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# A Weighted Generalization of Hardy–Hilbert-Type Inequality Involving Two Partial Sums

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**Abstract:** In this paper, we address Hardy–Hilbert-type inequality by virtue of constructing weight coefficients and introducing parameters. By using the Euler–Maclaurin summation formula, Abel's partial summation formula, and differential mean value theorem, a new weighted Hardy–Hilbert-type inequality containing two partial sums can be proven, which is a further generalization of an existing result. Based on the obtained results, we provide the equivalent statements of the best possible constant factor related to several parameters. Also, we illustrate how the inequalities obtained in the main results can generate some new Hardy–Hilbert-type inequalities.

**Keywords:** Hardy–Hilbert-type inequality; partial sum; parameter; Abel's partial summation formula

**MSC:** 26D15; 26D10; 26A42

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## 1. Introduction

The famous Hardy–Hilbert inequality is as follows (see [1]):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}} \tag{1}$$

where $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, the constant factor $\frac{\pi}{\sin(\pi/p)}$ being the best possible.

Krnić and Pečarić [2] provided a parameterized extension of the Hardy–Hilbert inequality, as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}} \tag{2}$$

where $\lambda_i \in (0,2]$ $(i = 1,2), \lambda_1+\lambda_2 = \lambda \in (0,4]$, the constant factor $B(\lambda_1,\lambda_2)$ being the best possible. Here, $B(u,v)$ is the beta function, which is defined as follows:

$$B(u,v) := \int_0^{\infty}\frac{x^{u-1}}{(1+x)^{u+v}}dx \ (u,v > 0)$$

By introducing the notion of partial sums, Adiyasuren et al. [3] presented an unusual extension of the Hardy–Hilbert inequality, as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < \lambda_1\lambda_2 B(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty}m^{-p\lambda_1-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{-q\lambda_2-1}B_n^q\right)^{\frac{1}{q}} \tag{3}$$

where $\lambda_i \in (0,1]\cap(0,\lambda)$ $(i = 1,2), \lambda_1+\lambda_2 = \lambda \in (0,2]$, and the constant factor

$\lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ is the best possible. The partial sums, $A_m := \sum_{i=1}^{m} a_i$ and $B_n := \sum_{k=1}^{n} b_k$ $(m,n \in \{1,2,\cdots\})$, satisfy $A_m = o(e^{tm})$, $B_n = o(e^{tn})$ $(t>0; m,n\to\infty)$, $0 < \sum_{m=1}^{\infty} m^{-p\lambda_1-1}A_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{-q\lambda_2-1}B_n^q < \infty$.

Huang, Wu and Yang [4] established an analogous version of inequality (3), which contains one partial sum $B_n$ and three parameters $\eta,\ \eta_1,\ \eta_2$.

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n-\eta)^\lambda} &< \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}(k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}} \\ &\times\Big[\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}(n-\eta_2)^{-q\hat{\lambda}_2-1}B_n^q\Big]^{\frac{1}{q}},\end{aligned} \tag{4}$$

where $\frac{1}{p}+\frac{1}{q}=1,\ p>1, \lambda\in(0,2], \lambda_1\in(0,\frac{3}{2}]\cap(0,\lambda+1),\ \lambda_2\in(0,\frac{1}{2}]\cap(0,\lambda)$, $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_2}{q}+\frac{\lambda_1}{p}$, $\eta_i\in(0,\frac{1}{4}]$ $(i=1,2)$, $\eta=\eta_1+\eta_2$, $B_n := \sum_{k=1}^{n} b_k$, $B_n = o(e^{t(n-\eta_2)})$ $(t>0,\ n\to\infty)$.

Liao, Wu, and Yang [5] applied a double power function to the weight coefficient and established the following inequality containing one partial sum in the right-hand side of the last series:

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\beta)^\lambda} &< \lambda(\tfrac{1}{\beta}k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(\tfrac{1}{\alpha}k_{\lambda+1}(\lambda_1))^{\frac{1}{q}} \\ &\times\Big[\sum_{m=1}^{\infty}m^{p(1-\alpha\hat{\lambda}_1)-1}a_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}n^{q[1-\beta(1+\hat{\lambda}_2)]-1}B_n^q\Big]^{\frac{1}{q}},\end{aligned} \tag{5}$$

where $\frac{1}{p}+\frac{1}{q}=1,\ p>1, \alpha,\beta\in(0,1], \lambda\in(0,5], \lambda_1\in(0,\frac{2}{\alpha}]\cap(0,\lambda+1),\ \lambda_2\in(0,\frac{2}{\beta}-1]\cap(0,\lambda+1)$,
$\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_2}{q}+\frac{\lambda_1}{p}$, $k_{\lambda+1}(\lambda_i) := B(\lambda_i,\lambda+1-\lambda_i)$ $(i=1,2)$, $B_n := \sum_{k=1}^{n} b_k$, $B_n = o(e^{tn^\beta})$ $(t>0,\ n\to\infty)$.

Following the result of [5], Gu and Yang [6] addressed the further extension of Inequality (5) by imbedding two partial sums in the right-hand side of the series, as follows:

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\beta)^\lambda} &< \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}(\tfrac{1}{\beta}k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(\tfrac{1}{\alpha}k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} \\ \times\Big[\sum_{m=1}^{\infty}&m^{p[1-\alpha(1+\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}A_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}n^{q[1-\beta(1+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}B_n^q\Big]^{\frac{1}{q}},\end{aligned} \tag{6}$$

where $\frac{1}{p}+\frac{1}{q}=1,\ p>1, \alpha,\beta\in(0,1], \lambda\in(0,4], \lambda_1\in(0,\frac{2}{\alpha}-1]\cap(0,\lambda+1)$, $\lambda_2\in(0,\frac{2}{\beta}-1]\cap(0,\lambda+1)$, $k_\lambda(\lambda_i) := B(\lambda_i,\lambda-\lambda_i)$ $(i=1,2)$, $A_m := \sum_{i=1}^{m} a_i$, $B_n := \sum_{k=1}^{n} b_k$ $(m,n\in\{1,2,\cdots\})$, $A_m = o(e^{tm^\alpha})$, $B_n = o(e^{tn^\beta})$ $(t>0,\ m,n\to\infty)$.

Inspired by the aforementioned studies [3–6], in this article, we construct a new weighted generalized version of Hardy–Hilbert inequality involving two partial sums, which has a different configuration of weight coefficients compared with the above inequalities (3)–(6). At the end of the paper, we show that our main finding is a generalization of the above-mentioned results obtained by Adiyasuren et al. [3]. Moreover, based on the obtained results, we are able to uncover the equivalent conditions of the best possible constant factor associated with several parameters. Also, we illustrate how the inequalities obtained in the main results can generate some new Hardy–Hilbert-type inequalities.

## 2. Preliminaries and Lemmas

In this section, we present several lemmas that are necessary to prove our main results. Below, we denote the set of conditions using (C1), which is repeated in subsequent sections.

(C1) $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda \in (0, 4], \alpha, \beta \in (0, 1], \lambda_1 \in (-1, \frac{2}{\alpha} - 1] \cap (-1, \lambda + 1)$,
$\lambda_2 \in (-1, \frac{2}{\beta} - 1] \cap (-1, \lambda + 1), \hat{\lambda}_1 := \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}, \hat{\lambda}_2 := \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$.
$a_m, b_n \geq 0 (m, n \in \mathrm{N} = \{1, 2, \cdots\}), A_m := \sum_{j=1}^{m} a_j, B_n := \sum_{k=1}^{n} b_k, A_m = o(e^{tm^{\alpha}})$,
$B_n = o(e^{tn^{\beta}})$ $(t > 0; m, n \to \infty)$, $0 < \sum_{m=1}^{\infty} m^{-p\alpha\hat{\lambda}_1 - 1} A_m^p < \infty, 0 < \sum_{n=1}^{\infty} n^{-q\beta\hat{\lambda}_2 - 1} B_n^q < \infty$.

In order to estimate the weight coefficients, we first introduce the following results related to Euler–Maclaurin summation formula.

**Lemma 1.** *(see [4,5,7]) (i) If* $(-1)^i \frac{d^i}{dt^i} g(t) > 0, t \in [m, \infty) (m \in \mathrm{N})$ *with*
$g^{(i)}(\infty) = 0$ $(i = 0, 1, 2, 3)$, $P_i(t), B_i$ $(i \in N)$ *are the Bernoulli functions and the Bernoulli numbers of i-order. Then:*

$$\int_m^{\infty} P_{2q-1}(t) g(t) dt = -\varepsilon_q \frac{B_{2q}}{2q} g(m) \quad (0 < \varepsilon_q < 1; q = 1, 2, \cdots). \tag{7}$$

*In particular, for* $q = 1$, *in view of* $B_2 = \frac{1}{6}$, *we have:*

$$-\frac{1}{12} g(m) < \int_m^{\infty} P_1(t) g(t) dt < 0; \tag{8}$$

*for* $q = 2$, *in view of* $B_4 = -\frac{1}{30}$, *we have:*

$$0 < \int_m^{\infty} P_3(t) g(t) dt < \frac{1}{120} g(m). \tag{9}$$

*(ii) If* $f(t)(> 0) \in C^3[m, \infty), f^{(i)}(\infty) = 0 (i = 0, 1, 2, 3)$, *then we have the following Euler–Maclaurin summation formula:*

$$\sum_{k=m}^{\infty} f(k) = \int_m^{\infty} f(t) dt + \frac{1}{2} f(m) + \int_m^{\infty} P_1(t) f'(t) dt, \tag{10}$$

$$\int_m^{\infty} P_1(t) f'(t) dt = -\frac{1}{12} f'(m) + \frac{1}{6} \int_m^{\infty} P_3(t) f'''(t) dt. \tag{11}$$

*Next, we establish the inequalities for weight coefficients by means of Lemma 1.*

**Lemma 2.** *For* $s \in (0, 6]$, $s_2 \in (0, \frac{2}{\beta}] \cap (0, s)$, $k_s(s_2) := B(s_2, s - s_2)$, *we define the following weight coefficient:*

$$\varpi_s(s_2, m) := m^{\alpha(s - s_2)} \sum_{n=1}^{\infty} \frac{\beta n^{\beta s_2 - 1}}{(m^{\alpha} + n^{\beta})^s} \ (m \in \mathrm{N}) \tag{12}$$

Then, we have:

$$0 < k_s(s_2)(1 - O(\frac{1}{m^{\alpha s_2}})) < \varpi_s(s_2, m) < k_s(s_2) \ (m \in \mathrm{N}). \tag{13}$$

where $O(\frac{1}{m^{\alpha s_2}}) := \frac{1}{k_s(s_2)} \int_0^{\frac{1}{m^{\alpha}}} \frac{u^{s_2 - 1}}{(1+u)^s} du > 0$.

**Proof.** For fixed $m \in \mathrm{N}$, we define a function $g(m, t)$ using
$g(m, t) := \frac{\beta t^{\beta s_2 - 1}}{(m^{\alpha} + t^{\beta})^s} (t > 0)$.

Using (10), we obtain:

$$\sum_{n=1}^{\infty} g(m,n) = \int_1^\infty g(m,t)dt + \tfrac{1}{2}g(m,1) + \int_1^\infty P(t)1g'(m,t)dt$$
$$= \int_0^\infty g(m,t)dt - h(m),$$
$$h(m) := \int_0^1 g(m,t)dt - \tfrac{1}{2}g(m,1) - \int_1^\infty P_1(t)g'(m,t)dt.$$

It follows from the function $g(m,t)$ that $-\frac{1}{2}g(m,1) = \frac{-\beta}{2(m^\alpha+1)^s}$. Moreover, integration by parts yields the following:

$$\int_0^1 g(m,t)dt = \beta\int_0^1 \frac{t^{\beta s_2-1}}{(m^\alpha+t^\beta)^s}dt \overset{u=t^\beta}{=} \int_0^1 \frac{u^{s_2-1}}{(m^\alpha+u)^s}du$$
$$= \frac{1}{s_2}\int_0^1 \frac{du^{s_2}}{(m^\alpha+u)^s} = \frac{1}{s_2}\frac{u^{s_2}}{(m^\alpha+u)^s}|_0^1 + \frac{s}{s_2}\int_0^1 \frac{u^{s_2}}{(m^\alpha+u)^{s+1}}du$$
$$= \frac{1}{s_2}\frac{1}{(m^\alpha+1)^s} + \frac{s}{s_2(s_2+1)}\int_0^1 \frac{du^{s_2+1}}{(m^\alpha+u)^{s+1}}$$
$$> \frac{1}{s_2}\frac{1}{(m^\alpha+1)^s} + \frac{s}{s_2(s_2+1)}\Big[\frac{u^{s_2+1}}{(m^\alpha+u)^{s+1}}\Big]_0^1 + \frac{s(s+1)}{s_2(s_2+1)(m^\alpha+1)^{s+2}}\int_0^1 u^{s_2+1}du$$
$$= \frac{1}{s_2}\frac{1}{(m^\alpha+1)^s} + \frac{\lambda}{s_2(s_2+1)}\frac{1}{(m^\alpha+1)^{s+1}} + \frac{s(s+1)}{s_2(s_2+1)(s_2+2)}\frac{1}{(m^\alpha+1)^{s+2}},$$
$$-g'(m,t) = -\frac{\beta(\beta s_2-1)t^{\beta s_2-2}}{(m^\alpha+t^\beta)^s} + \frac{\beta^2 s t^{\beta+\beta s_2-2}}{(m^\alpha+t^\beta)^{s+1}}$$
$$= -\frac{\beta(\beta s_2-1)t^{\beta s_2-2}}{(m^\alpha+t^\beta)^s} + \frac{\beta^2 s(m^\alpha+t^\beta-m^\alpha)t^{\beta s_2-2}}{(m^\alpha+t^\beta)^{s+1}} = \frac{\beta(\beta s-\beta s_2+1)t^{\beta s_2-2}}{(m^\alpha+t^\beta)^s} - \frac{\beta^2 s m^\alpha t^{\beta s_2-2}}{(m^\alpha+t^\beta)^{s+1}},$$

Note that, for $0 < s_2 \le \frac{2}{\beta}, 0 < \beta \le 1, s_2 < s \le 6$, it holds that:

$$(-1)^i\frac{d^i}{dt^i}\Big[\frac{t^{\beta s_2-2}}{(m^\alpha+t^\beta)^s}\Big] > 0, (-1)^i\frac{d^i}{dt^i}\Big[\frac{t^{\beta s_2-2}}{(m^\alpha+t^\beta)^{s+1}}\Big] > 0\ (i = 0,1,2,3).$$

Hence, using (8)–(11), we acquire:

$$\beta(\beta s-\beta s_2+1)\int_1^\infty P_1(t)\frac{t^{\beta s_2-2}}{(m^\alpha+t^\beta)^s}dt > -\frac{\beta(\beta s-\beta s_2+1)}{12(m^\alpha+1)^s},$$
$$-\beta^2 m^\alpha s\int_1^\infty P_1(t)\frac{t^{\beta s_2-2}}{(m^\alpha+t^\beta)^{s+1}}dt$$
$$= \frac{\beta^2 m^\alpha s}{12(m^\alpha+1)^{s+1}} - \frac{\beta^2 m^\alpha s}{6}\int_1^\infty P_3(t)\Big[\frac{t^{\beta s_2-2}}{(m^\alpha+t^\beta)^{s+1}}\Big]''dt$$
$$> \frac{\beta^2 m^\alpha s}{12(m^\alpha+1)^{s+1}} - \frac{\beta^2 m^\alpha s}{720}\Big[\frac{t^{\beta s_2-2}}{(m^\alpha+t^\beta)^{s+1}}\Big]''_{t=1}$$
$$> \frac{\beta^2(m^\alpha+1-1)s}{12(m^\alpha+1)^{s+1}} - \frac{\beta^2(m^\alpha+1)s}{720}\Big[\frac{(s+1)(s+2)\beta^2}{(m^\alpha+1)^{s+3}} + \frac{\beta(s+1)(5-\beta-2\beta s_2)}{(m^\alpha+1)^{s+2}} + \frac{(2-\beta s_2)(3-\beta s_2)}{(m^\alpha+1)^{s+1}}\Big]$$
$$= \frac{\beta^2 s}{12(m^\alpha+1)^s} - \frac{\beta^2 s}{12(m^\alpha+1)^{s+1}}$$
$$-\frac{\beta^2 s}{720}\Big[\frac{(s+1)(s+2)\beta^2}{(m^\alpha+1)^{s+2}} + \frac{\beta(s+1)(5-\beta-2\beta s_2)}{(m^\alpha+1)^{s+1}} + \frac{(2-\beta s_2)(3-\beta s_2)}{(m^\alpha+1)^s}\Big].$$

Note that $h(m) > \frac{1}{(m^\alpha+1)^s}h_1 + \frac{\lambda}{(m^\alpha+1)^{s+1}}h_2 + \frac{s(s+1)}{(m^\alpha+1)^{s+2}}h_3$, where

$$h_1 := \frac{1}{s_2} - \frac{\beta}{2} - \frac{\beta-\beta^2 s_2}{12} - \frac{\beta^2 s(2-\beta s_2)(3-\beta s_2)}{720},$$
$$h_2 := \frac{1}{s_2(s_2+1)} - \frac{\beta^2}{12} - \frac{\beta^3(s+1)(5-\beta-2\beta s_2)}{720},$$

and $h_3 := \frac{1}{s_2(s_2+1)(s_2+2)} - \frac{\beta^4(s+2)}{720}$. We find:

$$h_1 \ge \frac{1}{s_2} - \frac{\beta}{2} - \frac{\beta-\beta^2 s_2}{12} - \frac{s\beta^2(2-\beta s_2)(3-\beta s_2)}{720} = \frac{g(s_2)}{720 s_2},$$

where we define the function $g(\sigma)(\sigma \in (0, \frac{2}{\beta}])$ by

$$g(\sigma) := 720 - (420\beta + 6s\beta^2)\sigma + (60\beta^2 + 5s\beta^3)\sigma^2 - s\beta^4\sigma^3.$$

Thus, we deduce that for $\beta \in (0, 1], s \in (0, 6]$,

$$\begin{aligned} g'(\sigma) &= -(420\beta + 6s\beta^2) + 2(60\beta^2 + 5s\beta^3)\sigma - 3\beta^4\sigma^2 \\ &\leq -420\beta - 6s\beta^2 + 2(60\beta^2 + 5s\beta^3)\tfrac{2}{\beta} = (14s\beta - 180)\beta < 0, \end{aligned}$$

and then it follows that $h_1 \geq \frac{g(s_2)}{720s_2} \geq \frac{g(2/\beta)}{720s_2} = \frac{1}{6s_2} > 0$. We find that for $s_2 \in (0, \frac{2}{\beta}]$,

$$h_2 > \frac{\beta^2}{6} - \frac{\beta^2}{12} - \frac{5(s+1)\beta^2}{720} = (\frac{1}{12} - \frac{s+1}{140})\beta^2 > 0,$$

and $h_3 \geq (\frac{1}{24} - \frac{s+2}{720})\beta^3 > 0\ (0 < s \leq 6)$.

Hence, we have $h(m) > 0$. Now, setting $u = m^{-\alpha}t^{\beta}$, we obtain the following:

$$\begin{aligned} \varpi_s(s_2, m) &= m^{\alpha(s-s_2)} \sum_{n=1}^{\infty} g(m, n) < m^{\alpha(s-s_2)} \int_0^{\infty} g(m, t)dt \\ &= \beta m^{\alpha(s-s_2)} \int_0^{\infty} \frac{t^{\beta s_2 - 1}dt}{(m^{\alpha} + t^{\beta})^s} = \int_0^{\infty} \frac{u^{s_2-1}du}{(1+u)^s} = B(s_2, s - s_2) = k_s(s_2). \end{aligned}$$

On the other hand, using (10), we have:

$$\begin{aligned} \sum_{n=1}^{\infty} g(m, n) &= \int_1^{\infty} g(m, t)dt + \tfrac{1}{2}g(m, 1) + \int_1^{\infty} P_1(t)g'(m, t)dt \\ &= \int_1^{\infty} g(m, t)dt + H(m), \\ H(m) &:= \tfrac{1}{2}g(m, 1) + \int_1^{\infty} P_1(t)g'(m, t)dt. \end{aligned}$$

We obtain $\frac{1}{2}g(m, 1) = \frac{\beta}{2(m^{\alpha}+1)^s}$, and

$$g'(m, t) = -\frac{\beta(\beta s - \beta s_2 + 1)t^{\beta s_2 - 2}}{(m^{\alpha} + t^{\beta})^s} + \frac{\beta^2 s m^{\alpha} t^{\beta s_2 - 2}}{(m^{\alpha} + t^{\beta})^{s+1}}.$$

For $s_2 \in (0, \frac{2}{\beta}] \cap (0, s), 0 < s \leq 6$, by (7), we find
$-\beta(\beta s - \beta s_2 + 1)\int_1^{\infty} P_1(t)\frac{t^{\beta s_2 - 2}}{(m^{\alpha}+t^{\beta})^s}dt > 0$, and

$$\beta^2 m^{\alpha} s \int_1^{\infty} P_1(t)\frac{t^{\beta s_2 - 2}}{(m^{\alpha} + t^{\beta})^{s+1}}dt > -\frac{\beta^2 m^{\alpha} s}{12(m^{\alpha} + 1)^{s+1}} > -\frac{\beta^2 s}{12(m^{\alpha} + 1)^s}.$$

Hence, we obtain:

$$H(m) > \frac{\beta}{2(m^{\alpha} + 1)^s} - \frac{\beta^2 s}{12(m^{\alpha} + 1)^s} \geq \frac{\beta}{2(m^{\alpha} + 1)^s} - \frac{6\beta}{12(m^{\alpha} + 1)^s} = 0,$$

and then we deduce:

$$\begin{aligned} \varpi_s(s_2, m) &= m^{\alpha(s-s_2)} \sum_{n=1}^{\infty} g(m, n) > m^{\alpha(s-s_2)} \int_1^{\infty} g(m, t)dt \\ &= m^{\alpha(s-s_2)} \int_0^{\infty} g(m, t)dt - m^{\alpha(s-s_2)} \int_0^1 g(m, t)dt \\ &= k_s(s_2)\left[1 - \frac{1}{k_s(s_2)} \int_0^{\frac{1}{m^{\alpha}}} \frac{u^{s_2-1}}{(1+u)^s}du\right] > 0, \end{aligned}$$

We now set $O(\frac{1}{m^{\alpha s_2}}) = \frac{1}{k_s(s_2)}\int_0^{\frac{1}{m^\alpha}} \frac{u^{s_2-1}}{(1+u)^s}du$, satisfying

$$0 < \int_0^{\frac{1}{m^\alpha}} \frac{u^{s_2-1}}{(1+u)^s}du < \int_0^{\frac{1}{m^\alpha}} u^{s_2-1}du = \frac{1}{s_2 m^{\alpha s_2}}.$$

The two-side inequalities in (13) are derived. This proves Lemma 2. □

Next, we address an extended Hardy–Hilbert inequality, which is essential for proofing our main results in the next section.

**Lemma 3.** *Under the assumption (C1), we have the following Hardy–Hilbert-type inequality:*

$$\begin{aligned} I_{\lambda+2} := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{m^{\alpha-1}n^{\beta-1}}{(m^\alpha+n^\beta)^{\lambda+2}}A_m B_n &< \left(\frac{1}{\beta}k_{\lambda+2}(\lambda_2+1)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k_{\lambda+2}(\lambda_1+1)\right)^{\frac{1}{q}} \\ &\times\left(\sum_{m=1}^{\infty} m^{-p\alpha\hat{\lambda}_1-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-q\beta\hat{\lambda}_2-1}B_n^q\right)^{\frac{1}{q}} \end{aligned} \quad (14)$$

**Proof.** Based on the results found using Lemma 2, and by means of the principle of symmetry, for $s_1 \in (0, \frac{2}{\alpha}] \cap (0, s), s \in (0, 6]$, we can obtain the following inequalities for another weight coefficient:

$$\begin{aligned} 0 < k_s(s_1)(1 - O(\tfrac{1}{n^{\beta s_1}})) &< \omega_s(s_1, n) \\ := n^{\beta(s-s_1)} \sum_{m=1}^{\infty} \frac{\alpha m^{\alpha s_1-1}}{(m^\alpha+n^\beta)^s} &< k_s(s_1) = B(s_1, s-s_1)(n \in \mathrm{N}), \end{aligned} \quad (15)$$

where $O(\frac{1}{n^{\beta s_1}}) := \frac{1}{k_s(s_1)}\int_0^{\frac{1}{n^\beta}} \frac{u^{s_1-1}}{(1+u)^s}du > 0$.

By utilizing Hölder's inequality [8], we obtain:

$$\begin{aligned} I_{\lambda+2} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(m^\alpha+n^\beta)^{\lambda+2}}\left[\frac{m^{\alpha(-\lambda_1)/q}(\beta n^{\beta-1})^{1/p}}{n^{\beta(-\lambda_2)/p}(\alpha m^{\alpha-1})^{1/q}}m^{\alpha-1}A_m\right]\left[\frac{n^{\beta(-\lambda_2)/p}(\alpha m^{\alpha-1})^{1/q}}{m^{\alpha(-\lambda_1)/q}(\beta n^{\beta-1})^{1/p}}n^{\beta-1}B_n\right] \\ &\le \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\beta m^{p(\alpha-1)}}{(m^\alpha+n^\beta)^{\lambda+2}} \frac{m^{\alpha(-\lambda_1)(p-1)}n^{\beta-1}A_m^p}{n^{\beta(-\lambda_2)}(\alpha m^{\alpha-1})^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\alpha n^{q(\beta-1)}}{(m^\alpha+n^\beta)^{\lambda+2}} \frac{n^{\beta(-\lambda_2)(q-1)}m^{\alpha-1}B_n^q}{m^{\alpha(-\lambda_1)}(\beta n^{\beta-1})^{q-1}}\right]^{\frac{1}{q}} \\ &= \left(\frac{1}{\beta}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}\right)^{\frac{1}{q}}\left(\sum_{m=1}^{\infty} \varpi_{\lambda+2}(\lambda_2+1, m)m^{-p\alpha\hat{\lambda}_1-1}A_m^p\right)^{\frac{1}{p}} \\ &\times\left(\sum_{n=1}^{\infty} \omega_{\lambda+2}(\lambda_1+1, n)n^{-q\beta\hat{\lambda}_2-1}B_n^q\right)^{\frac{1}{q}}. \end{aligned}$$

Further, using (13) and (15) (for $s = \lambda + 2, s_i = \lambda_i + 1 (i = 1, 2)$), together with the assumption condition (C1), we derive the desired Inequality (14). Lemma 3 is proved. □

In the following, we prove two inequalities related to the partial sums in preparation for establishing a Hardy–Hilbert-type inequality involving partial sums.

**Lemma 4.** *Under the assumption (C1), for $t > 0$, the following inequalities hold true:*

$$\sum_{m=1}^{\infty} e^{-tm^\alpha}a_m \le t\alpha \sum_{m=1}^{\infty} e^{-tm^\alpha}m^{\alpha-1}A_m, \quad (16)$$

$$\sum_{n=1}^{\infty} e^{-tn^\beta}b_n \le t\beta \sum_{n=1}^{\infty} e^{-tn^\beta}n^{\beta-1}B_n. \quad (17)$$

**Proof.** In view of $A_m e^{-tm^\alpha} = o(1)(m \to \infty)$, applying Abel's summation using a parts formula provides:

$$\sum_{m=1}^{\infty} e^{-tm^\alpha} a_m = \lim_{m\to\infty} A_m e^{-tm^\alpha} + \sum_{m=1}^{\infty} A_m \left[e^{-tm^\alpha} - e^{-t(m+1)^\alpha}\right]$$
$$= \sum_{m=1}^{\infty} A_m \left[e^{-tm^\alpha} - e^{-t(m+1)^\alpha}\right].$$

Set $g(x) = e^{-tx^\alpha}, x \in [m, m+1]$. We conclude that $g'(x) = -t\alpha x^{\alpha-1} e^{-tx^\alpha}$, and for $\alpha \in (0,1], h(x) := x^{\alpha-1} e^{-tx^\alpha}$ is decreasing in $[m, m+1]$.

By using the differential mean value theorem, we have

$$\begin{aligned}\sum_{m=1}^{\infty} e^{-tm^\alpha} a_m &= -\sum_{m=1}^{\infty} A_m (g(m+1) - g(m)) \\ &= -\sum_{m=1}^{\infty} A_m g'(m+\theta) = t\alpha \sum_{m=1}^{\infty} h(m+\theta) A_m \\ &\leq t\alpha \sum_{m=1}^{\infty} h(m) A_m = t\alpha \sum_{m=1}^{\infty} m^{\alpha-1} e^{-tm^\alpha} A_m (\theta \in (0,1)),\end{aligned}$$

which leads to Inequality (16). In the same way as above, we can derive the inequality (17). Lemma 4 is proved. □

## 3. Main Results

**Theorem 1.** *Under the assumption (C1), we have the following Hardy–Hilbert-type inequality involving two partial sums:*

$$\begin{aligned}I := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(m^\alpha + n^\beta)^\lambda} &< \alpha\beta \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} \left(\frac{1}{\beta} k_{\lambda+2}(\lambda_2+1)\right)^{\frac{1}{p}} \left(\frac{1}{\alpha} k_{\lambda+2}(\lambda_1+1)\right)^{\frac{1}{q}} \\ &\times \left(\sum_{m=1}^{\infty} m^{-p\alpha\hat{\lambda}_1 - 1} A_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{-q\beta\hat{\lambda}_2 - 1} B_n^q\right)^{\frac{1}{q}}\end{aligned} \tag{18}$$

*In particular, for $\lambda_1 + \lambda_2 = \lambda$, we have:*

$$0 < \sum_{m=1}^{\infty} m^{-p\alpha\lambda_1 - 1} A_m^p < \infty, 0 < \sum_{n=1}^{\infty} n^{-q\beta\lambda_2 - 1} B_n^q < \infty,$$

*and the following inequality:*

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(m^\alpha + n^\beta)^\lambda} &< \alpha^{\frac{1}{p}} \beta^{\frac{1}{q}} \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} B(\lambda_1+1, \lambda_2+1) \\ &\times \left(\sum_{m=1}^{\infty} m^{-p\alpha\lambda_1 - 1} A_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{-q\beta\lambda_2 - 1} B_n^q\right)^{\frac{1}{q}}.\end{aligned} \tag{19}$$

**Proof.** By virtue of the expression

$$\frac{1}{(m^\alpha + n^\beta)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-(m^\alpha + n^\beta)t} dt$$

and using (16) and (17), it follows that

$$\begin{aligned} I &= \tfrac{1}{\Gamma(\lambda)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m b_n \int_0^{\infty} t^{\lambda-1} e^{-(m^{\alpha}+n^{\beta})t} dt \\ &= \tfrac{1}{\Gamma(\lambda)}\int_0^{\infty} t^{\lambda-1}\left(\sum_{m=1}^{\infty} e^{-m^{\alpha}t} a_m\right)\left(\sum_{n=1}^{\infty} e^{-n^{\beta}t} b_n\right) dt \\ &\le \tfrac{1}{\Gamma(\lambda)}\int_0^{\infty} t^{\lambda-1}\left(t\alpha\sum_{m=1}^{\infty} e^{-m^{\alpha}t} m^{\alpha-1} A_m\right)\left(t\beta\sum_{n=1}^{\infty} e^{-n^{\beta}t} n^{\beta-1} B_n\right) dt \\ &= \tfrac{\alpha\beta}{\Gamma(\lambda)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\alpha-1} n^{\beta-1} A_m B_n \int_0^{\infty} t^{\lambda+1} e^{-(m^{\alpha}+n^{\beta})t} dt \\ &= \alpha\beta\tfrac{\Gamma(\lambda+2)}{\Gamma(\lambda)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tfrac{m^{\alpha-1} n^{\beta-1}}{(m^{\alpha}+n^{\beta})^{\lambda+2}} A_m B_n. \end{aligned}$$

Furthermore, by virtue of Inequality (14), we deduce Inequality (18). This proves that Theorem 1 is complete. □

In the following theorems, we provide some equivalent statements of the best possible constant factor related to several parameters in (18).

**Theorem 2.** *For $\lambda_1 + \lambda_2 = \lambda \in (0, 4]$, if $\lambda_1 \in (0, \frac{2}{\alpha} - 1] \cap (0, \lambda)$, $\lambda_2 \in (0, \frac{2}{\beta} - 1] \cap (0, \lambda)$, then the constant factor*

$$\alpha\beta\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}\left(\frac{1}{\beta}k_{\lambda+2}(\lambda_2+1)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k_{\lambda+2}(\lambda_1+1)\right)^{\frac{1}{q}}$$

*in (18) is the best possible.*

**Proof.** We first prove that the constant factor

$$\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}B(\lambda_1+1, \lambda_2+1)(= \alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\lambda_1\lambda_2 B(\lambda_1, \lambda_2))$$

in (19) is the best possible in the condition.

For any $0 < \varepsilon < \min\{p\lambda_1, q\lambda_2\}$, we set:

$$\widetilde{a}_m := m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-1}, \widetilde{b}_n := n^{\beta(\lambda_2 - \frac{\varepsilon}{q})-1} (m, n \in \mathrm{N})$$

Note that $0 < \lambda_1 - \frac{\varepsilon}{p} \le \frac{2}{\alpha} - 1, 0 < \alpha(\lambda_1 - \frac{\varepsilon}{p}) \le 2 - \alpha < 2$, according to the result (2.2.24) described in [7], we have:

$$\begin{aligned} \widetilde{A}_m &:= \sum_{i=1}^{m} \widetilde{a}_i = \sum_{i=1}^{m} i^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-1} = \int_1^m t^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-1} dt \\ &+ \tfrac{1}{2}[m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-1} + 1] + \tfrac{\varepsilon_0}{12}[\alpha(\lambda_1 - \tfrac{\varepsilon}{p}) - 1][m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-2} - 1] \\ &= \tfrac{1}{\alpha(\lambda_1 - \frac{\varepsilon}{p})}(m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})} + c_1 + O(m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-1})1) \\ &\le \tfrac{m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})}}{\alpha(\lambda_1 - \frac{\varepsilon}{p})}(1 + |c_1| m^{-\alpha(\lambda_1 - \frac{\varepsilon}{p})} + |O(m^{-1})1|)(\varepsilon_0 \in (0,1); m \in \mathrm{N}, \mathrm{m} \to \infty). \end{aligned}$$

Using the same method, for $0 < \beta(\lambda_2 - \frac{\varepsilon}{q}) < 2$, we obtain:

$$\widetilde{B}_n := \sum_{k=1}^{n} \widetilde{b}_k \le \frac{n^{\beta(\lambda_2 - \frac{\varepsilon}{q})}}{\beta(\lambda_2 - \frac{\varepsilon}{q})}(1 + |c_2| n^{-\beta(\lambda_2 - \frac{\varepsilon}{q})} + |O_2(n^{-1})|)(n \in \mathrm{N}; n \to \infty)$$

$c_i (i = 1, 2)$ are constants. We deduce that $\widetilde{A}_m = o(e^{tm^{\alpha}}), \widetilde{B}_n = o(e^{tn^{\beta}}) (t > 0; m, n \to \infty)$.

If there is a constant $M(\leq \alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}B(\lambda_1+1,\lambda_2+1))$ such that (19) is valid when we replace $\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}B(\lambda_1+1,\lambda_2+1)$ by $M$, then from the particular substitution of $a_m=\widetilde{a}_m$, $b_n=\widetilde{b}_n$, $A_m=\widetilde{A}_m$ and $B_n=\widetilde{B}_n$ in (19), we have

$$< M\left(\sum_{m=1}^{\infty} m^{-p\alpha\lambda_1-1}\widetilde{A}_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-q\beta\lambda_2-1}\widetilde{B}_n^q\right)^{\frac{1}{q}} \tag{20}$$

For $a(x)\to 0(x\to\infty)$, we obtain:

$$\lim_{x\to\infty}\frac{(1+a(x))^p-1}{a(x)}=\lim_{x\to\infty}\frac{p(1+a(x))^{p-1}a'(x)}{a'(x)}=\lim_{x\to\infty}p(1+a(x))^{p-1}=p,$$

and then $(1+a(x))^p=1+O(a(x))(x\to\infty)$. So, we have:

$$\begin{aligned}&(1+|c_1|m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})}+|O_1(m^{-1})|)^p\\ &=1+O(|c_1|m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})}+|O_1(m^{-1})|)(m\in\mathrm{N};m\to\infty).\end{aligned}$$

Thus, we acquire:

$$\begin{aligned}&\sum_{m=1}^{\infty} m^{-p\alpha\lambda_1-1}\widetilde{A}_m^p\leq\left[\tfrac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p\sum_{m=1}^{\infty} m^{-\alpha\varepsilon-1}\left(1+|c_1|m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})}+|O(m^{-1})1|\right)^p\\ &=\left[\tfrac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p\sum_{m=1}^{\infty} m^{-\alpha\varepsilon-1}[1+O(|c_1|m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})}+|O(m^{-1})1|)]\\ &=\left[\tfrac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p\left[\sum_{m=2}^{\infty} m^{-\alpha\varepsilon-1}+1+\sum_{m=1}^{\infty}O(|c_1|m^{-\alpha(\lambda_1+\frac{\varepsilon}{q})-1}+|O(m^{-\alpha\varepsilon-2})1|)\right]\\ &=\left[\tfrac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p\left(\sum_{m=2}^{\infty} m^{-\alpha\varepsilon-1}+O_1(1)\right)\\ &<\left[\tfrac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p\left(\int_1^{\infty}x^{-\alpha\varepsilon-1}dx+O_1(1)\right)=\left[\tfrac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p(\tfrac{1}{\alpha\varepsilon}+O_1(1)).\end{aligned}$$

Similarly, we have:

$$\sum_{n=1}^{\infty} n^{-q\beta\lambda_2-1}\widetilde{B}_n^q<\left[\frac{1}{\beta(\lambda_2-\frac{\varepsilon}{q})}\right]^q(\frac{1}{\beta\varepsilon}+O_2(1)).$$

Furthermore, we obtain;

$$\widetilde{I}<\frac{M}{\varepsilon}\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\frac{1}{\beta(\lambda_2-\frac{\varepsilon}{q})}\left(\frac{1}{\alpha}+\varepsilon O_1(1)\right)^{\frac{1}{p}}\left(\frac{1}{\beta}+\varepsilon O_2(1)\right)^{\frac{1}{q}}.$$

By virtue of (15) (for $s=\lambda\in(0,4]$, $s_1=\lambda_1-\frac{\varepsilon}{p}\in(0,\frac{2}{\alpha}-1]\cap(0,\lambda)$), we still have:

$$\begin{aligned}&\widetilde{I}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\tfrac{m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}}{(m^\alpha+n^\beta)^\lambda}n^{\beta(\lambda_2-\frac{\varepsilon}{q})-1}=\tfrac{1}{\alpha}\sum_{n=1}^{\infty}n^{-\beta\varepsilon-1}\left[n^{\beta(\lambda_2+\frac{\varepsilon}{q})}\sum_{m=1}^{\infty}\tfrac{\alpha m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}}{(m^\alpha+n^\beta)^\lambda}\right]\\ &\geq\tfrac{1}{\alpha}k_\lambda(\lambda_1-\tfrac{\varepsilon}{p})\sum_{n=1}^{\infty}n^{-\beta\varepsilon-1}\left(1-O(\tfrac{1}{n^{\beta(\lambda_1-\frac{\varepsilon}{p})}})\right)\\ &=\tfrac{1}{\alpha}k_\lambda(\lambda_1-\tfrac{\varepsilon}{p})\left(\sum_{n=1}^{\infty}n^{-\beta\varepsilon-1}-\sum_{n=1}^{\infty}O(\tfrac{1}{n^{\beta(\lambda_1+\frac{\varepsilon}{q})+1}})\right)\\ &>\tfrac{1}{\alpha}k_\lambda(\lambda_1-\tfrac{\varepsilon}{p})\left(\int_1^{\infty}y^{-\beta\varepsilon-1}dy-O_3(1)\right)\\ &=\tfrac{1}{\alpha}B(\lambda_1-\tfrac{\varepsilon}{p},\lambda_2+\tfrac{\varepsilon}{p})\left(\tfrac{1}{\beta\varepsilon}-O_3(1)\right).\end{aligned}$$

From the above results, it follows that:

$$\frac{1}{\alpha}B(\lambda_1-\tfrac{\varepsilon}{p},\lambda_2+\tfrac{\varepsilon}{p})\Big(\tfrac{1}{\beta}-\varepsilon O_3(1)\Big)$$
$$<\varepsilon\widetilde{I}<M\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\frac{1}{\beta(\lambda_2-\frac{\varepsilon}{q})}\Big(\tfrac{1}{\alpha}+\varepsilon O_1(1)\Big)^{\frac{1}{p}}\Big(\tfrac{1}{\beta}+\varepsilon O_2(1)\Big)^{\frac{1}{q}}.$$

Setting $\varepsilon\to 0^+$, by means of the continuity of the beta function, we find

$$\frac{1}{\alpha\beta}B(\lambda_1,\lambda_2)\leq M\frac{1}{\alpha\beta\lambda_1\lambda_2}\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\beta}\right)^{\frac{1}{q}},$$

i.e.,

$$\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}B(\lambda_1+1,\lambda_2+1)=\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\lambda_1\lambda_2B(\lambda_1,\lambda_2)\leq M.$$

Consequently, $M=\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}B(\lambda_1+1,\lambda_2+1)$ is the best possible constant factor in (19), which implies that the constant factor in (18)

$$\alpha\beta\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}(\frac{1}{\beta}k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(\frac{1}{\alpha}k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

is the best possible. This proves Theorem 2. □

**Theorem 3.** *If the constant factor in (18)*

$$(\frac{1}{\beta}k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(\frac{1}{\alpha}k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

*is the best possible, then for*

$$\lambda-\lambda_1-\lambda_2\leq\min\left\{p(\frac{2}{\alpha}-1-\lambda_1),q(\frac{2}{\beta}-1-\lambda_2)\right\}$$

*we have* $\lambda_1+\lambda_2=\lambda$.

**Proof.** For $\hat{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}=\frac{\lambda-\lambda_1-\lambda_2}{p}+\lambda_1,\hat{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\frac{\lambda-\lambda_1-\lambda_2}{q}+\lambda_2$, we observe that

$\hat{\lambda}_1+\hat{\lambda}_2=\lambda$, and $0<\hat{\lambda}_1,\hat{\lambda}_2<\frac{\lambda}{p}+\frac{\lambda}{q}=\lambda$.

Now, for $\lambda-\lambda_1-\lambda_2\leq p(\frac{2}{\alpha}-1-\lambda_1)$, we have $\hat{\lambda}_1\leq\frac{2}{\alpha}-1$; for $\lambda-\lambda_1-\lambda_2\leq q(\frac{2}{\beta}-1-\lambda_2)$, we have $\hat{\lambda}_2\leq\frac{2}{\beta}-1$.

Substitution of $\lambda_i=\hat{\lambda}_i$ $(i=1,2)$ in (19), we still have:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(m^\alpha+n^\beta)^\lambda}<\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}B(\hat{\lambda}_1+1,\hat{\lambda}_2+1)$$
$$\times\left(\sum_{m=1}^{\infty}m^{-p\alpha\hat{\lambda}_1-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{-q\beta\hat{\lambda}_2-1}B_n^q\right)^{\frac{1}{q}}. \quad (21)$$

By applying Hölder's inequality [8], we deduce that:

$$
\begin{aligned}
&B(\hat{\lambda}_1+1,\hat{\lambda}_2+1) = k_{\lambda+2}(\tfrac{\lambda-\lambda_2}{p}+\tfrac{\lambda_1}{q}+1)\\
&= \int_0^\infty \frac{1}{(1+u)^{\lambda+2}} u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}} du = \int_0^\infty \frac{1}{(1+u)^{\lambda+2}} \left(u^{\frac{\lambda-\lambda_2}{p}}\right)\left(u^{\frac{\lambda_1}{q}}\right) du\\
&\geq \left[\int_0^\infty \frac{1}{(1+u)^{\lambda+2}} u^{\lambda-\lambda_2} du\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{1}{(1+u)^{\lambda+2}} u^{\lambda_1} du\right]^{\frac{1}{q}} \\
&= \left[\int_0^\infty \frac{1}{(1+v)^{\lambda+2}} v^{\lambda_2} dv\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{1}{(1+u)^{\lambda+2}} u^{\lambda_1} du\right]^{\frac{1}{q}}\\
&= (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}.
\end{aligned}
\tag{22}
$$

If the constant factor $\alpha\beta\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}\,(\frac{1}{\beta}k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(\frac{1}{\alpha}k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$ in (18) is the best possible, then, by comparison with the constant factors in (18) and (21), we have the following inequality:

$$
\begin{aligned}
&\alpha\beta\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}(\tfrac{1}{\beta}k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(\tfrac{1}{\alpha}k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}\\
&\quad\leq \alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}B(\hat{\lambda}_1+1,\hat{\lambda}_2+1)
\end{aligned}
$$

it follows that:

$$
B(\hat{\lambda}_1+1,\hat{\lambda}_2+1) \geq (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} \tag{23}
$$

Hence, with the aid of (22), we obtain:

$$
B(\hat{\lambda}_1+1,\hat{\lambda}_2+1) = (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}
$$

and (22) takes the form of equality.

We observe that the equality holds in (22) if and only if there exist constants $A$ and $B$, such that they are not both zero satisfying (see [8]): $Au^{\lambda-\lambda_2} = Bu^{\lambda_1}$ *a.e.* in $\mathrm{R}_+$. Without loss of generality, let $A \neq 0$, one has $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ *a.e.* in $\mathrm{R}_+$, that is $\lambda-\lambda_2-\lambda_1 = 0$. Hence, $\lambda_1+\lambda_2 = \lambda$. The proof of Theorem 3 is complete. □

**Remark 1.** *Putting $\alpha = \beta = 1$ in Inequality (19) with an application of the identity:*

$$
\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}B(\lambda_1+1,\lambda_2+1) = \lambda_1\lambda_2 B(\lambda_1,\lambda_2)
$$

*we obtain Inequality (3). Hence, Inequalities (18) and (19) are generalizations of Inequality (3) obtained by Adiyasuren et al. in an earlier paper [3].*

**Remark 2.** *As a direct application of the main result in Theorem 1, we can derive more Hardy–Hilbert-type inequalities from special cases of the parameters.*

*(i)* *Choosing $\alpha = \beta = \frac{1}{2}, \lambda_i \in (0,3] \cap (0,\lambda)\,(i=1,2)$ in (19), we obtain*

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(\sqrt{m}+\sqrt{n})^{\lambda}} < \frac{\lambda_1\lambda_2}{2}B(\lambda_1,\lambda_2)\\
&\times\left(\sum_{m=1}^{\infty} m^{-\frac{p\lambda_1}{2}-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-\frac{q\lambda_2}{2}-1}B_n^q\right)^{\frac{1}{q}}.
\end{aligned}
\tag{24}
$$

*(ii)* *Choosing $\alpha = \beta = \frac{1}{3}\ \lambda_i \in (0,4] \cap (0,\lambda)\,(i=1,2)$ in (19), we obtain:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(\sqrt[3]{m}+\sqrt[3]{n})^{\lambda}}<\frac{\lambda_1\lambda_2}{3}B(\lambda_1,\lambda_2) \times\left(\sum_{m=1}^{\infty}m^{-\frac{p\lambda_1}{3}-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{-\frac{q\lambda_2}{3}-1}B_n^q\right)^{\frac{1}{q}}. \quad (25)$$

(iii) *Taking* $\alpha=\beta=\lambda=1$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$ *in (19), we obtain:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}\left(\sum_{m=1}^{\infty}m^{-p}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{-q}B_n^q\right)^{\frac{1}{q}}. \quad (26)$$

(iv) *Taking* $\alpha=\beta=\lambda=1$, $\lambda_1=\frac{1}{p}$, $\lambda_2=\frac{1}{q}$ *in (19), we obtain:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{pq\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty}m^{-2}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{-2}B_n^q\right)^{\frac{1}{q}}. \quad (27)$$

(v) *Taking* $p=q=2$ *in Inequalities (26) and (27) provides:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{4}\left(\sum_{m=1}^{\infty}m^{-2}A_m^2\sum_{n=1}^{\infty}n^{-2}B_n^q\right)^{\frac{1}{2}}. \quad (28)$$

It is worth noting that the constant factors in the above inequalities are the best possible.

## 4. Conclusions

In this study, by using the idea of constructing weight coefficients and introducing parameters, we establish a new weighted generalization of a Hardy–Hilbert-type inequality involving two partial sums. Our main result is stated in Theorem 1. In order to illustrate the innovation of the current results, in Remark 1, we show that Inequalities (18) and (19) asserted by Theorem 1 are generalizations of the results presented by Adiyasuren et al. [3]. In Theorem 2 and Theorem 3, we propose and prove two results on equivalent statements of the best possible constant factor related to several parameters, which reveals the essential characteristics of this type of Hardy–Hilbert inequality. In Remark 2, we show some applications of the main results in establishing new inequalities. With regards to the future orientation of research, we notice that more and more researchers have paid attention to the extension of classical inequalities via fractional calculus in recent years [9–12]. We hope that the present results and some existing results about Hardy–Hilbert-type inequalities can be extended to the case of fractional calculus.

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