mathematics

MDPI

Article

Global Convergence of Algorithms Based on Unions of Non-Expansive Maps

Alexander J. Zaslavski

Department of Mathematics, Technion–Israel Institute of Technology, Haifa 32000, Israel; ajzasl@technion.ac.il

Abstract: In his recent research, M. K. Tam (2018) considered a framework for the analysis of iterative algorithms which can be described in terms of a structured set-valued operator. At each point in the ambient space, the value of the operator can be expressed as a finite union of values of single-valued para-contracting operators. He showed that the associated fixed point iteration is locally convergent around strong fixed points. In the present paper we generalize the result of Tam and show the global convergence of his algorithm for an arbitrary starting point. An analogous result is also proven for the Krasnosel'ski–Mann iterations.

Keywords: convergence analysis; fixed point; non-expansive mapping; para-contracting operator

MSC: 47H04; 47H10

check for updates

Citation: Zaslavski, A.J. Global Convergence of Algorithms Based on Unions of Non-Expansive Maps. *Mathematics* **2023**, *11*, 3213. https://doi.org/10.3390/math11143213

Academic Editors: Wei-Shih Du, Marko Kostić, Vladimir E. Fedorov and Manuel Pinto

Received: 17 May 2023
Revised: 19 July 2023
Accepted: 20 July 2023
Published: 21 July 2023

1. Introduction

The study of the fixed point theory of non-expansive operators [1–9] has been a rapidly growing area of research since Banach's classical result [10] on the existence of a unique fixed point for a strict contraction. Numerous developments have taken place in this area including, in particular, studies of feasibility, common fixed point problems and variational inequalities, which find important applications in engineering, medical and the natural sciences. See [1,7–9,11–16] and the references therein. In [17], a framework was suggested for the analysis of iterative algorithms, determined by a structured set-valued operator. For such algorithms it was shown in [17] that the associated fixed point iteration is locally convergent around strong fixed points. In [18], an analogous result was obtained for Krasnosel'ski–Mann iterations. In the present paper we generalize the main result of [17] and show the global convergence of the algorithm for an arbitrary starting point. An analogous result is also proven for the Krasnosel'ski–Mann iterations.

2. Global Convergence of Iterates

Let (X, ρ) be a metric space and $C \subset X$ be its non-empty, closed set. For each $x \in X$ and $r > 0$, put

$$B(x, r) = \{y \in X : \ \rho(x, y) \le r\}.$$

For each $x \in X$ and non-empty set $D \subset X$, set

$$\rho(x, D) = \inf\{\rho(x, y) : \ y \in D\}.$$

For each mapping $S : C \to C$, define

$$\mathrm{Fix}(S) = \{x \in C : \ S(x) = x\}.$$

Fix

$$\theta \in C.$$

Suppose that the following assumption holds:

(A1) For each $M > 0$, the set $B(\theta, M) \cap C$ is compact.

Assume that m is a natural number, $T_i : C \to C$, $i = 1, \dots, m$ are continuous operators and that the following assumption holds:

(A2) For each $i \in \{1, \dots, m\}$, $z \in \mathrm{Fix}(T_i)$, $x \in C$ and $y \in C \setminus \mathrm{Fix}(T_i)$, we have

$$\rho(z, T_i(x)) \leq \rho(z, x)$$

and

$$\rho(z, T_i(y)) < \rho(z, y).$$

Note that operators satisfying (A2) are called para-contractions [19].

Assume that for every point $x \in X$, a non-empty set

$$\phi(x) \subset \{1, \dots, m\} \tag{1}$$

is given. In other words,

$$\phi : X \to 2^{\{1,\dots,m\}} \setminus \{\emptyset\}.$$

Suppose that the following assumption holds:

(A3) For each $x \in C$ there exists $\delta > 0$ such that for each $y \in B(x, \delta) \cap C$,

$$\phi(y) \subset \phi(x).$$

Define

$$T(x) = \{T_i(x) : \ i \in \phi(x)\} \tag{2}$$

for each $x \in C$,

$$\bar{F}(T) = \{z \in C : \ T_i(z) = z, \ i = 1, \dots, m\} \tag{3}$$

and

$$F(T) = \{z \in C : \ z \in T(z)\}. \tag{4}$$

Assume that

$$\bar{F}(T) \neq \emptyset.$$

Denote by $\mathrm{Card}(D)$ the cardinality of a set D. For each $z \in R^1$, set

$$\lfloor z \rfloor = \max\{i : \ i \text{ is an integer and } i \leq z\}.$$

In the following we suppose that the sum over an empty set is zero.

We study the asymptotic behavior of sequences of iterates $x_{t+1} \in T(x_t)$, where $t = 0, 1, \dots$. In particular, we are interested in their convergence to a fixed point of T. This iterative algorithm was introduced in [17], also containing its application to sparsity-constrained minimisation.

The following result, which is proven in Section 4, shows that almost all iterates of our set-valued mappings are approximated solutions of the corresponding fixed point problem. Many results of this type are reported in [8,9].

Theorem 1. *Assume that $M > 0$, $\epsilon \in (0, 1)$ and that*

$$\bar{F}(T) \cap B(\theta, M) \neq \emptyset. \tag{5}$$

Then an integer $Q \geq 1$ exists such that for each sequence $\{x_i\}_{i=0}^{\infty} \subset C$ which satisfies

$$\rho(x_0, \theta) \leq M$$

and

$$x_{t+1} \in T(x_t) \text{ for each integer } t \geq 0$$

the inequality

$$\rho(x_t, \theta) \leq 3M$$

holds for all integers $t \geq 0$*,*

$$Card(\{t \in \{0, 1, \ldots, \} : \rho(x_t, x_{t+1}) > \epsilon\}) \leq Q$$

and $\lim_{t\to\infty} \rho(x_t, x_{t+1}) = 0$.

The following global convergence result is proven in Section 5.

Theorem 2. *Assume a sequence* $\{x_t\}_{t=0}^{\infty} \subset C$ *and that for each integer* $t \geq 0$*,*

$$x_{t+1} \in T(x_t).$$

Then

$$x_* = \lim_{t\to\infty} x_t$$

and a natural number t_0 *exist such that for each integer* $t \geq t_0$

$$\phi(x_t) \subset \phi(x_*)$$

and if an integer $i \in \phi(x_t)$ *satisfies* $x_{t+1} = T_i(x_t)$*, then*

$$T_i(x_*) = x_*.$$

Theorem (2) generalizes the main result of [17], which establishes a local convergence of the iterative algorithm for iterates starting from a point belonging to a neighborhood of a strong fixed point belonging to the set $\bar{F}(T)$.

3. An Auxiliary Result

Lemma 1. *Assume that* $M, \epsilon > 0$ *and that* $z_* \in C$ *satisfies*

$$T_i(z_*) = z_*, \ i = 1, \ldots, m. \tag{6}$$

Then $\delta > 0$ *exists such that for each* $s \in \{1, \ldots, m\}$ *and each* $x \in C \cap B(\theta, M)$ *satisfying*

$$\rho(x, T_s(x)) > \epsilon \tag{7}$$

the inequality

$$\rho(z_*, T_s(x)) \leq \rho(z_*, x) - \delta \tag{8}$$

is true.

Proof. Let $s \in \{1, \ldots, m\}$. It is sufficient to show that $\delta > 0$ exists such that for each $x \in C \cap B(\theta, M)$ satisfying (7), Inequality (8) is true. Assume the contrary, then for each integer $k \geq 1$, there exists

$$x_k \in C \cap B(\theta, M) \tag{9}$$

such that

$$\rho(x_k, T_s(x_k)) > \epsilon \tag{10}$$

and

$$\rho(z_*, T_s(x_k)) > \rho(z_*, x_k) - k^{-1}. \tag{11}$$

In view of (A1) and (9), extracting a subsequence and re-indexing, we may assume without loss of generality that there exists

$$x_* = \lim_{k\to\infty} x_k. \tag{12}$$

From (9)–(12) and the continuity of T_s,

$$\rho(x_*, \theta) \le M,$$

$$\rho(x_*, T_s(x_*)) = \lim_{k \to \infty} \rho(x_k, T_s(x_k)) \ge \epsilon$$

and

$$\rho(z_*, T_s(x_*)) \ge \rho(z_*, x_*).$$

This contradicts (6) and (A2). The contradiction reached proves Lemma 1. □

4. Proof of Theorem 1

From (5), there exists

$$z_* \in B(\theta, M) \cap \bar{F}(T). \tag{13}$$

Lemma 1 implies that $\delta \in (0, \epsilon)$ exists such that the following property holds:
(a) for each $s \in \{1, \dots, m\}$ and each $x \in C \cap B(z_*, 2M)$ satisfying

$$\rho(x, T_s(x)) > \epsilon$$

we have

$$\rho(z_*, T_s(x)) \le \rho(z_*, x) - \delta.$$

Choose a natural number

$$Q \ge 2M\delta^{-1}. \tag{14}$$

Assume that $\{x_i\}_{i=0}^{\infty} \subset C$,

$$\rho(x_0, \theta) \le M \tag{15}$$

and that for each integer $t \ge 0$,

$$x_{t+1} \in T(x_t). \tag{16}$$

Let $t \ge 0$ be an integer. From (2) and (16), $s \in \{1, \dots, m\}$ exists such that

$$x_{t+1} = T_s(x_t). \tag{17}$$

Assumption (A2) and Equations (3), (13) and (17) imply that

$$\rho(z_*, x_{t+1}) = \rho(z_*, T_s(x_t)) \le \rho(z_*, x_t). \tag{18}$$

Since t is an arbitrary non-negative integer, Equations (13), (15) and (18) imply that for each integer $i \ge 0$,

$$\rho(z_*, x_i) \le \rho(z_*, x_0) \le 2M \tag{19}$$

and

$$\rho(x_i, \theta) \le 3M.$$

Assume that

$$\rho(x_{t+1}, x_t) > \epsilon. \tag{20}$$

Property (a) and Equations (17), (19) and (20) imply that

$$\rho(z_*, x_{t+1}) = \rho(z_*, T_s(x_t)) \le \rho(z_*, x_t) - \delta.$$

Thus, we have shown that the following property holds:
(b) if an integer $t \ge 0$ satisfies (20), then

$$\rho(z_*, x_{t+1}) \le \rho(z_*, x_t) - \delta.$$

Assume that $n \geq 1$ is an integer. Property (b) and Equations (18)–(20) imply that

$$2M \geq \rho(z_*, x_0) \geq \rho(z_*, x_0) - \rho(z_*, x_{n+1})$$

$$= \sum_{t=0}^{n} (\rho(z_*, x_t) - \rho(z_*, x_{t+1}))$$

$$\geq \sum \{\rho(z_*, x_t) - \rho(z_*, x_{t+1}) : \ t \in \{0, \dots, n\}, \ \rho(x_t, x_{t+1}) > \epsilon\}$$

$$\geq \delta \mathrm{Card}(\{t \in \{0, \dots, n\} : \ \rho(x_t, x_{t+1}) > \epsilon\})$$

, and in view of (14),

$$\mathrm{Card}(\{t \in \{0, \dots, n\} : \ \rho(x_t, x_{t+1}) > \epsilon\}) \leq 2M\delta^{-1} \leq Q.$$

Since n is an arbitrary natural number, we conclude that

$$\mathrm{Card}(\{t \in \{0, 1, \dots\} : \ \rho(x_t, x_{t+1}) > \epsilon\}) \leq Q.$$

Since ϵ is any element of $(0, 1)$, Theorem 1 is proven.

5. Proof of Theorem 2

In view of Theorem 1, the sequence $\{x_t\}_{t=0}^{\infty}$ is bounded. In view of (A1), it has a limit point $x_* \in C$ and a subsequence $\{x_{t_k}\}_{k=0}^{\infty}$ such that

$$x_* = \lim_{k \to \infty} x_{t_k}. \tag{21}$$

In view of (A3) and (21), we may assume without loss of generality that

$$\phi(x_{t_k}) \subset \phi(x_*), \ k = 1, 2, \dots \tag{22}$$

and that

$$\widehat{p} \in \phi(x_*)$$

exists such that

$$x_{t_k+1} = T_{\widehat{p}}(x_{t_k}), \ k = 1, 2, \dots. \tag{23}$$

It follows from Theorem 1, the continuity of $T_{\widehat{p}}$ and Equations (21) and (23) that

$$T_{\widehat{p}}(x_*) = \lim_{k \to \infty} T_{\widehat{p}}(x_{t_k}) = \lim_{k \to \infty} x_{t_k+1} = \lim_{k \to \infty} x_{t_k} = x_*. \tag{24}$$

Set

$$I_1 = \{i \in \phi(x_*) : \ T_i(x_*) = x_*\}, \ I_2 = \phi(x_*) \setminus I_1. \tag{25}$$

In view of (24) and (25),

$$\widehat{p} \in I_1.$$

Fix $\delta_0 \in (0, 1)$, such that

$$\rho(x_*, T_i(x_*)) > 2\delta_0, \ i \in I_2. \tag{26}$$

Assumption (A3), the continuity of $T_i, \ i = 1, \dots, m$ and (26) imply that $\delta_1 \in (0, \delta_0)$ exists such that for each $x \in B(x_*, \delta_1) \cap C$,

$$\phi(x) \subset \phi(x_*), \tag{27}$$

$$\rho(x, T_i(x)) > \delta_0, \ i \in I_2. \tag{28}$$

Theorem 1 implies that an integer $q_1 \geq 1$ exists such that for each integer $t \geq q_1$,

$$\rho(x_t, x_{t+1}) \leq \delta_0/2. \tag{29}$$

Assume that

$$\epsilon \in (0, \delta_1), \tag{30}$$

$$t \geq q_1 \tag{31}$$

is an integer and that

$$\rho(x_t, x_*) \leq \epsilon. \tag{32}$$

It follows from (27), (28), (30) and (32) that

$$\phi(x_t) \subset \phi(x_*) \tag{33}$$

and

$$\rho(x_t, T_i(x_t)) > \delta_0, \ i \in I_2. \tag{34}$$

In view of (33),

$$s \in \phi(x_*)$$

exists such that

$$x_{t+1} = T_s(x_t). \tag{35}$$

From (29), (31) and (35),

$$\rho(x_t, T_s(x_t)) = \rho(x_t, x_{t+1}) \leq \delta_0/2. \tag{36}$$

It follows from (25), (34) and (36) that

$$s \in I_1, \ T_s(x_*) = x_*.$$

Combined with Assumption (A2) and Equations (32) and (35), this implies that

$$\rho(x_{t+1}, x_*) = \rho(T_s(x_t), x_*) \leq \rho(x_t, x_*) \leq \epsilon.$$

Thus, we have shown that if $t \geq q_1$ is an integer and (32) holds, then (33) is true and if $s \in \phi(x_*)$ and (35) holds, then $s \in I_1$ and $\rho(x_{t+1}, x_*) \leq \epsilon$.

By induction and (21), we obtain that

$$\rho(x_i, x_*) \leq \epsilon$$

for all sufficiently large natural numbers i. Since ϵ is an arbitrary element of $(0, \delta_1)$, we conclude that

$$\lim_{t \to \infty} x_t = x_*$$

and Theorem 2 are proven.

6. Krasnosel'ski-Mann Iterations

Assume that $(X, \|\cdot\|)$ is a normed space and that $\rho(x, y) = \|x - y\|, \ x, y \in X$. We use the notation, definitions and assumptions introduced in Section 2. In particular, we assume that Assumptions (A1)–(A3) hold. Suppose that the set C is convex and denoted by $Id : X \to X$ the identity operator: $Id(x) = x$, $x \in X$. Let

$$\kappa \in (0, 2^{-1}).$$

We consider the Krasnosel'ski-Mann iteration associated with our set-valued mapping T and obtain the global convergence result (see Theorem (4) below), which generalizes the local convergence result of [18] for iterates starting from a point belonging to a neighborhood of a strong fixed point belonging to the set $\bar{F}(T)$.

The following result is proven in Section 7.

Theorem 3. *Assume that $M > 0$, $\epsilon \in (0,1)$ and that*

$$\bar{F}(T) \cap B(\theta, M) \neq \emptyset \tag{37}$$

Then there exists an integer $Q \geq 1$ such that for each

$$\{\lambda_t\}_{t=0}^{\infty} \subset (\kappa, 1-\kappa) \tag{38}$$

and each sequence $\{x_i\}_{i=0}^{\infty} \subset C$ which satisfies

$$\|x_0 - \theta\| \leq M$$

and

$$x_{t+1} \in (1-\lambda_t)x_t + \lambda_t T(x_t) \text{ for each integer } t \geq 0 \tag{39}$$

the inequality

$$\|x_t - \theta\| \leq 3M$$

holds for all integers $t \geq 0$,

$$Card(\{t \in \{0,1,\ldots,\} : \|x_t - x_{t+1}\| > \epsilon\}) \leq Q$$

and $\lim_{t\to\infty} \|x_t - x_{t+1}\| = 0$.

The following result is proven in Section 8.

Theorem 4. *Assume that*

$$\{\lambda_t\}_{t=0}^{\infty} \subset (\kappa, 1-\kappa)$$

and that a sequence $\{x_t\}_{t=0}^{\infty} \subset C$ satisfies (39). Then

$$x_* = \lim_{t\to\infty} x_t$$

and a natural number t_0 exist such that for each integer $t \geq t_0$

$$\phi(x_t) \subset \phi(x_*)$$

and if an integer $i \in \phi(x_t)$ satisfies

$$x_{t+1} = \lambda_t T_i(x_t) + (1-\lambda)x_t,$$

then

$$T_i(x_*) = x_*.$$

7. Proof of Theorem 3

From (37), there exists

$$z_* \in B(\theta, M) \cap \bar{F}(T). \tag{40}$$

Lemma 1 implies that $\delta \in (0,\epsilon)$ exists such that the following property holds:

(c) for each $s \in \{1,\ldots,m\}$ and each $x \in C \cap B(z_*, 2M)$ satisfying

$$\rho(x, T_s(x)) > \epsilon$$

we have

$$\rho(z_*, T_s(x)) \leq \rho(z_*, x) - \delta.$$

Choose a natural number

$$Q \geq 2M\delta^{-1}\kappa^{-1}. \tag{41}$$

Assume that (38) holds and that a sequence $\{x_i\}_{i=0}^{\infty} \subset C$ satisfies (39) and

$$\|x_0 - \theta\| \leq M. \tag{42}$$

Let $t \geq 0$ be an integer. From (2) and (39), $s \in \{1, \ldots, m\}$ exists such that

$$x_{t+1} = \lambda_t T_s(x_t) + (1 - \lambda_t)x_t. \tag{43}$$

Assumption (A2) and Equations (3), (40) and (43) imply that z_* is a fixed point of T_s and that

$$\|x_{t+1} - z_*\| = \|\lambda_t T_s(x_t) + (1 - \lambda_t)x_t - z_*\|$$
$$\leq \lambda_t \|T_s(x_t) - z_*\| + (1 - \lambda_t)\|x_t - z_*\| \leq \|z_* - x_t\|. \tag{44}$$

Since t is an arbitrary non-negative integer, Equations (40), (42) and (44) imply that for each integer $i \geq 0$,

$$\|z_* - x_i\| \leq \|z_* - x_0\| \leq 2M$$

and

$$\|x_i - \theta\| \leq 3M.$$

Assume that

$$\|x_{t+1} - x_t\| > \epsilon. \tag{45}$$

It follows from (38), (43) and (45) that

$$\epsilon < \|x_{t+1} - x_t\| = \|\lambda_t T_s(x_t) + (1 - \lambda_t)x_t - x_t\| = \lambda_t \|T_s(x_t) - x_t\|$$

and

$$\|T_s(x_t) - x_t\| \geq \epsilon \lambda_t^{-1} \geq \epsilon(1 - \kappa)^{-1}. \tag{46}$$

Property (c) and Equation (46) imply that

$$\|z_* - T_s(x_t)\| \leq \|z_* - x_t\| - \delta. \tag{47}$$

From (38), (43) and (47),

$$\|x_{t+1} - z_*\| = \|\lambda_t T_s(x_t) + (1 - \lambda_t)x_t - z_*\|$$
$$\leq \lambda_t \|T_s(x_t) - z_*\| + (1 - \lambda_t)\|x_t - z_*\|$$
$$\leq \lambda_t(\|x_t - z_*\| - \delta) + (1 - \lambda_t)\|x_t - z_*\|$$
$$\leq \|x_t - z_*\| - \lambda_t \delta \leq \|x_t - z_*\| - \delta\kappa. \tag{48}$$

Thus, we have shown that the following property holds:
(d) if an integer $t \geq 0$ satisfies (45), then

$$\|z_* - x_{t+1}\| \leq \|z_* - x_t\| - \delta\kappa.$$

Assume that $n \geq 1$ is an integer. Property (d) and Equations (40), (42) and (44) imply that

$$2M \geq \|z_* - x_0\| \geq \|z_* - x_0\| - \|z_* - x_{n+1}\|$$
$$= \sum_{t=0}^{n} (\|z_* - x_t\| - \|z_* - x_{t+1}\|)$$
$$\geq \sum \{\|z_* - x_t\| - \|z_* - x_{t+1}\| : \ t \in \{0, \ldots, n\}, \ \|x_t - x_{t+1}\| > \epsilon\}$$
$$\geq \delta\kappa \mathrm{Card}(\{t \in \{0, \ldots, n\} : \ \|x_t - x_{t+1}\| > \epsilon\}),$$

and in view of (41),

$$\mathrm{Card}(\{t \in \{0, \dots, n\} : \|x_t - x_{t+1}\| > \epsilon\}) \leq 2M(\delta\kappa)^{-1} \leq Q.$$

Since n is an arbitrary natural number, we conclude that

$$\mathrm{Card}(\{t \in \{0, 1, \dots\} : \|x_t - x_{t+1}\| > \epsilon\}) \leq Q.$$

Since ϵ is any element of $(0, 1)$, we can obtain

$$\lim_{t \to \infty} \|x_t - x_{t+1}\| = 0.$$

Theorem 3 is thus proven.

8. Proof of Theorem 4

In view of Theorem (3), the sequence $\{x_t\}_{t=0}^{\infty}$ is bounded. In view of (A1), it has a limit point $x_* \in C$ and a subsequence $\{x_{t_k}\}_{k=0}^{\infty}$ such that

$$x_* = \lim_{k \to \infty} x_{t_k}. \tag{49}$$

In view of (A3) and Equations (38), (39) and (49), extracting a subsequence and re-indexing, we may assume without loss of generality that

$$\phi(x_{t_k}) \subset \phi(x_*), \ k = 1, 2, \dots \tag{50}$$

and that

$$\widehat{p} \in \phi(x_*)$$

exists such that

$$x_{t_k+1} = \lambda_{t_k} T_{\widehat{p}}(x_{t_k}) + (1 - \lambda_{t_k}) x_{t_k}, \ k = 1, 2, \dots \tag{51}$$

and that there exists

$$\lambda_* = \lim_{k \to \infty} \lambda_{t_k} \in [\kappa, 1 - \kappa]. \tag{52}$$

It follows from Theorem (3), the continuity of $T_{\widehat{p}}$ and Equations (49), (51) and (52) that

$$\begin{aligned} &\lambda_* T_{\widehat{p}}(x_*) + (1 - \lambda_*) x_* \\ &= \lim_{k \to \infty} (\lambda_{t_k} T_{\widehat{p}}(x_{t_k}) + (1 - \lambda_{t_k}) x_{t_k}) \\ &= \lim_{k \to \infty} x_{t_k+1} = \lim_{k \to \infty} x_{t_k} = x_*. \end{aligned} \tag{53}$$

Set

$$I_1 = \{i \in \phi(x_*) : T_i(x_*) = x_*\}, \ I_2 = \phi(x_*) \setminus I_1. \tag{54}$$

In view of (53) and (54),

$$\widehat{p} \in I_1.$$

Fix $\delta_0 \in (0, 1)$ such that

$$\|x_* - T_i(x_*)\| > 2\delta_0, \ i \in I_2. \tag{55}$$

Assumption (A3), the continuity of $T_i, \ i = 1, \dots, m$ and (55) imply that $\delta_1 \in (0, \delta_0)$ exists such that for each $x \in B(x_*, \delta_1) \cap C$,

$$\phi(x) \subset \phi(x_*), \tag{56}$$

$$\|x - T_i(x)\| > \delta_0, \ i \in I_2. \tag{57}$$

Theorem (3) implies that an integer $q_1 \geq 1$ exists such that for each integer $t \geq q_1$,

$$\|x_t - x_{t+1}\| \leq \kappa\delta_0/2. \quad (58)$$

Assume that

$$\epsilon \in (0, \delta_1), \quad (59)$$

$$t \geq q_1 \quad (60)$$

is an integer and that

$$\|x_t - x_*\| \leq \epsilon. \quad (61)$$

It follows from (56), (57), (59) and (61) that

$$\phi(x_t) \subset \phi(x_*) \quad (62)$$

and

$$\|x_t - T_i(x_t)\| > \delta_0, \ i \in I_2. \quad (63)$$

In view of (39),

$$s \in \phi(x_t) \subset \phi(x_*)$$

exists such that

$$x_{t+1} = \lambda_t T_s(x_t) + (1 - \lambda_t)x_t. \quad (64)$$

From (38), (58) and (64),

$$\kappa\delta_0/2 \geq \|x_{t+1} - x_t\| = \lambda_t \|T_s(x_t) - x_t\|$$

and

$$\|x_t - T_s(x_t)\| \leq \kappa\delta_0(2\lambda_t)^{-1} \leq \delta_0/2. \quad (65)$$

It follows from (54), (56), (57), (59), (61) and (65) that

$$s \in I_1, \ T_s(x_*) = x_*.$$

Combined with Assumption (A2) and Equations (39), (61) and (64), this implies that

$$\|x_{t+1} - x_*\| = \|\lambda_t T_s(x_t) + (1 - \lambda_t)x_t - x_*\|$$
$$\leq \lambda_t \|T_s(x_t) - x_*\| + (1 - \lambda_t)\|x_t - x_*\|$$
$$\leq \|x_t - x_*\| \leq \epsilon.$$

Thus, we have shown that if $t \geq q_1$ is an integer and (61) holds, then $\|x_{t+1} - x_*\| \leq \epsilon$. By induction and (49), we can obtain that

$$\|x_i - x_*\| \leq \epsilon$$

for all sufficiently large natural numbers i. Since ϵ is an arbitrary element of $(0, \delta_1)$, we can conclude that

$$\lim_{t \to \infty} x_t = x_*$$

and Theorem (4) are proven.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Gibali, A. A new split inverse problem and an application to least intensity feasible solutions. *Pure Appl. Funct. Anal.* **2017**, *2*, 243–258.
2. Goebel, K.; Kirk, W.A. *Topics in Metric Fixed Point Theory*; Cambridge University Press: Cambridge, UK, 1990.
3. Goebel, K.; Reich, S. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*; Marcel Dekker: New York, NY, USA; Basel, Switzerland, 1984.
4. Khamsi, M.A.; Kozlowski, W.M. *Fixed Point Theory in Modular Function Spaces*; Birkhäuser: Cham, Switzerland; Springer: Cham, Switzerland, 2015.
5. Kopecka, E.; Reich, S. A note on alternating projections in Hilbert space. *J. Fixed Point Theory Appl.* **2012**, *12*, 41–47. [CrossRef]
6. Reich, S.; Zaslavski, A.J. *Genericity in Nonlinear Analysis, Developments in Mathematics*; Springer: New York, NY, USA, 2014.
7. Qin, X.; Cho, S.Y.; Yao, J.-C. Weak and strong convergence of splitting algorithms in Banach spaces. *Optimization* **2020**, *69*, 243–267. [CrossRef]
8. Zaslavski, A.J. *Approximate Solutions of Common Fixed Point Problems*; Springer Optimization and Its Applications; Springer: Cham, Switzerland, 2016.
9. Zaslavski, A.J. *Algorithms for Solving Common Fixed Point Problems*; Springer Optimization and Its Applications; Springer: Cham, Switzerland, 2018.
10. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
11. Abbas, M.I.; Ragusa, M.A. Solvability of Langevin equations with two Hadamard fractional derivatives via Mittag–Leffler functions. *Appl. Anal.* **2020**, *101*, 3231–3245. [CrossRef]
12. Censor, Y.; Zaknoon, M. Algorithms and convergence results of projection methods for inconsistent feasibility problems: A review. *Pure Appl. Func. Anal.* **2018**, *3*, 565–586.
13. Gibali, A.; Reich, S.; Zalas, R. Outer approximation methods for solving variational inequalities in Hilbert space. *Optimization* **2017**, *66*, 417–437. [CrossRef]
14. Harbau, M.H.; Ali, B.; Hybrid subgradient method for pseudomonotone equilibrium problem and fixed points of relatively nonexpansive mappings in Banach spaces. *Filomat* **2022**, *6*, 3515–3525. [CrossRef]
15. Shukla, R.; Panicker, R. Approximating fixed points of enriched nonexpansive mappings in geodesic spaces. *J. Funct. Spaces* **2022**, *2022*, 6161839. [CrossRef]
16. Takahashi, W.; Xu, H.K.; Yao, J.-C. Iterative methods for generalized split feasibility problems in Hilbert spaces. *Set-Valued Var. Anal.* **2015**, *23*, 205–221. [CrossRef]
17. Tam, M.K. Algorithms based on unions of nonexpansive maps. *Optim. Lett.* **2018**, *12*, 1019–1027. [CrossRef]
18. Dao, M.N.; Tam, M.K. Union averaged operators with applications to proximal algorithms for min-convex functions. *J. Optim. Theory Appl.* **2019**, *181*, 61–94. [CrossRef]
19. Elsner, L.; Koltracht, I.; Neumann, M. Convergence of sequential and asynchronous nonlinear paracontractions. *Numer. Math.* **1992**, *62*, 305–319. [CrossRef]