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Dynamics of Non-Autonomous Stochastic Semi-Linear Degenerate Parabolic Equations with Nonlinear Noise

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Abstract: In the present paper, we aim to study the long-time behavior of a stochastic semi-linear degenerate parabolic equation on a bounded or unbounded domain and driven by a nonlinear noise. Since the theory of pathwise random dynamical systems cannot be applied directly to the equation with nonlinear noise, we first establish the existence of weak pullback mean random attractors for the equation by applying the theory of mean-square random dynamical systems; then, we prove the existence of (pathwise) pullback random attractors for the Wong–Zakai approximate system of the equation. In addition, we establish the upper semicontinuity of pullback random attractors for the Wong–Zakai approximate system of the equation under consideration driven by a linear multiplicative noise.

Keywords: stochastic degenerate parabolic equation; nonlinear noise; pullback random attractor; Wong–Zakai approximation; upper semicontinuity

MSC: 37L55; 35B40; 37B55; 35B41

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Citation: Liu, X.; Li, Y. Dynamics of Non-Autonomous Stochastic Semi-Linear Degenerate Parabolic Equations with Nonlinear Noise. *Mathematics* **2023**, *11*, 3158. https://doi.org/10.3390/math11143158

Academic Editor: Nicolae Herisanu

Received: 16 June 2023
Revised: 12 July 2023
Accepted: 12 July 2023
Published: 18 July 2023

1. Introduction

We consider the following stochastic semi-linear degenerate parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(\sigma(x)\nabla u) + \lambda u + f(x,u) = g(t,x) + h(t,x,u)\dfrac{dW}{dt}, & t > \tau, \\ u(\tau,x) = u_\tau(x), & \tau \in \mathbb{R}, \\ u(t,x)|_{\partial\mathcal{O}} = 0, & t > \tau, \end{cases} \tag{1}$$

where $\mathcal{O} \subseteq \mathbb{R}^N (N \geq 2)$ is an arbitrary (bounded or unbounded) domain, λ is a positive constant, W is a two-sided Hilbert space valued cylindrical Wiener process or a two-side real-valued Wiener process, the drift term f and diffusion term h are nonlinear functions with respect to u, the given function $g(t,x) \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathcal{O}))$. In addition, the variable non-negative coefficient $\sigma(x)$ is allowed to have at most a finite number of (essential) zeros at some points, which is understood the degeneracy of (1). As in [1,2], we assume that the non-negative function $\sigma(x) : \mathcal{O} \to \mathbb{R}^+ \cup \{0\}$ satisfies the following hypotheses:

$(\mathcal{H}_\alpha)$ $\sigma \in L^1_{\text{loc}}(\mathcal{O})$ and for some $\alpha \in (0,1)$, $\liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\mathcal{O}}$, when the domain $\mathcal{O}$ is bounded;

$(\mathcal{H}_\beta)$ σ satisfies condition $(\mathcal{H}_\alpha)$ and $\liminf_{|x|\to\infty} |x|^{-\beta}\sigma(x) > 0$ for some $\beta > 2$, when the domain $\mathcal{O}$ is unbounded.

The conditions $(\mathcal{H}_\alpha)$ and $(\mathcal{H}_\beta)$ indicate that the diffusion coefficient $\sigma(x)$ is extremely irregular.

One of the most important things in studying evolution partial differential equations is to investigate the long-time behavior of solutions of the equations. In this process, attractors are the ideal objects. At present, abundant results, both in an abstract context and concrete models, have been established for the deterministic infinite-dimensional dynamical systems, see, e.g., monographs [3–5] and papers [1,6–10]. However, when one considers the random influences on the systems under investigation, which are always presented as stochastic partial differential equations, and tries to establish the existence of attractors for them, the theory on deterministic infinite-dimensional systems cannot be applied directly. On the one hand, the stochastic dynamical systems are non-autonomous, and one cannot obtain a uniform (with regard to stochastic time symbol) absorbing set as the deterministic case as in, e.g., [4]; on the other hand, owing to the influences of stochastic driving systems, one cannot obtain the fixed invariant set for stochastic dynamical systems in general.

In order to overcome these drawbacks, Flandoli et al. in [11,12] introduced the theory of pathwise random dynamical systems and (pathwise) random attractors for the autonomous stochastic equations, in which the random attractor is a family of compact sets depending on random parameters and has some invariant properties under the action of the random dynamical system. Recent theories in [13,14] are related to non-autonomous pathwise random dynamical systems and pullback random attractors for non-autonomous stochastic equations, where the pullback random attractor is a family of compact sets depending on both random parameters and deterministic time symbols. Up to now, there have been many results on the existence and uniqueness of random attractors, and one can refer to [15–18] for the autonomous stochastic equations and [17,19–22] for the non-autonomous stochastic equations. In addition, for the result about random attractors for Equation (1) with linear noise, see, e.g., [15,17,18,23–25].

However, when one investigates the dynamics of stochastic evolution equations driven by nonlinear noise, the existence of random attractors cannot be established directly, since the serious challenge is that the existence of a random dynamical system is unknown in general for these kinds of systems. As far as it is known, up to now, there are two ways to overcome this difficulty in some sense. One method is to investigate the dynamic behavior of the Wong–Zakai approximate system corresponding to the original equation. For example, Lu and Wang in [26] revealed the existence of a pullback random attractor for the Wong–Zakai approximate system of a stochastic reaction–diffusion equation with the nonlinear noise in some bounded spatial domain, and later Wang et al. in [27] extended the result of [26] to unbounded domains by using the method of tail estimates. Chen et al. in [28] proved the existence of the pullback attractor for the fractional nonclassical diffusion equations with delay driven by additive white noise on unbounded domains, and investigated the approximations of those random attractors as the correlation time of the colored noise approaches zero. Another method is established by Kloeden et al. in [29] and Wang in [30], that is, they extended the concept of pathwise random attractors to mean context and established the corresponding existence theory of mean random attractors for random dynamical systems. Wang [31] proved the existence and uniqueness of weak pullback mean random attractors of lattice plate equations on the entire integer set with nonlinear damping driven by infinite-dimensional nonlinear noise. There are some relevant works, see, e.g., [32,33].

The first purpose of this article is to establish the existence of weak pullback mean random attractors for Equation (1) by using the theory of [30]. Toward this end, we first need to confirm the existence and uniqueness of a solution for Equation (1). For the existence of solutions to be a stochastic parabolic-type equation, e.g., a stochastic reaction–diffusion reaction, one can refer to [30,32,34,35]. Unlike reference [30], the existence of a solution for Equation (1) cannot be obtained directly by using the abstract result (Theorem 4.2.4) in [36] since the drift term $f(x,u)$ is allowed to be a polynomial growth of arbitrary order with respect to u in this article. We aim to prove the existence and uniqueness of the solution for Equation (1) by using the approach of [32], in which the author proves existence of solutions for stochastic reaction–diffusion equations involving drift term $f(x,t,u)$ with polynomial growth of any order and nonlinear diffusion term $\sigma(t,u)$, and the embedding $H^k(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $2 \le p \le \frac{2N}{N-2k}$ $(N \ge 2k)$ plays an essential role in this proof. Hence, we show the embedding result of the corresponding Sobolev space with weight $\sigma(x)$ in Section 2. In Section 3, we show that the solution generates a mean random dynamical system and establish the existence of weak pullback random attractors for Equation (1). We shall remark that since the mean random dynamical system is defined on the Banach space $L^p(\Omega, X)$ consisting of all Bochner-integrable functions and corresponding probability space $(\Omega, \mathcal{F}, P)$ lacks some topological structure, we only obtain the weakly compact property and weakly attracting property of mean random attractors for (1) in $L^2(\Omega, X)$.

The second goal is to investigate the dynamic behavior of the Wong–Zakai approximate system for Equation (1). We prove the existence of a pullback random attractor for the Wong–Zakai approximate system for Equation (1) with nonlinear diffusion term $h(t, x, u)$, which is allowed to be polynomial growth, and we also show that the pullback random attractor of Wong–Zakai approximation for Equation (1) converges to the attractor of Equation (1) as the size of approximation tends to zero, when $h(t, x, u)$ is equal to u. This work will be performed in Section 4. We remark that when we prove the pullback asymptotic compactness, we use the method of weighted Sobolev spaces to overcome the non-compactness of the usual Sobolev embeddings in the case of an unbounded domain, which is different from that of [26].

In what follows in this article, the constant C represents some positive constant and may change from line to line.

2. Preliminaries

2.1. Functional Setting

In this subsection, we introduce some function spaces and present some embedding results, which will be used in our proof.

Throughout this article, we let $(X, \|\cdot\|_X)$ be a separable Banach space and $L^p(\Omega, \mathcal{F}; X)$ $(1 < p < \infty)$ be the Banach space consisting of all strongly measurable and Bochner-integrable functions Ψ from Ω to X such that

$$\|\Psi\|_{L^p(\Omega,\mathcal{F};X)} = (\int_\Omega \|\Psi\|_X^p \mathrm{d}P)^{\frac{1}{p}} < +\infty. \tag{2}$$

Denote by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$ the complete filtered probability space satisfying the usual condition, i.e., $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub-σ-algebras of $\mathcal{F}$ that contains all P-null sets. We use $L^p(\Omega, \mathcal{F}_t; X)$ to represent the subspace of $L^p(\Omega, \mathcal{F}; X)$, which consists of all functions belonging to $L^p(\Omega, \mathcal{F}; X)$ and being strongly $\mathcal{F}_t$-measurable. For simplicity of notation, we denote by $\|\cdot\|$ the norm in $L^2(\mathcal{O})$ and $L^2(\Omega, \mathcal{F}_t; L^2(\mathcal{O}))$.

To investigate Equation (1), we introduce the weighted Sobolev space $D_0^{1,2}(\mathcal{O}, \sigma)$ defined by the completion of $\mathcal{C}_0^\infty(\mathcal{O})$ with norm $\|\cdot\|_{D_0^{1,2}(\mathcal{O},\sigma)}$,

$$\|u\|_{D_0^{1,2}(\mathcal{O},\sigma)} := (\int_\mathcal{O} \sigma(x)|\nabla u|^2 dx)^{\frac{1}{2}}. \tag{3}$$

And one can easily check that $D_0^{1,2}(\mathcal{O}, \sigma)$ is a Hilbert space with the inner product $(\cdot, \cdot)_\sigma$

$$(u, v)_\sigma := \int_\mathcal{O} \sigma(x)\nabla u \cdot \nabla v dx. \tag{4}$$

If condition $(\mathcal{H}_\alpha)$ (or $(\mathcal{H}_\beta)$ on an unbounded domain) holds, the operator $A = -\mathrm{div}$ $(\sigma(x)\nabla u)$ is positive and self-adjoint with a domain defined by

$$D(A) := \{u \in D_0^{1,2}(\mathcal{O}, \sigma) : \ Au \in L^2(\mathcal{O})\}.$$

Furthermore, one can easily observe that if σ satisfies $(\mathcal{H}_\alpha)$ and $(\mathcal{H}_\beta)$, then there exists a finite set $A = \{a_1, a_2, \cdots, a_k\} \subseteq \bar{\mathcal{O}}$ and δ, $r > 0$ such that the balls $B_i = B_r(a_i)$, $i = 1, 2, \cdots, k$, are disjoint and

$$\sigma(x) \geq \delta|x - a_i|^\alpha \text{ for } x \in B_i \cap \Omega,\ i = 1, 2, \cdots, k, \tag{5}$$

$$\sigma(x) \geq \delta \text{ for } x \in \Omega \setminus \cup_i B_i, \tag{6}$$

and moreover, if Ω is unbounded, then there exists $R > 0$ such that

$$\sigma(x) \geq \delta|x|^\beta \text{ for } x \in \Omega,\ |x| > R. \tag{7}$$

The following spaces will also be needed:

- $D^p(A) := \{u \in D_0^{1,2}(\mathcal{O}, \sigma) : Au \in L^p(\mathcal{O})\}$;
- $D_0^{-1}(\mathcal{O}, \sigma)$:= the dual space of $D_0^{1,2}(\mathcal{O}, \sigma)$;
- $H_0^m(\mathcal{O}, \sigma)$:= the closure of $C_0^\infty(\mathcal{O})$ with norm $\|\cdot\|_{H^m(\mathcal{O},\sigma)}$, defined by

$$\|u\|^2_{H^m(\mathcal{O},\sigma)} := \sum_{1 \leq |\kappa| \leq m} \int_{\mathcal{O}} \sigma(x)|D^\kappa u|^2 dx + \int_{\mathcal{O}} |u|^2 dx,\ m \in \mathbb{N}^+,$$

where $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_N)$ is a multi-index of order $|\kappa| = \kappa_1 + \kappa_2 + \cdots + \kappa_N$.

Lemma 1 ([37])**.** *There exists a constant c_1 such that the following inequality holds for all $u \in C_0^\infty(\mathbb{R}^N)$,*

$$\Big(\int_{\mathbb{R}^N} |u|^{2_\alpha^*} dx\Big)^{\frac{1}{2_\alpha^*}} \leq c_1 \Big(\sum_{|\kappa|=m} \int_{\mathbb{R}^N} |x|^\alpha |D^\kappa u|^2 dx\Big)^{\frac{1}{2}},$$

where $2_\alpha^* = \frac{2N}{N+2\alpha-2m}$ with $N - 2m \geq 0$.

Lemma 2. *Let $\sigma(x)$ satisfy assumption $(\mathcal{H}_\alpha)$ (or $(\mathcal{H}_\beta)$ on unbounded domain). Then, there exists a constant c_2 such that*

$$\Big(\int_{\mathcal{O}} |u|^{2_\alpha^*} dx\Big)^{\frac{1}{2_\alpha^*}} \leq c_2 \Big(\sum_{|\kappa|=m} \int_{\mathcal{O}} \sigma(x)|D^\kappa u|^2 dx\Big)^{\frac{1}{2}}, \text{ for every } u \in C_0^\infty(\mathcal{O}).$$

Proof of Lemma 2. By using Lemma 1, the Rellich–Kondrachov Theorem, and the General Sobolev inequality, we can obtain the conclusion of Lemma 2 in a similar way as in the proof of Proposition 2.5 in [2]. We omit the process here. □

The following embedding results play an important role in our proof in Sections 3 and 4.

Lemma 3 ([2])**.** *Let $\sigma(x)$ satisfy assumption $(\mathcal{H}_\alpha)$ (or $(\mathcal{H}_\beta)$ on unbounded domain). Then, it holds the compact embedding $D_0^{1,2}(\mathcal{O}, \sigma) \hookrightarrow\hookrightarrow L^2(\mathcal{O})$.*

Lemma 4. *Let $\sigma(x)$ satisfy assumption $(\mathcal{H}_\alpha)$ (or $(\mathcal{H}_\beta)$ on unbounded domain). Then, it holds the continuous embedding*

$$H_0^m(\mathcal{O}, \sigma) \hookrightarrow L^p(\mathcal{O}), \text{ for } 2 \leq p \leq 2_\alpha^*.$$

Proof of Lemma 4. Note that $2_\alpha^* > 2$ for $\alpha \in (0, 1)$. Then, we can find, by the interpolation theorem and Lemma 2, that

$$\|u\|_{L^p(\mathcal{O})} \leq C\|u\|^\theta \|u\|^{1-\theta}_{L^{2_\alpha^*}(\mathcal{O})} \leq C\|u\|_{H^m(\mathcal{O},\sigma)}, \text{ for any } u \in H_0^m(\mathcal{O}, \sigma),$$

where $\theta = \frac{2(2_\alpha^* - p)}{p(2_\alpha^* - 2)}$. The proof is completed. □

2.2. Theory of Random Attractors

In this subsection, we introduce some definitions and known results about weak pullback mean random attractors and pullback random attractors.

Definition 1. *A family of mappings $\Phi = \{\Phi(t,\tau) : t \in \mathbb{R}^+, \tau \in \mathbb{R}\}$ is called a mean random dynamical system on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$ if the following conditions hold for all $\tau \in \mathbb{R}$ and $t, s \in \mathbb{R}^+$:*

(i) $\Phi(t,\tau)$ *maps* $L^p(\Omega, \mathcal{F}_\tau; X)$ *to* $L^p(\Omega, \mathcal{F}_{t+\tau}; X)$;
(ii) $\Phi(0,\tau)$ *is the identity operator on* $L^p(\Omega, \mathcal{F}_\tau; X)$;
(iii) $\Phi(t+s,\tau) = \Phi(t,\tau+s) \circ \Phi(s,\tau)$.

Let $\mathcal{D} = \{\mathcal{D}(\tau) \subseteq L^p(\Omega, \mathcal{F}_\tau; X) : \tau \in \mathbb{R}\}$ be a family of nonempty bounded sets and $\mathcal{D}_0$ be a collection of such families satisfying some conditions. The collection $\mathcal{D}_0$ is said to be inclusion-closed if $\mathcal{D} = \{\mathcal{D}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$, then every family $O = \{O(\tau) : O(\tau) \subseteq \mathcal{D}(\tau), \tau \in \mathbb{R}\} \in \mathcal{D}_0$.

Definition 2. *A family of sets $K = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$ is called a $\mathcal{D}_0$-pullback absorbing set for Φ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$ if for every $\tau \in \mathbb{R}$ and $\mathcal{D} \in \mathcal{D}_0$, there exists $T = T(\tau, \mathcal{D}) > 0$ such that*

$$\Phi(t, \tau - t)(\mathcal{D}(\tau - t)) \subseteq K(\tau), \quad \forall t \geq T.$$

Moreover, if $K(\tau)$ is a weakly compact nonempty subset of $L^p(\Omega, \mathcal{F}_\tau; X)$ for each $\tau \in \mathbb{R}$, then $K = \{K(\tau) : \tau \in \mathbb{R}\}$ is said to be a weakly compact $\mathcal{D}_0$-pullback absorbing set for Φ.

Definition 3. *A family of sets $K = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$ is said to be a $\mathcal{D}_0$-pullback weakly attracting set of Φ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$ if for each $\tau \in \mathbb{R}, \mathcal{D} \in \mathcal{D}_0$ and every weak neighborhood $\mathcal{N}^w(K(\tau))$ of $K(\tau)$ in $L^p(\Omega, \mathcal{F}_\tau; X)$, there exists some $T = T(\tau, \mathcal{D}, \mathcal{N}^w(K(\tau))) > 0$ such that*

$$\Phi(t, \tau - t)(\mathcal{D}(\tau - t)) \subseteq \mathcal{N}^w(K(\tau)), \quad \forall t \geq T.$$

Definition 4. *We say a family $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$ is a weak $\mathcal{D}_0$-pullback mean random attractor for Φ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$ if it satisfies the following properties:*

- *Weak compactness: for any $\tau \in \mathbb{R}$, $\mathcal{A}(\tau)$ is a weakly compact subset of $L^p(\Omega, \mathcal{F}_\tau; X)$.*
- *Pullback weak attraction: for any $\tau \in \mathbb{R}$, $\mathcal{A}(\tau)$ is a $\mathcal{D}_0$-pullback weakly attracting set of Φ.*
- *Minimality: for any $\tau \in \mathbb{R}$, the family $\mathcal{A}$ is the minimal element of $\mathcal{D}_0$ in the sense that if $B = \{B(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$ is another weakly compact $\mathcal{D}_0$-pullback weakly attracting set of Φ, then $\mathcal{A}(\tau) \subseteq B(\tau)$.*

The following result about the existence and uniqueness of weak $\mathcal{D}_0$-pullback mean random attractors for Φ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$ comes from [30].

Lemma 5. *Suppose that $\mathcal{D}_0$ is an inclusion-closed collection of some families of nonempty bounded subsets of $L^p(\Omega, \mathcal{F}; X)$ and Φ is a weak mean random dynamical system on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$. If Φ possesses a weakly compact $\mathcal{D}_0$-pullback absorbing set $K \in \mathcal{D}_0$ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$, then Φ possesses a unique weak $\mathcal{D}_0$-pullback mean random attractor $\mathcal{A} \in \mathcal{D}_0$ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$, which is given by*

$$\mathcal{A}(\tau) = \Omega^w(K, \tau) = \bigcap_{r \geq 0} \overline{\bigcup_{t \geq r} \Phi(t, \tau - t, K(\tau - t))}^{w}, \ \forall \tau \in \mathbb{R},$$

where the closure is taken with respect to the weak topology of $L^p(\Omega, \mathcal{F}_\tau; X)$.

Denote by $\mathcal{D} = \{\mathcal{D}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ a family of nonempty bounded subsets of X and $\mathcal{D}_1$ a collection of such families satisfying some conditions. Let $(\Omega, \mathcal{F}, \mathrm{P}, \{\theta_t\}_{t\in\mathbb{R}})$ be a metric dynamical system. We now introduce the pathwise random dynamical system as in [11,14,38].

Definition 5. *A mapping $\Psi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \mapsto X$ is said to be a continuous pathwise random dynamical system (or a continuous cocycle) on X over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ if the following conditions hold for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$,*

(i) $\Psi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \mapsto X$ *is* $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$*-measurable;*
(ii) $\Psi(0, \tau, \omega, \cdot)$ *is the identity operator on* X*;*
(iii) $\Psi(t + s, \tau, \omega, \cdot) = \Psi(t, \tau + s, \theta_s\omega, \cdot) \circ \Psi(s, \tau, , \omega, \cdot)$*;*
(iv) $\Psi(t, \tau, \omega, \cdot) : X \mapsto X$ *is continuous.*

Definition 6. *A family $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ is said to be a $\mathcal{D}_1$-pullback absorbing set for a cocycle Ψ if for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $\mathcal{D} \in \mathcal{D}_1$, there exists some $T = T(\tau, \mathcal{D}, \omega) > 0$ such that*

$$\Psi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \quad \text{for all} \quad t \geq T.$$

Moreover, if for every $\tau \in \mathbb{R}$, and $\omega \in \Omega$, $K(\tau, \omega)$ is a closed nonempty subset of X and is measurable in ω with respect to $\mathcal{F}$, then K is said to be a closed measurable $\mathcal{D}_1$-pullback absorbing set for Ψ.

Definition 7. *We say that cocycle Ψ is $\mathcal{D}_1$-pullback asymptotically compact in X if for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence*

$$\{\Psi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \quad \text{has a convergent subsequence in } X,$$

as $t_n \to +\infty$, and $x_n \in B(\tau - t_n, \theta_{-t_n}\omega)$ with $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$.

Definition 8. *A family $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ is said to be a $\mathcal{D}_1$-pullback random attractor for Ψ if the following properties hold for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$:*

(i) *Measurability and Compactness: $\mathcal{A}$ is measurable in ω with respect to $\mathcal{F}$ and $\mathcal{A}(\tau, \omega)$ is compact in X;*
(ii) *Invariance: $\mathcal{A}$ is invariant in the sense that $\Psi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_t\omega)$, $\forall t \geq 0$;*
(iii) *Pullback attracting: $\mathcal{A}$ attracts $\mathcal{D}_1$ in the sense that for any $\mathcal{D} \in \mathcal{D}_1$,*

$$\lim_{t\to+\infty} \mathrm{dist}_X(\Psi(t, \tau - t, \theta_{-t}\omega, \mathcal{D}(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0,$$

where dist_X is the Hausdorff semi-distance in X.

3. Mean Random Attractors for Stochastic Semi-linear Degenerate Parabolic Equation

Let U be a separable Hilbert space and $L_2(U, L^2(\mathcal{O}))$ be the Hilbert space consisting of all Hilbert–Schmidt operators from U to $L^2(\mathcal{O})$ with norm $\|\cdot\|_{L_2(U,L^2(\mathcal{O}))}$. We consider the following non-autonomous stochastic semi-linear degenerate parabolic equation defined on any bounded or unbounded domain $\mathcal{O} \subseteq \mathbb{R}^N$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{div}(\sigma(x)\nabla u) + \lambda u + f(x,u) = g(t,x) + h(t,u)\dfrac{dW}{dt}, & t > \tau, \\ u(\tau, x) = u_\tau(x), & \tau \in \mathbb{R}, \\ u(t,x)|_{\partial\mathcal{O}} = 0, & t > \tau, \end{cases} \tag{8}$$

where W is a two-sided U-valued cylindrical Wiener process defined on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$, while $\sigma(x)$, λ and $g(t,x)$ are the same as described in Section 1. In this section, the stochastic term in Equation (8) is understood in the sense of Itô integration. Since the Itô integral is martingale, it is convenient for us to take the expectation and find the existence of a weak pullback mean random attractor.

Let $\mathcal{O}$ be a bounded domain (or an unbounded domain) and let the non-negative function $\sigma(x)$ satisfy $(\mathcal{H}_\alpha)$ (or $(\mathcal{H}_\beta)$). We assume that $f : \mathcal{O} \times \mathbb{R} \mapsto \mathbb{R}$ is a smooth nonlinear function such that $f(x, 0) = 0$ and for all $x \in \mathcal{O}$ and $s \in \mathbb{R}$,

$$f'(x,s) \geq -\phi_1(x), \tag{9}$$

$$f(x,s)s \geq a_1|s|^p - \phi_2(x), \tag{10}$$

$$|f(x,s)| \leq a_2|s|^{p-1} + \phi_3(x), \tag{11}$$

where a_1, a_2, a_3, $p > 2$ are positive constants, and $\phi_1(x) \in L^\infty(\mathcal{O})$ with $\phi_1(x) \geq 0$, $\phi_2(x) \in L^1(\mathcal{O})$, $\phi_3(x) \in L^{p_1}(\mathcal{O})$ with $\frac{1}{p} + \frac{1}{p_1} = 1$, $f'(x,s)$ denotes the derivation with respect to the second variable s. We also assume $f(x,s)$ is locally Lipschitz continuous in u, i.e., for each bounded interval $I \subseteq \mathbb{R}$, there is $a_I > 0$ such that

$$|f(x,s_1) - f(x,s_2)| \leq a_I|s_1 - s_2|, \ \forall x \in \mathcal{O}, \ s_1, \ s_2 \in I. \tag{12}$$

Assume $h : \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \mapsto L_2(U, L^2(\mathcal{O}))$ satisfies the following conditions:

($\mathbf{A_1}$) For any $t \in \mathbb{R}$, $\omega \in \Omega$ and $s \in L^2(\mathcal{O})$, there are positive constants $a_3 < \frac{1}{2}\lambda$ and L such that

$$\|h(t,\omega,s)\|^2_{L_2(U,L^2(\mathcal{O}))} \leq a_3\|s\|^2 + L. \tag{13}$$

($\mathbf{A_2}$) For each $r > 0$, there is a positive constant a_r depending on r such that for every $t \in \mathbb{R}$, $\omega \in \Omega$, and s_1, $s_2 \in L^2(\mathcal{O})$ with $\|s_1\| \leq r$ and $\|s_2\| \leq r$,

$$\|h(t,\omega,s_1) - h(t,\omega,s_2)\|^2_{L_2(U,L^2(\mathcal{O}))} \leq a_r\|s_1 - s_2\|^2. \tag{14}$$

Moreover, we suppose that for each given $s \in L^2(\mathcal{O})$, $\sigma(\cdot,\cdot,s) : \mathbb{R} \times \Omega \mapsto L_2(U, L^2(\mathcal{O}))$ is progressively measurable.

We now show that the solution of Equation (8) can define a mean random dynamical system. The definition of the solution for Equation (8) is given as follows in this case.

Definition 9. *Let $u_\tau \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$ and $T > \tau$. A $L^2(\mathcal{O})$-valued $\mathcal{F}_t$-adapted stochastic process u is called a solution of* (8) *on $[\tau, T]$ with initial data u_τ if*

$$u \in L^2(\Omega, C([\tau,T]; L^2(\mathcal{O}))) \cap L^2(\Omega \times [\tau,T]; D_0^{1,2}(\mathcal{O},\sigma)) \cap L^p(\Omega \times [\tau,T]; L^p(\mathcal{O}))$$

and P-a.s. satisfies

$$(u(t),\zeta) + \int_\tau^t (\sigma(x)\nabla u, \nabla\zeta)ds + \lambda\int_\tau^t (u,\zeta)ds + \int_\tau^t\int_{\mathcal{O}} f(u)\zeta dxds = \int_\tau^t (g(s),\zeta)ds$$
$$+ \int_\tau^t (h(s,u)dW(s),\zeta), \ \forall t \in [\tau,T], \ \zeta \in D_0^{1,2}(\mathcal{O},\sigma) \cap L^p(\mathcal{O}).$$

Using Lemmas 3 and 4, we can obtain the following result in a similar way that has been used in [32].

Lemma 6. *Let $T > \tau$ and $u_\tau \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$. If conditions* (9)–(14) *hold, then there exists a unique solution to Equation* (8) *in the sense of Definition* 9*. Additionally,*

$$E(\sup_{t\in[\tau,T]} \|u(t)\|^2) < \infty. \tag{15}$$

Note that $u \in L^2(\Omega, C([\tau, T]; L^2(\mathcal{O})))$ for all $T > \tau$, which implies that $u \in C([\tau, \infty); L^2(\Omega, L^2(\mathcal{O})))$. Thus, we can define the mean random dynamical system Φ for Equation (8) on $L^2(\Omega, \mathcal{F}; L^2(\mathcal{O}))$ by

$$\Phi(t, \tau, u_\tau) = u(t + \tau, \tau, u_\tau), \quad t > 0, \tau \in \mathbb{R},$$

where $u_\tau \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$ and u is the solution of system (8) with initial data u_τ.

Let $\mathcal{D} = \{\mathcal{D}(\tau) \subseteq L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) : \tau \in \mathbb{R}\}$ be a family of nonempty bounded sets. A family $\mathcal{D}$ is said to be tempered if for any $\nu > 0$, there is

$$\lim_{\tau \to -\infty} \sup_{u \in D(\tau)} e^{\nu\tau} \|u\|^2 = 0. \tag{16}$$

We denote by $\mathcal{D}_0$ the collection of all tempered families of nonempty bounded subsets of $L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$, that is,

$$\mathcal{D}_0 = \{\mathcal{D} = \{\mathcal{D}(\tau) \subseteq L^p(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) : \mathcal{D}(\tau) \neq \emptyset, \text{ bounded}, \tau \in R\} : \mathcal{D} \text{ satisfies (16)}\}.$$

From now on, we assume

$$\int_{-\infty}^{\tau} e^{\lambda s} \|g(s, \cdot)\|^2 ds < +\infty, \quad \forall \tau \in \mathbb{R}. \tag{17}$$

To find the existence of tempered random attractors, we further assume

$$\lim_{\tau \to -\infty} e^{\nu\tau} \int_{-\infty}^{0} e^{\lambda s} \|g(s + \tau, \cdot)\|^2 ds = 0, \quad \forall \nu > 0. \tag{18}$$

To investigate the existence of weak $\mathcal{D}_0$-pullback mean random attractors for Equation (8), we need the uniform estimate of solutions, and by the following result, we can construct a weakly compact $\mathcal{D}_0$-pullback absorbing set for Φ.

We present a Gronwall-type lemma, which is a convenient tool for subsequential discussions. The reader may refer to [39] for the detailed proof.

Lemma 7. *Let $g(t)$ be an integrable function, and $f(t)$ be an absolutely continuous function that satisfies the differential inequality*

$$\frac{d}{dt} f(t) \leq k f(t) + g(t).$$

Then,

$$f(t) \leq f(\alpha) e^{k(t-\alpha)} + \int_{\alpha}^{t} g(s) e^{k(t-s)} ds. \tag{19}$$

Lemma 8. *Suppose (9)–(14) and (17) hold. Then, for every $\tau \in \mathbb{R}$ and $\mathcal{D} \in \mathcal{D}_0$, there exists some $T = T(\tau, \mathcal{D}) > 0$ such that for all $t \geq T$ and $u_{\tau - t} \in \mathcal{D}(\tau - t)$, the solution u to Equation (8) satisfies*

$$E(\|u(\tau, \tau - t, u_{\tau - t})\|^2) \leq M + M \int_{-\infty}^{0} e^{\lambda s} \|g(s + \tau)\|^2 ds, \tag{20}$$

where M is a positive constant independent of τ and $\mathcal{D}$.

Proof of Lemma 8 . By the Itô formula, we obtain from (8) that for each $r \geq \tau - t$,

$$
\begin{aligned}
&\|u(r,\tau-t,u_{\tau-t})\|^2+2\int_{\tau-t}^{r}\|u(s,\tau-t,u_{\tau-t})\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}ds+2\lambda\int_{\tau-t}^{r}\|u(s,\tau-t,u_{\tau-t})\|^2ds\\
&+2\int_{\tau-t}^{r}\int_{\mathcal{O}}f(x,u(s,\tau-t,u_{\tau-t}))u(s,\tau-t,u_{\tau-t})dxds\\
=&\|u_{\tau-t}\|^2+2\int_{\tau-t}^{r}(g(s),u(s,\tau-t,u_{\tau-t}))ds+\int_{\tau-t}^{r}\|h(s,u(s,\tau-t,u_{\tau-t}))\|^2_{L_2(U,L^2(\mathcal{O}))}ds\\
&+2\int_{\tau-t}^{r}(u(s,\tau-t,u_{\tau-t}),h(s,u(s,\tau-t,u_{\tau-t}))dW(s)),
\end{aligned}
\tag{21}
$$

Taking the expectation on both sides of (21), we find, for almost all $r\geq\tau-t$, that

$$
\begin{aligned}
&E(\|u(r,\tau-t,u_{\tau-t})\|^2)+2\int_{\tau-t}^{r}E(\|u(s,\tau-t,u_{\tau-t})\|^2_{D_0^{1,2}(\mathcal{O},\sigma)})ds\\
&+2\lambda\int_{\tau-t}^{r}E(\|u(s,\tau-t,u_{\tau-t})\|^2)ds\\
&+2\int_{\tau-t}^{r}E\Big(\int_{\mathcal{O}}f(x,u(s,\tau-t,u_{\tau-t}))u(s,\tau-t,u_{\tau-t})dx\Big)ds\\
=&E(\|u_{\tau-t}\|^2)+2\int_{\tau-t}^{r}E(g(s),u(s,\tau-t,u_{\tau-t}))ds\\
&+\int_{\tau-t}^{r}E(\|h(s,u(s,\tau-t,u_{\tau-t}))\|^2_{L_2(U,L^2(\mathcal{O}))})ds.
\end{aligned}
\tag{22}
$$

Thus, for almost all $r\geq\tau-t$, we have

$$
\begin{aligned}
&\frac{d}{dr}E(\|u(r,\tau-t,u_{\tau-t})\|^2)+2E(\|u(r,\tau-t,u_{\tau-t})\|^2_{D_0^{1,2}(\mathcal{O},\sigma)})\\
&+2\lambda E(\|u(r,\tau-t,u_{\tau-t})\|^2)+2E\Big(\int_{\mathcal{O}}f(x,u(r,\tau-t,u_{\tau-t}))u(r,\tau-t,u_{\tau-t})dx\Big)\\
=&\ 2E(g(r),u(r,\tau-t,u_{\tau-t}))\\
&+E(\|h(r,u(r,\tau-t,u_{\tau-t}))\|^2_{L_2(U,L^2(\mathcal{O}))}).
\end{aligned}
\tag{23}
$$

Now, we estimate each item on the right-hand side of (23). By (10), we find that

$$
\begin{aligned}
&\int_{\mathcal{O}}f(u(r,\tau-t,u_{\tau-t}))u(r,\tau-t,u_{\tau-t})dx\\
\geq&a_1\int_{\mathcal{O}}|u(r,\tau-t,u_{\tau-t})|^pdx-\|\phi_2\|_{L^1(\mathcal{O})},
\end{aligned}
\tag{24}
$$

which implies

$$
\begin{aligned}
&2E\Big(\int_{\mathcal{O}}f(u(r,\tau-t,u_{\tau-t}))u(r,\tau-t,u_{\tau-t})dx\Big)\\
\geq&2a_1E(\|u(r,\tau-t,u_{\tau-t})\|^p_{L^p(\mathcal{O})})-2\|\phi_2\|_{L^1(\mathcal{O})}.
\end{aligned}
\tag{25}
$$

Note that

$$
\begin{aligned}
&(g(r),u(r,\tau-t,u_{\tau-t}))\\
\leq&\frac{\lambda}{4}\|u(r,\tau-t,u_{\tau-t})\|^2+\frac{1}{\lambda}\|g(r)\|^2,
\end{aligned}
\tag{26}
$$

which implies that

$$
\begin{aligned}
&2E(g(r),u(r,\tau-t,u_{\tau-t}))\\
\leq&\frac{\lambda}{2}E(\|u(r,\tau-t,u_{\tau-t})\|^2)+\frac{2}{\lambda}\|g(r)\|^2.
\end{aligned}
\tag{27}
$$

We deduce from (13) and (23)–(27) that, for almost all $r \geq \tau - t$,

$$\frac{d}{dr}E(\|u(r,\tau-t,u_{\tau-t})\|^2) + \lambda E(\|u(r,\tau-t,u_{\tau-t})\|^2)$$
$$\leq \frac{2}{\lambda}\|g(r)\|^2 + 2\|\phi_2\|_{L^1(\mathcal{O})} + L. \tag{28}$$

Applying Gronwall's inequality to (28), we obtain

$$E(\|u(r,\tau-t,u_{\tau-t})\|^2)$$
$$\leq e^{\lambda(\tau-t-r)}E(\|u_{\tau-t})\|^2) + \frac{1}{\lambda}(2\|\phi_2\|_{L^1(\mathcal{O})} + L) + e^{-\lambda r}\int_{\tau-t}^{r} e^{\lambda s}\frac{2}{\lambda}\|g(s)\|^2 ds. \tag{29}$$

Then, we find that

$$E(\|u(\tau,\tau-t,u_{\tau-t})\|^2)$$
$$\leq e^{-\lambda t}E(\|u_{\tau-t}\|^2) + \frac{1}{\lambda}(2\|\phi_2\|_{L^1(\mathcal{O})} + L) + e^{-\lambda\tau}\frac{2}{\lambda}\int_{-\infty}^{\tau} e^{\lambda s}\|g(s)\|^2 ds. \tag{30}$$

Since $u_{\tau-t} \in \mathcal{D}(\tau-t)$ and $\mathcal{D} = \{\mathcal{D}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$, we obtain

$$e^{-\lambda\tau}e^{\lambda(\tau-t)}E(\|u_{\tau-t}\|^2) \to 0 \quad as \quad t \to +\infty.$$

Therefore, there exists $T = T(\tau, \mathcal{D}) > 0$ such that for all $t \geq T$,

$$e^{-\lambda t}E(\|u_{\tau-t}\|^2) \leq 1. \tag{31}$$

By (30) and (31), we find, for all $t \geq T$, that there exists some positive constant M independent of τ and $\mathcal{D}$ such that

$$E(\|u(\tau,\tau-t,u_{\tau-t})\|^2) \leq M + M\int_{-\infty}^{0} e^{\lambda s}\|g(s+\tau)\|^2 ds.$$

This completes the proof. □

Corollary 1. *Let (9)–(14), (17) and (18) hold. Then, the mean random dynamical system Φ for Equation (8) possesses a weakly compact $\mathcal{D}_0$-pullback absorbing set $K_0 = \{K_0(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$, which is given by*

$$K_0(\tau) = \{u \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) : E(\|u\|^2) \leq \mathcal{R}_0(\tau)\}, \tag{32}$$

where

$$\mathcal{R}_0(\tau) := M + M\int_{-\infty}^{0} e^{\lambda s}\|g(s+\tau)\|^2 ds \tag{33}$$

with M being the same constant as in Lemma 8.

Proof of Corollary 1. We know that for each $\tau \in \mathbb{R}$, $K_0(\tau)$ in (32) is a bounded and closed convex subset of $L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$, and therefore it is weakly compact in $L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$. Lemma 8 indicates that for every $\tau \in \mathbb{R}$ and $\mathcal{D} \in \mathcal{D}_0$, there exists $T = T(\tau, D) > 0$ such that

$$\Phi(t, \tau-t, D(\tau-t)) \subseteq K_0(\tau), \ \forall t \geq T. \tag{34}$$

In addition, from (18) and (33), we obtain for any $\nu > 0$

$$\lim_{\tau\to-\infty} \sup_{u\in K_0(\tau)} e^{\nu\tau}\|u\| = 0,$$

that is $K_0 \in \mathcal{D}_0$. Hence, K_0 is a weakly compact $\mathcal{D}_0$-pullback absorbing set for Φ. □

Theorem 1. *Suppose* (9)–(14), (17) *and* (18) *hold. Then, the mean random dynamical system* Φ *for problem* (8) *possesses a unique weak* $\mathcal{D}_0$*-pullback mean random attractor* $\bar{\mathcal{A}}_0 = \{\bar{\mathcal{A}}_0(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$ *in* $L^2(\Omega, \mathcal{F}; L^2(\mathcal{O}))$ *over* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$.

Proof of Theorem 1. From Lemma 5 and Corollary 1, we can easily find the existence and uniqueness of weak $\mathcal{D}_0$-pullback mean random attractor $\bar{\mathcal{A}}_0 \in \mathcal{D}_0$ of Φ for Equation (8). □

4. Wong–Zakai Approximations of Stochastic Semi-Linear Degenerate Parabolic Equation

In this section, we consider the following stochastic semi-linear degenerate parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{div}(\sigma(x)\nabla u) + \lambda u + f(x,u) = g(t,x) + h(t,x,u) \circ \dfrac{dW}{dt}, & t > \tau, \\ u(\tau, x) = u_\tau(x), & \tau \in \mathbb{R}, \\ u(x,t)|_{\partial\mathcal{O}} = 0, & t > \tau. \end{cases} \tag{35}$$

Here, $W = \omega(t)$ is a two-sided real-valued Wiener process on a probability space and the other terms are the same as described in Section 1. The symbol " $\circ$ " indicates that the stochastic term in Equation (35) is understood in the sense of Stratonovich's integration.

We remark that, in this section, we consider the stochastic term of Equation (35) in the sense of Stratonovich's integration because the Stratonovich's interpretation is more appropriate than Itô's when we consider the pathwise dynamical behavior (fixed any $\omega \in \Omega$) of the Wong–Zakai approximate system corresponding to the equation (see [40] for details).

4.1. Random Dynamical Systems for Wong–Zakai Approximations

In this subsection, we first define a continuous cocycle Ψ for Wong–Zakai approximate system of Equation (35), and then prove that there exists a unique pullback random attractor for the cocycle Ψ.

Let $\mathcal{O}$ be a bounded domain (or an unbounded domain) and let the non-negative function $\sigma(x)$ satisfy $(\mathcal{H}_\alpha)$ (or $(\mathcal{H}_\beta)$). In what follows, we assume that $f : \mathcal{O} \times \mathbb{R} \mapsto \mathbb{R}$ is a smooth nonlinear function such that for all $x \in \mathcal{O}$ and $s \in \mathbb{R}$,

$$f(x,s)s \geq \alpha_1|s|^p - \beta_1(x), \tag{36}$$

$$|f(x,s)| \leq \alpha_2|s|^{p-1} + \beta_2(x), \tag{37}$$

$$f'(x,s) \geq \alpha_3|s|^{p-2} - \beta_3(x), \tag{38}$$

where $p > 2$, α_1, α_2, α_3 are positive numbers, $\beta_1(x) \in L^1(\mathcal{O})$, $\beta_2(x) \in L^{p_1}(\mathcal{O})$ with $\frac{1}{p_1} + \frac{1}{p} = 1$, $\beta_3(x) \in L^\infty(\mathcal{O})$. Let h be a continuous function and for all $t, s \in \mathbb{R}$, $x \in \mathcal{O}$, satisfy

$$|h(t,x,s)| \leq \psi_1(t,x)|s|^{q-1} + \psi_2(t,x), \tag{39}$$

$$|\frac{\partial}{\partial s}h(t,x,s)| \leq \psi_3(t,x)|s|^{q-2} + \psi_4(t,x), \tag{40}$$

where $2 \leq q < p, \psi_1 \in L_{\mathrm{loc}}^{\frac{p}{p-q}}(\mathbb{R}; L_{\mathrm{loc}}^{\frac{p}{p-q}}(\mathcal{O}))$ and $\psi_2 \in L_{\mathrm{loc}}^{p_1}(\mathbb{R}; L^{p_1}(\mathcal{O}))$, and $\psi_3, \psi_4 \in L^\infty(\mathbb{R}; L^\infty(\mathcal{O}))$.

In the sequel, let $(\Omega, \mathcal{F}, \mathrm{P})$ be the classical Wiener probability space, where

$$\Omega = C_0(\mathbb{R}, \mathbb{R}) := \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\} \tag{41}$$

with the open compact topology. The Brownian motion has the form $W(t,\omega) = \omega(t)$. Consider the Wiener shift θ_t on the probability space $(\Omega, \mathcal{F}, P)$ defined by

$$\theta_t\omega(\cdot) = \omega(t+\cdot) - \omega(t). \tag{42}$$

Then, from [41], we find that $(\Omega, \mathcal{F}, \mathrm{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system and there exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\widetilde{\Omega} \subseteq \Omega$ of full P measure such that for each $\omega \in \widetilde{\Omega}$,

$$\frac{\omega(t)}{t} \to 0 \qquad \text{as} \qquad \mathrm{t} \to \pm\infty. \tag{43}$$

For brevity, we identify the space $\widetilde{\Omega}$ with Ω. For any given $\delta \neq 0$, define the random variable $\mathcal{G}_\delta$ by

$$\mathcal{G}_\delta(\omega) = \frac{\omega(\delta)}{\delta}, \qquad \forall\, \omega \in \Omega. \tag{44}$$

We obtain from (42) and (44) that

$$\mathcal{G}_\delta(\theta_t\omega) = \frac{\omega(t+\delta) - \omega(t)}{\delta} \quad \text{and} \quad \int_0^t \mathcal{G}_\delta(\theta_s\omega)ds = \int_t^{t+\delta} \frac{\omega(s)}{\delta}ds + \int_\delta^0 \frac{\omega(s)}{\delta}ds. \tag{45}$$

By the continuity of ω and (45), the following result has been proved in [26].

Lemma 9. *Let $\tau \in \mathbb{R}$, $T > 0$, and $\omega \in \Omega$. Then, for each $\epsilon > 0$, there is a constant $\delta' = \delta'(\epsilon, \tau, \omega, T) > 0$ such that for every $0 < |\delta| < \delta'$ and $t \in [\tau, \tau + T]$,*

$$|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds - \omega(t)| < \epsilon. \tag{46}$$

Let us consider the Wong–Zakai approximate system of Equation (35):

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\mathrm{div}(\sigma(x)\nabla u)) + \lambda u + f(x,u) = g(t,x) + h(t,x,u)\mathcal{G}_\delta(\theta_t\omega), & t > \tau, \\ u(\tau, x) = u_\tau(x), & \tau \in \mathbb{R}, \\ u(x,t)|_{\partial\mathcal{O}} = 0, & t > \tau. \end{cases} \tag{47}$$

Notice that system (47) can be viewed as a deterministic equation parameterized by $\omega \in \Omega$. Let assumptions (36)–(40) hold, and then by the Galerkin method similar to [1], we can prove that for any $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\mathcal{O})$, Equation (47) possesses a unique solution

$$u(\cdot, \tau, \omega, u_\tau) \in C([\tau,\infty); L^2(\mathcal{O})) \cap L^2_{\mathrm{loc}}((0,\infty); D_0^{1,2}(\mathcal{O},\sigma)) \cap L^p_{\mathrm{loc}}((0,\infty); L^p(\mathcal{O})). \tag{48}$$

In addition, the solution $u(\cdot, \tau, \omega, u_\tau)$ is continuous in $u_\tau \in L^2(\mathcal{O})$ and is $(\mathcal{F}, \mathcal{B}(L^2(\mathcal{O})))$-measurable in $\omega \in \Omega$. Hence, we can define a continuous cocycle $\Psi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \mapsto L^2(\mathcal{O})$ by

$$\Psi(t, \tau, \omega, u_\tau) = u(t+\tau, \tau, \theta_{-\tau}\omega, u_\tau), \quad \forall\ \tau \in \mathbb{R},\, t > 0,\, \omega \in \Omega,\, u_\tau \in L^2(\mathcal{O}). \tag{49}$$

Let $\mathcal{D}_1 = \{\mathcal{D}_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of $L^2(\mathcal{O})$. A family $\mathcal{D}_1$ is said to be tempered if for any $\nu > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there is

$$\lim_{t\to-\infty} \sup_{u\in D(\tau+t,\theta_t\omega)} e^{\nu t}\|u\| = 0.$$

We denote by $\mathcal{D}_1$ the class of all tempered families of nonempty bounded subsets of $L^2(\mathcal{O})$.

Now, we commit to proving the existence of $\mathcal{D}_1$-pullback random attractors for the cocycle Ψ corresponding to Equation (47) in $L^2(\mathcal{O})$.

Lemma 10. *Suppose* (17), (18) *and* (36)–(40) *hold. Then, the continuous cocycle* Ψ *of problem* (47) *possesses a closed measurable* $\mathcal{D}_1$*-pullback absorbing set* $K_1 = \{K_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$*, which is given by*

$$K_1(\tau,\omega) = \{u \in L^2(\mathcal{O}) : \|u\|^2 \leq R(\tau,\omega)\}, \tag{50}$$

where

$$R(\tau,\omega) = M_1 + M_1 \int_{-\infty}^{0} e^{\lambda s}(\|g(s+\tau)\|^2 + |\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1})ds \tag{51}$$

with M_1 *is a positive constant independent of* τ, ω *and* $\mathcal{D}_1$.

Proof of Lemma 10. We first prove that, for any given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $K_1(\tau,\omega)$ given by (50) is a pullback absorbing set for the cocycle Ψ. Taking the inner product of Equation (47) with u in $L^2(\mathcal{O})$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|u\|^2_{D_0^{1,2}(\mathcal{O},\sigma)} + \lambda\|u\|^2 + \int_{\mathcal{O}} f(x,u)udx \\ =&(g,u) + \mathcal{G}_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u)udx. \end{aligned} \tag{52}$$

By (36), we find that

$$\int_{\mathcal{O}} f(x,u)udx \geq \alpha_1 \int_{\mathcal{O}} |u|^p dx - \|\beta_1\|_{L^1(\mathcal{O})}. \tag{53}$$

By (39) and Young's inequality, we obtain

$$\begin{aligned} &\mathcal{G}_\delta(\theta_t\omega)\int_{\mathcal{O}} h(t,x,u)udx \\ \leq&|\mathcal{G}_\delta(\theta_t\omega)|\int_{\mathcal{O}}(|\psi_1(t,x)||u|^q + |\psi_2(t,x)||u|)dx \\ \leq&\frac{\alpha_1}{2}\int_{\mathcal{O}}|u|^p dx + C\int_{\mathcal{O}}|\psi_1(t,x)\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-q}}dx + C\int_{\mathcal{O}}|\psi_2(t,x)\mathcal{G}_\delta(\theta_t\omega)|^{p_1}dx \\ \leq&\frac{\alpha_1}{2}\int_{\mathcal{O}}|u|^p dx + C|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-q}}\|\psi_1(t)\|^{\frac{p}{p-q}}_{L^{\frac{p}{p-q}}} + C|\mathcal{G}_\delta(\theta_t\omega)|^{p_1}\|\psi_2(t)\|^{p_1}_{L^{p_1}} \\ \leq&\frac{\alpha_1}{2}\int_{\mathcal{O}}|u|^p dx + C|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-q}} + C|\mathcal{G}_\delta(\theta_t\omega)|^{p_1}. \end{aligned} \tag{54}$$

From Cauchy's inequality, we have

$$(g(t,x),u) \leq \frac{\lambda}{2}\|u\|^2 + \frac{1}{2\lambda}\|g\|^2. \tag{55}$$

Therefore, it follows easily from (52)–(55) that

$$\begin{aligned} &\frac{d}{ds}\|u\|^2 + 2\|u\|^2_{D_0^{1,2}(\mathcal{O},\sigma)} + \lambda\|u\|^2 + \alpha_1\|u\|^p_{L^p(\mathcal{O})} \\ \leq&2\|\beta_1\|_{L^1(\mathcal{O})} + \frac{1}{\lambda}\|g\|^2 + C|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + C|\mathcal{G}_\delta(\theta_s\omega)|^{p_1}. \end{aligned} \tag{56}$$

Multiplying (56) by $e^{\lambda s}$, replacing ω by $\theta_{-\tau}\omega$ and then integrating with respect to s over $(\tau - t, \tau)$ with $t \geq 0$, we find that

$$
\begin{aligned}
&\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+2\int_{\tau-t}^{\tau}e^{\lambda(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}ds\\
&+\alpha_1\int_{\tau-t}^{\tau}e^{\lambda(s-\tau)}\|u\|^p_{L^p}ds\\
\leq& e^{-\lambda t}\|u_{\tau-t}\|^2+\frac{2\|\beta_1\|_{L^1(\mathcal{O})}}{\lambda}+\frac{1}{\lambda}\int_{\tau-t}^{\tau}e^{\lambda(s-\tau)}\|g(s)\|^2ds\\
&+C\int_{\tau-t}^{\tau}e^{\lambda(s-\tau)}(|\mathcal{G}_\delta(\theta_{s-\tau}\omega)|^{\frac{p}{p-q}}+|\mathcal{G}_\delta(\theta_{s-\tau}\omega)|^{p_1})ds\\
\leq& e^{-\lambda t}\|u_{\tau-t}\|^2+\frac{2\|\beta_1\|_{L^1(\mathcal{O})}}{\lambda}+\frac{1}{\lambda}\int_{-\infty}^{0}e^{\lambda s}\|g(s+\tau)\|^2ds\\
&+C\int_{-\infty}^{0}e^{\lambda s}(|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}}+|\mathcal{G}_\delta(\theta_s\omega)|^{p_1})ds.
\end{aligned}\tag{57}
$$

The last two integrals in (57) are well-defined due to (17), (43), (45) and the continuity of ω. For every $u_{\tau-t}\in D_1(\tau-t,\theta_{-t}\omega)$ and $D_1=\{D_1(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}_1$, we have

$$\limsup_{t\to+\infty}e^{-\lambda t}\|u_{\tau-t}\|^2=0.\tag{58}$$

Hence, there exists some $T_1=T_1(\sigma,\tau,\omega,D_1)>0$ such that for all $t\geq T_1$,

$$
\begin{aligned}
&\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+2\int_{\tau-t}^{\tau}e^{\lambda(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}ds\\
&+\alpha_1\int_{\tau-t}^{\tau}e^{\lambda(s-\tau)}\|u\|^p_{L^p}ds\\
\leq&1+\frac{2\|\beta_1\|_{L^1(\mathcal{O})}}{\lambda}+\frac{1}{\lambda}\int_{-\infty}^{0}e^{\lambda s}\|g(s+\tau)\|^2ds+C\int_{-\infty}^{0}e^{\lambda s}(|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}}+|\mathcal{G}_\delta(\theta_s\omega)|^{p_1})ds\\
\leq&M_1+M_1\int_{-\infty}^{0}e^{\lambda s}(\|g(s+\tau)\|^2+|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}}+|\mathcal{G}_\delta(\theta_s\omega)|^{p_1})ds,
\end{aligned}\tag{59}
$$

where M_1 is a positive constant independent of τ, ω and D_1. Then, by (59), we find that, for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and every $D_1\in\mathcal{D}_1$, $K_1(\tau,\omega)$ given by (50) satisfies

$$\Psi(t,\tau-t,\theta_{-t}\omega,D_1(\tau-t,\theta_{-t}\omega))\subseteq K_1(\tau,\omega).$$

We next prove that $K_1\in\mathcal{D}_1$. Let ν be an arbitrary positive constant. Then, for each $\tau\in\mathbb{R}$ and $\omega\in\Omega$, we can find from (51) that

$$
\begin{aligned}
&e^{\nu t}\|K_1(\tau+t,\theta_t\omega)\|^2\leq e^{\nu t}R(\tau+t,\theta_t\omega)\\
=&M_1e^{\nu t}+M_1e^{\nu t}\int_{-\infty}^{0}e^{\lambda s}\Big(\|g(s+t+\tau)\|^2+|\mathcal{G}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}}+|\mathcal{G}_\delta(\theta_{s+t}\omega)|^{p_1}\Big)ds.
\end{aligned}\tag{60}
$$

First, we can find from (18) that

$$
\begin{aligned}
&\lim_{t\to-\infty}e^{\nu t}\int_{-\infty}^{0}e^{\lambda s}\|g(s+\tau+t)\|^2ds\\
=&\lim_{t\to-\infty}e^{-\lambda\tau}e^{\nu t}\int_{-\infty}^{\tau}e^{\lambda s}\|g(s+t)\|^2ds=0.
\end{aligned}\tag{61}
$$

Let $\tilde{\nu} = \min\{\lambda, \nu\}$, and then we can find from (43) and (45) that for any $t \leq 0$,

$$\begin{aligned} & e^{\nu t} \int_{-\infty}^{0} e^{\lambda s} (|\mathcal{G}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_{s+t}\omega)|^{p_1}) ds \\ \leq & \int_{-\infty}^{t} e^{\tilde{\nu} s} (|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1}) ds. \end{aligned} \tag{62}$$

Note that

$$\int_{-\infty}^{0} e^{\tilde{\nu} s} (|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1}) ds < +\infty,$$

which implies that

$$\int_{-\infty}^{t} e^{\tilde{\nu} s} (|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1}) ds \to 0 \text{ as } t \to -\infty. \tag{63}$$

It follows from (60)–(63) that K_1 is tempered, i.e., $K_1 \in \mathcal{D}_1$. Moreover, since for each $\tau \in \mathbb{R}$, $R(\tau, \cdot) : \Omega \mapsto \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable, then $K_1(\tau, \cdot)$ is also measurable. Hence, $K_1 \in \mathcal{D}_1$ is a closed measurable $\mathcal{D}_1$-pullback absorbing set for Ψ. The proof is completed. □

Lemma 11. *Let (36)–(40) hold. Then, for each $\tau \in \mathbb{R}$, $t > \tau$, $\omega \in \Omega$ and for each bounded sequence $\{u_{0,n}\}_{n=1}^{\infty} \subseteq L^2(\mathcal{O})$, the sequence $\{u(t, \tau, \omega, u_{0,n})\}_{n=1}^{\infty}$ possesses a convergent subsequence in $L^2(\mathcal{O})$.*

Proof of Lemma 11. Taking $T > t$, and integrating (56) over $[\tau, T]$, we can find that

$$\{u(\cdot, \tau, \omega, u_{0,n})\}_{n=1}^{\infty} \text{ is bounded in } L^p\big((\tau, T); L^p(\mathcal{O})\big) \cap L^2\big((\tau, T); D_0^{1,2}(\mathcal{O}, \sigma)\big). \tag{64}$$

We can also infer from (37), (39) and (64) that, for $s \in [\tau, T]$,

$$\begin{aligned} \{f(\cdot, u(\cdot, \tau, \omega, u_{0,n}))\}_{n=1}^{\infty} \text{ and } \{h(\cdot, \cdot, u(\cdot, \tau, \omega, u_{0,n}))\mathcal{G}_\delta(\theta_s\omega)\}_{n=1}^{\infty} \\ \text{are bounded in } L^{p_1}((\tau, T); L^{p_1}(\mathcal{O})). \end{aligned} \tag{65}$$

Then, it follows from (64), (65), and Equation (47), that

$$\{\frac{\partial}{\partial t} u(\cdot, \tau, \omega, u_{0,n})\}_{n=1}^{\infty} \text{ is bounded in } L^2\big((\tau, T); D_0^{-1,2}(\mathcal{O}, \sigma)\big) + L^{p_1}\big((\tau, T); L^{p_1}(\mathcal{O})\big). \tag{66}$$

By Lemma 3, we note that the embedding $D_0^{1,2}(\mathcal{O}, \sigma) \hookrightarrow L^2(\mathcal{O})$ is compact (in both cases of bounded and unbounded domain). Then, we can find from (64), (66) and the Aubin–Lions compactness lemma that there exist some $w \in L^2((\tau, T); L^2(\mathcal{O}))$ and a subsequence of $\{u(s, \tau, \omega, u_{0,n})\}_{n=1}^{\infty}$ such that

$$u(\cdot, \tau, \omega, u_{0,n_k}) \to w \text{ in } L^2((\tau, T); L^2(\mathcal{O})). \tag{67}$$

By choosing a further subsequence (re-labeled the same), we infer from (67) that

$$u(s, \tau, \omega, u_{0,n_k}) \to w(s) \text{ in } L^2(\mathcal{O}), \text{ a.e. } s \in [\tau, T]. \tag{68}$$

Finally, since $t \in (\tau, T)$, we can by the continuity of solutions on initial data in $L^2(\mathcal{O})$ and (68) obtain

$$u(t, \tau, \omega, u_{0,n_k}) = u(t, s, \omega, u(s, \tau, \omega, u_{0,n_k})) \to u(t, s, \omega, w(s)),$$

i.e., $u(t, \tau, \omega, u_{0,n})$ possesses a convergent subsequence in $L^2(\mathcal{O})$. We complete the proof. □

Lemma 12. *Suppose* (17) *and* (36)–(40) *hold. Then, the continuous cocycle* Ψ *for Equation* (47) *is* $\mathcal{D}_1$*-pullback asymptotically compact in* $L^2(\mathcal{O})$.

Proof of Lemma 12. For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D_1 = \{D_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, $t_n \to +\infty$ as $n \to \infty$ and $u_{0,n} \in D_1(\tau - t_n, \theta_{-t_n}\omega)$, we shall prove the sequence $\Psi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n})$ has a convergent subsequence in $L^2(\mathcal{O})$. Note that $t_n \to +\infty$ as $n \to \infty$ and $u_{0,n} \in D_1(\tau - t_n, \theta_{-t_n}\omega)$. We can find from Lemma 10 that there exist $N_1 = N_1(\tau, \omega, D_1) > 0$ such that for all $n \geq N_1$ that

$$\|u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})\| \leq C(\tau, \omega), \tag{69}$$

which implies that

$$\{u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})\}_{n=1}^{\infty} \text{ is bounded in } L^2(\mathcal{O}). \tag{70}$$

It follows from (70) and Lemma 11 that the sequence

$$\{u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})\}_{n=1}^{\infty} \text{ is precompact in } L^2(\mathcal{O}),$$

which along with $\Psi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n}) = u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})$, it implies the result. □

Theorem 2. *Suppose* (17), (18) *and* (36)–(40) *hold. Then, the continuous cocycle* Ψ *associated with system* (47) *possesses a unique* $\mathcal{D}_1$*-pullback random attractor* $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ *in* $L^2(\mathcal{O})$.

Proof of Theorem 2. From Lemmas 10 and 12 as well as ([27], Proposition 2.1), the existence of unique $\mathcal{D}_1$-pullback random attractor $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ follows. □

4.2. Stochastic Semi-Linear Degenerate Parabolic Equation Driven by linear Multiplicative Noise

In this subsection, we discuss the following stochastic semi-linear degenerate parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(\sigma(x)\nabla u) + \lambda u + f(x,u) = g(t,x) + u \circ \dfrac{dW}{dt}, & t > \tau, \\ u(\tau, x) = u_\tau(x), & \tau \in \mathbb{R}, \\ u(x,t)|_{\partial\mathcal{O}} = 0, & t > \tau, \end{cases} \tag{71}$$

and consider the following Wong–Zakai approximate system for Equation (71):

$$\begin{cases} \dfrac{\partial u_\delta}{\partial t} - \operatorname{div}(\sigma(x)\nabla u_\delta) + \lambda u_\delta + f(x,u_\delta) = g(t,x) + u_\delta \mathcal{G}_\delta(\theta_t\omega), & t > \tau, \\ u_\delta(\tau, x) = u_{\delta,\tau}(x), & \tau \in \mathbb{R}, \\ u_\delta(x,t)|_{\partial\mathcal{O}} = 0, & t > \tau. \end{cases} \tag{72}$$

We will investigate the relations between the solutions of Equations (71) and (72). To this end, we need to transform the stochastic Equation (71) into a pathwise deterministic one. Let

$$v(t, \tau, \omega) = e^{-\omega(t)} u(t, \tau, \omega), \tag{73}$$

with

$$v_\tau = e^{-\omega(\tau)} u_\tau.$$

Then, by (71) and (73), we obtain

$$\begin{cases} \dfrac{\partial v}{\partial t} + Av + \lambda v + e^{-\omega(t)} f(x,u) = e^{-\omega(t)} g(t,x), & t > \tau, \\ u(\tau,x) = u_\tau(x), & \tau \in \mathbb{R}, \\ u(x,t)|_{\partial\mathcal{O}} = 0, & t > \tau, \end{cases} \tag{74}$$

where $Av = -\text{div}(\sigma(x)\nabla v)$. We also introduce a similar transform for Equation (72) as we did for Equation (71). Let

$$v_\delta(t,\tau,\omega,v_{\delta,\tau}) = e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} u_\delta(t,\tau,\omega,u_{\delta,\tau}) \tag{75}$$

with

$$v_{\delta,\tau} = e^{-\int_0^\tau \mathcal{G}_\delta(\theta_r\omega)dr} u_{\delta,\tau}.$$

Then, we have

$$\begin{cases} \dfrac{\partial v_\delta}{\partial t} + Av_\delta + \lambda v_\delta + e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} f(x,u_\delta) = e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} g(t,x), & t > \tau, \\ v(\tau,x) = v_{\delta,\tau}(x), & \tau \in \mathbb{R}, \\ v(x,t)|_{\partial\mathcal{O}} = 0, & t > \tau. \end{cases} \tag{76}$$

For any $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $v_\tau \in L^2(\mathcal{O})$, let (36)–(38) hold. Then, by the classic Galerkin method, we can obtain the existence and uniqueness of solution $v(\cdot,\tau,\omega,v_\tau) \in C([\tau,\infty),L^2(\mathcal{O}))$ for system (74). In addition $v(\cdot,\tau,\omega,v_\tau)$ is continuous in $v_\tau \in L^2(\mathcal{O})$ and is $(\mathcal{F},\mathcal{B}(L^2(\mathcal{O})))$-measurable in $\omega \in \Omega$. Thus, we can define a continuous cocycle $\widetilde{\Psi}_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \mapsto L^2(\mathcal{O})$ for system (71) by

$$\widetilde{\Psi}_0(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau) = e^{\omega(t)-\omega(-\tau)} v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau). \tag{77}$$

Similarly, we can also define a continuous cocycle $\widetilde{\Psi}_\delta(t,\tau,\omega,u_{\delta,\tau})$ for system (72).

Lemma 13. *Assume* (17), (18) *and* (36)–(38) *hold. Then, the continuous cocycle* $\widetilde{\Psi}_0$ *for system* (71) *possesses a closed measurable* $\mathcal{D}_1$*-pullback absorbing set* $\widetilde{B}_0 = \{\widetilde{B}_0(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, *which is given by*

$$\widetilde{B}_0(\tau,\omega) = \{u \in L^2(\mathcal{O}) : \|u\|^2 \le \widetilde{R}_0(\tau,\omega)\}, \tag{78}$$

where

$$\widetilde{R}_0(\tau,\omega) = 4\int_{-\infty}^0 e^{\frac{3}{2}\lambda s - 2\omega(s)} \left(\frac{1}{\lambda}\|g(s+\tau)\|^2 + \|\beta_1\|_{L^1(\mathcal{O})}\right) ds. \tag{79}$$

Proof of Lemma 13. Taking the inner product of Equation (74) with $v(t,\tau,\omega) = e^{-\omega(t)} u(t,\tau,\omega)$, we have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|v\|^2_{D_0^{1,2}(\mathcal{O},\sigma)} + \lambda\|v\|^2 + e^{-\omega(t)} \int_{\mathcal{O}} f(x,u) v dx = e^{-\omega(t)}(g,v). \tag{80}$$

It follows from (36) and (73) that

$$e^{-\omega(t)} \int_{\mathcal{O}} f(x,u) v dx \geqslant \alpha_1 e^{-2\omega(t)} \|u\|^p_{L^p(\mathcal{O})} - \|\beta_1\|_{L^1(\mathcal{O})} e^{-2\omega(t)}. \tag{81}$$

By Cauchy's inequality, we obtain

$$e^{-\omega(t)}(g,v) \leqslant \frac{\lambda}{4}\|v\|^2 + \frac{1}{\lambda} e^{-2\omega(t)} \|g\|^2. \tag{82}$$

Then, combining (80)–(82), we have

$$\frac{d}{ds}\|v\|^2 + 2\|v\|^2_{D_0^{1,2}(\mathcal{O},\sigma)} + \frac{3}{2}\lambda\|v\|^2 + 2\alpha_1 e^{-2\omega(s)}\|u\|^p_{L^p(\mathcal{O})} \leqslant 2\|\beta_1\|_{L^1(\mathcal{O})}e^{-2\omega(s)} + \frac{2}{\lambda}e^{-2\omega(s)}\|g\|^2. \tag{83}$$

Multiplying $e^{\frac{3}{2}\lambda s}$ on both sides of (83) and then integrating with respect to s over $[\tau - t, \tau]$ with $t > 0$, we obtain

$$\begin{aligned}&\|v(\tau,\tau-t,\omega,v_{\tau-t}\|^2 + 2\int_{\tau-t}^{\tau} e^{\frac{3}{2}\lambda(s-\tau)}\|v\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}ds + 2\alpha_1\int_{\tau-t}^{\tau} e^{\frac{3}{2}\lambda(s-\tau)-2\omega(s)}\|u\|^p_{L^p(\mathcal{O})}ds \\ &\leqslant 2\|\beta_1\|_{L^1(\mathcal{O})}\int_{\tau-t}^{\tau} e^{\frac{3}{2}\lambda(s-\tau)-2\omega(s)}ds + \frac{2}{\lambda}\int_{\tau-t}^{\tau} e^{\frac{3}{2}\lambda(s-\tau)-2\omega(s)}\|g(s)\|^2ds + \|v_{\tau-t}\|^2 e^{-\frac{3}{2}\lambda t}.\end{aligned} \tag{84}$$

Replacing ω in (84) by $\theta_{-\tau}\omega$ and using

$$u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}) = e^{(\omega(-\tau+s)-\omega(-\tau))}v(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}), \tag{85}$$

we find that

$$\begin{aligned}&\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 e^{2\omega(-\tau)} + 2\alpha_1\int_{\tau-t}^{\tau} e^{\frac{3}{2}\lambda(s-\tau)-2(\omega(-\tau+s)-\omega(-\tau))}\|u\|^p_{L^p(\mathcal{O})}ds \\ &+ 2\int_{\tau-t}^{\tau} e^{\frac{3}{2}\lambda(s-\tau)}e^{-2\omega(s-\tau)+2\omega(-\tau)}\|u\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}ds \\ &\leq 2\|\beta_1\|_{L^1(\mathcal{O})}\int_{\tau-t}^{\tau} e^{\frac{3}{2}\lambda(s-\tau)-2(\omega(-\tau+s)-\omega(-\tau))}ds + e^{2\omega(-\tau)-2\omega(-t)}\|u_{\tau-t}\|^2 e^{-\frac{3}{2}\lambda t} \\ &+ \frac{2}{\lambda}\int_{\tau-t}^{\tau} e^{\frac{3}{2}\lambda(s-\tau)-2(\omega(-\tau+s)-\omega(-\tau))}\|g(s)\|^2ds.\end{aligned} \tag{86}$$

Then, from (86), we obtain

$$\begin{aligned}&\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 + 2\alpha_1\int_{-\infty}^{0} e^{\frac{3}{2}\lambda s-2\omega(s)}\|u\|^p_{L^p(\mathcal{O})}ds \\ &+ 2\int_{-\infty}^{0} e^{\frac{3}{2}\lambda s}e^{-2\omega(s)}\|u\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}ds \\ &\leq 2\|\beta_1\|_{L^1(\mathcal{O})}\int_{-\infty}^{0} e^{\frac{3}{2}\lambda s-2\omega(s)}ds + \frac{2}{\lambda}\int_{-\infty}^{0} e^{\frac{3}{2}\lambda s-2\omega(s)}\|g(s+\tau)\|^2ds \\ &+ e^{-2\omega(-t)}e^{-\frac{3}{2}\lambda t}\|u_{\tau-t}\|^2.\end{aligned} \tag{87}$$

By (17) and (43), we have

$$2\int_{-\infty}^{0} e^{\frac{3}{2}\lambda s-2\omega(s)}(\frac{1}{\lambda}\|g(s+\tau)\|^2 + \|\beta_1\|_{L^1(\mathcal{O})})ds < \infty. \tag{88}$$

Note that if $u_{\tau-t} \in D_1(\tau-t,\theta_{-t}\omega)$ and $D_1 \in \mathcal{D}_1$, then by (43) we have

$$\limsup_{t\to+\infty} e^{-2\omega(-t)}e^{-\frac{3}{2}\lambda t}\|u_{\tau-t}\|^2 = 0. \tag{89}$$

Then, there exists some $T_4 = T_4(\tau,\omega,D_1) > 0$ such that for all $t \geq T_4$,

$$e^{-2\omega(-t)}e^{-\frac{3}{2}\lambda t}\|u_{\tau-t}\|^2 \leq 2\int_{-\infty}^{0} e^{\frac{3}{2}\lambda s-2\omega(s)}(\frac{1}{\lambda}\|g(s+\tau)\|^2 + \|\beta_1\|_{L^1(\mathcal{O})})ds, \tag{90}$$

which along with (77) implies that

$$\widetilde{\Psi}_0(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega)) \subseteq \widetilde{B}_0(\tau,\omega),\ \forall t \geq T_4,$$

where $\widetilde{B}_0(\tau,\omega)$ is given by (78). In addition, by (18), (43) and the continuity of $\omega(t)$, we can easily find that $\widetilde{B}_0$ is tempered, that is, $\widetilde{B}_0 \in \mathcal{D}_1$. Hence, $\widetilde{B}_0 \in \mathcal{D}_1$ is a closed measurable $\mathcal{D}_1$-pullback absorbing set for $\widetilde{\Psi}_0$. The proof is completed. □

Theorem 3. *Suppose* (17), (18) *and* (36)–(38) *hold. Then, the continuous cocycle $\widetilde{\Psi}_0$ for system* (71) *is $\mathcal{D}_1$-pullback asymptotically compact and possesses a unique $\mathcal{D}_1$-pullback random attractor $\widetilde{\mathcal{A}}_0 = \{\widetilde{\mathcal{A}}_0(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ in $L^2(\mathcal{O})$.*

Proof of Theorem 3. The proof of $\mathcal{D}_1$-pullback asymptotical compactness of cocycle Ψ_0 in $L^2(\mathcal{O})$ is similar to that of Lemma 12. And then by ([27], Proposition 2.1) and Lemma 13, we can easily find that the cocycle Ψ_0 possesses a unique $\mathcal{D}_1$-pullback random attractor $\mathcal{A}_0$. □

Lemma 14. *Suppose* (17), (18) *and* (36)–(38) *hold. Then, the continuous cocycle $\widetilde{\Psi}_\delta$ for Equation* (72) *possesses a closed measurable $\mathcal{D}_1$-pullback absorbing set $\widetilde{B}_\delta = \{\widetilde{B}_\delta(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$,*

$$\widetilde{B}_\delta(\tau,\omega) = \{u_\delta \in L^2(\mathcal{O}) : \|u_\delta\|^2 \le \widetilde{R}_\delta(\tau,\omega)\}, \tag{91}$$

where

$$\widetilde{R}_\delta(\tau,\omega) = 4\int_{-\infty}^{0} e^{\frac{3}{2}\lambda s} e^{2\int_s^0 \mathcal{G}_\delta(\theta_r\omega)dr}\Big(\frac{1}{\lambda}\|g(s+\tau)\|^2 + \|\beta_1\|_{L^1(\mathcal{O})}\Big)ds. \tag{92}$$

In addition, we have for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$

$$\lim_{\delta\to 0} \widetilde{R}_\delta(\tau,\omega) = \widetilde{R}_0(\tau,\omega), \tag{93}$$

where $\widetilde{R}_0(\tau,\omega)$ is given by (79).

Proof of Lemma 14. By (76), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_\delta\|^2 + \|v_\delta\|^2_{D_0^{1,2}(\mathcal{O},\sigma)} + \lambda\|v_\delta\|^2 + \int_{\mathcal{O}} e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} f(x,u_\delta)v_\delta dx \\ = e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}(g,v_\delta).\end{aligned} \tag{94}$$

By (36) and (75), we obtain

$$\begin{aligned}&-\int_{\mathcal{O}} e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} f(x,u_\delta)v_\delta dx \\ &\le -\alpha_1 e^{-2\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}\|u_\delta\|^p_{L^p(\mathcal{O})} + \|\beta_1\|_{L^1(\mathcal{O})} e^{-2\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}.\end{aligned} \tag{95}$$

By Cauchy's inequality, we obtain

$$e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}(g,v_\delta) \le \frac{\lambda}{4}\|v_\delta\|^2 + \frac{1}{\lambda}e^{-2\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}\|g\|^2. \tag{96}$$

Then, it follows from (94)–(96) that

$$\begin{aligned}\frac{d}{ds}\|v_\delta\|^2 + 2\|v_\delta\|^2_{D_0^{1,2}(\mathcal{O},\sigma)} + \frac{3}{2}\lambda\|v_\delta\|^2 + 2\alpha_1 e^{-2\int_0^s \mathcal{G}_\delta(\theta_r\omega)dr}\|u_\delta\|^p_{L^p(\mathcal{O})} \\ \le 2e^{-2\int_0^s \mathcal{G}_\delta(\theta_r\omega)dr}\Big(\frac{1}{\lambda}\|g\|^2 + \|\beta_1\|_{L^1(\mathcal{O})}\Big).\end{aligned} \tag{97}$$

For all $\tau \in \mathbb{R}, t \in \mathbb{R}^+$ and $\omega \in \Omega$, multiplying $e^{\frac{3}{2}\lambda s}$ and then integrating with respect to s from $\tau - t$ to τ, we have

$$\begin{aligned}&\|v_\delta(\tau,\tau-t,\omega,v_{\delta,\tau-t})\|^2+2\int_{\tau-t}^{\tau}e^{\frac{3}{2}\lambda(s-\tau)}\|v_\delta\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}ds\\&+2\alpha_1\int_{\tau-t}^{\tau}e^{\frac{3}{2}\lambda(s-\tau)}e^{-2\int_0^s\mathcal{G}_\delta(\theta_r\omega)dr}\|u_\delta\|^p_{L^p(\mathcal{O})}ds\\&\le e^{-\frac{3}{2}\lambda t}\|v_{\delta,\tau-t}\|^2+2\int_{\tau-t}^{\tau}e^{\frac{3}{2}\lambda(s-\tau)}e^{-2\int_0^s\mathcal{G}_\delta(\theta_r\omega)dr}(\frac{1}{\lambda}\|g(s)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})ds.\end{aligned}\tag{98}$$

Replacing ω in (98) by $\theta_{-\tau}\omega$, we obtain

$$\begin{aligned}&\|v_\delta(\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2+2\int_{\tau-t}^{\tau}e^{\frac{3}{2}\lambda(s-\tau)}\|v_\delta\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}ds\\&+2\alpha_1\int_{\tau-t}^{\tau}e^{\frac{3}{2}\lambda(s-\tau)}e^{-2\int_0^s\mathcal{G}_\delta(\theta_{r-\tau}\omega)dr}\|u_\delta\|^p_{L^p(\mathcal{O})}ds\\&\le e^{-\frac{3}{2}\lambda t}\|v_{\delta,\tau-t}\|^2+2\int_{\tau-t}^{\tau}e^{\frac{3}{2}\lambda(s-\tau)}e^{-2\int_0^s\mathcal{G}_\delta(\theta_{r-\tau}\omega)dr}(\frac{1}{\lambda}\|g(s)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})ds.\end{aligned}\tag{99}$$

By (75) and (99) we obtain

$$\begin{aligned}&\|u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2\\&\le e^{-\frac{3}{2}\lambda t}e^{2\int_{\tau-t}^{\tau}\mathcal{G}_\delta(\theta_{r-\tau}\omega)dr}\|u_{\delta,\tau-t}\|^2\\&\quad+2\int_{\tau-t}^{\tau}e^{\frac{3}{2}\lambda(s-\tau)}e^{2\int_s^{\tau}\mathcal{G}_\delta(\theta_{r-\tau}\omega)dr}(\frac{1}{\lambda}\|g(s)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})ds\\&\le e^{-\frac{3}{2}\lambda t}e^{2\int_{-t}^{0}\mathcal{G}_\delta(\theta_r\omega)dr}\|u_{\delta,\tau-t}\|^2\\&\quad+2\int_{-\infty}^{0}e^{\frac{3}{2}\lambda s+2\int_s^0\mathcal{G}_\delta(\theta_r\omega)dr}(\frac{1}{\lambda}\|g(s+\tau)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})ds.\end{aligned}\tag{100}$$

By (17), (43) and (45) and the continuity of $\omega(t)$, we obtain

$$2\int_{-\infty}^{0}e^{\frac{3}{2}\lambda s+2\int_s^0\mathcal{G}_\delta(\theta_r\omega)dr}(\frac{1}{\lambda}\|g(s+\tau)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})ds<\infty.\tag{101}$$

Note that if $u_{\delta,\tau-t}\in D_1(\tau-t,\theta_{-t}\omega)$ and $D_1\in\mathcal{D}_1$, then by (43), (45) and the continuity of $\omega(t)$, we obtain

$$\limsup_{t\to+\infty}e^{-\frac{3}{2}\lambda t}e^{2\int_{-t}^{0}\mathcal{G}_\delta(\theta_r\omega)dr}\|u_{\delta,\tau-t}\|^2=0,\tag{102}$$

which implies that there exists $T_5=T_5(\tau,\omega,D_1,\delta)>0$ such that for all $t\ge T_5$,

$$\begin{aligned}&e^{-\frac{3}{2}\lambda t}e^{2\int_{-t}^{0}\mathcal{G}_\delta(\theta_r\omega)dr}\|u_{\delta,\tau-t}\|^2\\&\le 2\int_{-\infty}^{0}e^{\frac{3}{2}\lambda s+2\int_s^0\mathcal{G}_\delta(\theta_r\omega)dr}(\frac{1}{\lambda}\|g(s+\tau)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})ds.\end{aligned}\tag{103}$$

By (100)–(103), we obtain

$$\|u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2\le 4\int_{-\infty}^{0}e^{\frac{3}{2}\lambda s+2\int_s^0\mathcal{G}_\delta(\theta_r\omega)dr}(\frac{1}{\lambda}\|g(s+\tau)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})ds.\tag{104}$$

In other words, we obtain for all $t\ge T_5$,

$$u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-t}\omega))\subseteq\widetilde{B}_\delta(\tau,\omega),\tag{105}$$

where $\widetilde{B}_\delta(\tau,\omega)$ is given by (91). In addition, $\widetilde{B}_\delta$ is tempered due to (18), (43) and (45). Therefore, $\widetilde{B}_\delta$ is a closed measurable $\mathcal{D}_1$-pullback absorbing set of Ψ_δ. The proof of (93) is similar to that of ([26], Lemma 3.7) and the details are omitted here. □

Theorem 4. *Suppose (17), (18) and (36)–(38) hold. Then, the cotinuous cocycle $\widetilde{\Psi}_\delta$ for Equation (72) is $\mathcal{D}_1$-pullback asymptotically compact and possesses a unique $\mathcal{D}_1$-pullback random attractor $\widetilde{\mathcal{A}}_\delta = \{\widetilde{\mathcal{A}}_\delta(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ in $L^2(\mathcal{O})$.*

Proof of Theorem 4. As similar to Lemma 12, one can prove that the cocycle $\widetilde{\Psi}_\delta$ in $L^2(\mathcal{O})$ is $\mathcal{D}_1$-pullback asymptotical compactness. And then by Lemma 14, we find the cocycle $\widetilde{\Psi}_\delta$ satisfies all conditions of ([27], Proposition 2.1), so the cocycle $\widetilde{\Psi}_\delta$ possesses a unique $\mathcal{D}_1$-pullback random attractor $\widetilde{\mathcal{A}}_\delta$. □

Now, we show that the solution of Equation (72) converges to the solution of Equation (71) as $\delta \to 0$. Toward this end, we further assume the following assumption holds: there exists some $\alpha_4 > 0$ such that for all $x \in \mathcal{O}$, $s \in \mathbb{R}$,

$$|f'(x,s)| \le \alpha_4(1 + |s|^{p-2}). \tag{106}$$

Lemma 15. *Suppose (17), (18) and (36)–(38) hold. Let u and u_δ be the solutions of Equation (71) and Equation (72), respectively, with initial data u_τ and $u_{\delta,\tau}$. If $u_{\delta,\tau} \to u_\tau$ in $L^2(\mathcal{O})$ as $\delta \to 0$, then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$, there exists some $\tilde{\delta}_0 = \tilde{\delta}_0(\tau,\omega,T) > 0$ such that for any $0 < |\delta| < \tilde{\delta}_0$ and $t \in [\tau,\tau+T]$, $u_\delta(t,\tau,\omega,u_{\delta,\tau}) \to u(t,\tau,\omega,u_\tau)$ in $L^2(\mathcal{O})$.*

Proof of Lemma 15. Let $\xi = v_\delta - v$ and then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\xi\|^2 + \|\xi\|^2_{D_0^{1,2}(\mathcal{O},\sigma)} + \lambda\|\xi\|^2 \\
&= \int_{\mathcal{O}} (e^{-\omega(t)} f(x,u) - e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} f(x,u_\delta))\xi dx + (e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} - e^{-\omega(t)})(g(t),\xi).
\end{aligned} \tag{107}$$

By using (37)–(38) and (106), we have

$$\begin{aligned}
&(e^{-\omega(t)} f(x,u) - e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} f(x,u_\delta))\xi \\
=&\big(e^{-\omega(t)} f(x,e^{\omega(t)}v) - e^{-\omega(t)} f(x,v_\delta e^{\omega(t)})\big)\xi + \big(e^{-\omega(t)} f(x,v_\delta e^{\omega(t)}) - e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} f(x,v_\delta e^{\omega(t)})\big)\xi \\
&+ \big(e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} f(x,v_\delta e^{\omega(t)}) - e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} f(x,v_\delta e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr})\big)\xi \\
=&f'(x,e^{\omega(t)}v + \theta_1 e^{\omega(t)}v_\delta)(e^{-\omega(t)}(ve^{\omega(t)} - v_\delta e^{\omega(t)}))\xi + f(x,v_\delta e^{\omega(t)})(e^{-\omega(t)} - e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr})\xi \\
&+ e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} v_\delta(e^{\omega(t)} - e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}) f'(x,e^{\omega(t)}v_\delta + \theta_2 v_\delta e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr})\xi \\
=&-f'(x,e^{\omega(t)}v + \theta_1 e^{\omega(t)}v_\delta)\xi^2 + f(x,v_\delta e^{\omega(t)})\xi(e^{-\omega(t)} - e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}) \\
&+ v_\delta(e^{\omega(t)-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} - 1) f'(x,e^{\omega(t)}v_\delta + \theta_2 v_\delta e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr})\xi \\
\le& \mid \beta_3 \mid |\xi|^2 + \big(\alpha_2 e^{(p-1)\omega(t)}|v_\delta|^{p-1}|\xi| + |\beta_2||\xi|\big)\big|e^{-\omega(t)} - e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}\big| \\
&+ \alpha_4\Big|1 - e^{\omega(t)-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}\Big|\Big(|v_\delta|^{p-1}\Big|e^{\omega(t)} + e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}\Big|^{p-2}|\xi| + |v_\delta||\xi|\Big),
\end{aligned} \tag{108}$$

where $\theta_1, \theta_2 \in (0,1)$. From Lemma 9, we find that for any $\epsilon > 0$, there exists some $\tilde{\delta}_1 = \tilde{\delta}_1(\epsilon,\tau,\omega,T) > 0$ such that

$$|1 - e^{\omega(t)-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}| < \epsilon,\ |e^{-\omega(t)} - e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}| < \epsilon,\ \forall 0 < |\delta| < \tilde{\delta}_1,\ t \in [\tau,\tau+T]. \tag{109}$$

It follows from (108) and (109) that

$$\int_{\mathcal{O}} (e^{-\omega(t)} f(x,u) - e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} f(x,u_\delta))\xi dx \le C\|\xi\|^2 + C\epsilon(\|v_\delta\|^p_{L^p(\mathcal{O})} + \|v\|^p_{L^p(\mathcal{O})} + 1). \tag{110}$$

By Cauchy's inequality, we have

$$(e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} - e^{-\omega(t)})(g(t), \xi) \leq |e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} - e^{-\omega(t)}|(\frac{1}{2}\|g\|^2 + \frac{1}{2}\|\xi\|^2). \tag{111}$$

Combing (107)–(111), we obtain

$$\frac{d}{dt}\|\xi\|^2 \leq C\|\xi\|^2 + C\epsilon(\|v_\delta\|^p_{L^p(\mathcal{O})} + \|v\|^p_{L^p(\mathcal{O})} + \|g\|^2 + 1). \tag{112}$$

Applying Gronwall's inequality to (112), we find for all $0 < |\delta| < \tilde{\delta}_1$ and $t \in [\tau, \tau + T]$,

$$\begin{aligned} \|\xi(t)\|^2 \leq & e^{C(t-\tau)}\|\xi(\tau)\|^2 + C\epsilon e^{C(t-\tau)} \int_\tau^t \Big(1 + \|v_\delta(s,\tau,\omega,v_{\delta,\tau})\|^p_{L^p(\mathcal{O})} \\ & + \|v(s,\tau,\omega,v_\tau)\|^p_{L^p(\mathcal{O})} + \|g(s)\|^2\Big)ds. \end{aligned} \tag{113}$$

By (73), (75), (84), (98) and (113), we find that there exists some $\tilde{\delta}_2 \in (0, \tilde{\delta}_1)$ and $\tilde{c_1} = \tilde{c}_1(\tau, T, \omega) > 0$ such that for all $0 < |\delta| < \tilde{\delta}_2$ and $t \in [\tau, \tau + T]$,

$$\begin{aligned} & \|v_\delta(t,\tau,\omega,v_{\delta,\tau}) - v(t,\tau,\omega,v_\tau)\|^2 \\ \leq & e^{\tilde{c}_1(t-\tau)}\|v_{\delta,\tau} - v_\tau\|^2 + \tilde{c}_1\epsilon e^{\tilde{c}_1(t-\tau)}\Big(1 + \|v_\tau\|^2 + \|v_{\delta,\tau}\|^2 + \int_\tau^t \|g(s)\|^2 ds\Big). \end{aligned} \tag{114}$$

Using (73) and (75) again, we obtain

$$\begin{aligned} & \|u_\delta(t,\tau,\omega,u_{\delta,\tau}) - u(t,\tau,\omega,u_\tau)\| \\ \leq & \|v_\delta(t,\tau,\omega,v_{\delta,\tau}) - v(t,\tau,\omega,v_\tau)\|\Big|e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}\Big| + \Big|e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} - e^{\omega(t)}\Big|\|v(t,\tau,\omega,v_\tau)\|. \end{aligned} \tag{115}$$

Note that $u_{\delta,\tau} = v_{\delta,\tau}e^{\int_0^\tau \mathcal{G}_\delta(\theta_r\omega)dr}$ and $u_\tau = v_\tau e^{\omega(\tau)}$. Then by the continuity of $\omega(t)$, (46), (84), (114) and (115), we can obtain the desired convergence. □

Lemma 16. *Suppose* (17), (18) *and* (36)–(38) *hold. For any given* $\tau \in \mathbb{R}$, $T > 0$ *and* $\omega \in \Omega$, *if* $\delta_n \to 0$ *and* $u_n \in \widetilde{\mathcal{A}}_{\delta_n}(\tau, \omega)$, *then the sequence* $\{u_n\}_{n=1}^\infty$ *has a convergent subsequence in* $L^2(\mathcal{O})$.

Proof of Lemma 16. By using Lemma 3 and a similar method as that of Lemma 3.10 in [26], we can obtain the result. □

Theorem 5. *Suppose* (17), (18), (36) *and* (37) *hold. Then, for any given* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *the following relationship holds:*

$$\lim_{\delta\to 0} \mathrm{d}_{L^2(\mathcal{O})}(\widetilde{\mathcal{A}}_\delta(\tau,\omega), \widetilde{\mathcal{A}}_0(\tau,\omega)) = 0.$$

Proof of Theorem 5. By Lemmas 13 and 14, we find that, for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\delta\to 0}\|\widetilde{B}_\delta(\tau,\omega)\|^2 = \|\widetilde{B}_0(\tau,\omega)\|^2 \leq \widetilde{B}_0(\tau,\omega),$$

where $\widetilde{B}_0(\tau,\omega)$ is given by (79) and $\widetilde{B}_0 = \{\widetilde{B}_0(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$. Let $\delta \to 0$ and $u_\delta \to u_\tau$, and then from Lemma 15, we find, for every $\tau \in \mathbb{R}$, $t \in \mathbb{R}^+$, and $\omega \in \Omega$, that $\Psi_\delta(\tau,t,\omega,u_{\delta,\tau}) \to \Psi_0(\tau,t,\omega,u_\tau)$ in $L^2(\mathcal{O})$. Then, by Lemma 16 and Theorem 3.1 in [21], we can obtain the result. □

5. Conclusions

In this paper, the long-term dynamical behavior of a class of random semilinear degenerate parabolic equations driven by nonlinear noise over bounded or unbounded regions is studied. Using the theory established by Kloeden et al. and Wang, we prove the existence and uniqueness of the weak pullback mean random attractor of this equation, and prove the existence and uniqueness of the pullback random attractor of the Wong–Zakai approximation system. In addition, the upper semicontinuity of the pullback random attractor of the Wong–Zakai approximation system of the equation driven by linear multiplicative noise is established. In the future, we will investigate how to discretize the system for numerical simulation while the discretized system still retains the dynamics of the original system so that it can be applied to practical problems.

Author Contributions: Validation, Y.L.; Writing—original draft, X.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by "the Fundamental Research Funds for the Central Universities" and NSFC (Grant 12271141).

Conflicts of Interest: The authors declare no conflict of interest.

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