mathematics

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A New Approach to Soft Continuity

Sandeep Kaur [1], Tareq M. Al-shami [2,3,*], Alkan Özkan [4] and M. Hosny [5]

[1] Department of Mathematics, Akal University, Talwandi Sabo, Bathinda 151302, India
[2] Department of Mathematics, Sana'a University, Sana'a P.O. Box 1247, Yemen
[3] Department of Engineering Mathematics & Physics, Faculty of Engineering & Technology, Future University, New Cairo 11835, Egypt
[4] Department of Mathematics, Faculty of Arts and Sciences, Iğdır University, Iğdır 18300, Turkey
[5] Department of Mathematics, Faculty of Science for Girls, King Khalid University, Abha 61471, Saudi Arabia
* Correspondence: t.alshami@su.edu.ye or tareqalshami83@gmail.com

Abstract: The concept of continuity in topological spaces has a very important place. For this reason, a great deal of work has been done on continuity, and many generalizations of continuity have been obtained. In this work, we seek to find a new approach to the study of soft continuity in soft topological spaces in connection with an induced mapping based on soft sets. By defining the $*$-image of a soft set, we define an induced soft mapping and present its related properties. To elaborate on the obtained results and relationships, we furnish a number of illustrative examples.

Keywords: soft sets; soft point; soft topological spaces; induced soft mappings; soft continuous mappings

MSC: 54A40; 03E72; 54C99

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Citation: Kaur, S.; Al-shami, T.M.; Özkan, A.; Hosny, M. A New Approach to Soft Continuity. *Mathematics* **2023**, *11*, 3164. https://doi.org/10.3390/math11143164

Academic Editor: Francisco Gallego Lupiañez

Received: 13 June 2023
Revised: 11 July 2023
Accepted: 15 July 2023
Published: 19 July 2023

1. Introduction

Most empirical problems in technological areas such as engineering, computer science, and social sciences which deal with unreliability and vagueness require solutions arrived at via scientific methods rather than long-established approaches. To cope with such issues, a new mathematical tool called soft set theory was put forward by Molodtsov in [1]. The research and applications of this set theory have been growing exponentially in various directions, resulting in great theoretical and technological successes [2–7].

Thereafter, Maji et al. [8] focused on abstract research of soft set operators with applications in decision-making problems. Later, the theory of soft topological spaces defined over an initial universe with a predetermined set of parameters was proposed by Shabir and Naz in [9]; their work centered on the theoretical studies of soft topological spaces. Majumdar and Samanta [10] presented mappings on soft sets and their application in medical diagnosis. Kharal and Ahmed [11] brought up the view of soft mapping with properties; subsequently, soft continuity of soft mappings was instigated in [12]. Recently, Al-shami [13] adopted a novel approach to define soft mappings using the idea of soft points. He showed the advantages of this approach to simplify computations and move classical concepts of crisp mappings to soft frames. Many works devoted to studying soft continuity and its characterizations can be found in the literature reviews provided in [14–30]).

Mappings in any mathematical discipline are known as structure-preserving operations. Because these concepts establish the relationship between two or more domains under certain rules, they are useful tools to consider when developing models for many problems. Different fields such as mathematics, computer science, chemistry, psychology, and logic have shaped the concept of transformation according to their specific conditions and characteristics.

Induced mappings are transformations that help us to study the relationships between different mathematical structures, including

- Any type of structure
- A constraint on a given transformation
- Reduction of an operator to a transformation.

Any of the above can lead to useful induced mappings. Kalinowski [31] explained that an induced mapping is linear if and only if the matrices maintain their order, then in [32] generalized the results in [31] by removing any constraints on the mapping. Liu and Zhang [33] characterized the general form of all mappings f induced by f_{ij} while preserving rank-1 matrices over a field. In particular, similarity-preserving nonlinear mappings were studied by Du et al. [34]. Yang et al. [35] characterized induced mappings that preserve similarity and the inverse of matrices. Kahn [36] considered Postnikov systems from a geometric point of view. The main result is that Postnikov systems have an induced mapping with a mapping of spaces $f : X \to Y$. To develop the algebraic topology of mappings between Hilbert spaces, Kato [37] presented an induced Hilbert Clifford algebra and constructed an induced mapping between the K-theory of the Higson–Kasparov–Trout Clifford algebra and the induced Clifford algebra. In his thesis, Nguyen [38] provided Galois cohomological interpretations for induced mappings that occur naturally in short exact sequences, which can be used to classify isomorphism classes of algebraic objects over a field. Macías and Macías [39] studied induced mappings between n-fold pseudo-hyperspace suspensions for a given mapping between a continual. For a continuous metric X space, Gómez-Rueda et al. [40] presented the induced transform $f_n : F_n(X) \to F_n(X)$ for a mapping $f : X \to X$. Higuera and Illanes [41] studied the dynamical properties of f_n mappings. Kwietniak and Misiurewicz [42] studied chaotic systems of induced mappings f_n. Another type of induced mapping, denoted by $\psi^\#$, was provided by Arkhangel'skii and Ponomarev [43].

The above studies suggest that the induced mappings used in the literature are insufficient for the study of induced mappings constructed on soft mappings. The present study aims to fill these gaps in the literature through its methodological approach and the obtained results.

In this study, we describe a new approach to studying soft continuous mappings using an induced mapping based on soft sets. We pick up the idea of our proposed soft-induced mapping from the mapping introduced by Arkhangel'skii and and Ponomarev in [43], which is defined as $\psi^\# : P(U) \to P(V)$ given by $\psi^\#(M) = \{v \in V | \psi^{-1}(v) \subset M\}$ for any subset M of U, where ψ is a mapping from U to V and $P(U)$ and $P(V)$ are the collections of all subsets of U and V, respectively. We begin by introducing the definition of soft fibers by using soft points which, makes it possible to define the $*$-image of a soft set. Consequently, this gives rise to an induced mapping $(\psi, g)^* : S(U, E) \to S(V, F)$, where E and F are the set of parameters and $S(U, E)$ and $S(V, F)$ are the collections of all soft sets over U and V, respectively. Using this, we prove a number of new characterizations of soft continuous mappings.

2. Preliminaries

In this section, we present definitions and results related to soft set theory which help us to prove our results in the next section.

Definition 1 ([1])**.** *Assume U and E are respectively the initial universal set and a set of parameters, where $P(U)$ is the power set of U. A soft set M_E over U is a mapping provided by $M : E \to P(U)$, that is, a parameterized family of subsets of the universe U.*

Definition 2 ([8])**.** *A soft set M_E over U is called:*

1. *A null soft set, symbolized by Φ_E, if $M(e) = \emptyset$ for all $e \in E$;*
2. *An absolute soft set, symbolized by $\widetilde{U}$, if $M(e) = U$ for all $e \in E$.*

Definition 3 ([8]). *Let M_A and N_B be soft sets over the common universe U. Their (soft) union, symbolized by $M_A \cup N_B$, is a soft set H_C, where $C = A \cup B$ and H is defined for all $e \in C$ by*

$$H(e) = \begin{cases} M(e), \text{ if } e \in A - B, \\ N(e), \text{ if } e \in B - A, \\ M(e) \cup N(e), \text{ if } e \in A \cap B. \end{cases}$$

Definition 4 ([44]). *The (soft) intersection of two soft sets M_A and N_B over a common universe U, denoted $M_A \cap N_B$, is a soft set H_C, where $C = A \cap B \neq \emptyset$ and H is defined for all $e \in C$ by $H(e) = M(e) \cap N(e)$.*

Definition 5 ([8]). *Let M_A, $N_B \in S(U, E)$. We say that M_A is a (soft) subset of N_B (denoted by $M_A \subset N_B$) if $A \subset B$ and $M(e) \subset N(e)$ for all $e \in A$.*

For our main results, we will make use of the elementary concept of soft points provided in [45].

Definition 6. *A soft point is a soft set P_E over X, denoted by P_e^u, provided by $P(e) = \{u\}$ for an element $e \in E$ and $P(e') = \emptyset$ for all $e' \neq e$.*

A soft point P_e^u belongs to a soft set M_E, denoted by $P_e^u \widetilde{\in} M_E$, if $u \in M(e)$.

Two soft points $P_{e_1}^{u_1}$, $P_{e_2}^{u_2}$ are said to be:

1. Equal if $e_1 = e_2$ and $u_1 = u_2$;
2. Not equal if $e_1 \neq e_2$ or $u_1 \neq u_2$.

The family of all soft points over U will be denoted by $SP(U, E)$.

Every soft set can be written as the union of all soft points belonging to it, i.e., $M_E = \bigcup_{P_e^u \widetilde{\in} M_E} P_e^u$.

Definition 7 ([9]). *A soft topology on U is a subfamily $\tau \subseteq S(U, E)$ containing Φ_E, $\widetilde{U}$ and satisfying the conditions that the union of the arbitrary and the intersection of a finite number of soft sets of τ is part of τ.*

The triplet (U, τ, E) is called a soft topological space over U where the members of τ are called soft open sets. A soft closed set in (U, τ, E) is the soft complement [44] of a soft open set for which the soft complement of a soft set M_E, denoted as M_E^c, is defined by $M^c(e) = U - M(e)$ for all $e \in E$.

Definition 8 ([45]). *Let M_E be a soft set over U. Then, the soft closure of M_E, denoted by $\overline{M_E}$, is the smallest soft closed set containing M_E and the soft interior of M_E, denoted by M_E°, is the largest soft open set that is contained in M_E.*

Theorem 1 ([27]). *In a soft topological space (U, τ, E), $(\overline{M_E})^c = ((M_E)^c)^o$; thus, $\overline{(M_E)^c} = ((M_E)^o)^c$ for any soft subset M_E of $\widetilde{U}$.*

We now discuss the important concept of soft mappings between the families $S(U, E)$ and $S(V, F)$ of soft sets over U and V, respectively.

Definition 9 ([11]). *1. A mapping $(\psi, g) : S(U, E) \to S(V, F)$ corresponding to the mappings $\psi : U \to V$ and $g : E \to F$ is a soft mapping if the soft image of a soft set M_E, denoted by $(\psi, g)(M_E)$, is a soft set over V, which is provided by a mapping $(\psi, g)(M_E) : F \to P(V)$ defined as*

$$(\psi, g)(M_E)(f) = \begin{cases} \bigcup_{e \in g^{-1}(f)} \psi(M(e)), & if\ g^{-1}(f) \neq \emptyset. \\ \emptyset, & if\ g^{-1}(f) = \emptyset. \end{cases}$$

for all $f \in F$.

2. *For a soft set N_F in $S(V,F)$, the soft inverse image of N_F (denoted by $(\psi,g)^{-1}(N_F)$) is a soft set over U provided by a mapping $(\psi,g)^{-1}(N_F): E \to P(U)$ defined as $(\psi,g)^{-1}(N_F)(e) = \psi^{-1}(N(g((e)))$ for all e in E.*

A soft mapping (ψ,g) is surjective (injective) [45] if and only if both ψ and g are surjective (injective).

Proposition 1 ([45]). *Let $(\psi,g): S(U,E) \to S(V,F)$ be a soft mapping. Then, for all soft sets (M_E, O_E) in U and (N_F, P_F) in V it holds that:*

1. *$M_E \subset O_E$ implies $(\psi,g)(M_E) \subset (\psi,g)(O_E)$;*
2. *$N_F \subset P_F$ implies $(\psi,g)^{-1}(N_F) \subset (\psi,g)^{-1}(P_F)$;*
3. *$(\psi,g)(\psi,g)^{-1}(N_F) \subset N_F$ equality holds if (ψ,g) is surjective;*
4. *$(\psi,g)^{-1}(\psi,g)(M_E) \subset M_E$ equality holds if (ψ,g) is injective.*

Definition 10 ([12]). *Let (U,τ_1,E) and (V,τ_2,F) be two soft topological spaces. A soft mapping $(\psi,g): S(U,E) \to S(V,F)$ is soft continuous if and only if $(\psi,g)^{-1}(N_F)$ is a soft closed set in U for every soft closed set N_F in V or $(\psi,g)^{-1}(G_F)$ is a soft open set in U for every soft open set G_F in V.*

Theorem 2 ([21]). *A soft mapping $(\psi,g): S(U,E) \to S(V,F)$ is said to be soft continuous if and only if $(\psi,g)(\overline{M_E}) \subset \overline{(\psi,g)(M_E)}$ for every soft set M_E in $\widetilde{U}$.*

Throughout this paper, U refers to a universal set and E is a set of parameters. Moreover, (U,τ_1,E) and (V,τ_2,F) stand for soft topological spaces and $(\psi,g): S(U,E) \to S(V,F)$ stands for a soft mapping corresponding to the (classical) mappings $\psi: U \to V$ and $g: E \to F$.

3. Soft Induced Mappings

In this section, we begin by introducing the concept of fibers in connection with soft sets with the following example.

Definition 11. *Let $(\psi,g): S(U,E) \to S(V,F)$ be a soft mapping and let P_f^v be any soft point in V. A soft fiber $(\psi,g)^{-1}(P_f^v)$ is a soft set in U defined by*

$$(\psi,g)^{-1}(P_f^v)(e) = \psi^{-1}(P(g(e)) \\ = \begin{cases} \psi^{-1}(v) & \text{if } g(e) = f \\ \emptyset & \text{if } g(e) \neq f \end{cases}$$

Example 1. *Take the following sets $U = \{u_1,u_2\}$, $V = \{v_1,v_2\}$, $E = \{e_1,e_2\}$, and $F = \{f_1,f_2\}$. Let (ψ,g) be a soft mapping where $\psi: U \to V$ and $g: E \to F$ are the mappings defined by $\psi(u_1) = \psi(u_2) = v_1$, $g(e_1) = f_1$ and $g(e_2) = f_2$. Take $P_{f_1}^{v_1}$ as a soft point in V; then, the soft fiber $(\psi,g)^{-1}(P_{f_1}^{v_1})$ is provided by a soft set defined as $(\psi,g)^{-1}(P_{f_1}^{v_1})(e_1) = \{u_1,u_2\}$ and $(\psi,g)^{-1}(P_{f_1}^{v_1})(e_2) = \emptyset$.*

Next, we introduce the concept of the $*$-image of a soft set, which is used to define a soft induced mapping $(\psi,g)^*: S(U,E) \to S(V,F)$ corresponding to the mappings $\psi: U \to V$ and $g: E \to F$.

To do this, let M_E be a soft set in U with a parameter set E. Then, the $*$-image of M_E will be a soft set in V defined as the collection of all soft points P_f^v in V, which has an inverse image under the soft mapping (ψ, g) contained in M_E; in other words, it is a collection of all soft points P_f^v in V such that

$$\begin{aligned} &(\psi, g)^{-1}(P_f^v) \subset M_E \\ \text{i.e., } &(\psi, g)^{-1}(P_f^v)(e) \subset M(e), \text{for every } e \text{ in } E. \\ \text{i.e., } &\begin{cases} \psi^{-1}(v) \subset M(e) & if\, g(e) = f \\ V & if\, g(e) \neq f \end{cases} \end{aligned}$$

Now, we are in a position to show how the $*$-image of a soft set is calculated by the following definition.

Definition 12. *Let $\psi : U \to V$ and $g : E \to F$ be two mappings. Then, the $*$-image of a soft set M_E, denoted by $(\psi, g)^*(M_E)$, is a soft set in V, where $(\psi, g)^*(M_E) : F \to P(V)$ is defined by*

$$(\psi, g)^*(M_E)(f) = \begin{cases} \{v|\psi^{-1}(v) \subset M(e)\} & if\, e \in g^{-1}(f) \neq \emptyset \\ V & if\, g^{-1}(f) = \emptyset \end{cases}.$$

In the next example, we illustrate the above definition of soft induced mapping.

Example 2. *Consider $U = \{u_1, u_2\}$, $V = \{v_1, v_2\}$, $E = \{e_1, e_2\}$, and $F = \{f_1, f_2\}$. Let (ψ, g) be a soft mapping where $\psi : U \to V$ and $g : E \to F$ are mappings defined by $\psi(u_1) = v_2$, $\psi(u_2) = v_1$, $g(e_1) = f_1$, and $g(e_2) = f_2$. Assume that $M_E = \{(e_1, \{u_1\}), (e_2, \{u_2\})\}$ is a soft set in $S(U, E)$; then, the $*$-image of M_E is a soft set provided by $(\psi, g)^*(M_E)(f_1) = \{v|\psi^{-1}(v) \subset M(e_1)\} = \{v|\psi^{-1}(v) \subset u_1\} = v_2$. Similarly, $(\psi, g)^*(M_E)(f_2) = v_1$. Therefore,*

$$(\psi, g)^*(M_E) = \{(f_1, \{v_2\}), (f_2, \{v_1\})\}.$$

Remark 1. *For the surjective mapping (ψ, g), we have $(\psi, g)^*(M_E) \subset (\psi, g)(M_E)$ for any soft set M_E in U. It is clear to see that for any $f \in F$ and $e \in g^{-1}(f)$,*

$$\{v|\psi^{-1}(v) \subset M(e)\} \subset \bigcup_{e \in g^{-1}(f)} \psi(M(e)) \quad if\, \psi \text{ is on}$$

$$\begin{bmatrix} y \in \{v|\psi^{-1}(v) \subset M(e)\} & \Longrightarrow \psi^{-1}(y) \subset M(e) \\ & \Longrightarrow y \in \psi(M(e)) \quad if\, \psi \text{ is on} \\ & \Longrightarrow y \in \cup\psi(M(e)) \end{bmatrix}$$

The next example supports this remark.

Example 3. *Consider the soft mapping (ψ, g) in Example 2; for a soft set $M_E = \{(e_1, \{u_1\}), (e_2, \{u_1\})\}$ in $S(U, E)$, we have*

$$(\psi, g)(M_E) = \{(f_1, \{v_1\}), (f_2, \{v_1\})\} \text{ and}$$

$$(\psi, g)^*(M_E) = \{(f_1, \{v_2\}), (f_2, \{v_2\})\}.$$

That is, $(\psi, g)(M_E)$ is not a soft subset of $(\psi, g)^*(M_E)$ and $(\psi, g)^*(M_E)$ is not a soft subset of $(\psi, g)(M_E)$.

Next, we present several properties satisfied by the soft mapping $(\psi, g)^*$ that we will need for our next results.

Proposition 2. *Let $(\psi, g) : S(U, E) \to S(V, F)$ be a soft mapping, let M_E, N_E be soft sets in U, and let G_F be a soft set in V. Then:*

1. $(\psi,g)^*(M_E) \subset (\psi,g)^*(N_E)$ *if* $M_E \subset N_E$;
2. *For any soft set* M_E *in* U, $(\psi,g)^{-1}((\psi,g)^*(M_E)) \subset M_E$ *equality holds if* (ψ,g) *is injective;*
3. *For any soft set* G_F *in* V, $G_F \subset ((\psi,g)^*((\psi,g)^{-1}(G_F)))$ *equality holds if* (ψ,g) *is surjective;*
4. $(\psi,g)^*(\emptyset_E) = ((\psi,g)(U_E))^c$.

Proof. 1. Let $g^{-1}(f) \neq \emptyset$; then, for every $e \in g^{-1}(f)$,

$$\begin{aligned}(\psi,g)^*(M_E)(f) &= \{v|\psi^{-1}(v) \subset M(e)\}\\ &\subseteq \{v|\psi^{-1}(v) \subset N(e)\}\\ &= (\psi,g)^*(N_E)(f)\end{aligned}$$

and the result obviously holds for $g^{-1}(f) = \emptyset$.

2. For each $e \in E$,

$$\begin{aligned}(\psi,g)^{-1}((\psi,g)^*(M_E)(e)) &= \psi^{-1}((\psi,g)^*(M_E)(g(e)))\\ &= \psi^{-1}(\{v|\psi^{-1}(v) \subset M(e)\})\\ &\subset M(e)\end{aligned}$$

which is proved.

3. Let $g^{-1}(f) \neq \emptyset$; then,

$$\begin{aligned}(\psi,g)^*((\psi,g)^{-1}(G_F))(f) &= \{v|\psi^{-1}(v) \subset \psi^{-1}(G(g(e)))\} \quad \text{for every } e \in g^{-1}(f)\\ &= \{v|\psi^{-1}(v) \subset \psi^{-1}(G(f))\}\\ &\supset G(f) = G_F(f)\end{aligned}$$

For $g^{-1}(f) = \emptyset$, $(\psi,g)^*((\psi,g)^{-1}(G_F))(f) = V$ and
Hence, $G_F \subset (\psi,g)^*((\psi,g)^{-1}(G_F))$.

4. Let $g^{-1}(f) \neq \emptyset$; then.

$$\begin{aligned}((\psi,g)(U_E))^c(g) &= V - (\psi,g)(U_E)(g)\\ &= V - \bigcup_{e \in g^{-1}(f)} \psi(U(e))\\ &= V - \psi(U)\\ &= \{v|\psi^{-1}(v) = \emptyset\}\\ &= \{v|\psi^{-1}(v) \subset \emptyset(e)\}\\ &= (\psi,g)^*(\emptyset_E)(f)\end{aligned}$$

which is obvious for $g^{-1}(f) = \emptyset$.
□

The following lemma provides another very useful property of the soft mapping $(\psi,g)^*$ in terms of the complement and the intersection of soft sets.

Lemma 1. *Let* (ψ,g) *be a soft mapping and let* M_E, N_E *be soft sets in* U. *Then,*

(a) $(\psi,g)^*(M_E^c) = ((\psi,g)(M_E))^c$

(b) $(\psi,g)^*(M_E \cap N_E) = (\psi,g)^*(M_E) \cap (\psi,g)^*(N_E)$.

Proof. (a) Assume $g^{-1}(f) \neq \emptyset$, then,

$$(\psi, g)^*(M_E^c)(f) = \{v|\psi^{-1}(v) \subset M^c(e)\}$$
$$= \{v|\psi^{-1}(v) \subset U - M(e)\} = A \text{ (say)}$$

for all $e \in g^{-1}(f)$

$$((\psi, g)(M_E))^c(f) = V - (\psi, g)(M_E)(f)$$
$$= V - \bigcup_{e \in g^{-1}(f)} \psi(M(e)) = B \text{ (say)}$$

Which is enough to prove $A = B$.

Let $y \in A = \{v|\psi^{-1}(v) \subset U - M(e)\}$; then, $\psi^{-1}(y) \subset U - M(e)$ and $y \in \psi(U)$ with $y \notin \psi(M(e))$ for every $e \in g^{-1}(f)$, which further implies $y \in V - \bigcup_{e \in g^{-1}(f)} \psi(M(e))$. Hence, $y \in B$.

Now, assume $y \in B$; then, $y \in V - \bigcup_{e \in g^{-1}(f)} \psi(M(e))$, which implies $y \in V$ and $y \notin \psi(M(e))$ for all $e \in g^{-1}(f)$.

Now, if $y \in V$, then $\psi^{-1}(y) \subset \psi^{-1}(V) = U$; on the other hand, $\psi^{-1}(y) \not\subset M(e)$, as for $z \in \psi^{-1}(y)$ we have $\psi(z) = y \notin \psi(M(e))$, which implies $z \notin M(e)$. Therefore, $\psi^{-1}(y) \not\subset M(e)$, which provides $\psi^{-1}(y) \subset U - M(e)$; thus, $y \in \{v|\psi^{-1}(v) \subset U - M(e)\} = A$.

Hence, for $g^{-1}(f) \neq \psi$ we have

$$(\psi, g)^*(M_E^c) = ((\psi, g)(M_E))^c$$

Now, if $g^{-1}(f) = \psi$, then

$$(\psi, g)^*(M_E^c)(f) = V = ((\psi, g)(M_E))^c(f)$$

.

(b) Assume $g^{-1}(f) \neq \psi$. Then,

$$(\psi, g)^*(M_E \cap N_E)(f) = \{v|\psi^{-1}(v) \subset M(e) \cap N(e)\} \text{ for all } e \in g^{-1}(f)$$

$$= \{v|\psi^{-1}(v) \subset M(e)\} \cap \{v|\psi^{-1}(v) \subset N(e)\}$$
$$= (\psi, g)^*(M_E)(f) \cap (\psi, g)^*(N_E)(f)$$

The result is obvious if $g^{-1}(f) = \psi$. □

The following example shows that $(\psi, g)^*(M_E \cup N_E)$ need not be equal to $(\psi, g)^*(M_E) \cup (\psi, g)^*(N_E)$, that is, the union need not be preserved by $(\psi, g)^*$.

Example 4. *Take $U = \{u_1, u_2\}$, $V = \{v_1, v_2\}$, $E = \{e_1, e_2\}$, and $F = \{f_1, f_2\}$. Let $\psi : U \to V$ and $g : E \to F$ be mappings defined by $\psi(u_1) = \psi(u_2) = v_1$, $g(e_1) = f_1$ and $g(e_2) = f_2$. For a soft set $M_E = \{(e_1, \{u_1\}), (e_2, \{u_1\})\}$ in $S(U, E)$ and $N_E = \{(e_1, \{u_2\}), (e_2, \{u_2\})\}$ in $S(U, E)$, we have*

$$(\psi, g)^*(M_E) = (\psi, g)^*(N_E) = \{(f_1, \{v_2\}), (f_2, \{v_2\})\}.$$

On the other hand, $M_E \cup N_E = \{(e_1, \{u_1, u_2\}), (e_2, \{u_1, u_2\})\}$, which gives

$$(\psi, g)^*(M_E \cup N_E) = \{(f_1, \{v_1, v_2\}), (f_2, \{v_1, v_2\})\},$$

whereas $(\psi, g)^*(M_E) \cup (\psi, g)^*(N_E) = \{(f_1, \{v_2\}), (f_2, \{v_2\})\}$.

The next result provides another relation between the $*$-image of the soft mapping (ψ, g) and the inverse image of (ψ, g), which will be useful in our next result.

Proposition 3. *For a soft set M_E in U and G_F in V, $(\psi, g)^{-1}(G_F) \subset M_E$ if and only if $G_F \subset (\psi, g)^*(M_E)$.*

Proof. Let $(\psi, g)^{-1}(G_F) \subset M_E$; then, per Part 3 of Proposition 2,

$$G_F \subset (\psi, g)^*(\psi, g)^{-1}(G_F) \subset (\psi, g)^*(M_E).$$

Now, let $G_F \subset (\psi, g)^*(M_E)$; then, per Part 2 of Proposition 2, $(\psi, g)^{-1}(\psi, g)^*(M_E) \subset M_E$ which implies $(\psi, g)^{-1}(G_F) \subset (\psi, g)^{-1}(\psi, g)^*(M_E) \subset M_E$. □

The following theorem provides a representation of a soft continuous mapping in connection with the interior of soft sets.

Theorem 3. *Let $(\psi, g) : S(U, E) \to S(V, F)$ be a soft mapping; then, (ψ, g) is soft continuous if and only if for every soft set M_E in U we have $((\psi, g)^*(M_E))^\circ \subset ((\psi, g)^*(M_E^\circ))$.*

Proof. To prove this result, it is enough to prove that the soft continuity of (ψ, g) is equivalent to $((\psi, g)^*(M_E^c))^\circ \subset ((\psi, g)^*(M_E^c)^\circ)$ for any soft set M_E in U.

Let (ψ, g) be a soft continuous mapping and let M_E be a soft set in U; then,

$$\begin{aligned}
((\psi, g)^*(M_E^c))^\circ &= \overline{((\psi, g)^*(M_E^c))^c}^c \text{ per Theorem 1;} \\
&= \overline{((\psi, g)(M_E))^c)^c}^c \text{ per Lemma 1;} \\
&= \overline{(\psi, g)(M_E)}^c; \\
&\subset [(\psi, g)(\overline{M_E})]^c \text{ due to the soft continuity of} (\psi, g), \text{Theorem 2;} \\
&= (\psi, g)^*(\overline{M_E}^c) \text{ per Lemma 1;} \\
&= (\psi, g)^*((M_E^c)^\circ) \text{ per Theorem 1}
\end{aligned}$$

which is true for the arbitrary soft set M_E of U. Hence, (ψ, g) is soft continuous if and only if $(\psi, g)^*(M_E))^\circ \subset (\psi, g)^*(M_E)^\circ)$ for every soft set M_E in U. □

Next, we define a soft set M_E^* induced by the soft set M_E in U, which is defined as the inverse image of the $*$-image of the soft set M_E.

Definition 13. *Let $(\psi, g) : S(U, E) \to S(V, F)$ be a soft mapping. Define a soft set $M_E^* = (\psi, g)^{-1}(\psi, g)^*(M_E)$ in U, which is defined by*

$$M_E^*(e) = (\psi, g)^{-1}(\psi, g)^*(M_E)(e) = \psi^{-1}((\psi, g)^*(M_E))(g(e)) = \psi^{-1}\{v|\psi^{-1}(v) \subset M(e)\}$$

for every $e \in E$ and $e \in g^{-1}(g(e))$.

From Part 2 of Proposition 2, it is clear that $M_E^* \subset M_E$.

The following example illustrates the above definition.

Example 5. *Taking the soft set $M_E = \{(e_1, \{u_1, u_2\}), (e_2, \{u_2\})\}$ in Example 2, the soft set M_E^* is provided by $M_E^*(e_1) = \psi^{-1}\{v|\psi^{-1}(v) \subset M(e_1)\} = \psi^{-1}\{v|\psi^{-1}(v) \subset \{u_1, u_2\}\} = \psi^{-1}\{v_1, v_2\} = \{u_1, u_2\}$ and $M_E^*(e_2) = \psi^{-1}\{v|\psi^{-1}(v) \subset M(e_2)\} = \psi^{-1}\{v|\psi^{-1}(v) \subset \{u_2\}\} = \psi^{-1}\{v_2\} = \emptyset$. Therefore, $M_E^* = \{(e_1, \{u_1, u_2\}), (e_2, \emptyset)\}$.*

Several properties of the soft set M_E^* are presented in the next proposition.

Proposition 4. For a soft mapping $(\psi, g) : S(U, E) \to S(V, F)$:

1. $\widetilde{U}^* = \widetilde{U}$ and $\Phi_E{}^* = \Phi_E$

2. $(\psi,g)^*(M^*_E) = (\psi,g)^*(M_E)$ for every soft set M_E in U.

Proof. 1. The proof is obvious.

2. Assume $e \in g^{-1}(f) \neq \emptyset$; then, $(\psi,g)^*(M^*_E)(f) = \{v|\psi^{-1}(v) \subset M^*(e)\} = \{v|\psi^{-1}(v) \subset \psi^{-1}\{v|\psi^{-1}(v) \subset M(e)\}\} = \{v|\psi^{-1}(v) \subset M(e)\} = (\psi,g)^*(M_E)$.
□

To demonstrate Part 2 of the above proposition, we continue with Example 7.

Example 6. *In the above Example 7, we have* $M_E = \{(e_1,\{u_1,u_2\}),(e_2,\{u_2\})\}$ *and* $M^*_E = \{(e_1,\{u_1,u_2\}),(e_2,\emptyset)\}$. *It can be easily seen that* $(\psi,g)^*(M^*_E) = (\psi,g)^*(M_E) = \{(f_1,\{v_1,v_2\}),(f_2,\{v_2\})\}$.

Remark 2. For the surjective mapping (ψ,g) we have $(\psi,g)^*(M_E) = (\psi,g)(M^*_E)$ for any soft set M_E in U. It is clear from definition of M^*_E and Part 3 of by Proposition 2 that $(\psi,g)(M^*_E) = (\psi,g)(\psi,g)^{-1}(\psi,g)^*(M_E) = (\psi,g)^*(M_E)$.

In the next theorem, we present another characterization of a soft continuous mapping with respect to the previously defined soft set M^*_E.

Theorem 4. *Let* $(\psi,g) : S(U,E) \to S(V,F)$ *be a surjective soft mapping; then,* (ψ,g) *is soft continuous if and only if* M^*_E *is soft open in* U *whenever* $(\psi,g)(M^*_E)$ *is soft open in* V *for every soft set* M_E *in* U.

Proof. Let (ψ,g) be a soft continuous mapping and let $(\psi,g)(M^*_E)$ be a soft open set in V for a soft set M_E in U. Then, according to the soft continuity of (ψ,g) we can find that $(\psi,g)^{-1}(\psi,g)(M^*_E)$ is soft open in U. Using the definition of M^*_E, we have $(\psi,g)^{-1}(\psi,g)(M^*_E) = (\psi,g)^{-1}\ (\psi,g)(\psi,g)^{-1}(\psi,g)^*(M_E)$. Now, because $(\psi,g)^{-1}(\psi,g)(\psi,g)^{-1} = (\psi,g)^{-1}$, we have

$$(\psi,g)^{-1}(\psi,g)(M^*_E) = (\psi,g)^{-1}(\psi,g)(\psi,g)^{-1}(\psi,g)^*(M_E) = (\psi,g)^{-1}(\psi,g)^*(M_E) = M^*_E.$$

It is clear that M^*_E is soft open in U, as $(\psi,g)^{-1}(\psi,g)(M^*_E)$ is soft open in U. Hence, the proof is completed.

Conversely, let N_F be soft open in V and assume that $(\psi,g)^{-1}(N_F) = M_E$.

Now, because (ψ,g) is surjective, $N_F = (\psi,g)^*(\psi,g)^{-1}(N_F)$, where N_F is soft open. Therefore, $(\psi,g)^*(\psi,g)^{-1}(N_F)$ is soft open in V; thus, $(\psi,g)^{-1}(N_F)$ is soft open per our hypothesis. Hence, (ψ,g) is soft continuous. □

As an instance of the above theorem, we have the following.

Example 7. *Consider the sets* $U = V = \{u_1,u_2,u_3\}$, $E = F = \{0,1\}$. *Let* $(\psi,g) : S(U,E) \to S(V,F)$ *be a soft mapping where* $\psi : U \to V$ *is a mapping defined by* $\psi(u_1) = u_1, \psi(u_2) = u_3, \psi(u_3) = u_2$ *and* $g : E \to F$ *is an identity mapping. Assume that* $\tau = \{\Phi_E, \widetilde{U}, M_{1_E} = \{(0,\{u_1\}),(1,\{u_3\})\}, M_{2_E} = \{(0,\{u_2\}),(1,\{u_1\})\}, M_{3_E} = \{(0,\{u_3\}),(1,\{u_2\})\}, M_{4_E} = \{(0,\{u_1,u_2\}),(1,\{u_1,u_3\})\}, M_{5_E} = \{(0,\{u_1,u_3\}),(1,\{u_2,u_3\})\}, M_{6_E} = \{(0,\{u_2,u_3\}),(1,\{u_1,u_2\})\}\}$ *and* $\tau^* = \{\Phi_F, \widetilde{V}, N_{1_F} = \{(0,\{u_1\}),(1,\emptyset)\}, N_{2_F} = \{(0,\{u_2\}),(1,\emptyset)\}, N_{3_F} = \{(0,\{u_3\}),(1,\emptyset)\}, N_{4_F} = \{(0,\{u_1,u_2\}),(1,\emptyset)\}, N_{5_F} = \{(0,\{u_1,u_3\}),(1,\emptyset)\}, N_{6_F} = \{(0,\{u_2,u_3\}),(1,\emptyset)\}, N_{7_F} = \{(0,\{u_1,u_2,u_3\}),(1,\emptyset)\}\}$ *are soft topological spaces of* $S(U,E)$ *and* $S(V,F)$, *respectively. Clearly,* (ψ,g) *is not soft continuous. Taking a soft set* $M_E = \{(0,\{u_1,u_2\}),(1,\emptyset)\}$ *in* U, *we have* $M^*_E = \{(0,\{u_1,u_2\}),(1,\emptyset)\}$ *and* $(\psi,g)(M^*_E) = \{(0,\{u_1,u_3\}),(1,\emptyset)\}$. *It is easy to see that* $(\psi,g)(M^*_E)$ *is soft open in* V *and that* M^*_E *is not soft open in* U.

4. Conclusions

Soft set theory as a general mathematical tool for dealing with uncertainty was first introduced in 1999 by Molodtsov [1]. While many scientists have worked on the properties and applications of soft set theory, the concept of induced mappings on this set has not yet been studied. Mappings are known as structure-preserving concepts in any mathematical discipline. Therefore, in this manuscript we have presented the concept of an induced mapping defined on a soft set, and have discussed its main properties. In addition, we have elucidated the interrelationships between induced soft mappings and soft mappings as well as applied to characterize soft continuity.

This work represents a beginning point for soft-induced mathematical notions and constructions that are based on induced map-theoretic operations. Furthermore, this soft-induced mapping enables us to prove new representations of soft continuous mapping, contributing to the understanding of the algebraic structure of soft sets. Hence, we think that in the future soft-induced mapping structures will be actively studied.

Author Contributions: Conceptualization, S.K. and T.M.A.-s.; Methodology, S.K., T.M.A.-s. and A.Ö.; Formal Analysis, S.K., T.M.A.-s., A.Ö. and M.H.; Investigation, S.K., T.M.A.-s., A.Ö. and M.H.; Writing—Original Draft Preparation, S.K. and A.Ö.; Writing—Review and Editing, T.M.A.-s.; Funding Acquisition, T.M.A.-s. and M.H. All authors have read and agreed to the published version of the manuscript.

Funding: The fourth author extends her appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through a research groups program under grant RGP2/310/44.

Data Availability Statement: No data were used to support this study.

Conflicts of Interest: The authors declare no conflict of interest.

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