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Enhanced Effectiveness in Various Ladder Graphs Based on the *F*-Centroidal Meanness Criterion

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Abstract: Graph labeling allows for the representation of additional attributes or properties associated with the vertices, edges, or both of graphs. This can provide a more comprehensive and detailed representation of the system being modeled, allowing for a richer analysis and interpretation of the graph. Graph labeling in ladder graphs has a wide range of applications in engineering, computer science, physics, biology, and other fields. It can be applied to various problem domains, such as image processing, wireless sensor networks, VLSI design, bioinformatics, social network analysis, transportation networks, and many others. The versatility of ladder graphs and the ability to apply graph labeling to them make them a powerful tool for modeling and analyzing diverse systems. If a function Υ is an injective vertex assignment in $\{1, 2, \ldots q+1\}$ and the inductive edge assignment function Υ^* in $\{1, 2, \ldots q\}$ is expressed as a graph with q edges, defined as $\Upsilon^*(uv) = \left\lfloor \frac{2\,[\Upsilon(u)^2 + \Upsilon(u)\Upsilon(v) + \Upsilon(v)^2]}{3\,[\Upsilon(u)+\Upsilon(v)]} \right\rfloor$, then the function is referred to as *F*-centroidal mean labeling. This is known as the *F*-centroidal mean criterion. Here, we have determined the *F*-centroidal mean criteria of the graph ladder, slanting ladder, triangular ladder, $TL_n \circ S_m$, $SL_n \circ S_m$ for $m \leq 2$, double-sided step ladder, D_n^*, and diamond ladder.

Keywords: labeling; *F*-centroidal mean labeling; *F*-centroidal mean graph

MSC: 05C78; 05C12; 05C38; 05C90

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1. Introduction

Ladder graphs can be used for image segmentation, where the task is to partition an image into distinct regions based on their characteristics. Graph labeling can be employed to assign labels to the vertices or edges of a ladder graph, representing the image pixels or their relationships. Ladder graphs with labeled edges can model the adjacency of pixels in an image, and the labels can represent attributes such as color, intensity, or texture. Image segmentation using ladder graphs and graph labeling can have applications in computer vision, medical imaging, and image analysis for engineering tasks such as object recognition, image understanding, and pattern recognition. Graph labeling can be applied to represent different characteristics of the sensor nodes or the links between them, such as node locations, sensing capabilities, or communication strengths. Labeled ladder graphs can help in designing efficient routing algorithms, optimizing network performance, and managing sensor networks for various applications, including environmental monitoring, smart grids, and industrial automation. Also, ladder graphs with graph labeling can be employed in VLSI design, where complex digital circuits are implemented on integrated circuits. Graph labeling can represent the characteristics of circuit components, such as gates, flip-flops, or interconnects, and their relationships in the circuit. Labeled ladder graphs can be used for tasks such as circuit optimization, layout generation, and logic synthesis, enabling engineers to design and optimize VLSI circuits for various applications, including microprocessors, digital signal processing, and communication systems. Ladder graphs with graph labeling can be utilized in bioinformatics, which is the application of computational techniques to analyze biological data. Graph labeling can represent biological entities, such as DNA sequences, protein interactions, or metabolic pathways, and their relationships in a biological system. Labeled ladder graphs can be used for tasks such as gene expression analysis, protein–protein interaction prediction, and metabolic pathway reconstruction, helping researchers in understanding biological processes and designing bio-informatics algorithms for biological data analysis. Moreover, it can be employed in social network analysis, which involves studying the relationships between individuals in a social network. Graph labeling can represent the attributes or characteristics of individuals, such as age, gender, occupation, or interests, and the relationships between them, such as friendships, collaborations, or influence. Labeled ladder graphs can be used for tasks such as community detection, sentiment analysis, and information diffusion analysis in social networks, enabling researchers to gain insights into social structures, behaviors, and dynamics.

We adhere to the notations and terminology in [1,2]. The ladder graph L_n is defined with $L_n = P_n \times K_2$, where P_n is a path containing n nodes and K_2 is a two-vertex complete graph. The slanting ladder SL_n is a graph created by combining the paths $u_1, u_2, \ldots u_n$ and $v_1, v_2, \ldots v_n$ with $u_{\alpha+1}, 1 \leq \alpha \leq n-1$. The triangular ladder TL_n, for $n \geq 2$ is a graph formed by merging two pathways using $u_1, u_2, \ldots u_n$ and $v_1, v_2, \ldots v_n$ by combining the edges $u_\alpha v_\alpha$, $1 \leq \alpha \leq n$ and $u_\alpha v_{\alpha+1}$, $1 \leq \alpha \leq n-1$. We include [3] for an in-depth investigation of graph labeling. The authors discussed the FRSM for the line graphs of $[P_n; S_1]$ and $S(P_n \circ S_1)$ in [4]. C_{exp} average assignments and $(1,1,1)$ face labelings for generalised prism are described in [5,6], respectively. In [7,8], Alanazi et al. talked about the classical meanness of the double-sided step graph. The super (a,d)-edge-anti magic total characteristics of graphs have been highlighted by Dafik Slamin et al. in [9]. In [10], Moussa and Badr demonstrated the odd gracefulness of ladder graphs. The authors of [11,12] stressed the importance of the edge even graceful labeling. Deb and Limaye talked about the elegant labelings of triangular snakes in [13], and Diefenderfer et al. examined the prime vertex labelings of various graph families in [14]. We highlighted some results in ladder graphs according to the F-centroidal meanness property, which was inspired by such remarkable investigation into the subject of F-centroidal mean graph assignments in [15,16]. If a function Υ is an injective vertex assignment in $\{1,2,\ldots q+1\}$ and inductive edge assignment function Υ^* in $\{1,2,\ldots q\}$ is expressed as a graph with q edges, defined as

$$\Upsilon^*(uv) = \left\lfloor \frac{2\left[\Upsilon(u)^2 + \Upsilon(u)\Upsilon(v) + \Upsilon(v)^2\right]}{3\left[\Upsilon(u) + \Upsilon(v)\right]} \right\rfloor,$$

then the function is referred to as F-centroidal mean labeling. This is known as the F-centroidal mean criterion. With regard to our criteria, Figure 1 highlights the F-centroidal mean labeling of cycle C_4. The node and link assignment sets of C_4 are $\{1,2,4,5\}$ and $\{1,2,3,4\}$. After the assignments of C_4, it obeys the conditions for F-centroidal mean requirements.

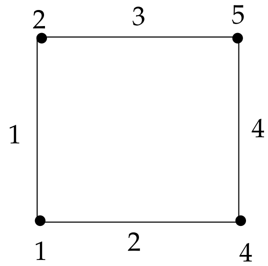

Figure 1. An assignment of nodes and links of C_4 based on the F-centroidal mean criterion.

2. Main Results

Based on the definition of the F-centroidal mean requirement, the injective node assignment is $\{1, 2, 3, \ldots, q+1\}$ and the generated bijective link assignment is $\{1, 2, 3, \ldots, q\}$; we will discuss the F-centroidal meanness of the graphs ladder, slanting ladder, triangular ladder, $TL_n \circ S_m$, $SL_n \circ S_m$ for $m \leq 2$, double-sided step ladder, D_n^*, and diamond ladder.

Theorem 1. *The ladder graph L_n permits the F-centroidal mean requirement for $n \geq 1$.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of the ladder graph L_n.

Then, the following description of $\Upsilon : V(L_n) \to \{1, 2, 3, \ldots, 3n-1\}$ is provided.

$$\Upsilon(u_\alpha) = 3\alpha - 1, \text{ for } 1 \leq \alpha \leq n \text{ and}$$
$$\Upsilon(v_\alpha) = 3\alpha - 2, \text{ for } 1 \leq \alpha \leq n.$$

After that, the generated line assignment Υ^* is accomplished.

$$\Upsilon^*(u_\alpha u_{\alpha+1}) = 3\alpha, \text{ for } n-1 \geq \alpha \geq 1,$$
$$\Upsilon^*(v_\alpha v_{\alpha+1}) = -1 + 3\alpha, \text{ for } n-1 \geq \alpha \geq 1 \text{ and}$$
$$\Upsilon^*(u_\alpha v_\alpha) = 3\alpha - 2, \text{ for } 1 \leq \alpha \leq n.$$

As a result, the graph permits the F-centroidal mean requirement. □

Theorem 2. *The slanting ladder graph SL_n permits the F-centroidal mean requirement for $n \geq 2$.*

Proof. Let the vertex set of SL_n be $\{u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n\}$ and the edge set of SL_n.

Then, the following description of $\Upsilon : V(SL_n) \to \{1, 2, 3, \ldots, 3n-2\}$ is provided.

$$\Upsilon(u_\alpha) = -4 + 3\alpha, \text{ for } n \geq \alpha \geq 2,$$
$$\Upsilon(v_\alpha) = 3\alpha, \text{ for } n-1 \geq \alpha \geq 1 \text{ and}$$
$$\Upsilon(v_n) = -2 + 3n.$$
$$\Upsilon(u_1) = 1.$$

After that, the generated line assignment Υ^* is accomplished.

$$\Upsilon^*(v_\alpha v_{\alpha+1}) = 1 + 3\alpha, \text{ for } n-2 \geq \alpha \geq 1,$$
$$\Upsilon^*(v_{n-1} v_n) = -3 + 3n \text{ and}$$
$$\Upsilon^*(v_\alpha u_{\alpha+1}) = -1 + 3\alpha, \text{ for } n-1 \geq \alpha \geq 1,$$
$$\Upsilon^*(u_\alpha u_{\alpha+1}) = \begin{cases} 1, & \alpha = 1 \\ -3 + 3\alpha, & n-1 \geq \alpha \geq 2. \end{cases}$$

As a result, the graph permits the F-centroidal mean requirement. □

Theorem 3. *The triangular ladder graph TL_n permits the F-centroidal mean requirement for $n \geq 2$.*

Proof. Let $\{u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of TL_n.

Then, the following description of $\Upsilon : V(TL_n) \to \{1, 2, 3, \ldots, 4n-2\}$ is provided.

$$\begin{aligned} \Upsilon(v_\alpha) &= 4\alpha - 4, \text{ for } 2 \leq \alpha \leq n, \\ \Upsilon(v_1) &= 1 \text{ and} \\ \Upsilon(u_\alpha) &= 4\alpha - 2, \text{ for } 1 \leq \alpha \leq n. \end{aligned}$$

After that, the generated line assignment Υ^* is accomplished.

$$\begin{aligned} \Upsilon^*(u_\alpha u_{\alpha+1}) &= 4\alpha, \text{ for } 1 \leq \alpha \leq n-1, \\ \Upsilon^*(u_\alpha v_\alpha) &= 4\alpha - 3, \text{ for } 1 \leq \alpha \leq n, \\ \Upsilon^*(u_\alpha v_{\alpha+1}) &= 4\alpha - 1, \text{ for } 1 \leq \alpha \leq n \text{ and} \\ \Upsilon^*(v_\alpha v_{\alpha+1}) &= 4\alpha - 2, \text{ for } 1 \leq \alpha \leq n-1. \end{aligned}$$

As a result, the graph permits the F-centroidal mean requirement. □

Theorem 4. *The graph $TL_n \circ S_m$ permits the F-centroidal mean requirement for $n \geq 2$ and $m \leq 2$.*

Proof. Let $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ be the vertices of the triangular ladder TL_n. Let m represent the number of nodes in the graph S_m. Let $u_1^{(\alpha)}, u_2^{(\alpha)}, u_3^{(\alpha)}, \ldots, u_m^{(\alpha)}$ and $v_1^{(\alpha)}, v_2^{(\alpha)}, v_3^{(\alpha)}, \ldots, v_m^{(\alpha)}$ be the pendant vertices attached at each u_α and v_α, respectively, for $1 \leq \alpha \leq n$.

Case i. $m = 1$

Assume that $n \geq 3$.

Then, the following description of $\Upsilon : V(TL_n \circ S_1) \longrightarrow \{1, 2, 3, \ldots, 6n-2\}$ is provided.

$$\begin{aligned} \Upsilon(u_\alpha) &= 6\alpha - 3, \text{ for } 1 \leq \alpha \leq n, \\ \Upsilon(v_\alpha) &= \begin{cases} 5, & \alpha = 2 \\ -4 + 6\alpha, & \alpha \text{ is odd and } n \geq \alpha \geq 1, \\ -5 + 6\alpha, & \alpha \text{ is even and } n \geq \alpha \geq 4, \end{cases} \\ \Upsilon(u_1^{(\alpha)}) &= \begin{cases} 7, & \alpha = 1 \\ 6\alpha - 2, & 2 \leq \alpha \leq n \text{ and} \end{cases} \\ \Upsilon(v_1^{(\alpha)}) &= \begin{cases} 11, & \alpha = 2 \\ -5 + 6\alpha, & \alpha \text{ is odd and } n \geq \alpha \geq 1, \\ -4 + 6\alpha, & \alpha \text{ is even and } n \geq \alpha \geq 4. \end{cases} \end{aligned}$$

After that, the generated line assignment Υ^* is accomplished.

$$\begin{aligned} \Upsilon^*(u_\alpha u_{\alpha+1}) &= 6\alpha, \text{ for } 1 \leq \alpha \leq n-1, \\ \Upsilon^*(u_\alpha v_\alpha) &= \begin{cases} 5\alpha - 3, & 1 \leq \alpha \leq 2 \\ 6\alpha - 4, & 3 \leq \alpha \leq n, \end{cases} \\ \Upsilon^*(u_\alpha u_1^{(\alpha)}) &= \begin{cases} 5, & \alpha = 1 \\ 6\alpha - 3, & 2 \leq \alpha \leq n\,, \end{cases} \\ \Upsilon^*(v_\alpha v_1^{(\alpha)}) &= \begin{cases} 7\alpha - 6, & 1 \leq \alpha \leq 2 \\ -5 + 6\alpha, & 3 \leq \alpha \leq n \end{cases} \\ \Upsilon^*(u_\alpha v_{\alpha+1}) &= \begin{cases} 4, & \alpha = 1 \\ 6\alpha - 1, & 2 \leq \alpha \leq n-1 \text{ and} \end{cases} \end{aligned}$$

$$\Upsilon^*(v_\alpha v_{\alpha+1}) = \begin{cases} 3, & \alpha = 1 \\ -2+6\alpha, & 2 \le \alpha \le n-1. \end{cases}$$

Case ii. $m = 2$

Then, the following description of $\Upsilon : V(TL_n \circ S_2) \longrightarrow \{1, 2, 3, \ldots, 8n-2\}$ is provided.

$$\Upsilon(u_\alpha) = 8\alpha - 3, \text{ for } 1 \le \alpha \le n,$$
$$\Upsilon(v_\alpha) = \begin{cases} 7\alpha - 5, & 1 \le \alpha \le 2 \\ 8\alpha - 6, & 3 \le \alpha \le n \end{cases}$$
$$\Upsilon\left(u_1^{(\alpha)}\right) = 8\alpha - 4, \text{ for } 1 \le \alpha \le n,$$
$$\Upsilon\left(u_2^{(\alpha)}\right) = 8\alpha - 2, \text{ for } 1 \le \alpha \le n,$$
$$\Upsilon(v_1^{(\alpha)}) = \begin{cases} 7\alpha - 6, & \alpha = 2 \\ 8\alpha - 10, & 3 \le \alpha \le n \end{cases} \text{ and}$$
$$\Upsilon^*\left(v_2^{(\alpha)}\right) = 8\alpha - 5, \text{ for } 1 \le \alpha \le n.$$

After that, the generated line assignment Υ^* is accomplished.

$$\Upsilon^*(u_\alpha u_{\alpha+1}) = 8\alpha + 1, \text{ for } 1 \le \alpha \le n-1,$$
$$\Upsilon^*(v_\alpha v_{\alpha+1}) = 8\alpha - 2, \text{ for } 1 \le \alpha \le n,$$
$$\Upsilon^*\left(u_\alpha u_1^{(\alpha)}\right) = 8\alpha - 4, \text{ for } 1 \le \alpha \le n,$$
$$\Upsilon^*\left(u_\alpha u_2^{(\alpha)}\right) = 8\alpha - 3, \text{ for } 1 \le \alpha \le n,$$
$$\Upsilon^*(v_\alpha v_1^{(\alpha)}) = \begin{cases} 1, & \alpha = 1 \\ 8\alpha - 8, & 2 \le \alpha \le n, \end{cases}$$
$$\Upsilon^*\left(v_\alpha v_2^{(\alpha)}\right) = 8\alpha - 6, \text{ for } 1 \le \alpha \le n,$$
$$\Upsilon^*(u_\alpha v_\alpha) = 8\alpha - 5, \text{ for } 1 \le \alpha \le n \textit{ and}$$
$$\Upsilon^*(u_\alpha v_{\alpha+1}) = 8\alpha - 1, \text{ for } 1 \le \alpha \le n-1.$$

As a result, the graph permits the F-centroidal mean requirement.

Figure 2 demonstrates the assignment of nodes and links of $TL_2 \circ S_1$ based on the F-centroidal mean criterion.

An F-centroidal mean labeling of $TL_2 \circ S_1$ for $n = 2$ is thus obtained. □

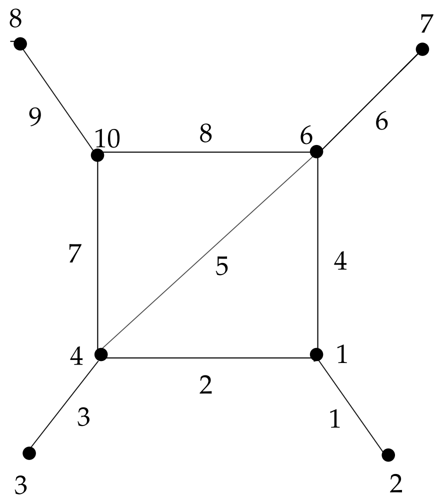

Figure 2. An assignment of nodes and links of $TL_2 \circ S_1$ based on the F-centroidal mean criterion.

Theorem 5. *The graph $SL_n \circ S_m$ permits the F-centroidal mean requirement for $n \geq 2$ and $m \leq 2$.*

Proof. Let $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ be the vertices of the slanting ladder SL_n.

Case i. $m = 1$

Then, the following description of $\Upsilon : V(SL_n \circ S_1) \longrightarrow \{1, 2, 3, \ldots, 5n-2\}$ is provided.

$$\Upsilon(u_\alpha) = \begin{cases} \alpha + 1, & 1 \leq \alpha \leq 2 \\ 5\alpha - 6, & 3 \leq \alpha \leq n, \end{cases}$$

$$\Upsilon(v_\alpha) = \begin{cases} 5\alpha, & n-1 \geq \alpha \geq 1 \\ -2 + 5\alpha, & n = \alpha, \end{cases}$$

$$\Upsilon(u_1^{(\alpha)}) = \begin{cases} -2 + 3\alpha, & 1 \leq \alpha \leq 2 \\ -7 + 5\alpha, & 3 \leq \alpha \leq n \text{ and} \end{cases}$$

$$\Upsilon(v_1^{(\alpha)}) = \begin{cases} 1 + 5\alpha, & n-1 \geq \alpha \geq 1 \\ -3 + 5\alpha, & n = \alpha. \end{cases}$$

After that, the generated line assignment Υ^* is accomplished.

$$\Upsilon^*(u_\alpha u_{\alpha+1}) = \begin{cases} 2, & 1 = \alpha \\ -4 + 5\alpha, & n-1 \geq \alpha \geq 2, \end{cases}$$

$$\Upsilon^*(v_\alpha v_{\alpha+1}) = \begin{cases} 2 + 5\alpha, & n-2 \leq \alpha \leq 1 \\ 1 + 5\alpha, & n-1 = \alpha, \end{cases}$$

$$\Upsilon^*(u_\alpha u_1^{(\alpha)}) = \begin{cases} 1, & \alpha = 1 \\ 5\alpha - 7, & 2 \leq \alpha \leq n, \end{cases}$$

$$\Upsilon^*(u_\alpha u_{\alpha-1}) = -6 + 5\alpha, \text{ for } 2 \leq \alpha \leq n,$$

$$\Upsilon^*(v_\alpha v_1^{(\alpha)}) = \begin{cases} 5\alpha, & 1 \leq \alpha \leq n-1 \\ 5\alpha - 3, & \alpha = n \text{ and} \end{cases}$$

Case ii. $m = 2$

Then, the following description of $\Upsilon : V(SL_n \circ S_2) \longrightarrow \{1, 2, 3, \ldots, 7n-2\}$ is provided.

$$\Upsilon(u_\alpha) = \begin{cases} 2, & \alpha = 1 \\ 5, & \alpha = 2 \\ 7\alpha - 6, & 3 \leq \alpha \leq n, \end{cases}$$

$$\Upsilon(v_\alpha) = \begin{cases} 7, & \alpha = 1 \\ 7\alpha - 2, & 2 \leq \alpha \leq n, \end{cases}$$

$$\Upsilon(u_1^{(\alpha)}) = \begin{cases} 3\alpha - 2, & 1 \leq \alpha \leq 2 \\ 7\alpha - 7, & 3 \leq \alpha \leq n, \end{cases}$$

$$\Upsilon(u_2^{(\alpha)}) = \begin{cases} 3\alpha, & 1 \leq \alpha \leq 2 \\ 7\alpha - 4, & 3 \leq \alpha \leq n, \end{cases}$$

$$\Upsilon(v_1^{(\alpha)}) = \begin{cases} 3\alpha + 5, & 1 \leq \alpha \leq 2 \\ 7\alpha - 5, & 3 \leq \alpha \leq n \text{ and} \end{cases}$$

$$\Upsilon(v_2^{(\alpha)}) = \begin{cases} 10, & 1 = \alpha \\ -1 + 7\alpha, & n-1 \geq \alpha \geq 2 \\ -3 + 7\alpha, & n = \alpha. \end{cases}$$

After that, the generated line assignment Υ^* is accomplished.

$$\Upsilon^*(u_\alpha u_{\alpha+1}) = \begin{cases} 7\alpha - 4, & 1 \le \alpha \le 2 \\ 7\alpha - 3, & 3 \le \alpha \le n-1, \end{cases}$$

$$\Upsilon^*(v_\alpha v_{\alpha+1}) = \begin{cases} 9, & \alpha = 1 \\ 7\alpha + 1, & 2 \le \alpha \le n-1, \end{cases}$$

$$\Upsilon^*(u_\alpha u_1^{(\alpha)}) = \begin{cases} 3\alpha - 2, & 1 \le \alpha \le 2 \\ 7\alpha - 7, & 3 \le \alpha \le n, \end{cases}$$

$$\Upsilon^*(u_\alpha u_2^{(\alpha)}) = \begin{cases} 3\alpha - 1, & 1 \le \alpha \le 2 \\ 7\alpha - 5, & 3 \le \alpha \le n, \end{cases}$$

$$\Upsilon^*(v_\alpha v_1^{(\alpha)}) = \begin{cases} 3 + 4\alpha, & 2 \ge \alpha \ge 1 \\ -4 + 7\alpha, & n \ge \alpha \ge 3, \end{cases}$$

$$\Upsilon^*(v_\alpha v_2^{(\alpha)}) = \begin{cases} 8, & 1 = \alpha \\ -2 + 7\alpha, & n-1 \ge \alpha \ge 2, \\ -3 + 7\alpha, & n = \alpha \ \ and \end{cases}$$

$$\Upsilon^*(u_\alpha v_{\alpha-1}) = 7\alpha - 8, \text{ for } 2 \le \alpha \le n.$$

As a result, the graph permits the F-centroidal mean requirement. □

Theorem 6. *The double-sided step ladder graph $2ST_{2n}$ permits the F-centroidal mean requirement for $n \ge 1$.*

Proof. Let $u_{1,1}, u_{1,2}, u_{1,3}, \ldots, u_{1,2n}, u_{2,1}, u_{2,2}, u_{2,3}, \ldots, u_{2,2n}, u_{3,1}, u_{3,2}, u_{3,3}, \ldots, u_{3,2n-2}, u_{4,1}, u_{4,2}, u_{4,3}, \ldots, u_{4,2n-4}, \ldots, u_{n+1,1}, u_{n+1,2}$ be the vertices of the double-sided step ladder graph $2ST_{2n}$.

Assume that $n \ge 2$.

Then, the following description of $\Upsilon : V(2ST_{2n}) \longrightarrow \{1, 2, 3, \ldots 2n^2 + 3n\}$ is provided.

$$\Upsilon(u_{\alpha,\beta}) = \begin{cases} \alpha + \beta^2 - 1, & 1 \le \alpha \le 2 \text{ and } \\ & 1 \le \beta \le n+1 \\ \alpha + (n+1)^2 - (n+1-\beta)(3n+2-\beta) - 1, & 1 \le \alpha \le 2 \text{ and } \\ & n+2 \le \beta \le 2n, \end{cases}$$

$$\Upsilon\big(u_{\alpha,j}\big) = \Upsilon(u_{\alpha-1,\beta+1}) + 1, \text{ for } 3 \le \alpha \le n \ \text{ and } \ 1 \le \beta \le 2n - (2\alpha - 4) \ \text{ and }$$
$$\Upsilon\big(u_{n+1,\beta}\big) = \Upsilon(u_{n,\beta+1}) + 1, \text{ for } 1 \le \alpha \le 2.$$

After that, the generated line assignment Υ^* is accomplished.

$$\Upsilon^*(u_{\alpha,\beta} u_{\alpha,\beta+1}) = \begin{cases} \alpha + \beta^2 + \beta - 1, & 1 \le \alpha \le 2 \text{ and } \\ & 1 \le \beta \le n \\ \alpha + (n+1)^2 - & \\ (n-\beta)(3n+2-\beta) - (n+1) - 1, & 1 \le \alpha \le 2 \text{ and } \\ & n+1 \le \beta \le 2n-1, \end{cases}$$

For $3 \leq \alpha \leq n$,

$$\begin{aligned}
\Upsilon^*\left(u_{\alpha,\beta}u_{\alpha,\beta+1}\right) &= \Upsilon^*(u_{\alpha-1,\beta+1}u_{\alpha-1,\beta+2}) + 1, \text{ for } 1 \leq \beta \leq 2n - (2\alpha - 3). \\
\Upsilon^*(u_{n+1,1}u_{n+1,2}) &= \Upsilon^*(u_{n,2}u_{n,3}) + 1, \\
\Upsilon^*(u_{\alpha,1}u_{\alpha+1,1}) &= \begin{cases} \beta^2, & 1 \leq \beta \leq n+1 \\ (n+1)^2 - (n+1-j)(3n+2-\beta), & n+2 \leq \beta \leq 2n, \end{cases} \\
\Upsilon^*(u_{2,\beta}u_{3,\beta-1}) &= \begin{cases} \beta^2 + 1, & 2 \leq \beta \leq n+1 \\ (n+1)^2 - (n+1-\beta)(3n+2-\beta) + 1, & n+2 \leq \beta \leq 2n-1, \end{cases}
\end{aligned}$$

For $3 \leq \alpha \leq n-1$,

$$\begin{aligned}
\Upsilon^*\left(u_{\alpha,\beta}\, u_{\alpha+1,\beta-1}\right) &= \Upsilon^*(u_{\alpha-1,\beta+1}\, u_{\alpha,\beta}) + 1, \ for\ 2 \leq \beta \leq 2n - (2\alpha - 3) \text{ and} \\
\Upsilon^*\left(u_{n,\beta}u_{n+1,\beta-1}\right) &= \Upsilon^*(u_{n-1,\beta+1}u_{n-1,\beta}) + 1, \text{ for } 1 \leq \beta \leq 2.
\end{aligned}$$

For $n = 1$, the graph $2ST_{2n}$ is a cycle C_4 and its F-centroidal meanness is shown in Figure 1.

As a result, the graph permits the F-centroidal mean requirement. □

Theorem 7. *The graph D_n^* permits the F-centroidal mean requirement for $n \geq 2$.*

Proof. Let $V(D_n^*) = \{a_{\alpha,\beta} : 1 \leq \alpha \leq n, \beta = 1,2,3,4\}$ and $E(D_n^*) = \{a_{\alpha,1}a_{\alpha+1,1}, a_{\alpha,3}a_{\alpha+1,3} : 1 \leq \alpha \leq n-1\} \cup \{a_{\alpha,1}a_{\alpha,2}, a_{\alpha,2}a_{\alpha,3}, a_{\alpha,3}a_{\alpha,4}, a_{\alpha,4}a_{\alpha,1} : 1 \leq \alpha \leq n\}$ be the vertex set and edge set of the graph D_n^*.

Then, the following description of $\Upsilon : V(D_n^*) \to \{1,2,3,\ldots,6n-1\}$ is provided.

For $1 \leq \alpha \leq n$,

$$\begin{aligned}
\Upsilon(a_{\alpha,1}) &= 6\alpha - 4, \\
\Upsilon(a_{\alpha,2}) &= 6\alpha - 5, \\
\Upsilon(a_{\alpha,3}) &= 6\alpha - 3 \text{ and} \\
\Upsilon(a_{\alpha,4}) &= 6\alpha - 1.
\end{aligned}$$

After that, the generated line assignment Υ^* is accomplished.

For $1 \leq \alpha \leq n-1$,

$$\begin{aligned}
\Upsilon^*(a_{\alpha,1}a_{\alpha+1,1}) &= 6\alpha - 1 \text{ and} \\
\Upsilon^*(a_{\alpha,3}a_{\alpha+1,3}) &= 6\alpha.
\end{aligned}$$

For $1 \leq \alpha \leq n$,

$$\begin{aligned}
\Upsilon^*(a_{\alpha,1}a_{\alpha,2}) &= 6\alpha - 5, \\
\Upsilon^*(a_{\alpha,2}a_{\alpha,3}) &= 6\alpha - 4, \\
\Upsilon^*(a_{\alpha,3}a_{\alpha,4}) &= 6\alpha - 2 \text{ and} \\
\Upsilon^*(a_{\alpha,4}a_{\alpha,1}) &= 6\alpha - 3.
\end{aligned}$$

As a result, the graph permits the F-centroidal mean requirement. □

Theorem 8. *The diamond ladder graph Dl_n permits the F-centroidal mean requirement for any $n \geq 1$.*

Proof. Let $V(Dl_n) = \{x_\alpha, y_\alpha : 1 \leq \alpha \leq n\} \cup \{z_\alpha : 1 \leq \alpha \leq 2n\}$ and $E(Dl_n) = \{x_\alpha x_{\alpha+1}, y_\alpha y_{\alpha+1} : 1 \leq \alpha \leq n-1\} \cup \{x_\alpha y_\alpha : 1 \leq \alpha \leq n\} \cup \{x_\alpha z_{2\alpha-1}, x_\alpha z_{2\alpha}, y_\alpha z_{2\alpha-1}, y_\alpha z_{2\alpha} : 1 \leq \alpha \leq n\} \cup \{z_{2\alpha}z_{2\alpha+1} : 1 \leq \alpha \leq n-1\}$.

Then, the following description of $\Upsilon : V(Dl_n) \to \{1,2,3,\ldots,8n-2\}$ is provided.

$$\Upsilon(x_\alpha) = 8\alpha - 6, \text{ for } 1 \leq \alpha \leq n,$$
$$\Upsilon(y_\alpha) = 8\alpha - 4, \text{ for } 1 \leq \alpha \leq n,$$
$$\Upsilon(z_\alpha) = \begin{cases} 1, & \alpha = 1 \\ 4\alpha - 2, & 2 \leq \alpha \leq 2n \text{ and } \alpha \text{ is even} \\ 4\alpha - 4, & 2 \leq \alpha \leq 2n \text{ and } \alpha \text{ is odd.} \end{cases}$$

After that, the generated line assignment Υ^* is accomplished.

$$\Upsilon^*(x_\alpha x_{\alpha+1}) = 8\alpha - 2, \text{ for } 1 \leq \alpha \leq n-1,$$
$$\Upsilon^*(y_\alpha y_{\alpha+1}) = 8\alpha, \text{ for } 1 \leq \alpha \leq n-1,$$
$$\Upsilon^*(x_\alpha y_\alpha) = 8\alpha - 5, \text{ for } 1 \leq \alpha \leq n,$$
$$\Upsilon^*(z_{2\alpha} z_{2\alpha+1}) = 8\alpha - 1, \text{ for } 1 \leq \alpha \leq n-1,$$
$$\Upsilon^*(x_\alpha z_{2\alpha-1}) = 8\alpha - 7, \text{ for } 1 \leq \alpha \leq n,$$

$$\Upsilon^*(x_\alpha z_{2i}) = 8\alpha - 4, \text{ for } 1 \leq \alpha \leq n,$$
$$\Upsilon^*(y_\alpha z_{2\alpha-1}) = 8\alpha - 6, \text{ for } 1 \leq \alpha \leq n \text{ and}$$
$$\Upsilon^*(y_\alpha z_{2\alpha}) = 8\alpha - 3, \text{ for } 1 \leq \alpha \leq n.$$

As a result, the graph permits the *F*-centroidal mean requirement. □

3. Conclusions

In general, graph labeling has many practical applications in various fields of science and technology, and its versatility and power make it an important tool for analyzing and understanding complex systems. The ladder graphs with graph labeling can be applied in various engineering applications, including image processing, wireless sensor networks, VLSI design, bioinformatics, and social network analysis, to model, analyze, and optimize complex systems based on labeled graph representations. An *F*-centroidal meanness of various ladder graphs is discussed in detailed. Using alternative graph operations, similar results can be found for a variety of cyclic ladder, wheel, butterfly, and various step ladder graphs. In future work, we will study the necessary and sufficient conditions for some ladder-related graph to be an *F*-centroidal mean graph.

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