found in [3,9,12,38,40].

Theorem 1 ([38])**.** *Let* $\{a_i \mid i \in I\} \subset L^X$. *Then,* $\beta(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \beta(a_i)$.

Theorem 2 ([38])**.** *Let* $\{a_i \mid i \in I\} \subset L^X$. *Then,* $\alpha(\bigwedge_{i \in I} a_i) = \bigcup_{i \in I} \alpha(a_i)$.

Definition 2 ([38,41])**.** *Let* (X, δ) *be an L-topological space,* $x_\lambda \in M$ *and* $A \in \delta'$. *If* $x_\lambda \nleq A$, *then* A *is called a closed R-neighborhood of* x_λ. *Let* $B \in L^X$; *if there exists a closed R-neighborhood* A *of* α *such that* $B \leq A$, *then* B *is called an R-neighborhood of* α. *Meanwhile,* B' *is called a Q-neighborhood of* α.

3. *L*-Quasi-Metric on L^X

In the section, by comparing the above **(I)–(IV)** with the classical metric in general topology (see Definition 1), we can, first of all, define a kind of metric on L^X as follows.

Definition 3. *A mapping* $p : M \times M \to [0, +\infty)$ *is called an L-quasi-metric on* L^X *if it satisfies the following properties:*

(A1) if $a \geq b$*, then* $p(a,b) = 0$*;*

(A2) (triangle inequality) $p(a,c) \leq p(a,b) + p(b,c)$*.*

An L-quasi-metric p *is called an L-pseudo-metric on* L^X *if it still satisfies the following property:*

(A3) $\forall a, b \in M$, $\exists y \not\leq b'$ *s.t.* $p(a,y) < r \Leftrightarrow \exists x \not\leq a'$ *s.t.* $p(b,x) < r$*.*

An L-pseudo-metric p *is called an L-metric on* L^X *if it still satisfies the following property:*

(A4) if $p(a,b) = 0$*, then* $a \geq b$*.*

Definition 4. *Given a mapping* p*:* $M \times M \to [0, +\infty)$*. For* $A \in L^X$*, define* $D_r(A) = \bigvee\{b \in M \mid \exists a \ll A, p(a,b) < r\}$*,* $D_{-r}(A) = \bigvee\{a \in M \mid D_r(a) \leq A\}$ *and* $D_{-r} * D_{-s}(A) = \bigvee\{a \in M \mid D_r(a) \leq D_{-s}(A)\}$*.*

Theorem 3. *If* p *is an L-quasi-metric on* L^X*, then* $D_r(a) = \bigvee_{s<r} D_s(a)$*.*

Proof. Obviously, when $s < r$, $D_s(a) \leq D_r(a)$. Thus, $D_r(x_\alpha) \geq \bigvee_{s<r} D_s(x_\alpha)$. Conversely, let $c \ll D_r(a)$. Then, by the definition of $D_r(a)$ and the way below relation, there exist $e \in M$ and $h_e \ll a$ such that $c \leq e$ and $p(h_e, e) < r$, respectively. Because $c \leq e$, according to the conditions (A1) and (A2) in Definition 3, we can obtain $p(h_e, c) \leq p(h_e, e) < r$. Take s with $p(h_e, c) < s < r$. Then, $c \leq D_s(a)$, and, consequently, $D_r(a) \leq \bigvee_{s<r} D_s(a)$, as desired. □

Theorem 4. *If* p *is an L-quasi-metric on* L^X*, then* $D_{-r}(A) = \bigwedge_{s<r} D_{-s}(A)$*.*

Proof. Clearly, $D_{-r}(A) \leq \bigwedge_{s<r} D_{-s}(A)$. Conversely, let $c \ll \bigwedge_{s<r} D_{-s}(A)$. Then, by the way below relation for each $s < r$, there is $a \in M$ such that $c \leq a$ and $D_s(a) \leq A$. According to the conditions (A1) and (A2) in Definition 3, we can obtain $D_s(c) \leq D_s(a)$, and then we can assert $D_s(c) \leq A$. Consequently, $\bigvee_{s<r} D_s(c) \leq A$. By Theorem 3, we have $D_r(c) \leq A$. Hence, $c \leq D_{-r}(A)$, and then $\bigwedge_{s<r} D_{-s}(A) \leq D_{-r}(A)$, as desired. □

Theorem 5. *If* p *is an L-quasi-metric on* L^X*, then* $D_{-r-s}(A) \leq D_{-r}*D_{-s}(A)$*.*

Proof. By the definitions of $D_{-r-s}(A)$ and $D_{-r} * D_{-s}(A)$, we need to prove $\{a \in M \mid D_{r+s}(a) \leq A\} \subseteq \{b \in M \mid D_r(b) \leq D_{-s}(A)\}$. This proof is as follows. Let $D_{r+s}(a) \leq A$. Then, we need to check $D_r(a) \leq D_{-s}(A)$. Because

$$D_r(a) \leq D_{-s}(A) \Leftrightarrow \bigvee\{e \mid \exists a_e \ll a, p(a_e, e) < r\}$$

$$\leq \bigvee\{c \in M \mid D_s(c) \leq A\},$$

if $e \in \{e \mid \exists a_e \ll a, p(a_e, e) < r\}$, then $e \leq \bigvee\{c \in M \mid D_s(c) \leq A\}$, which is equivalent to proving that, for any $g \ll e$, it holds that $D_s(g) \leq A$. In fact, let $u \in \{u \mid \exists g_u \ll g, p(g_u, u) < s\}$. Since $p(a_e, e) < r$, it holds that $p(a_e, g_u) < r$ by the conditions (A1) and (A2) in Definition 3. Hence, $p(a_e, u) \leq p(a_e, g_u) + p(g_u, u) < r + s$. Because of $D_{r+s}(a) \leq A$, we have $u \leq A$, and so $D_s(g) \leq A$, as desired. □

Theorem 6. *If* p *is an L-quasi-metric on* L^X*, then* $\{D_{-r}(\alpha) \mid \alpha \in J, r \in [0, +\infty)\}$ *is a co-topological base, and the co-topology is denoted by* ψ_p*.*

Proof. Let $\mathcal{T}$ be the family of all any intersections of elements of $\{D_{-r}(a) \mid a \in J, r \in [0, +\infty)\}$. Now, we check that $\mathcal{T}$ is a co-topology.

Let $\lambda, \mu \in J$ (about J, see Section 2) and $s, r \in [0, +\infty)$. We need to prove $D_{-r}(\lambda) \vee D_{-s}(\mu) = K \in \mathcal{T}$. Case 1: when $r = 0$ and $s = 0$, we can obtain $K = \underline{0}$. Therefore,

$K \in \mathcal{T}$. Case 2: if $K = \underline{1}$, then, in view of $K = \wedge\emptyset$, we have $K \in \mathcal{T}$. Case 3: if $K \neq \underline{1}$ or $\underline{0}$, then, by Theorem 2, we can obtain $\alpha(D_{-r}(\lambda)) \supseteq \alpha(K)$ and $\alpha(D_{-s}(\mu)) \supseteq \alpha(K)$. Let $\gamma \in \alpha(K) \subseteq \alpha(D_{-r}(\lambda)) \cap \alpha(D_{-s}(\mu))$. Then, by Theorem 4, we may take two numbers e_1, e_2 with $r > e_1 > 0$, $s > e_2 > 0$, such that $\gamma \in \alpha(D_{-e_1}(\lambda))$ and $\gamma \in \alpha(D_{-e_2}(\mu))$. Therefore, we can obtain $\gamma \geq D_{-e_1}(\lambda)$ and $\gamma \geq D_{-e_2}(\mu)$, respectively. Let t_γ=inf$\{r - e_1, s - e_2\}$. Then, by Theorem 5, we have

$$\gamma \geq D_{-t_\gamma}(\gamma) \geq D_{-t_\gamma}(D_{-e_1}(\lambda)) = D_{-t_\gamma} * (D_{-e_1}(\lambda))$$
$$\geq D_{-t_\gamma - e_1}(\lambda) \geq D_{-r}(\lambda).$$

Similarly, it holds that $\gamma \geq D_{-s}(\mu)$. Hence, $K = \wedge\alpha(K) \geq \wedge\{D_{-t_\gamma}(\gamma) \mid \gamma \in \alpha(K)\} \geq D_{-r}(\lambda) \vee D_{-s}(\mu) = K$. Consequently, $K = \wedge\{D_{-t_\gamma}(\gamma) \mid \gamma \in \alpha(K)\}$. $\mathcal{T} = \psi_p$, as desired. □

Theorem 7. *If p is an L-quasi-metric on L^X and the co-topology is ψ_p, then $\psi_p(\alpha) = \{D_{-r}(\beta) \mid r > 0, \alpha \not\leq \beta\}$ is a Q-neighborhood base of α.*

Proof. Given $\alpha \not\leq \beta$, owing to $D_{-r}(\beta) \leq \beta$, we have $\alpha \not\leq D_{-r}(\beta)$. In addition, by Theorem 6, we can assert that $D_{-r}(\beta)$ is a closed set. Therefore, each element of $\psi_p(\alpha)$ is a Q-neighborhood of α. Conversely, let $A \in \psi_p$, satisfying $\alpha \not\leq A$. Then, by Theorem 6 and the definition of $\alpha(A)$, we can obtain

$$A = \wedge\{D_{-r_i}(\alpha_i) \mid i \in \Gamma, \alpha_i \in \alpha(A)\}$$
$$= \wedge\{\alpha_i \mid i \in \Gamma, \alpha_i \in \alpha(A)\}.$$

It follows that there must exist some α_i such that $\alpha \not\leq \alpha_i$. As a result, we have $\alpha \not\leq D_{-r_i}(\alpha_i)$. Additionally, in view of $A \leq D_{-r_i}(\alpha_i)$, we can assert that $\psi_p(a)$ is a Q-neighborhood base of α. □

Theorem 8. *Suppose that p is a mapping from $M \times M$ to $[0, +\infty)$. Then, $D_{-r}(\bigwedge\limits_{i\in\Gamma} a_i) = \bigwedge\limits_{i\in\Gamma} D_{-r}(a_i)$.*

Proof. If $\Gamma = \emptyset$, then it is straightforward. Thus, we might as well set $\Gamma \neq \emptyset$. Obviously, by the definition of D_{-r}, we have $D_{-r}(a_i) \geq D_{-r}(\bigwedge\limits_{i\in\Gamma} a_i)$ for each $i \in \Gamma$. Thus, $\bigwedge\limits_{i\in\Gamma} D_{-r}(x_{\alpha_i}) \geq D_{-r}(\bigwedge\limits_{i\in\Gamma} x_{\alpha_i})$. Conversely, let $h \ll \bigwedge\limits_{i\in\Gamma} D_{-r}(a_i)$. Then, for each $i \in \Gamma$, there exists $a \in M$ such that $h \leq a$ and $D_r(a) \leq a_i$. Hence, $D_r(h) \leq a_i$, and then $D_r(h) \leq \bigwedge\limits_{i\in\Gamma} a_i$ and $h \leq D_{-r}(\bigwedge\limits_{i\in\Gamma} a_i)$. Because h is arbitrary, it is true that $\bigwedge\limits_{i\in\Gamma} D_{-r}(a_i) \leq D_{-r}(\bigwedge\limits_{i\in\Gamma} a_i)$, as desired. □

Corollary 1. *Let p be an L-quasi-metric on L^X and let ψ_p be the co-topology. If $B \in L^X$, then $D_{-r}(B) \in \psi_p$.*

Proof. Let $B = \bigwedge\limits_{i\in\Gamma}\{a_i \mid i \in \Gamma, a_i \in J\}$. Then, by Theorem 8, we have $D_{-r}(B) = \bigwedge\limits_{i\in\Gamma} D_{-r}(a_i)$, so that $D_{-r}(B) \in \psi_p$. □

Theorem 9. *Let p be an L-quasi-metric on L^X. Then, $D_{-r}(\alpha) \geq A$ if and only if $D_r(A) \leq \alpha$.*

Proof. (*Sufficiency*). Let $e \in \{b \in M \mid \exists a \ll A, p(a,b) < r\}$. Then, there is $f_e \ll A$ such that $p(f_e, e) < r$. Take $c \in M$ with $f_e \ll c \ll A$. Then, $p(c,e) \leq p(f_e, e) < r$. Hence, $e \leq D_r(c)$. Since $c \ll A \leq D_{-r}(\alpha)$, there exists $d \in M$ such that $c \ll d$ and $D_r(d) \leq \alpha$. Because of $c \ll d$, by the definition of D_r, we can obtain $D_r(c) \leq D_r(d)$, and then we have $D_r(c) \leq \alpha$. Hence, $e \leq \alpha$. Therefore, we have

$$D_r(A) = \bigvee\{b \in M \mid \exists a \ll A, p(a,b) < r\} \leq \alpha.$$

(*Necessity*). For any $e \ll A$, we can deduce $D_r(e) \leq D_r(A) \leq \alpha$, and then $e \leq D_{-r}(\alpha) = \bigvee\{e \in M \mid D_r(e) \leq \alpha\}$. Consequently, $A \leq D_{-r}(\alpha)$, as desired. □

Theorem 10. *Let p be an L-quasi-metric on L^X and let ψ_p be the co-topology. Then,* $\overline{A} = \bigwedge_{r>0} D_r(A)$.

Proof. Let $a \ll A$. Then, $D_r(a) \leq D_r(A)$. Thus $D_{-r}(D_r(A)) = \bigvee\{a \in M \mid D_r(a) \leq D_r(A)\} \geq A$. Thus, we have $D_r(A) \geq D_{-r}(D_r(A)) \geq A$ for each $r > 0$. Therefore, $\bigwedge_{r>0} D_r(A) \geq \overline{A}$. Conversely, if $a \not\leq \overline{A}$, then there exist $\alpha \in L^X$ and $r > 0$ such that $a \not\leq \alpha$ and $D_{-r}(\alpha) \geq A$. By Theorem 9, we can obtain $D_r(A) \leq \alpha$. Since $a \not\leq \alpha$, we have $a \not\leq D_r(A)$. Consequently, $D_r(A) \leq \overline{A}$, so that $\bigwedge_{r>0} D_r(A) \leq \overline{A}$, as desired. □

4. Some Properties of Spheres in *L*-Quasi-Metric Space

In this section, we investigate some relationships between several spheres which are defined by using an L-quasi-metric on L^X and show some related properties about L-quasi-metrics by using the following spheres, which play a crucial role in characterizing metric-induced topology.

Definition 5. *Given a mapping $p : M \times M \to [0, +\infty)$, for $a, b \in M$ and $r \in [0, +\infty)$, we define the following:*

$$U_r(a) = \bigvee\{c \in M \mid p(a, c) < r\};$$

$$B_r(a) = \bigvee\{c \in M \mid p(a, c) \leq r\};$$

$$Q_r(b) = \bigvee\{c \in M \mid p(c, b) > r\};$$

$$P_r(b) = \bigvee\{c \in M \mid p(c, b) \geq r\}.$$

Theorem 11. *Let p be an L-quasi-metric on L^X. Then,* (1) $U_r(b) = \bigvee_{s<r} B_s(b)$; (2) $B_r(b) = \bigwedge_{r<s} U_s(b)$.

Proof. (1). If $s < r$, then $B_s(b) \leq U_r(b)$. Thus, $\bigvee_{s<r} B_s(b) \leq U_r(b)$. Conversely, let $c \ll U_r(b)$. Then, by the way below relation and (A2) in Definition 3, we can obtain $p(b, c) < r$. Taking s with $p(b, c) < s < r$, we have $c \leq U_s(b)$, and then $c \leq \bigvee_{s<r} U_s(b)$. Consequently, $U_r(b) \leq \bigvee_{s<r} U_s(b) \leq \bigvee_{s<r} B_s(b)$.

(2). Obviously, $B_r(b) \leq \bigwedge_{r<s} U_s(b)$. Conversely, let $a \ll \bigwedge_{r<s} U_s(b)$. Then, for any $s > r$, we have $p(b, a) < s$. Because s is arbitrary, it is true that $p(b, a) \leq r$. Hence, $a \leq B_r(b)$. Consequently, $\bigwedge_{r<s} U_s(b) \leq B_r(b)$. □

Theorem 12. *Let p be an L-quasi-metric on L^X. Then,* $\bigwedge_{u<r} P_u(a) = P_r(a)$.

Proof. If $u < r$, then $P_r(a) \leq P_u(a)$. Thus, $P_r(a) \leq \bigwedge_{u<r} P_u(a)$. Conversely, let $c \ll \bigwedge_{u<r} P_u(a)$. Then, it holds that $c \ll P_u(a)$ for every $u < r$. Therefore, there exists $e \in M$ such that $c \leq e$ and $p(e, a) \geq u$, and then $p(c, a) \geq u$. Because u is arbitrary, we have $p(c, a) \geq r$, which implies $c \leq P_r(a)$. Therefore, $\bigwedge_{u<r} P_u(a) \leq P_r(a)$. □

Theorem 13. *If p is an L-quasi-metric on L^X and for any $b \in M$ there is $\bigwedge_{x \ll b} p(x, b) = 0$, then $p(b, a) = \bigwedge_{x \ll b} p(x, a)$.*

Proof. Let $p_1(b,a) = \bigwedge_{x \ll b} p(x,a)$. Since $x \ll b$, we have $x \leq b$. Therefore, by triangle inequality $p(x,a) \geq p(b,a)$, $p_1(b,a) = \bigwedge_{x \ll b} p(x,a) \geq p(b,a)$. Conversely, we have

$$p_1(b,a) = \bigwedge_{x \ll b} p(x,a) \leq \bigwedge_{x \ll b} (p(b,a) + p(x,b))$$
$$= p(b,a) + \bigwedge_{x \ll b} p(x,b).$$

In view of $\bigwedge_{x \ll b} p(x,b) = 0$, we can obtain $p_1(b,a) \leq p(b,a)$, as desired. □

Corollary 2. *Let p be an L-quasi-metric on L^X. Then, p is a Yang–Shi pseudo-metric if and only if, for each $a \in M$, it holds that $\bigwedge_{x \ll a} p(x,a) = 0$.*

Theorem 14. *If mapping $p : M \times M \longrightarrow [0,+\infty)$ satisfies the property* (E3)* *for each $a \in M$ and $r > 0$, $a \not\leq P_r(a)$, then, when $b \geq a$, $p(b,a) = 0$.*

Proof. If $p(b,a) \neq 0$, then there exists a number $r \in R^+$ such that $p(b,a) \geq r$, and then $b \leq P_r(a)$. Therefore, $a \leq P_r(a)$, which contradicts (E3)*. □

Theorem 15. *Let p be an L-quasi-metric on L^X. If $\bigwedge_{c \ll a} p(c,a) = \lambda > 0$, then $a \not\leq P_r(a)$ if and only if $r > \lambda$.*

Proof. Let $r > \lambda$. If $a \leq P_r(a)$, then, for each $x \ll a$, there exists $y \in M$ such that $x \leq y$ and $p(y,a) \geq r$. Therefore, $p(x,a) \geq p(y,a) \geq r$, so that $\lambda = \bigwedge_{x \ll a} p(x,a) \geq r$. This is a contradiction. Thus, $a \not\leq P_r(a)$.

Conversely, assume that $r \leq \lambda$. Then, $P_r(a) \geq P_\lambda(a)$. Since $a \not\leq P_r(a)$, it is true that $a \not\leq P_\lambda(a)$. In view of $\bigwedge_{c \ll a} p(c,a) = \lambda$, we have $p(c,a) \geq \lambda$ for any $c \ll a$, and then $c \leq P_\lambda(a)$. Therefore, $a \leq P_\lambda(a) \leq P_r(a)$. This is a contradiction. Thus, $r > \lambda$, as desired. □

A mapping $p : M \times M \longrightarrow [0,+\infty)$ is called a Yang pseudo-metric on L^X if it satisfies (A1)–(A3) and (E3)* [8]. Therefore, by Corollary 2 and Theorems 14 and 15, we have the following result.

Corollary 3. *p is a Yang–Shi pseudo-metric if and only if p is a Yang pseudo-metric on L^X.*

Theorem 16. *Let p be an L-quasi-metric on L^X. Then, the family $\{U_r(a) \mid a \in M, r \in [0,+\infty)\}$ is a basis for a topology which is called the metric topology induced by p and denoted by ζ_p.*

Proof. Let ζ_p be the set of arbitrary unions of the family. To prove that ζ_p is a topology, we only need to prove that the intersection of any two elements of ζ_p belongs to ζ_p.

Let $U_s(a), U_t(b) \in \zeta_p$ and let $A = U_s(a) \bigwedge U_t(b)$. Case 1: if $s = 0$ or $t = 0$, then it is easy to check $A \in \zeta_p$. Case 2: if $s \neq 0$ and $t \neq 0$, then $A \neq \underline{0}$. In this case, let $c \ll A$. Then, $c \ll U_s(a)$ and $c \ll U_t(b)$. Therefore, $p(a,c) < s$ and $p(b,c) < t$. Let $r_c = (s - p(a,c)) \wedge (t - p(b,c))$. Now, we prove $A = \bigvee_{c \ll A} U_{r_c}(c)$.

Clearly, $A \leq \bigvee_{c \ll A} U_{r_c}(c)$. Conversely, let $e \ll \bigvee_{c \ll A} U_{r_c}(c)$. Then, there exists $c \ll A$ such that $e \ll U_{r_c}(c)$, and then $p(c,e) < r_c$. Hence, we can obtain $p(c,e) < s - p(a,c)$ and $p(c,e) < t - p(b,c)$. Consequently, $p(a,e) < s$ and $p(b,e) < t$. Therefore, $e \leq U_s(a)$ and $e \leq U_t(b)$, so that $e \leq A$, as desired. □

Theorem 17. *Let p be an L-quasi-metric on L^X. Then, $A^\circ = \bigvee\{a \mid \exists r > 0, U_r(a) \leq A\}$.*

Proof. Let $T = \bigvee\{a \mid \exists r > 0, U_r(a) \leq A\}$. Obviously, $T \leq A^\circ$. Conversely, let $e \ll A^\circ$. Then, by Theorem 16, there exists $U_r(a)$ such that $e \ll U_r(a) \leq A$, and then $p(a,e) < r$. Let $s = r - p(a,e)$. Given $c \in M$ with $p(e,c) < s$. Then, $p(a,c) \leq p(a,e) + p(e,c) < r - s + p(e,c) < r$. Therefore, $U_s(e) \leq U_r(a) \leq A$, so that $e \leq T$, as desired. □

Theorem 18. *Given a mapping $p : M \times M \to [0, +\infty)$, where p satisfies* (A3) *(see Definition 3). Then, $U_r(b) = D_{-r}(b')'$.*

Proof. Let $p(b,a) < r$. Since for every $x \not\leq a'$ (i.e., $a \not\leq x'$), there is $z \ll x$ such that $z \not\leq a'$(i.e., $z \ll x$ and $a \not\leq z'$), there is $w \in M$ with $w \not\leq b'$ such that $p(z,w) < r$, from (A3). Therefore, it must hold that $x \not\ll D_{-r}(b')$. Otherwise, there exists $c \in M$ such that $x \leq c$ and $D_r(c) \leq b'$. Since $D_r(x) \leq D_r(c)$, we have $D_r(x) \leq b'$. In addition, from $z \ll x$ and $p(z,w) < r$, we can deduce $w \leq D_r(x)$, so that $w \leq b'$. However, this is a contradiction. In short, as long as $x \not\leq a'$, it is true that $x \not\ll D_{-r}(b')$. Thus, $D_{-r}(b') \leq a'$, i.e., $a \leq D_{-r}(b')'$. Thus, $U_r(b) \leq D_{-r}(b')'$.

Conversely, let $x \not\leq D_{-r}(b')$. Then, $D_r(x) \not\leq b'$. Thus, there is $e \in \{e \in M \mid \exists x_e \ll x, p(x_e, e) < r\}$ such that $e \not\leq b'$. By (A3) there exists $y \not\leq x'_e$ such that $p(b,y) < r$, and then $y \leq U_r(b)$. In view of $x_e \ll x$ and $y \not\leq x'_e$, we can obtain $y \not\leq x'$. Therefore, $U_r(b) \not\leq x'$, i.e., $x \not\leq U_r(b)'$. That is to say that, as long as $x \not\leq D_{-r}(b')$, it must hold that $x \not\leq U_r(b)'$. It follows that $U_r(b)' \leq D_{-r}(b')$, i.e., $D_{-r}(b')' \leq U_r(b)$, as desired. □

Theorem 19. *A mapping $p : M \times M \to [0, +\infty)$ satisfies* (A3) *if and only if it holds that $\bigvee\limits_{b \ll a} U_r(b) = U_r(a)$.*

Proof. If $U_r(b) \not\leq a'$, then there exists x with $p(b,x) < r$ such that $x \not\leq a'$. Because (A3) is equivalent to $U_r(a) \not\leq b' \Leftrightarrow U_r(b) \not\leq a'$ for any $a, b \in M$, we can obtain the following formulas:

$$U_r(a) \leq b' \Leftrightarrow U_r(b) \leq a' = \bigwedge_{x \ll a} x' \Leftrightarrow x \ll a, U_r(b) \leq x'$$

$$\Leftrightarrow x \ll a, U_r(x) \leq b' \Leftrightarrow \bigvee_{x \ll a} U_r(x) \leq b',$$

as desired. □

Corollary 4. *Suppose that mapping $p : M \times M \to [0, +\infty)$ satisfies* (A3). *Then, $U_r(b) = D_r(b)$.*

Proof. By Theorem 19, $U_r(b) = \bigvee\limits_{e \ll b} U_r(e) = \bigvee\{a \in M \mid \exists e \ll b, p(e,a) < r\} = D_r(b)$, as desired. □

Definition 6. *Suppose that mapping $p : M \times M \to [0, +\infty)$ satisfies (A3). Then, for $A \in L^X$ and $r > 0$, define $U_r(A) = \bigvee\limits_{a \ll A} U_r(a)$.*

Remark 1. *If $A = b \in M$, then, by Theorem 19, $U_r(A) = \bigvee\limits_{a \ll A} U_r(a)$. Furthermore, by Corollary 4 and Definition 6, we have $U_r(A) = D_r(A)$. As a result, if a mapping $p : M \times M \to [0, +\infty)$ satisfies (A3), then $U_r(A)$ and $D_r(A)$ are equivalent.*

5. *L*-Pseudo-Metric on L^X

In this section, we investigate L-pseudo-metric on L^X. In particular, the relationship between the two topologies: ψ_p and ζ_p, which have been presented in Theorem 6 and Theorem 16 respectively, are acquired below.

Theorem 20. *If p is an L-pseudo-metric on L^X, then $\psi_p = \zeta'_p$.*

Proof. By Theorem 18, $U_r(b) = D_{-r}(b')'$. Therefore, in view of Theorem 6 and Theorem 16, we can assert that the result is true, as desired. □

Corollary 5. *If p is an L-pseudo-metric on L^X, then $\overline{A} = \bigwedge_{r>0} U_r(A)$.*

Proof. It is easy to check the result by Theorem 10 and Remark 1. □

Theorem 21. *Let p be an L-pseudo-metric on L^X. Then, $\overline{A} = \bigvee\{b \in M \mid \exists$ a sequence $\{b_i \ll A \mid i \in N\}$ such that $p(b_i, b) \to 0\}$.*

Proof. Let $b \ll \overline{A}$. Since $\overline{A} = \bigwedge_{s>0} U_s(A)$, we have $b \ll U_{\frac{1}{k}}(A) = \bigvee_{c \ll A} U_{\frac{1}{k}}(c)$ for every $k \in N$. Therefore, there exists $b_k \ll A$ such that $b \ll U_{\frac{1}{k}}(b_k)$, so that $p(b_k, b) < \frac{1}{k}$.

Conversely, let $b \in \{b \in M \mid \exists$ a sequence $\{b_i \ll A \mid i \in N\}$ such that $p(b_i, b) \to 0\}$. Then, by Corollary 1, $D_{-r}(\alpha)$ is a Q-neighborhood of b for any $b \not\leq \alpha$ and $r > 0$. Now, we check $A \not\leq D_{-r}(\alpha)$.

By Theorem 9, we have $b \not\leq D_{-r}(\alpha) \Leftrightarrow D_r(b) \not\leq \alpha$ and $A \not\leq D_{-r}(\alpha) \Leftrightarrow D_r(A) \not\leq \alpha$. Thus, we need to prove this result: if $D_r(b) \not\leq \alpha$, then $D_r(A) \not\leq \alpha$, i.e., $D_r(A) \leq \alpha \Rightarrow D_r(b) \leq \alpha$. The proof is as follows.

Let $D_r(A) \leq \alpha$ and it be true that $D_r(b) = U_r(b) = \bigvee\{c \mid p(b, c) < r\}$. If $p(b, c) = s < r$, then

$$p(b_i, c) \leq p(b_i, b) + p(b, c) = p(b_i, b) + s.$$

Since $p(b_i, b) \to 0$, there exists $N_r \in N$ such that, when $i \geq N_r$, we have $p(b_i, c) < r$. Therefore, $c \leq D_r(A)$, so that $D_r(b) \leq D_r(A)$. Consequently, $D_r(b) \leq \alpha$, as desired. □

Theorem 22. *Let p be an L-pseudo-metric on L^X. Then, $P_r(a)$ is a closed set in ζ_p.*

Proof. By Corollary 5 and Remark 1, we prove $P_r(a) = \bigwedge_{t>0} D_t(P_r(a))$. In addition, when $t > s$, it is easy to see that $D_s(P_r(a)) \leq D_t(P_r(a))$. Hence, we have

$$\bigwedge_{t>0} D_t(P_r(a)) = \bigwedge_{r>s>0} D_s(P_r(a)).$$

Therefore, $P_r(a) \leq \bigwedge_{r>s>0} D_s(P_r(a))$. Conversely, let $h \ll \bigwedge_{r>s>0} D_s(P_r(a))$. Then, for any s with $r > s > 0$, it is true that $h \ll D_s(P_r(a))$. Thus, there exists $e \in M$ with $h \leq e$ and $b \in M$, such that $b \ll P_r(a)$ and $p(b, e) < s$, so that $p(b, a) \geq r$ and $p(b, h) < s$. Hence, $p(h, a) \geq p(b, a) - p(b, h) \geq r - s$, and then $h \leq P_{r-s}(a)$. By Theorem 12, we have $h \leq \bigwedge_{r>s>0} P_{r-s}(a) = P_r(a)$. Consequently, $\bigwedge_{r>s>0} D_s(P_r(a)) \leq P_r(a)$, as desired. □

Theorem 23. *Let p be an L-pseudo-metric on L^X; then, $\bigvee_{z \not\leq b'} P_\lambda(z)' \leq D_\lambda(b)$.*

Proof. Let $a \ll \bigvee_{z \not\leq b'} P_\lambda(z)'$. Then, there exists $e \in M$ such that $a \ll e \leq \bigvee_{z \not\leq b'} P_\lambda(z)'$. Therefore, $e' \geq \bigwedge_{z \not\leq b'} P_\lambda(z)$. Thus, for any $x \not\leq e'$, there exists $z \not\leq b'$ such that $x \not\leq P_\lambda(z)$, which implies $p(x, z) < \lambda$. According to (A3), there exists $y = y(x)$ such that $y \not\leq x'$ and $p(b, y) < \lambda$. Let $q = \bigvee\{y = y(x) \mid x \not\leq e'\}$. Then, $q \not\leq x'$, i.e., $x \not\leq q'$. That is to say that, as long as $x \not\leq e'$, it must hold that $x \not\leq q'$. Hence, $q' \leq e'$, i.e., $e \leq q$. Therefore, $a \ll e \leq q$. Thus, there exists $y = y(x)$ such that $a \leq y$, and then $p(b, a) \leq p(b, y) < \lambda$. It follows that $a \leq U_\lambda(b) = D_\lambda(b)$. Consequently, $\bigvee_{z \not\leq b'} P_\lambda(z)' \leq D_\lambda(b)$, as desired. □

Theorem 24. *If p is an L-pseudo-metric on L^X, then $B_r(b) = \bigwedge_{r<s} B_s(b)$.*

Proof. Obviously, $B_r(b) \leq \bigwedge_{r<s} B_s(b)$. Conversely, let $h \ll \bigwedge_{r<s} B_s(b)$. Then, for every $s > r$, it is true that $h \ll B_s(b)$. Thus, there exists $e \in M$ such that $e \geq h$ and $p(b,e) \leq s$, so that $p(b,h) \leq p(b,e) \leq s$. Because s is arbitrary, we have $p(b,h) \leq r$, and then $h \leq B_r(b)$. Therefore, $\bigwedge_{r<s} B_s(b) \leq B_r(b)$, as desired. □

Theorem 25. *If p is an L-pseudo-metric on L^X, then $\overline{B_r(b)} = B_r(b)$.*

Proof. We only need to prove $\overline{B_r(b)} \leq B_r(b)$. Let $h \ll \overline{B_r(b)} = \bigwedge_{s>0} D_s(B_r(b))$. Then, for every $s > 0$, it is true that $h \ll D_s(B_r(b))$. Hence, there exist $a \ll B_r(b)$ and $b \in M$ such that $h \leq b$ and $p(a,b) < s$, and then $p(a,h) < s$. Because $a \ll B_r(b)$, we can obtain $p(b,a) \leq r$. Hence, we have

$$p(b,h) \leq p(b,a) + p(a,h) < s + r.$$

It follows that $h \leq U_{r+s}(b) = D_{r+s}(b)$, and then $h \leq \bigwedge_{s>0} D_{r+s}(b)$. According to Theorem 11, we have $h \leq B_r(b)$. As a result, $\overline{B_r(b)} \leq B_r(b)$, as desired. □

Because $B_r(b)$ is a closed set, $\overline{U_r(b)} \leq B_r(b)$. In general, $\overline{U_r(b)} \neq B_r(b)$. Therefore, we give the following result.

Theorem 26. *Let p be an L-pseudo-metric on L^X. If there exists $c \in M$ such that $p(c,a) < r$ and $p(b,c) < s$ for any $a, b \in M$ satisfying $p(b,a) < r + s$, then $\overline{U_r(b)} = B_r(b)$.*

Proof. We only need to prove $B_r(b) \leq \overline{U_r(b)}$. Due to $U_r(b) = D_r(b)$, we need to prove $B_r(b) \leq \overline{D_r(b)}$. According to Theorem 11, we have

$$B_r(b) = \bigwedge_{s>0} D_{r+s}(b).$$

Let $a \ll D_{r+s}(b)$. Then, we have $p(b,a) < r + s$. Because there is $c \in M$ such that $p(c,a) < r$, $p(b,c) < s$ and $D_r(D_s(b)) = \bigvee_{c \ll D_s(b)} D_r(c)$, we can obtain $a \leq D_r(D_s(b))$. Therefore, $D_{r+s}(b) \leq D_r(D_s(b))$. As a result, we have $B_r(b) = \bigwedge_{s>0} D_{r+s}(b) = \bigwedge_{s>0} D_{s+r}(b) \leq \bigwedge_{s>0} D_s(D_r(b)) = \overline{D_r(b)} = \overline{U_r(b)}$, as desired. □

6. Further Properties about *L*-Pseudo-Metric

In this section, based on a class of spherical mappings, we acquire an equivalent characterization of L-pseudo-metric on L^X in terms of a class of mapping clusters.

Definition 7. *Given a mapping $p : M \times M \to [0, +\infty)$. For any $a, b \in M$, define $D_r^{-1}(b) = \bigwedge\{a' \mid D_r(a) \leq b'\}$ and $D_r \circ D_s(b) = \bigvee\{D_r(a) \mid a \leq D_s(b)\}$.*

Theorem 27. *If p is an L-pseudo-metric on L^X, then it satisfies the following properties:*

(1) $\bigvee_{r>0} D_r(b) = \underline{1}$;

(2) $b \leq D_r(b)$;

(3) $D_r(\bigvee_{i \in I} b_i) = \bigvee_{i \in I} D_r(b_i)$;

(4) $D_r(b) = \bigvee_{s<r} D_s(b)$;

(5) $D_r \circ D_s(b) \leq D_{r+s}(b)$;

(6) $D_r(b) = D_r^{-1}(b)$.

Proof. (1) and (2) are immediate, by definitions.

(3) Let $A = \bigvee_{i\in I} b_i$. Then, according to $\beta^*(\bigvee_{i\in I} b_i) = \bigcup_{i\in I} \beta^*(b_i)$, Remark 1 and the definition of $D_r(A)$, (3) is true.

(4) According to Theorem 11 and Corollary 4, it is easy to check that $D_r(b) = \bigvee_{s<r} B_s(b) = \bigvee_{s<r} D_s(b)$.

(5) We need to prove the following formulas:

$$D_r \circ D_s(b) = \vee\{D_r(a) \mid a \le D_s(b)\}$$

$$= \vee\{[\vee\{e \in M \mid \exists f \ll a, p(f,e) < r\} \mid a \le D_s(b)]\}$$

$$\le D_{r+s}(b).$$

In fact, let $a \le D_s(b) = \bigvee\{g \in M \mid \exists h \ll b, p(h,g) < s\}$ and let $e \in M$, satisfying that there exists $f \ll a$ such that $p(f,e) < r$. Then, there must exist g with $f \le g$ and h_g with $h_g \ll b$ such that $p(h_g, g) < s$. Hence,

$$p(h_g, e) \le p(h_g, f) + p(f,e) \le p(h_g, g) + p(f,e) < r + s.$$

Thus, $e \le D_{r+s}(b)$. Hence, $D_r(a) \le D_{r+s}(b)$. Consequently, $D_r \circ D_s(b) \le D_{r+s}(b)$.

(6) By Theorem 18 and Corollary 4, we can deduce $D_r^{-1}(b) = \bigwedge\{a' \mid D_r(a) \le b'\} = (D_{-r}(b'))' = U_r(b) = D_r(b)$. □

Theorem 28. *Suppose that the family* $\{D_r(a) \mid a \in M, r \in [0,+\infty)\}$ *satisfies the above properties* (1)–(6) *and we define* $p(a,b) = \bigvee\{r \mid b \not\le D_r(a)\}$. *Then, the following hold:*

(a) p *is a mapping from* $M \times M$ *to* $[0,+\infty)$;

(b) *As well as* (A1) *and* (A2), p *further satisfies* $p(a,b) = \bigvee_{c\ll b} p(a,c)$;

(c) p *is an L-fuzzy pseudo-metric on* L^X;

(d) $p(a,b) = \bigwedge\{r \mid b \le D_r(a)\}$.

Proof. First of all, we prove the following two conclusions:

(i) If $p(a,b) < r$, then $b \le D_r(a)$;

(ii) If $b \le D_r(a)$, then $p(a,b) \le r$.

(i) Suppose that $b \not\le D_r(a)$. Then, this means that, for any $s < r$, it holds that $b \not\le D_s(a)$. Therefore, $p(a,b) = \bigvee\{s \mid b \not\le D_s(a)\} \ge r$, so that (i) is true.

(ii) Suppose that $p(a,b) = \bigvee\{s \mid b \not\le D_s(a)\} > r$; then, there exists $s > r$ such that $b \not\le D_s(a)$. By the condition (4), we have $D_s(a) \ge D_r(a)$. Thus, $b \not\le D_r(a)$, and then (ii) holds.

(a) Let $b \ll \underline{1} = \bigvee_{r>0} D_r(a)$. Then, there exists r such that $b \le D_r(a)$. By (ii), we can obtain $p(a,b) \le r \in [0,+\infty)$. As for $p(a,b) \ge 0$, this is obvious from the definition.

(b) (A1). If $b \le a$, then, according to property (2), for each $r > 0$, there is $b \le D_r(a)$. In view of (ii), we can obtain $p(a,b) \le r$. Because r is arbitrary, we have $p(a,b) = 0$.

(A2). Let $a,b,c \in M$, $p(a,b) = r$ and $p(b,c) = t$. Then, for any $s > 0$, we have $p(a,b) < r+s$ and $p(b,c) < t+s$. Therefore, by (i), we know $b \le D_{r+s}(a)$ and $c \le D_{t+s}(b)$. By (3) and (5), we have

$$c \le D_{t+s}(b) \le D_{t+s}(D_{r+s}(a))$$

$$\le D_{t+s} \circ D_{r+s}(a) \le D_{r+t+2s}(a).$$

Therefore, by (ii), we can obtain $p(a,c) \le r+t+2s$. Because s is arbitrary, we have $p(a,c) \le r+t$. Consequently, $p(a,c) \le p(a,b) + p(b,c)$.

Next, we demonstrate that $p(a,b) = \bigvee_{c\ll b} p(a,c)$.

Let $c \ll b$. Then, by (A1) and (A2), $p(a,c) \leq p(a,b)$. If $p(a,b) = 0$, then $\bigvee_{c \ll b} p(a,c) = 0$. Thus, we might as well suppose $p(a,b) = r > 0$. For any $s \in (0,r)$, by (ii), we know $b \not\leq D_s(a)$, which implies that there exists $e \ll b$ such that $e \not\leq D_s(a)$, and then $p(a,e) \geq s$. Hence, $\bigvee_{c \ll b} p(a,c) \geq s$. Because s is arbitrary, we can assert that $\bigvee_{c \ll b} p(a,c) \geq r$. If $\bigvee_{c \ll b} p(a,c) > t > r$, then there exists $e \ll b$ such that $p(a,e) > t$. By (ii), we know $e \not\leq D_t(a)$. Thereby, $b \not\leq D_t(a)$, so that $p(a,b) = r > t$. This is a contradiction. As a result, we can assert that $\bigvee_{c \ll b} p(a,c) = r$.

(c) We need to prove (A3). Suppose $D_r^{-1}(b) \leq a'$. Then, by the definition of D_r^{-1}, we can obtain

$$D_r^{-1}(b) = \bigwedge\{e' \mid D_r(e) \leq b', e \in M\} \leq a'$$

$$\Leftrightarrow \bigvee\{e \mid D_r(e) \leq b', e \in M\} \geq a.$$

Thus, for any $h \ll a$, there exists e such that $h \leq e$ and $D_r(e) \leq b'$, and then $D_r(h) \leq D_r(e) \leq b'$. By the property (3), we know $D_r(a) = \bigvee_{h \ll a} D_r(h) \leq b'$. In view of the property (6), we can obtain $D_r^{-1}(a) \leq b'$.

Similarly, we can prove that $D_r^{-1}(b) \leq a'$ if $D_r^{-1}(a) \leq b'$. Therefore, $D_r^{-1}(b) \leq a' \Leftrightarrow D_r^{-1}(a) \leq b'$. By the property (6), we have $D_r(a) \not\leq b' \Leftrightarrow D_r(b) \not\leq a'$, which is equivalent to (A3).

(d) Let $\inf\{r \mid b \leq D_r(a)\} = t$. If $p(a,b) = \sup\{r \mid b \not\leq D_r(a)\} > t$, then there exists $r > t$ such that $b \not\leq D_r(a)$. By the property (4), $b \not\leq D_s(a)$ for any $s < r$. Thus, $t = \inf\{r \mid b \leq D_r(a)\} \geq r$. This is a contradiction. If $t > p(a,b)$, then there is a number u satisfying $t > u > p(a,b)$. Since $t = \inf\{r \mid b \leq D_r(a)\} > u$, we can assert that $b \not\leq D_u(a)$. Therefore, by conclusion (i), it holds that $p(a,b) \geq u$. This contradicts $p(a,b) < u$. Consequently, $p(a,b) = t$, so that $p(a,b) = \inf\{r \mid b \leq D_r(a)\}$, as desired. □

7. *L*-Metric on L^X

In this section, we shall show the relationship between L-pseudo-metric and L-metric on L^X. First of all we give the following concept.

Definition 8. *The space (X, δ) is claimed T_1 if and only if $b = \overline{b}$ for any $b \in M$.*

Theorem 29. *Let p be an L-fuzzy pseudo-metric on L^X. Then, (X, ζ_p) is T_1-space if and only if p satisfies* (A4).

Proof. Let $b \in M$ and let $h \ll \overline{b} = \bigwedge_{r>0} D_r(b)$. Then, for each $r > 0$, we can obtain $h \ll D_r(b) \leq B_r(b)$. Therefore, $p(b,h) = 0$. Hence, by (A4) we know $h \leq b$, and then $b = \bar{b}$.

Conversely, suppose that $p(b,a) = 0$. Then, for each $r > 0$, we have $a \leq D_r(b)$, and then we can obtain $a \leq \bigwedge_{r>0} D_r(b) = \overline{b} = b$. Therefore, p satisfies (A4). □

Corollary 6. *The space (X, δ) is L-metrizable if and only if it is T_1-space- and L-pseudo-metrizable.*

8. Applications

In this section, we further show some related applications of L-quasi (pseudo)-metric on L^X.

Theorem 30. *If p is an L-pseudo-metric on L^X and satisfies the property $p(a,b) = \bigwedge_{c \ll a} p(c,b)$, then the following apply:*

(a) $\bigvee_{z \not\leq b'} P_\lambda(z)' = D_\lambda(b)$;

(b) *The family* $\{P_r(b) \mid b \in M, r \in [0,+\infty)\}$ *is a closed topological base and the topology is denoted by* η_p*;*

(c) $\eta_p = \zeta'_p$.

Proof. First of all we prove the result: (i) $c \le P_r(b) \Leftrightarrow p(c,b) \ge r$ for any $c, b \in M$. In fact, we only need to prove $c \le P_r(b) \Rightarrow p(c,b) \ge r$. Let $h \ll c$. Then, there exists $e \in M$ such that $e \ge h$ and $p(e,b) \ge r$, so that $r \le p(e,b) \le p(e,h) + p(h,b) = p(h,b)$. Therefore, $p(c,b) = \bigwedge_{h \ll c} p(h,b) \ge r$.

(a) By Theorem 23, we only need to prove $\bigvee_{z \not\le b'} P_\lambda(z)' \ge D_\lambda(b)$. Let $a \ll D_\lambda(b)$. Then, $p(b,a) < \lambda$. In addition, for $\forall x \not\le a'$, i.e., $a \not\le x'$, by (A3), there exists $z \not\le b'$ such that $p(x,z) < \lambda$. By (i), we have $x \not\le P_\lambda(z)$, so that $x \not\le \bigwedge_{z \not\le b'} P_\lambda(z)$. Because $x \not\le a'$ implies $x \not\le \bigwedge_{z \not\le b'} P_\lambda(z)$, we can assert that $\bigwedge_{z \not\le b'} P_\lambda(z) \le a'$, i.e., $a \le \bigvee_{z \not\le b'} P_\lambda(z)'$. Hence, $\bigvee_{z \not\le b'} P_\lambda(z)' \ge D_\lambda(b)$.

(b) It needs to be proven that the intersection of any subset of $\{P_r(b) \mid r \in [0,+\infty), b \in M\}$ is a topology, i.e.,

$$\eta_p = \{\bigwedge_{i \in \Gamma} P_i(b_i) \mid \Gamma \subseteq [0,+\infty), b_i \in M\}$$

because $\bigwedge \emptyset = \underline{1}$ and $\bigwedge_{r>0} P_r(a) = \emptyset$ for any $a \in M$, $\underline{0}, \underline{1} \in \eta_p$. Secondly, let $A \subseteq \eta_p$ and $B \subseteq \eta_p$. Then, according to the definition of η_p, it is straightforward for $A \bigwedge B \in \eta_p$. Thus, we only need to prove that, for any $a, b \in M$ and any $r, s \in [0,+\infty)$, $P_r(a) \vee P_s(b)$ is the intersection of some elements in $\{P_r(a) \mid a \in M, r \in (0,+\infty)\}$. The proof is as follows.

Case 1: when $r = 0$ or $s = 0$, $P_r(a) = P_0(a) = \underline{1}$ or $P_s(b) = P_0(b) = \underline{1}$ is true. Therefore, $P_r(a) \vee P_s(b) = \underline{1} \in \eta_p$;

Case 2: when $r, s \in (0,+\infty)$ and we let $A = P_r(a) \vee P_s(b)$. Then, according to Theorem 22, we can assert that A is a closed set in ζ_p. Therefore, we have $A' = \bigvee_i D_{r_i}(c_i)$, i.e., $A = (\bigvee_i D_{r_i}(c_i))' = \bigwedge_i D_{r_i}(c_i)'$. By (a), we can obtain $A = \bigwedge_i \bigwedge_{z \not\le c'_i} P_{r_i}(z)$, as desired.

(c) By (b), we know that it is an open set for every $D_r(b)$ in η_p. By Theorem 20, it is a closed set for every $P_r(b)$ in ζ'_p, which implies $\eta_p = \zeta'_p$. □

Suppose that, for any $a \in M$, there exists a corresponding *Q-neighborhoods* base of a and the base is countable. Then, the space (X, δ) is called Q-C_I [41,42].

Theorem 31. *Suppose that p is an L-pseudo-metric on* L^X *and satisfies the property* $p(a,b) = \bigwedge_{c \ll a} p(c,b)$*. Then,* (1) $\{P_r(a)' \mid a \in M, r \in [0,+\infty)\}$ *is a* $Q-$ *neighborhoods base of a;* (2) *the space* (X, ζ_p) *is* $Q - C_I$.

Proof. (1) Let $A \in \zeta'_p$ satisfying $a \not\le A$, i.e., A is a closed R-neighborhood of a. Then, by Theorem 8, the family $\{P_r(b) \mid b \in M, r \in [0,+\infty)\}$ is a closed topology base for ζ_p. Therefore, $A = \bigwedge_{i \in \Gamma} P_{r_i}(b_{r_i})$. Since $a \not\le A$, there exists some $s \in \Gamma$ such that $a \not\le P_s(b)$. Let $p(a,b) = t$. Then, $p(a,b) = t < s$. Take any $e \in M$ satisfying $p(e,b) \ge s$. Since $s \le p(e,b) \le p(e,a) + p(a,b) = p(e,a) + t$, we have $s - t \le p(e,a)$, which implies $e \le P_{s-t}(a)$. Therefore, $P_s(b) \le P_{s-t}(a)$, so that $\{P_r(a)' \mid a \in M, r \in [0,+\infty)\}$ is a *Q-neighborhoods* base of a.

(2) Let B be an R-neighborhood of a and let Q^+ be the set of all rational numbers in $(0,+\infty)$. Then, for any $r > 0$, there exists $t \in Q^+$ with $0 < t < r$ such that $P_r(a) \le P_t(a)$. Therefore, we can assert that $\{P_t(a)', t \in Q^+\}$ is also a *Q-neighborhoods* base of a, so that ζ_p is $Q - C_I$. □

However, if p is an L-pseudo-metric on L^X and satisfies $p(a,b) = \bigvee_{c \ll b} p(a,c)$, then ζ_p is not $Q - C_I$.

Actually, in 1985, M.K. Luo [43] constructed an example of this kind of metric on I^X whose metric topology had no σ-locally finite base. Therefore, the topological space is not C_{II}, so that ζ_p was, of course, not $Q - C_I$.

9. Conclusions

In this paper, first, we put forward an L-quasi (pseudo)-metric on L^X and show a series of its related properties. Secondly, we present two topologies: ψ_p and ζ_p, generated by an L-quasi-metric with different spherical mappings and prove that $\psi_p = \zeta'_p$ if p is further an L-pseudo-metric on L^X. Thirdly, we characterize an equivalent form of the L-metric in terms of a class of mapping clusters and acquire a desired result. Finally, based on the L-metric, we assert that a Yang–Shi metric topology is $Q - C_I$, but, in general, an Erceg metric topology is not.

In future work, we will continue to investigate the Chen metric on L^X and study this kind of topological space whose topology has a σ-locally finite base. Beyond that, we also intend to inquire into some questions on the fuzzifying metric topology.

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