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Characterization Results of Solution Sets Associated with Multiple-Objective Fractional Optimal Control Problems

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Abstract: This paper investigates some duality results of a mixed type for a class of multiple objective fractional optimal control problems. More precisely, by considering the Wolfe- and Mond–Weir-type dualities, we formulate a robust mixed-type dual problem and, under suitable convexity assumptions of the involved functionals, we establish some equivalence results between the solution sets of the considered models. Essentially, we investigate robust weak, robust strong, and robust strict converse-type duality results. To the best of the authors' knowledge, robust duality results for such problems are new in the specialized literature.

Keywords: dual problem; optimal control; duality results; robust necessary efficiency conditions

MSC: 26B25; 49J20; 90C32

1. Introduction

In the research paper [1], Hanson applied the duality theory from mathematical programming to a new class of functions named invex functions. In this regard, Craven and Glover [2] established that invex functions are characterized as functions where the stationary/critical points become global minima. As a generalization of the work of Mond and Hanson [3], Mond and Smart [4] formulated some sufficiency and duality results in scalar variational control problems. Also, duality theorems have been stated for linear fractional variational problems by Aggarwal et al. [5]. Mukherjee and Rao [6] presented mixed dual problems associated with multiobjective variational problems and established dualities under ρ-invexity hypotheses. Historically, multiobjective variational problems governed by equality and inequality restrictions have been of great importance and interest (including conditions of optimality, dual problems, and various areas of applicability), and we have only failed to consider the following researchers: Zhian and Qingkai [7], Zalmai [8], Mititelu [9], Hachimi and Aghezzaf [10], Chen [11], Kim and Kim [12], and Nahak and Nanda [13]. Gulati et al. [14] studied optimality conditions and the associated duality for a class of multiobjective control problems. Arana-Jiménez et al. [15] investigated a necessary and sufficient condition for duality in some multiobjective variational problems. Khazafi et al. [16] discussed sufficiency and duality for multiobjective control problems under generalized (B, ρ)-type I functions. Zhang et al. [17] analyzed the sufficiency and duality for multiobjective variational control problems under G-invexity assumptions.

Recently, Das et al. [18] provided sufficient KKT-type second-order optimality conditions for a class of set-valued fractional minimax problems. Under contingent epi-derivative and generalized second-order cone convexity hypotheses, the authors formulated some duals for the considered problem. Khan and Al-Solamy [19] discussed, for a non-differentiable

minimax fractional programming problem, the optimality condition for an optimal solution and a dual model. Mititelu and Treanţă [20] formulated some efficiency conditions in vector control problems generated by multiple integrals. Sharma [21] presented a higher-order duality for variational control problems. Oliveira and Silva [22] studied sufficient optimality conditions for some multiobjective control problems. In the last decade, Treanţă [23] and his collaborators investigated some classes of multi-dimensional multiobjective variational control problems. In this direction, Treanţă and Mititelu [24] formulated duality results in multi-dimensional vector fractional control problems by considering (ρ, b)-quasiinvexity assumptions.

Most optimization problems that occur in practice have several objective functions that must be optimized simultaneously. This type of problem, of considerable interest, includes various branches of mathematical sciences, design engineering, and game theory. Because of the increasing complexity of the environment, the initial data often suffer from inaccuracy. For example, in the modeling of many processes in industry and economy in order to make decisions, it is not always possible to have complete information about the parameters and variables involved. Therefore, an adequate uncertainty framework is necessary to formulate the model, and new methods have to be adapted or developed to provide optimal or efficient solutions in a certain sense. In order to tackle the uncertainty in an optimization problem, robust and interval-valued optimization represents some growing branches of applied mathematics and may provide an alternative choice for considering the uncertainty. Over time, several researchers and mathematicians have been interested to obtain many solution procedures in interval analysis and robust control. In order to formulate necessary and sufficient optimality conditions and duality theorems for different types of robust and interval-valued variational problems, various approaches have been proposed.

In this paper, under the motivation of the above-mentioned research papers and by considering suitable convexity hypotheses for the involved integral-like functionals, a mixed-type dual model is developed for the multiple objective fractional optimal control problem determined by multiple integral functionals defined in Ritu et al. [25]. More specifically, this paper is essentially a natural continuation of the studies stated in Mititelu and Treanţă [20] and Ritu et al. [25]. In this regard, by using the robust necessary efficiency conditions established in Ritu et al. [25], we investigate robust weak, robust strong, and robust strict converse-type duality results. The limitations of the existing works and the main credits of this paper are the following: (i) the presence of mixed constraints involving partial derivatives, (ii) the presence of the uncertainty data both in the cost functionals but also in the constraint functionals, and (iii) the combination of parametric and robust approaches to study the considered class of problems.

2. Preliminaries

Let us start with the standard Euclidean spaces $\mathbb{R}^p, \mathbb{R}^q, \mathbb{R}^r$, and $\mathbb{R}^n$, and a compact set in $\mathbb{R}^p$, denoted by S. Define the multi-time variable $t = (t^\alpha)$, $\alpha = \overline{1, p}$, such that $t \in S$. Also, consider the space (denoted by A) of state functions with continuous first-order partial derivatives as $\lambda = (\lambda^i) : S \to \mathbb{R}^q$ and consider the continuous control functions in the space B as $\pi = (\pi^j) : S \to \mathbb{R}^r$. Additionally, we use the abbreviations: $\Upsilon := (t, \lambda(t), \pi(t))$, $dt := dt^1 \cdots dt^p$, $\lambda_\alpha(t) := \dfrac{\partial \lambda}{\partial t^\alpha}(t)$. Next, we formulate the rules that are considered for any two points $a, b \in \mathbb{R}^n$:

(i) $a < b \Leftrightarrow a_s < b_s$, $\forall s = \overline{1, n}$,
(ii) $a = b \Leftrightarrow a_s = b_s$, $\forall s = \overline{1, n}$,
(iii) $a \leqq b \Leftrightarrow a_s \leq b_s$, $\forall s = \overline{1, n}$,
(iv) $a \leq b \Leftrightarrow a_s \leq b_s$, $\forall s = \overline{1, n}$ and $a_s < b_s$ for some s.

The robust multiple objective fractional optimal control problem is formulated (see, also, Mititelu and Treanţă [20], Treanţă and Mititelu [24], and Ritu et al. [25]) as:

$$(\mathcal{P}) \qquad \min_{(\lambda,\pi)} \left\{ \frac{\int_S h(\mathrm{Y},\varsigma)dt}{\int_S z(\mathrm{Y},\gamma)dt} := \left(\frac{\int_S h_1(\mathrm{Y},\varsigma_1)dt}{\int_S z_1(\mathrm{Y},\gamma_1)dt}, \ldots, \frac{\int_S h_p(\mathrm{Y},\varsigma_p)dt}{\int_S z_p(\mathrm{Y},\gamma_p)dt} \right) \right\}$$

subject to

$$f(\mathrm{Y},\lambda_\alpha(t),\sigma) \leqq 0,$$

$$g(\mathrm{Y},\lambda_\alpha(t),\delta) := \lambda_\alpha(t) - \Theta_\alpha(\mathrm{Y},\delta) = 0, \quad \alpha = \overline{1,p}$$

$$t \in S, \ \lambda(t_0) = \lambda_0, \ \lambda(t_1) = \lambda_1,$$

where

$$h_\epsilon : S \times A \times B \times \mathcal{G}_\epsilon \to \mathbb{R}, \ \epsilon = \overline{1,p}, \quad h = (h_1,\ldots,h_p),$$

$$z_\epsilon : S \times A \times B \times Q_\epsilon \to \mathbb{R}, \ \epsilon = \overline{1,p}, \quad z = (z_1,\ldots,z_p),$$

$$f_l : J^1\left(S,\mathbb{R}^q\right) \times B \times \mathcal{T}_l \to \mathbb{R}, \ l = \overline{1,m}, \quad f = (f_1,\ldots,f_m),$$

$$g_s : J^1\left(S,\mathbb{R}^q\right) \times B \times \mathcal{M}_s \to \mathbb{R}, s = \overline{1,n}, \quad g = (g_1,\ldots,g_n),$$

are C^1-class functionals (almost everywhere); the jet bundle of first-order associated with S and $\mathbb{R}^q$ is stated as $J^1\left(S,\mathbb{R}^q\right)$; also, we assume $\int_S z_\epsilon(\mathrm{Y},\gamma_\epsilon)dt > 0, \epsilon = \overline{1,p}$, and $\varsigma = (\varsigma_\epsilon), \gamma = (\gamma_\epsilon), \sigma = (\sigma_l)$, and $\delta = (\delta_s)$ represent the uncertainty parameters of the compact convex sets $\mathcal{G} = (\mathcal{G}_\epsilon) \subset \mathbb{R}^p, Q = (Q_\epsilon) \subset \mathbb{R}^p, \mathcal{T} = (\mathcal{T}_l) \subset \mathbb{R}^m$, and $\mathcal{M} = (\mathcal{M}_s) \subset \mathbb{R}^n$.

The robust counterpart for $(\mathcal{P})$ is introduced as follows:

$$(\mathcal{RP}) \qquad \min_{(\lambda,\pi)} \frac{\int_S \max_{\varsigma \in \mathcal{G}} h(\mathrm{Y},\varsigma)dt}{\int_S \min_{\gamma \in Q} z(\mathrm{Y},\gamma)dt}$$

$$:= \min_{(\lambda,\pi)} \left(\frac{\int_S \max_{\varsigma_1 \in \mathcal{G}_1} h_1(\mathrm{Y},\varsigma_1)dt}{\int_S \min_{\gamma_1 \in Q_1} z_1(\mathrm{Y},\gamma_1)dt}, \ldots, \frac{\int_S \max_{\varsigma_p \in \mathcal{G}_p} h_p(\mathrm{Y},\varsigma_p)dt}{\int_S \min_{\gamma_p \in Q_p} z_p(\mathrm{Y},\gamma_p)dt} \right)$$

subject to

$$f(\mathrm{Y},\lambda_\alpha(t),\sigma) \leqq 0, \quad t \in S, \ \sigma \in \mathcal{T}$$

$$g(\mathrm{Y},\lambda_\alpha(t),\delta) = 0, \quad t \in S, \ \delta \in \mathcal{M}$$

$$\lambda(t_0) = \lambda_0, \ \lambda(t_1) = \lambda_1.$$

The feasible solution set of $(\mathcal{RP})$, known as the robust feasible solution set for $(\mathcal{P})$, is denoted as follows:

$$\mathcal{S} = \{(\lambda,\pi) \in A \times B : f(\mathrm{Y},\lambda_\alpha(t),\sigma) \leqq 0,$$

$$g(\mathrm{Y},\lambda_\alpha(t),\delta) = 0, \ \lambda(t_0) = \lambda_0, \ \lambda(t_1) = \lambda_1, \ t \in S, \ \sigma \in \mathcal{T}, \ \delta \in \mathcal{M}\}.$$

Next, we consider the following parametric scalar optimal control problem corresponding to $(\mathcal{P})$ as follows:

$$(\mathcal{P}_w) \qquad \min_{(\lambda,\pi)} \left\{ \int_S h_w(\mathrm{Y},\varsigma_w)dt - \mathrm{Y}_w^0 \int_S z_w(\mathrm{Y},\gamma_w)dt \right\}$$

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subject to

$$f(Y, \lambda_\alpha(t), \sigma) \leqq 0,$$

$$g(Y, \lambda_\alpha(t), \delta) = 0,$$

$$t \in S, \ \lambda(t_0) = \lambda_0, \ \lambda(t_1) = \lambda_1,$$

$$\int_S \left[h_\epsilon(Y, \varsigma_\epsilon) - Q_\epsilon^0 z_\epsilon(Y, \gamma_\epsilon) \right] dt \leq 0, \quad \epsilon = \overline{1, p}, \ \epsilon \neq w.$$

</div>

The robust counterpart associated to $(\mathcal{P}_w)$ is given by:

$$(\mathcal{RP}_w) \qquad \min_{(\lambda, \pi)} \left\{ \int_S \max_{\varsigma_w \in \mathcal{G}_w} h_w(Y, \varsigma_w) dt - Q_w^0 \int_S \min_{\gamma_w \in Q_w} z_w(Y, \gamma_w) dt \right\}$$

<div align="center">

subject to

$$(\lambda, \pi) \in \mathcal{S},$$

$$\int_S \left[h_\epsilon(Y, \varsigma_\epsilon) - Q_\epsilon^0 z_\epsilon(Y, \gamma_\epsilon) \right] dt \leq 0, \quad \epsilon = \overline{1, p}, \ \epsilon \neq w.$$

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Definition 1. *A feasible pair $(\bar{\lambda}, \bar{\pi})$ is named as a robust weak optimal solution for $(\mathcal{P}_w)$ if:*

$$\int_S \max_{\varsigma_w \in \mathcal{G}_w} h_w(\bar{Y}, \varsigma_w) dt - Q_w^0 \int_S \min_{\gamma_w \in Q_w} z_w(\bar{Y}, \gamma_w) dt$$

$$< \int_S \max_{\varsigma_w \in \mathcal{G}_w} h_w(Y, \gamma_w) dt - Q_w^0 \int_S \min_{\gamma_w \in Q_w} z_w(Y, \gamma_w) dt,$$

for all feasible pairs (λ, π).

Definition 2. *A feasible pair $(\bar{\lambda}, \bar{\pi})$ is named a robust optimal solution in $(\mathcal{P}_w)$ if*

$$\int_S \max_{\varsigma_w \in \mathcal{G}_w} h_w(\bar{Y}, \varsigma_w) dt - Q_w^0 \int_S \min_{\gamma_w \in Q_w} z_w(\bar{Y}, \gamma_w) dt$$

$$\leq \int_S \max_{\varsigma_w \in \mathcal{G}_w} h_w(Y, \gamma_w) dt - Q_w^0 \int_S \min_{\gamma_w \in Q_w} z_w(Y, \gamma_w) dt,$$

for all feasible pairs (λ, π).

Definition 3. *A vector functional $\int_S h(Y, \lambda_\alpha(t), \varsigma) dt$ is said to be convex at $(\bar{\lambda}, \bar{\pi}) \in A \times B$ if the inequality*

$$\int_S h(Y, \varsigma) dt - \int_S h(\bar{Y}, \varsigma) dt \geqq \int_S (\lambda - \bar{\lambda}) h_\lambda(\bar{Y}, \varsigma) dt + \int_S (\pi - \bar{\pi}) h_\pi(\bar{Y}, \varsigma) dt$$

$$+ \int_S (\lambda_\alpha - \bar{\lambda}_\alpha) h_{\lambda_\alpha}(\bar{Y}, \varsigma) dt$$

holds for all $(\lambda, \pi) \in A \times B$.

Definition 4. *A feasible pair $(\bar{\lambda}, \bar{\pi}) \in \mathcal{S}$ is named a robust weak efficient solution for $(\mathcal{P})$ if there does not exist $(\lambda, \pi) \in \mathcal{S}$ fulfilling*

$$\frac{\int_S \max_{\varsigma \in \mathcal{G}} h(Y, \varsigma) dt}{\int_S \min_{\gamma \in Q} z(Y, \gamma) dt} < \frac{\int_S \max_{\varsigma \in \mathcal{G}} h(\bar{Y}, \varsigma) dt}{\int_S \min_{\gamma \in Q} z(\bar{Y}, \gamma) dt}.$$

Definition 5. *A feasible pair $(\bar{\lambda}, \bar{\pi}) \in \mathcal{S}$ is named a robust efficient solution for $(\mathcal{P})$ if $(\lambda, \pi) \in \mathcal{S}$ does not exist satisfying*

$$\frac{\int_S \max_{\varsigma \in \mathcal{G}} h(Y, \varsigma) dt}{\int_S \min_{\gamma \in Q} z(Y, \gamma) dt} \leq \frac{\int_S \max_{\varsigma \in \mathcal{G}} h(\bar{Y}, \varsigma) dt}{\int_S \min_{\gamma \in Q} z(\bar{Y}, \gamma) dt}.$$

Theorem 1 ([25] Robust necessary efficiency conditions for $(\mathcal{P})$)**.** *Let $(\bar{\lambda}, \bar{\pi}) \in \mathcal{S}$ be a robust weak efficient solution to the considered robust multiple objective fractional optimal control problem $(\mathcal{P})$ and $\max_{\varsigma_w \in \mathcal{G}_w} h_w(Y, \varsigma_w) = h_w(Y, \bar{\varsigma}_w)$, $\min_{\gamma_w \in Q_w} z_w(Y, \gamma_w) = z_w(Y, \bar{\gamma}_w)$. Then, the scalars $\bar{\eta} = (\bar{\eta}_w) \in \mathbb{R}^p$, the piecewise differentiable functions $\bar{\rho} = (\bar{\rho}_l(t)) \in \mathbb{R}_+^m, \bar{\theta} = (\bar{\theta}_s(t)) \in \mathbb{R}^n$, and the parameters of uncertainty $\bar{\sigma} \in \mathcal{T}, \bar{\delta} \in \mathcal{M}$ exist, fulfilling*

$$\bar{\eta}^T \left[h_\lambda(\bar{Y}, \bar{\varsigma}) - Q^0 z_\lambda(\bar{Y}, \bar{\gamma}) \right] + \bar{\rho}^T f_\lambda(\bar{Y}, \bar{\lambda}_\alpha(t), \bar{\sigma}) + \bar{\theta}^T g_\lambda(\bar{Y}, \bar{\lambda}_\alpha(t), \bar{\delta})$$

$$-D_\alpha \left[\bar{\rho}^T f_{\lambda_\alpha}(\bar{Y}, \bar{\lambda}_\alpha(t), \bar{\sigma}) + \bar{\theta}^T g_{\lambda_\alpha}(\bar{Y}, \bar{\lambda}_\alpha(t), \bar{\delta}) \right] = 0,$$

$$\bar{\eta}^T \left[h_\pi(\bar{Y}, \bar{\varsigma}) - Q^0 z_\pi(\bar{Y}, \bar{\gamma}) \right] + \bar{\rho}^T f_\pi(\bar{Y}, \bar{\lambda}_\alpha(t), \bar{\sigma}) + \bar{\theta}^T g_\pi(\bar{Y}, \bar{\lambda}_\alpha(t), \bar{\delta}) = 0,$$

$$\bar{\rho}^T f(\bar{Y}, \bar{\lambda}_\alpha(t), \bar{\sigma}) = 0, \ \bar{\rho} \geqq 0,$$

$$\bar{\eta} \geq 0,$$

for all $t \in S$, excepting the discontinuity points.

3. Main Results: Mixed Robust Duality

In this section, by using the robust necessary efficiency conditions established in Ritu et al. [25], we investigate robust weak, robust strong, and robust strict converse-type duality results. More precisely, by considering the Wolfe- and Mond–Weir-type dualities, we formulate a robust mixed-type dual problem, and, under suitable convexity assumptions of the involved functionals, we establish some equivalence results between the solution sets of the considered models. The methodology used is based on several techniques from the calculus of variations, the Lagrange-Hamilton theory, and the distribution and control theory, which are appropriate in the study of the considered robust variational control problems. To the best of the authors' knowledge, the robust duality results for such types of problems are new in the specialized literature.

Further, by denoting $\Pi := (t, \iota(t), \kappa(t))$, we associate a Wolfe-type robust dual model for $(\mathcal{P})$, as follows:

$$(\mathcal{WD} - \mathcal{P}) \qquad \max_{(\iota(\cdot), \kappa(\cdot))} \int_S \left\{ \left[h(\Pi, \varsigma) - Q^0 z(\Pi, \gamma) \right] + \rho^T f(\Pi, \iota_\alpha, \sigma) e \right.$$

$$\left. + \theta^T g(\Pi, \iota_\alpha, \delta) e \right\} dt$$

subject to

$$\eta^T \left[h_\lambda(\Pi, \varsigma) - Q^0 z_\lambda(\Pi, \gamma) \right] + \rho^T f_\lambda(\Pi, \iota_\alpha, \sigma) + \theta^T g_\lambda(\Pi, \iota_\alpha, \delta)$$

$$-D_\alpha \left[\rho^T f_{\lambda_\alpha}(\Pi, \iota_\alpha, \sigma) + \theta^T g_{\lambda_\alpha}(\Pi, \iota_\alpha, \delta) \right] = 0, \tag{1}$$

$$\eta^T \left[h_\pi(\Pi, \varsigma) - Q^0 z_\pi(\Pi, \gamma) \right] + \rho^T f_\pi(\Pi, \iota_\alpha, \sigma) + \theta^T g_\pi(\Pi, \iota_\alpha, \delta) = 0, \tag{2}$$

$$\iota(t_0) = \lambda_0, \quad \iota(t_1) = \lambda_1, \tag{3}$$

$$\eta \geq 0, \quad e^T \eta = 1, \quad e = (1, \dots 1) \in \mathbb{R}^p. \tag{4}$$

The corresponding robust counterpart for $(\mathcal{WD} - \mathcal{P})$ is formulated as:

$$(\mathcal{RWD} - \mathcal{P}) \qquad \max_{(\iota(.),\kappa(.))} \int_S \{ \left[h(\Pi, \bar{\varsigma}) - Q^0 z(\Pi, \bar{\gamma}) \right] + \rho^T f(\Pi, \iota_\alpha, \bar{\sigma}) e$$

$$+ \theta^T g(\Pi, \iota_\alpha, \bar{\delta}) e \} dt$$

subject to

$$\eta^T \left[h_\lambda(\Pi, \bar{\varsigma}) - Q^0 z_\lambda(\Pi, \bar{\gamma}) \right] + \rho^T f_\lambda(\Pi, \iota_\alpha, \bar{\sigma}) + \theta^T g_\lambda(\Pi, \iota_\alpha, \bar{\delta})$$

$$- D_\alpha \left[\rho^T f_{\lambda_\alpha}(\Pi, \iota_\alpha, \bar{\sigma}) + \theta^T g_{\lambda_\alpha}(\Pi, \iota_\alpha, \bar{\delta}) \right] = 0,$$

$$\eta^T \left[h_\pi(\Pi, \bar{\varsigma}) - Q^0 z_\pi(\Pi, \bar{\gamma}) \right] + \rho^T f_\pi(\Pi, \iota_\alpha, \bar{\sigma}) + \theta^T g_\pi(\Pi, \iota_\alpha, \bar{\delta}) = 0,$$

$$\iota(t_0) = \lambda_0, \quad \iota(t_1) = \lambda_1,$$

$$\eta \geq 0, \quad e^T \eta = 1, \quad e = (1, ...1) \in \mathbb{R}^p,$$

for $\varsigma \in \mathcal{G}, \ \gamma \in \mathcal{Q}, \sigma \in \mathcal{T}, \delta \in \mathcal{M}$.

We denote $\mathcal{D}_w = \{ (\iota, \kappa, \eta, \rho, \theta, \varsigma, \gamma, \sigma, \delta) \text{ satisfying conditions (1)–(4)} \}$ to be the feasible solution set to $(\mathcal{RWD} - \mathcal{P})$, and we name it as the robust feasible solution set to $(\mathcal{WD} - \mathcal{P})$.

Definition 6. *A feasible point* $(\bar{\iota}, \bar{\kappa}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\varsigma}, \bar{\gamma}, \bar{\sigma}, \bar{\delta}) \in \mathcal{D}_w$ *is considered to be the robust weak efficient solution to* $(\mathcal{WD} - \mathcal{P})$*, if there does not exist* $(\iota, \kappa, \eta, \rho, \theta, \varsigma, \gamma, \sigma, \delta) \in \mathcal{D}_w$ *satisfying*

$$\int_S \{ \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] + \rho^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) e + \theta^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) e \} dt$$

$$< \int_S \{ \left[h(\Pi, \bar{\varsigma}) - Q^0 z(\Pi, \bar{\gamma}) \right] + \rho^T f(\Pi, \iota_\alpha, \bar{\sigma}) e + \theta^T g(\Pi, \iota_\alpha, \bar{\delta}) e \} dt,$$

where $\bar{\Pi} := (t, \bar{\iota}(t), \bar{\kappa}(t))$.

The Mond–Weir robust dual model (see Mond and Weir [26]) associated with $(\mathcal{P})$, considering data uncertainty in both the objective and constraint functionals, is given as follows:

$$(\mathcal{MWD} - \mathcal{P}) \qquad \max_{(\iota(\cdot),\kappa(\cdot))} \int_S \left[h(\Pi, \varsigma) - Q^0 z(\Pi, \gamma) \right] dt$$

subject to

$$\eta^T \left[h_\lambda(\Pi, \varsigma) - Q^0 z_\lambda(\Pi, \gamma) \right] + \rho^T f_\lambda(\Pi, \iota_\alpha, \sigma) + \theta^T g_\lambda(\Pi, \iota_\alpha, \delta)$$

$$- D_\alpha \left[\rho^T f_{\lambda_\alpha}(\Pi, \iota_\alpha, \sigma) + \theta^T g_{\lambda_\alpha}(\Pi, \iota_\alpha, \delta) \right] = 0, \tag{5}$$

$$\eta^T \left[h_\pi(\Pi, \varsigma) - Q^0 z_\pi(\Pi, \gamma) \right] + \rho^T f_\pi(\Pi, \iota_\alpha, \sigma) + \theta^T g_\pi(\Pi, \iota_\alpha, \delta) = 0, \tag{6}$$

$$\rho^T f(\Pi, \iota_\alpha, \sigma) \geqq 0, \tag{7}$$

$$g(\Pi, \iota_\alpha, \delta) = 0, \tag{8}$$

$$\iota(t_0) = \lambda_0, \quad \iota(t_1) = \lambda_1, \tag{9}$$

$$\eta \geq 0, \quad e^T \eta = 1, \quad e = (1, ...1) \in \mathbb{R}^p. \tag{10}$$

The corresponding robust counterpart for $(\mathcal{MWD} - \mathcal{P})$ is stated as:

$$(\mathcal{RMWD} - \mathcal{P}) \qquad \max_{(\iota(.),\kappa(.))} \int_S \left[h(\Pi, \bar{\varsigma}) - Q^0 z(\Pi, \bar{\gamma}) \right] dt$$

subject to

$$\eta^T \left[h_\lambda(\Pi, \bar{\varsigma}) - Q^0 z_\lambda(\Pi, \bar{\gamma}) \right] + \rho^T f_\lambda(\Pi, \iota_\alpha, \bar{\sigma}) + \theta^T g_\lambda(\Pi, \iota_\alpha, \bar{\delta})$$

$$-D_\alpha \left[\rho^T f_{\lambda_\alpha}(\Pi, \iota_\alpha, \bar{\sigma}) + \theta^T g_{\lambda_\alpha}(\Pi, \iota_\alpha, \bar{\delta}) \right] = 0,$$

$$\eta^T \left[h_\pi(\Pi, \bar{\varsigma}) - Q^0 z_\pi(\Pi, \bar{\gamma}) \right] + \rho^T f_\pi(\Pi, \iota_\alpha, \bar{\sigma}) + \theta^T g_\pi(\Pi, \iota_\alpha, \bar{\delta}) = 0,$$

$$\rho^T f(\Pi, \iota_\alpha, \sigma) \geqq 0,$$

$$g(\Pi, \iota_\alpha, \delta) = 0,$$

$$\iota(t_0) = \lambda_0, \quad \iota(t_1) = \lambda_1,$$

$$\eta \geq 0, \quad e^T \eta = 1, \quad e = (1, ...1) \in \mathbb{R}^p,$$

for $\varsigma \in \mathcal{G}, \ \gamma \in \mathcal{Q}, \sigma \in \mathcal{T}, \delta \in \mathcal{M}$.

We denote $\mathcal{D}_{mw} = \{(\iota, \kappa, \eta, \rho, \theta, \varsigma, \gamma, \sigma, \delta)$ fulfilling (5)–(10)$\}$ to be the feasible solution set to $(\mathcal{RMWD} - \mathcal{P})$, and we call it the robust feasible solution set to $(\mathcal{MWD} - \mathcal{P})$.

Definition 7. *A feasible point* $(\bar{\iota}, \bar{\kappa}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\varsigma}, \bar{\gamma}, \bar{\sigma}, \bar{\delta}) \in \mathcal{D}_{mw}$ *is named a robust weak efficient solution to* $(\mathcal{MWD} - \mathcal{P})$ *if* $(\iota, \kappa, \eta, \rho, \theta, \varsigma, \gamma, \sigma, \delta) \in \mathcal{D}_{mw}$ *does not exist satisfying*

$$\int_S \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] dt < \int_S \left[h(\Pi, \bar{\varsigma}) - Q^0 z(\Pi, \bar{\gamma}) \right] dt.$$

Next, we associate a mixed robust dual model for $(\mathcal{P})$, as follows:

$$(m\mathcal{D} - \mathcal{P}) \qquad \max_{(\iota(\cdot), \kappa(\cdot))} \int_S \{ \left[h(\Pi, \varsigma) - Q^0 z(\Pi, \gamma) \right] + \rho^T f(\Pi, \iota_\alpha, \sigma) e$$

$$+ \theta^T g(\Pi, \iota_\alpha, \delta) e \} dt$$

subject to

$$\eta^T \left[h_\lambda(\Pi, \varsigma) - Q^0 z_\lambda(\Pi, \gamma) \right] + \rho^T f_\lambda(\Pi, \iota_\alpha, \sigma) + \theta^T g_\lambda(\Pi, \iota_\alpha, \delta)$$

$$-D_\alpha \left[\rho^T f_{\lambda_\alpha}(\Pi, \iota_\alpha, \sigma) + \theta^T g_{\lambda_\alpha}(\Pi, \iota_\alpha, \delta) \right] = 0, \tag{11}$$

$$\eta^T \left[h_\pi(\Pi, \varsigma) - Q^0 z_\pi(\Pi, \gamma) \right] + \rho^T f_\pi(\Pi, \iota_\alpha, \sigma) + \theta^T g_\pi(\Pi, \iota_\alpha, \delta) = 0, \tag{12}$$

$$\iota(t_0) = \lambda_0, \quad \iota(t_1) = \lambda_1, \tag{13}$$

$$\eta \geq 0, \quad e^T \eta = 1, \quad e = (1, ...1) \in \mathbb{R}^p, \tag{14}$$

$$\rho^T f(\Pi, \iota_\alpha, \sigma) \geqq 0, \tag{15}$$

$$g(\Pi, \iota_\alpha, \delta) = 0. \tag{16}$$

The corresponding robust counterpart for $(m\mathcal{D} - \mathcal{P})$ is stated as:

$$(\mathcal{Rm\mathcal{D}} - \mathcal{P}) \qquad \max_{(\iota(.), \kappa(.))} \int_S \{ \left[h(\Pi, \bar{\varsigma}) - Q^0 z(\Pi, \bar{\gamma}) \right] + \rho^T f(\Pi, \iota_\alpha, \bar{\sigma}) e$$

$$+ \theta^T g(\Pi, \iota_\alpha, \bar{\delta}) e \} dt$$

subject to

$$\eta^T \left[h_\lambda(\Pi, \bar{\varsigma}) - Q^0 z_\lambda(\Pi, \bar{\gamma}) \right] + \rho^T f_\lambda(\Pi, \iota_\alpha, \bar{\sigma}) + \theta^T g_\lambda(\Pi, \iota_\alpha, \bar{\delta})$$

$$-D_\alpha \left[\rho^T f_{\lambda_\alpha}(\Pi, \iota_\alpha, \bar{\sigma}) + \theta^T g_{\lambda_\alpha}(\Pi, \iota_\alpha, \bar{\delta}) \right] = 0,$$

$$\eta^T \left[h_\pi(\Pi, \bar{\varsigma}) - Q^0 z_\pi(\Pi, \bar{\gamma}) \right] + \rho^T f_\pi(\Pi, \iota_\alpha, \bar{\sigma}) + \theta^T g_\pi(\Pi, \iota_\alpha, \bar{\delta}) = 0,$$

$$\iota(t_0) = \lambda_0, \quad \iota(t_1) = \lambda_1,$$

$$\eta \geq 0, \quad e^T \eta = 1, \quad e = (1, \dots 1) \in \mathbb{R}^p,$$

$$\rho^T f(\Pi, \iota_\alpha, \sigma) \geqq 0,$$

$$g(\Pi, \iota_\alpha, \delta) = 0,$$

for $\varsigma \in \mathcal{G}, \ \gamma \in \mathcal{Q}, \sigma \in \mathcal{T}, \delta \in \mathcal{M}$.

We denote $\mathcal{D}_m = \{(\iota, \kappa, \eta, \rho, \theta, \varsigma, \gamma, \sigma, \delta)$ satisfying conditions (11)–(16)$\}$ to be the feasible solution set to $(\mathcal{R}m\mathcal{D} - \mathcal{P})$, and we call it as the robust feasible solution set to $(m\mathcal{D} - \mathcal{P})$.

Definition 8. *A feasible point* $(\bar{\iota}, \bar{\kappa}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\varsigma}, \bar{\gamma}, \bar{\sigma}, \bar{\delta}) \in \mathcal{D}_m$ *is named as a robust weak efficient solution to* $(m\mathcal{D} - \mathcal{P})$, *if there does not exist* $(\iota, \kappa, \eta, \rho, \theta, \varsigma, \gamma, \sigma, \delta) \in \mathcal{D}_m$ *fulfilling*

$$\int_S \left\{ \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] + \rho^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma})e + \theta^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta})e \right\} dt$$

$$< \int_S \left\{ \left[h(\Pi, \bar{\varsigma}) - Q^0 z(\Pi, \bar{\gamma}) \right] + \rho^T f(\Pi, \iota_\alpha, \bar{\sigma})e + \theta^T g(\Pi, \iota_\alpha, \bar{\delta})e \right\} dt.$$

In the following, we establish a robust weak-type duality theorem for $(\mathcal{P})$.

Theorem 2 (Robust weak duality theorem)**.** *Let* $(\bar{\lambda}, \bar{\pi})$ *and* $(\bar{\iota}, \bar{\kappa}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\varsigma}, \bar{\gamma}, \bar{\sigma}, \bar{\delta})$ *be the robust feasible solutions of* $(\mathcal{P})$ *and* $(m\mathcal{D} - \mathcal{P})$, *respectively. Assume that* $\max_{\varsigma \in \mathcal{G}} h(\bar{Y}, \varsigma) = h(\bar{Y}, \bar{\varsigma})$ *and* $\min_{\gamma \in \mathcal{Q}} z(\bar{Y}, \gamma) = z(\bar{Y}, \bar{\gamma})$. *Further, if* $\int_S \bar{\eta}^T \left[h(., \bar{\varsigma}) - Q^0 z(., \bar{\gamma}) \right] dt$, $\int_S \bar{\rho}^T f(., \bar{\sigma}) dt$ *and* $\int_S \bar{\theta}^T g(., \bar{\delta}) dt$ *are convex at* $(\bar{\iota}, \bar{\kappa})$, *then the following inequality cannot hold:*

$$\int_S \left[h(\bar{Y}, \bar{\varsigma}) - Q^0 z(\bar{Y}, \bar{\gamma}) \right] dt$$

$$< \int_S \left\{ \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] + \bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma})e + \bar{\theta}^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta})e \right\} dt.$$

Proof. Assume on the contrary that

$$\int_S \left\{ \max_{\varsigma \in \mathcal{G}} h(\bar{Y}, \varsigma) - Q^0 \min_{\gamma \in \mathcal{Q}} z(\bar{Y}, \gamma) \right\} dt < \int_S \left\{ \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] \right.$$

$$\left. + \bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma})e + \bar{\theta}^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta})e \right\} dt$$

is fulfilled. Since $\max_{\varsigma \in \mathcal{G}} h(\bar{Y}, \varsigma) - Q^0 \min_{\gamma \in \mathcal{Q}} z(\bar{Y}, \gamma) = h(\bar{Y}, \bar{\varsigma}) - Q^0 z(\bar{Y}, \bar{\gamma})$, we obtain

$$\int_S \left[h(\bar{Y}, \bar{\varsigma}) - Q^0 z(\bar{Y}, \bar{\gamma}) \right] dt < \int_S \left\{ \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] \right.$$

$$\left. + \bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma})e + \bar{\theta}^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta})e \right\} dt$$

is satisfied. As $(\bar{\lambda}, \bar{\pi})$ is the robust feasible solution to the problem $(\mathcal{P})$, it implies

$$\int_S \left\{ \left[h(\bar{Y}, \bar{\varsigma}) - Q^0 z(\bar{Y}, \bar{\gamma}) \right] + \bar{\rho}^T f(\bar{Y}, \bar{\lambda}_\alpha, \bar{\sigma})e + \bar{\theta}^T g(\bar{Y}, \bar{\lambda}_\alpha, \bar{\delta})e \right\} dt$$

$$< \int_S \left\{ \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] + \bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma})e + \bar{\theta}^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta})e \right\} dt.$$

By considering that $\bar{\eta} \geq 0$ and $\bar{\eta}^T e = 1$, therefore, the above inequality can be written as

$$\int_S \left\{ \bar{\eta}^T \left[h(\bar{Y}, \bar{\varsigma}) - Q^0 z(\bar{Y}, \bar{\gamma}) \right] + \bar{\rho}^T f(\bar{Y}, \bar{\lambda}_\alpha, \bar{\sigma}) + \bar{\theta}^T g(\bar{Y}, \bar{\lambda}_\alpha, \bar{\delta}) \right\} dt$$

$$< \int_S \{\bar{\eta}^T \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] + \bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) + \bar{\theta}^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) \} dt. \quad (17)$$

Since $\int_S \bar{\eta}^T \left[h(., \bar{\varsigma}) - Q^0 z(., \bar{\gamma}) \right] dt$, $\int_S \bar{\rho}^T f(., \bar{\sigma}) dt$ and $\int_S \bar{\theta}^T g(., \bar{\delta}) dt$ are convex at $(\bar{\iota}, \bar{\kappa})$, we have

$$\int_S \{\bar{\eta}^T \left[h(\bar{Y}, \bar{\varsigma}) - Q^0 z(\bar{Y}, \bar{\gamma}) \right] - \bar{\eta}^T \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] \} dt \geqq$$

$$\int_S (\bar{\lambda} - \bar{\iota}) \bar{\eta}^T \left[\frac{\partial h}{\partial \lambda}(\bar{\Pi}, \bar{\varsigma}) - Q^0 \frac{\partial z}{\partial \lambda}(\bar{\Pi}, \bar{\gamma}) \right] dt$$

$$+ \int_S (\bar{\pi} - \bar{\kappa}) \bar{\eta}^T \left[\frac{\partial h}{\partial \pi}(\bar{\Pi}, \bar{\varsigma}) - Q^0 \frac{\partial z}{\partial \pi}(\bar{\Pi}, \bar{\gamma}) \right] dt, \quad (18)$$

$$\int_S \{\bar{\rho}^T f(\bar{Y}, \bar{\lambda}_\alpha, \bar{\sigma}) - \bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) \} dt \geqq \int_S (\bar{\lambda} - \bar{\iota}) \bar{\rho}^T f_\lambda(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) dt$$

$$+ \int_S (\bar{\lambda}_\alpha - \bar{\iota}_\alpha) \bar{\rho}^T f_{\lambda_\alpha}(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) dt + \int_S (\bar{\pi} - \bar{\kappa}) \bar{\rho}^T f_\pi(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) dt \quad (19)$$

and

$$\int_S \{\bar{\theta}^T g(\bar{Y}, \bar{\lambda}_\alpha, \bar{\delta}) - \bar{\rho}^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) \} dt \geqq \int_S (\bar{\lambda} - \bar{\iota}) \bar{\theta}^T g_\lambda(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) dt$$

$$+ \int_S (\bar{\lambda}_\alpha - \bar{\iota}_\alpha) \bar{\theta}^T g_{\lambda_\alpha}(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) dt + \int_S (\bar{\pi} - \bar{\kappa}) \bar{\theta}^T g_\pi(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) dt. \quad (20)$$

Adding the inequalities (18)–(20) and using the dual constraints for the dual feasible solution $(\bar{\iota}, \bar{\kappa}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\varsigma}, \bar{\sigma}, \bar{\delta})$, we obtain

$$\int_S \{\bar{\eta}^T \left[h(\bar{Y}, \bar{\varsigma}) - Q^0 z(\bar{Y}, \bar{\gamma}) \right] + \bar{\rho}^T f(\bar{Y}, \bar{\lambda}_\alpha, \bar{\sigma}) + \bar{\theta}^T g(\bar{Y}, \bar{\lambda}_\alpha, \bar{\delta}) \} dt$$

$$\geqq \int_S \{\bar{\eta}^T \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] + \bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) + \bar{\theta}^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) \} dt,$$

which is a contradiction to the inequality (17). This completes the proof. □

In the following, we establish a robust strong-type duality theorem for $(\mathcal{P})$.

Theorem 3 (Robust strong duality theorem). *Let $(\bar{\lambda}, \bar{\pi})$ be a robust weak efficient solution to $(\mathcal{P})$. Consider that $\max_{\varsigma \in \mathcal{G}} \{h(\bar{Y}, \varsigma) - Q^0 \min_{\gamma \in Q} z(\bar{Y}, \gamma) \} = h(\bar{Y}, \bar{\varsigma}) - Q^0 z(\bar{Y}, \bar{\gamma})$ and the constraint qualification conditions hold for $(\mathcal{P})$. Then, $\bar{\eta} \in \mathbb{R}^p$, $\bar{\rho} = (\bar{\rho}_l(t)) \in \mathbb{R}_+^m, \bar{\theta} = (\bar{\theta}_b(t)) \in \mathbb{R}^n$ exist as the piecewise smooth functions, and $\bar{\sigma} \in \mathcal{T}, \bar{\delta} \in \mathcal{M}, \bar{\varsigma} \in \mathcal{G} \ \bar{\gamma} \in Q$ as the parameters of uncertainty such that $(\bar{\lambda}, \bar{\pi}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\varsigma}, \bar{\gamma}, \bar{\sigma}, \bar{\delta})$ is a robust feasible solution to $(m\mathcal{D} - \mathcal{P})$. Moreover, if Theorem 3.1 holds, then $(\bar{\lambda}, \bar{\pi}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\varsigma}, \bar{\gamma}, \bar{\sigma}, \bar{\delta})$ is a robust weak efficient solution to $(m\mathcal{D} - \mathcal{P})$.*

Proof. As $(\bar{\lambda}, \bar{\pi})$ is a robust weak efficient solution to $(\mathcal{P})$, therefore, by Theorem 1, $\bar{\eta} \in \mathbb{R}^p$, $\bar{\rho} = (\bar{\rho}_l(t)) \in \mathbb{R}_+^m, \bar{\theta} = (\bar{\theta}_b(t)) \in \mathbb{R}^n$ exist as the piecewise differentiable functions and $\bar{\sigma} \in \mathcal{T}, \bar{\delta} \in \mathcal{M}, \bar{\varsigma} \in \mathcal{G} \ \bar{\gamma} \in Q$ as the parameters of uncertainty, such that the conditions (1)–(4) hold at $(\bar{\lambda}, \bar{\pi})$. Hence, $(\bar{\lambda}, \bar{\pi}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\varsigma}, \bar{\gamma}, \bar{\sigma}, \bar{\delta})$ is a robust feasible solution to $(m\mathcal{D} - \mathcal{P})$ and the corresponding objective function values are equal. Suppose conditions (1)–(4) hold at $(\bar{\lambda}, \bar{\pi})$ and $(\bar{\lambda}, \bar{\pi}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\varsigma}, \bar{\gamma}, \bar{\sigma}, \bar{\delta})$ is not a weak efficient solution to $(m\mathcal{D} - \mathcal{P})$. Thus, $(\iota, \kappa, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\varsigma}, \bar{\gamma}, \bar{\sigma}, \bar{\delta})$ exists satisfying

$$\int_S \{ \left[h(\bar{Y}, \bar{\varsigma}) - Q^0 z(\bar{Y}, \bar{\gamma}) \right] + \rho^T f(\bar{Y}, \bar{\lambda}_\alpha, \bar{\sigma}) e + \theta^T g(\bar{Y}, \bar{\lambda}_\alpha, \bar{\delta}) e \} dt$$

$$< \int_S \{ \left[h(\Pi, \bar{\varsigma}) - Q^0 z(\Pi, \bar{\gamma}) \right] + \rho^T f(\Pi, \iota_\alpha, \bar{\sigma}) e + \theta^T g(\Pi, \iota_\alpha, \bar{\delta}) e \} dt.$$

From Theorem 1, we obtain

$$\int_S \left[h(\bar{Y}, \bar{\varsigma}) - Q^0 z(\bar{Y}, \bar{\gamma}) \right] dt$$

$$< \int_S \left[h(\Pi, \varsigma) - Q^0 z(\Pi, \bar{\gamma}) \right] + \rho^T f(\Pi, \iota_\alpha, \bar{\sigma}) e + \theta^T g(\Pi, \iota_\alpha, \bar{\delta}) e \} dt.$$

Since $\max_{\varsigma \in \mathcal{G}} \{ h(\bar{Y}, \varsigma) - Q^0 \min_{\gamma \in Q} z(\bar{Y}, \gamma) \} = h(\bar{Y}, \bar{\xi}) - Q^0 z(\bar{Y}, \bar{\gamma})$, we have

$$\int_S \{ \max_{\varsigma \in \mathcal{G}} h(\bar{Y}, \varsigma) - Q^0 \min_{\gamma \in Q} z(\bar{Y}, \gamma) \} dt <$$

$$\int_S \{ \left[h(\Pi, \bar{\xi}) - Q^0 z(\Pi, \bar{\gamma}) \right] + \rho^T f(\Pi, \iota_\alpha, \bar{\sigma}) e + \theta^T g(\Pi, \iota_\alpha, \bar{\delta}) e \} dt,$$

which is a contradiction to Theorem 3.1. Hence, $(\bar{\lambda}, \bar{\pi}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\xi}, \bar{\gamma}, \bar{\sigma}, \bar{\delta})$ is a robust weak efficient solution in $(m\mathcal{D} - \mathcal{P})$. $\square$

Next, we establish a robust strict converse-type duality result for $(\mathcal{P})$.

Theorem 4 (Robust strict converse duality theorem). *Let $(\bar{\iota}, \bar{\kappa}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\xi}, \bar{\gamma}, \bar{\sigma}, \bar{\delta})$ be a robust feasible solution in $(m\mathcal{D} - \mathcal{P})$. Consider that $\max_{\varsigma \in \mathcal{G}} \{ h(\bar{Y}, \varsigma) - Q^0 \min_{\gamma \in Q} z(\bar{Y}, \gamma) \} = h(\bar{Y}, \bar{\xi}) - Q^0 z(\bar{Y}, \bar{\gamma})$ and $\int_S \bar{\eta}^T \left[h(., \bar{\xi}) - Q^0 z(., \bar{\gamma}) \right] dt$, $\int_S \bar{\rho}^T f(., \bar{\sigma}) dt$ and $\int_S \bar{\theta}^T g(., \bar{\delta}) dt$ are strictly convex at $(\bar{\iota}, \bar{\kappa})$. If $(\bar{\lambda}, \bar{\pi}) \in \mathcal{S}$ such that*

$$\int_S \left[h(\bar{Y}, \bar{\xi}) - Q^0 z(\bar{Y}, \bar{\gamma}) \right] dt$$

$$= \int_S \{ \left[h(\bar{\Pi}, \bar{\xi}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] + \bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) e + \bar{\theta}^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) e \} dt,$$

then, $(\bar{\lambda}, \bar{\pi})$ is a robust weak efficient solution in $(\mathcal{P})$.

Proof. As $(\bar{\lambda}, \bar{\pi}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\xi}, \bar{\gamma}, \bar{\sigma}, \bar{\delta})$ is a robust feasible solution in $(m\mathcal{D} - \mathcal{P})$, on multiplying the inequality (11) and (12) by $(\hat{\lambda} - \bar{\iota})$ and $(\hat{\pi} - \bar{\kappa})$, respectively, and then integrating, we obtain

$$\int_S (\hat{\lambda} - \bar{\iota}) \{ \bar{\eta}^T \left[\frac{\partial h}{\partial \lambda} (\bar{\Pi}, \bar{\xi}) - Q^0 \frac{\partial z}{\partial \lambda} (\bar{\Pi}, \bar{\gamma}) \right] + \bar{\rho}^T f_\lambda (\bar{\Pi}, \bar{\iota}_\alpha(t), \bar{\sigma})$$

$$+ \bar{\theta}^T g_\lambda (\bar{\Pi}, \bar{\iota}_\alpha(t), \bar{\delta}) - D_\alpha \left[\bar{\rho}^T f_{\lambda_\alpha} (\bar{\Pi}, \bar{\iota}_\alpha(t), \bar{\sigma}) + \bar{\theta}^T g_{\lambda_\alpha} (\bar{\Pi}, \bar{\iota}_\alpha(t), \bar{\delta}) \right] \} dt$$

$$+ \int_S (\hat{\pi} - \bar{\kappa}) \{ \bar{\eta}^T \left[\frac{\partial h}{\partial \pi} (\bar{\Pi}, \bar{\xi}) - Q^0 \frac{\partial z}{\partial \pi} (\bar{\Pi}, \bar{\gamma}) \right] + \bar{\rho}^T f_\pi (\bar{\Pi}, \bar{\iota}_\alpha(t), \bar{\sigma})$$

$$+ \bar{\theta}^T g_\pi (\bar{\Pi}, \bar{\iota}_\alpha(t), \bar{\delta}) \} dt = 0. \tag{21}$$

Now, we assume on the contrary that $(\bar{\lambda}, \bar{\pi})$ is not a robust weak efficient solution in $(\mathcal{P})$. Consequently, $(\hat{\lambda}, \hat{\pi}) \in \mathcal{S}$ exists such that

$$\int_S \{ \max_{\varsigma \in \mathcal{G}} h(\hat{Y}, \varsigma) - Q^0 \min_{\gamma \in Q} z(\hat{Y}, \gamma) \} dt$$

$$< \int_S \{ \max_{\varsigma \in \mathcal{G}} h(\bar{Y}, \varsigma) - Q^0 \min_{\gamma \in Q} z(\bar{Y}, \gamma) \} dt,$$

or, equivalently,

$$\int_S \left[h(\hat{Y}, \bar{\xi}) - Q^0 z(\hat{Y}, \bar{\gamma}) \right] dt < \int_S \left[h(\bar{Y}, \bar{\xi}) - Q^0 z(\bar{Y}, \bar{\gamma}) \right] dt.$$

By considering the hypothesis,

$$\int_S \left[h(\bar{Y}, \bar{\xi}) - Q^0 z(\bar{Y}, \bar{\gamma}) \right] dt$$

$$= \int_S \{ \left[h(\bar{\Pi}, \bar{\xi}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] + \bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) e + \bar{\theta}^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) e \} dt,$$

therefore, the above inequality yields

$$\int_S \left[h(\hat{Y}, \bar{\varsigma}) - Q^0 z(\hat{Y}, \bar{\gamma}) \right] dt < \int_S \{ \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] +$$

$$\bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) e + \bar{\theta}^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) e \} dt.$$

Since $\bar{\eta} > 0$, therefore

$$\int_S \bar{\eta}^T \left[h(\hat{Y}, \bar{\varsigma}) - Q^0 z(\hat{Y}, \bar{\gamma}) \right] dt < \int_S \{ \bar{\eta}^T \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] +$$

$$\bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) + \bar{\theta}^T g(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) \} dt. \tag{22}$$

By using the strict convexity property of $\int_S \bar{\eta}^T \left[h(., \bar{\varsigma}) - Q^0 z(,, \bar{\gamma}) \right] dt$ at $(\bar{\iota}, \bar{\kappa})$, we have

$$\int_S \{ \bar{\eta}^T \left[h(\hat{Y}, \bar{\varsigma}) - Q^0 z(\hat{Y}, \bar{\gamma}) \right] - \bar{\eta}^T \left[h(\bar{\Pi}, \bar{\varsigma}) - Q^0 z(\bar{\Pi}, \bar{\gamma}) \right] \} dt$$

$$> \int_S (\hat{\lambda} - \bar{\iota}) \bar{\eta}^T \left[\frac{\partial h}{\partial \lambda}(\bar{\Pi}, \bar{\varsigma}) - Q^0 \frac{\partial z}{\partial \lambda}(\bar{\Pi}, \bar{\gamma}) \right] dt$$

$$+ \int_S (\hat{\pi} - \bar{\kappa}) \bar{\eta}^T \left[\frac{\partial h}{\partial \pi}(\bar{\Pi}, \bar{\varsigma}) - Q^0 \frac{\partial z}{\partial \pi}(\bar{\Pi}, \bar{\gamma}) \right] dt,$$

which, together with the inequality (22) and feasibility of $(\bar{\iota}, \bar{\kappa})$, gives

$$\int_S (\hat{\lambda} - \bar{\iota}) \bar{\eta}^T \left[\frac{\partial h}{\partial \lambda}(\bar{\Pi}, \bar{\varsigma}) - Q^0 \frac{\partial z}{\partial \lambda}(\bar{\Pi}, \bar{\gamma}) \right] dt$$

$$+ \int_S (\hat{\pi} - \bar{\kappa}) \bar{\eta}^T \left[\frac{\partial h}{\partial \pi}(\bar{\Pi}, \bar{\varsigma}) - Q^0 \frac{\partial z}{\partial \pi}(\bar{\Pi}, \bar{\gamma}) \right] dt < 0. \tag{23}$$

Again, by the strict convexity property of $\int_S \bar{\rho}^T f(., \bar{\sigma}) dt$ at $(\bar{\iota}, \bar{\kappa})$, we have

$$\int_S \{ \bar{\rho}^T f(\hat{Y}, \bar{\lambda}_\alpha, \bar{\sigma}) - \bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) \} dt > \int_S (\hat{\lambda} - \bar{\iota}) \bar{\rho}^T f_\lambda(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) dt$$

$$+ \int_S (\hat{\lambda}_\alpha - \bar{\iota}_\alpha) \bar{\rho}^T f_{\lambda_\alpha}(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) dt + \int_S (\hat{\pi} - \bar{\kappa}) \bar{\rho}^T f_\pi(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) dt. \tag{24}$$

Also, as $(\hat{\lambda}, \hat{\pi})$ and $(\bar{\lambda}, \bar{\pi}, \bar{\eta}, \bar{\rho}, \bar{\theta}, \bar{\varsigma}, \bar{\sigma}, \bar{\delta})$ are robust feasible solutions in $(\mathcal{P})$ and $(m\mathcal{D} - \mathcal{P})$, respectively, we obtain

$$\int_S \bar{\rho}^T f(\hat{Y}, \bar{\lambda}_\alpha, \bar{\sigma}) dt - \int_S \bar{\rho}^T f(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) dt \leqq 0,$$

which, together with inequality (24), results in

$$\int_S (\hat{\lambda} - \bar{\iota}) \bar{\rho}^T f_\lambda(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) dt + \int_S (\hat{\lambda}_\alpha - \bar{\iota}_\alpha) \bar{\rho}^T f_{\lambda_\alpha}(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) dt$$

$$+ \int_S (\hat{\pi} - \bar{\kappa}) \bar{\rho}^T f_\pi(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\sigma}) dt < 0. \tag{25}$$

Similarly, since $\int_S \bar{\theta}^T g(., \lambda_\alpha, \bar{\delta}) dt$ is also strictly convex function, we obtain

$$\int_S (\hat{\lambda} - \bar{\iota}) \bar{\theta}^T g_\lambda(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) dt + \int_S (\hat{\lambda}_\alpha - \bar{\iota}_\alpha) \bar{\theta}^T g_{\lambda_\alpha}(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) dt$$

$$+ \int_S (\hat{\pi} - \bar{\kappa}) \bar{\theta}^T g_\pi(\bar{\Pi}, \bar{\iota}_\alpha, \bar{\delta}) dt < 0. \tag{26}$$

On adding the inequalities (23), (25), and (26), we obtain the following inequality:

$$\int_S (\hat{\lambda} - \bar{\iota}) \Big\{ \bar{\eta}^T \Big[\frac{\partial h}{\partial \lambda}(\bar{\Pi}, \bar{\varsigma}) - Q^0 \frac{\partial z}{\partial \lambda}(\bar{\Pi}, \bar{\gamma}) \Big] + \bar{\rho}^T f_\lambda(\bar{\Pi}, \bar{\iota}_\alpha(t), \bar{\sigma})$$

$$+ \bar{\theta}^T g_\lambda(\bar{\Pi}, \bar{\iota}_\alpha(t), \bar{\delta}) - D_\alpha \Big[\bar{\rho}^T f_{\lambda_\alpha}(\bar{\Pi}, \bar{\iota}_\alpha(t), \bar{\sigma}) + \bar{\theta}^T g_{\lambda_\alpha}(\bar{\Pi}, \bar{\iota}_\alpha(t), \bar{\delta}) \Big] \Big\} dt$$

$$+ \int_S (\hat{\pi} - \bar{\kappa}) \Big\{ \bar{\eta}^T \Big[\frac{\partial h}{\partial \pi}(\bar{\Pi}, \bar{\varsigma}) - Q^0 \frac{\partial z}{\partial \pi}(\bar{\Pi}, \bar{\gamma}) \Big] + \bar{\rho}^T f_\pi(\bar{\Pi}, \bar{\iota}_\alpha(t), \bar{\sigma})$$

$$+ \bar{\theta}^T g_\pi(\bar{\Pi}, \bar{\iota}_\alpha(t), \bar{\delta}) \Big\} dt < 0,$$

which contradicts the inequality (21). This completes the proof. □

Remark 1. *(i) In order to justify the main results derived in the paper, some illustrative applications and numerical simulations can be consulted by the reader in the recent research work of Jayswal et al. [27].*

(ii) Regarding the future research directions associated with this paper, we could mention the study of the case where the second-order partial derivatives are presented, as well as the situation when the involved functionals are not necessarily (strictly) convex.

4. Conclusions

This paper established three robust mixed-type duality theorems, namely, weak, strong, and strict converse dual. Based on Wolfe- and Mond–Weir-type dualities, we formulated a robust mixed-type dual problem and, under the suitable convexity assumptions of the involved functionals, we established some equivalence results between the solution sets of the considered models.

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