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Article

To the Problem of Discontinuous Solutions in Applied Mathematics

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Abstract: This paper addresses discontinuities in the solutions of mathematical physics that describe actual processes and are not observed in experiments. The appearance of discontinuities is associated in this paper with the classical differential calculus based on the analysis of infinitesimal quantities. Nonlocal functions and nonlocal derivatives, which are not specified, in contrast to the traditional approach to a point, but are the results of averaging over small but finite intervals of the independent variable are introduced. Classical equations of mathematical physics preserve the traditional form but include nonlocal functions. These equations are supplemented with additional equations that link nonlocal and traditional functions. The proposed approach results in continuous solutions of the classical singular problems of mathematical physics. The problems of a string and a circular membrane loaded with concentrated forces are used to demonstrate the procedure. Analytical results are supported with experimental data.

Keywords: differential calculus; applied mathematics; equations of mathematical physics; discontinuous and singular solutions

MSC: 34A35; 00A79; 35J67; 34M35

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1. Nonlocal Functions

Modern differential calculus was developed by I. Newton and G. Leibnitz in the 17th century. It is demonstrated in the following cubic polynomial:

$$y = a + bx + cx^2 + ex^3 \tag{1}$$

To introduce the derivative, we assume that y and x have some small increments, Δy and Δx. Then:

$$y + \Delta y = a + b(x + \Delta x) + c(x^2 + 2x\Delta x + \Delta x^2) + e(x^3 + 3x\Delta x^2 + 3x^2\Delta x + \Delta x^3)$$

Subtracting y in Equation (1) and dividing by Δx, we get:

$$\frac{\Delta y}{\Delta x} = b + c(2x + \Delta x) + d(3x^2 + 3x\Delta x + \Delta x^2)$$

To obtain the derivative, we need to eliminate Δx from the right-hand side of this equation. Naturally, we cannot put $\Delta x = 0$ because the resulting operation makes no sense. We also cannot neglect Δx because the operation becomes approximate in this case and is not acceptable in mathematics. Thus, we need to introduce the infinitesimal quantity dx, which is infinitely small but not zero. Then, we arrive at the following equation for the derivative:

$$\frac{dy}{dx} = b + 2cx + 3ex^2 \tag{2}$$

The other definition of the derivative that does not require the introduction of infinitesimal quantities was proposed by J. Landen in 1764 [1]. It introduces an interval $[x_1, x_2]$ and assumes that $x_1 \leq x \leq x_2$. We determine the derivative of the function in Equation (1) as:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{b(x_2 - x_1) + c(x_2^2 - x_1^2) + e(x_2^3 - x_1^3)}{x_2 - x_1} = b + c(x_1 + x_2) + e(x_1^2 + x_1x_2 + x_2^2)$$

Taking $x_1 = x_2 = x$, we get:

$$\frac{dy}{dx} = b + 2cx + 3ex^2$$

This result coincides with Equation (2) but does not require the introduction of infinitesimal quantities.

We further use the Landen definition of the derivative to construct nonlocal derivatives.

To introduce the nonlocal function, we consider a conventional function $u(x)$ as shown in Figure 1.

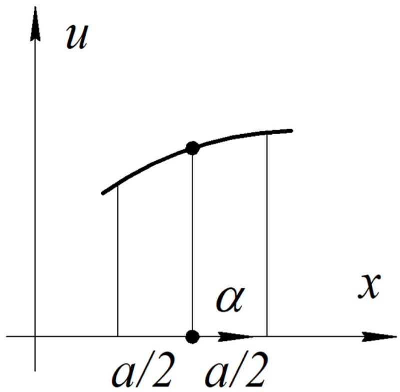

Figure 1. A function of one variable.

We assume that this function describes a real physical process and, according to the definition, is smooth and has traditional derivatives of any order. We introduce in the vicinity of point x the local coordinate α, such that $-a/2 \leq \alpha \leq a/2$ (Figure 1). Since function $u(x)$ is smooth, we decompose it in the vicinity of point $\alpha = 0$ into the Taylor series:

$$u(x, \alpha) = u(x) + \alpha u' + \frac{\alpha^2}{2!}u'' + \frac{\alpha^3}{3!}u''' \quad (3)$$

where $u' = du/dx$, and the series is restricted to the terms presented in Equation (3). We introduce the nonlocal function $U(x)$ as the average value of function $u(x, \alpha)$ on the interval $[-a/2, a/2]$, i.e.,:

$$U(x) = \frac{1}{a}\int_{-a/2}^{a/2} u(x, \alpha)d\alpha$$

Substituting Equation (3), we arrive at [2]:

$$U(x) = u(x) + \frac{a^2}{24}u''(x) \quad (4)$$

Thus, the nonlocal function depends not only on the value of the original function at a point but on the value of the second derivative as well. According to Landen, we should introduce the nonlocal derivative as:

$$\frac{Du}{Dx} = \frac{1}{a}[u(x, a/2) - u(x, -a/2)]$$

In contrast to the original Landen definition, the parameter a is small but finite. Substituting Equation (3) and taking into account Equation (4), we get:

$$\frac{Du}{Dx} = u' + \frac{a^2}{24}u''' + \cdots = \frac{dU}{dx} \tag{5}$$

Thus, the nonlocal derivative of the original function is the classical derivative of the nonlocal function.

In the following sections, the proposed approach is demonstrated on the classical problems of mathematical physics—the problems of a string and a membrane loaded with concentrated forces.

2. A String Loaded with a Concentrated Force

We consider a string loaded with the axial tensile force t and the transverse force P as in Figure 2.

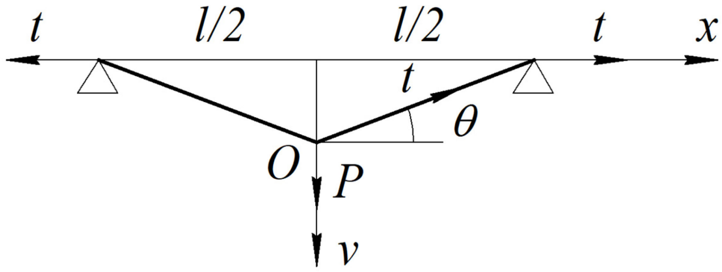

Figure 2. A string loaded with a concentrated force.

The equilibrium equation has the following form:

$$2t \sin\theta = P \tag{6}$$

For a relatively small deflection, $v(x)$, we can take $\sin\theta \approx \theta$, and Equation (6) reduces to:

$$\theta = \frac{P}{2t} \tag{7}$$

To obtain the classical solution, we put $\theta = -dv/dx = -v'$, and Equation (7) becomes:

$$v' = -\frac{P}{2t} \tag{8}$$

The solution of this equation is:

$$v = -\frac{Px}{2t} + C_0 \tag{9}$$

Satisfying the boundary condition, $v(x = l/2) = 0$ (Figure 2), we finally get the classical solution for the string deflection:

$$v = \frac{P}{2t}\left(\frac{l}{2} - x\right) \tag{10}$$

This solution has two points of discontinuity at $x = 0$ and $x = l/2$, at which the deflection derivative does not exist. For a real string, we have the obvious conditions of $v'(x = 0) = v'(x = l/2) = 0$, which cannot be satisfied with Equation (9). The experimental deflection of a steel string with diameter $d = 0.49$ mm, length $l = 124$ mm, and elastic modulus $E = 209.3$ GPa loaded with forces $t = 49$ N and $P = 1.96$ N is shown in Figure 3 with dots.

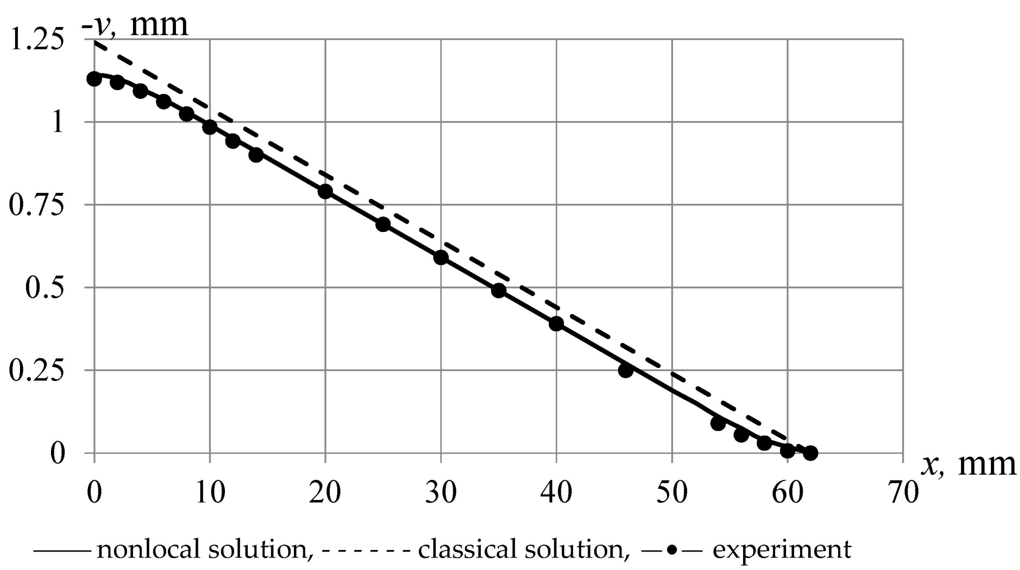

Figure 3. Distribution of the deflection over the string length.

The experimental facility for investigating the transverse displacements of a stretched string under the action of a transverse local force is shown in Figure 4. The equipment provides a constant controlled string tension. During the test, the transverse displacements of the string are measured with an accuracy of 0.1 mm using an optical system.

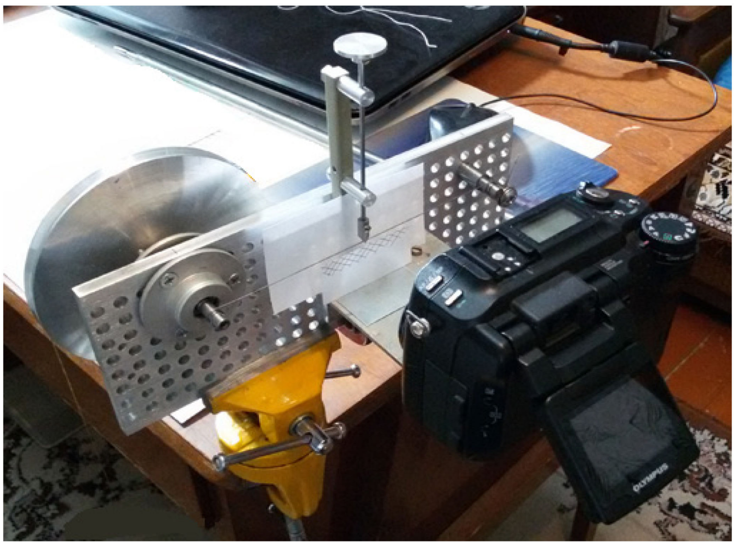

Figure 4. Testing of a string.

As can be seen, the classical solution in Equation (10) presented with the dashed line does not coincide with the experiment.

We obtain the nonlocal solution of the problem. Using Figure 1 and Equation (4), we introduce the nonlocal angle:

$$\Theta = \theta + \frac{a^2}{24}\theta''$$

where angle θ is averaged over some intervals $[-a/2, a/2]$. Changing θ to Θ in Equation (7), we arrive at the following equilibrium equation:

$$\theta + \frac{a^2}{24}\theta'' = \frac{P}{2t}$$

Since $\theta = -v'$, we get:

$$v' + \frac{a^2}{24}v''' = -\frac{P}{2t} \tag{11}$$

Integration yields:

$$v + \frac{a^2}{24}v'' = -\frac{Px}{2t} + C_0 \tag{12}$$

This equation allows us to clear out the procedure used to construct the nonlocal solution. The left-hand part contains the ordinary Helmhotz-type operator acting on the string deflection, whereas its right-hand part is the classical solution of the problem specified by Equation (9).

This conclusion is valid in the general case [2] in which the original function $u(x, y, z)$ can be found from the following equation:

$$u + \frac{a^2}{24}\Delta u = u_c \tag{13}$$

where Δ is the Laplace operator, and u_c is the classical solution. If $u(x, y, z)$ is a scalar function, the Laplace operator is invariant, and Equation (13) is valid in any coordinate frame.

Equation (11) includes parameter a, which is not known yet. To determine this parameter, we should take into account that Equation (10) describes the mathematical model of the string, which is the one-dimensional manifold. The real string is characterized with an elastic modulus and the shape and dimensions of the cross-section, which do not enter into the classical solution, Equation (10). We consider a more adequate physical model—a beam shown in Figure 5.

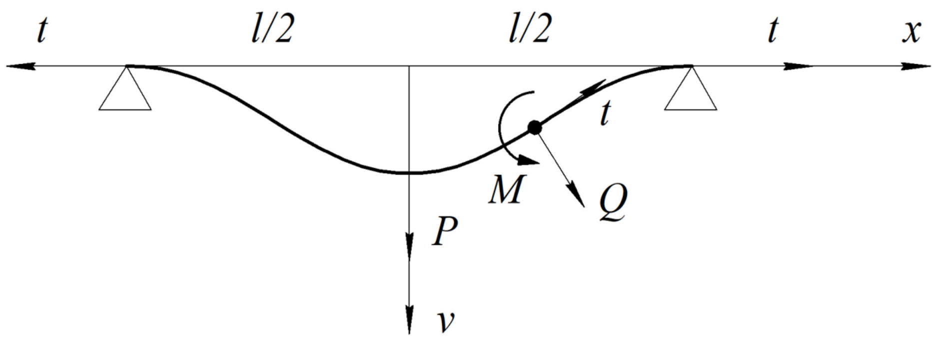

Figure 5. Stretching and bending of a beam.

In this model, the axial force t acting in the beam cross-section is supplemented with transverse force Q and bending moment M (Figure 5). A beam is described by the following equations:

$$2(tv' + Q) = -P,\ Q = M',\ M = -Dv''$$

which can be reduced to:

$$v' - \frac{D}{t}v''' = -\frac{Px}{2t}$$

where D is the bending stiffness. Matching this equation to Equation (11), we can conclude that $a^2 = -24D/t$. Since D and t cannot be negative, parameter a is imaginary. This result looks natural because the obtained results are based on a model of a homogeneous continuum that ignores the material's actual microstructure. Thus, we cannot expect that this model allows us to determine the actual value of the microstructural parameter. To avoid imaginary values, we present Equation (12) in the following form:

$$v - s^2v'' = -\frac{Px}{2t} + C_0 \tag{14}$$

in which:

$$s = \sqrt{D/t} \tag{15}$$

Thus, parameter s depends on the bending stiffness and the axial force and does not depend on the transverse force and the string length. Putting $D = 0$, we get $s = 0$, and Equation (14) reduces to the classical Equation (9).

The solution of Equation (14) is:

$$v = C_1 e^{-kx} + C_2 e^{kx} - \frac{Px}{2t} + C_0,\ k = \frac{1}{s},\ 0 \le x \le \frac{l}{2} \tag{16}$$

This solution specifies two boundary-layer effects in the vicinity of points $x = 0$ and $x = l/2$. For a long string, we can neglect the interaction of these effects and present Equation (16) as:

$$v = C_1 e^{-kx} + C_2 e^{-k(\frac{l}{2}-x)} - \frac{Px}{2t} + C_0$$

We consider the boundary condition at $x = 0$, i.e., $v'(0) = 0$. Taking $C_2 = 0$ and using this condition to determine C_1, we arrive at the solution that is valid for $0 \le x \le l/4$:

$$v = -\frac{P}{2tk} e^{-kx} - \frac{Px}{2t} + C_0 \tag{17}$$

We consider the part of the string corresponding to $l/4 \le x \le l/2$. To apply the boundary conditions $v(l/2) = v'(l/2) = 0$, we put $C_1 = 0$ and get:

$$v = C_2 e^{-k(\frac{l}{2}-x)} - \frac{Px}{2t} + C_0 \tag{18}$$

Using the boundary conditions, we can find C_2 and C_0 and present Equations (17) and (18) as:

$$v = -\frac{P}{2kt}\left(1 + e^{-kx}\right) + \frac{P}{2t}\left(\frac{l}{2} - x\right),\ \ 0 \le x \le \frac{l}{4}$$

$$v = -\frac{P}{2kt}\left(1 - e^{-k(\frac{l}{2}-x)}\right) + \frac{P}{2t}\left(\frac{l}{2} - x\right),\ \frac{l}{4} \le x \le \frac{l}{2}$$

For $x = l/4$, we can neglect the exponential terms in comparison with unity, and both solutions yield one and the same result. For the experimental string, the obtained solution is shown in Figure 3 with the solid line. As can be seen, it is in good agreement with the experiment.

3. A Circular Membrane Loaded with a Concentrated Force

A circular membrane loaded with in-plane forces t and a concentrated force P applied at the membrane center (Figure 6) is a classical singular problem of mathematical physics [3].

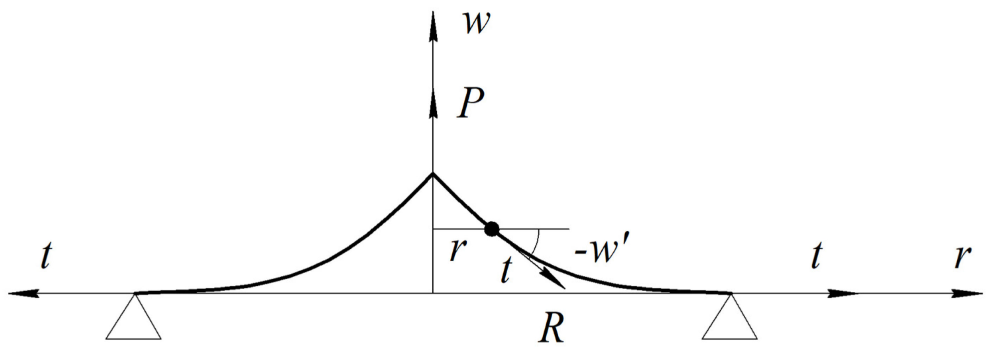

Figure 6. Circular membrane loaded with in-plane tensile forces t and a concentrated force P.

The membrane deflection satisfies the following equation [3]:

$$t\Delta w = -p(r), \;\; \Delta w = \frac{1}{r}(rw')', \;\; (\cdot)' = d(\cdot)/dr \tag{19}$$

For a concentrated force, $p(r) = P\delta(r)$, in which $\delta(r)$ is the delta function. The solution of Equation (19) that satisfies the boundary condition $w(r = R) = 0$ is [3]:

$$w_c = \frac{P}{2\pi t}\ln\frac{R}{r} \tag{20}$$

As can be seen, $w_c \to \infty$ if $r \to 0$. Moreover, the deflection is infinitely high for any force P, irrespective of how small it can be. Naturally, this result does not have a physical meaning.

To obtain the nonlocal solution of the problem, we apply Equation (13). Taking into account that $w(r)$ is a scalar function, we have [2]:

$$w + \frac{a^2}{24}\Delta w = w_c, \; \Delta w = \frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} \tag{21}$$

where w_c is specified by Equation (20). To determine parameter a, we assume, as earlier, that the membrane is a three-dimensional object—a circular plate with a finite thickness h, rather than a two-dimensional mathematical manifold. The plate deflection satisfies the following equation [4]:

$$w - \frac{D}{t}\Delta w = \frac{P}{2\pi t}\ln r \tag{22}$$

where $D = Eh^3/12(1-\nu^2)$, E is the elastic modulus, and ν is Poisson's ratio. Matching Equations (21) and (22), we can, as earlier, conclude that $a^2 = -24D/t$. Introducing parameter s in accordance with Equation (15), we arrive at:

$$s^2\Delta w - w = -\frac{P}{2\pi t}\ln\frac{R}{r}$$

Using the following notations:

$$\rho = r/s, R_s = R/s,$$

we finally have:

$$\frac{d^2 w}{d\rho^2} + \frac{1}{\rho}\frac{dw}{d\rho} - w = -\frac{P}{2\pi t}\ln\frac{R_s}{\rho} \tag{23}$$

The general solution of this equation can be written in terms of modified Bessel functions $I_0(\rho)$, $K_0(\rho)$ and the corresponding particular solution as:

$$w(\rho) = C_1 I_0(\rho) + C_2 K_0(\rho) + \frac{P}{2\pi t}\ln\frac{R_s}{\rho}$$

Using the boundary condition $w(\rho = R_s) = 0$, we can determine the constant C_1 and get:

$$w(\rho) = C_2\left[K_0(\rho) - \frac{K_0(R_s)}{I_0(R_s)}I_0(\rho)\right] + \frac{P}{2\pi t}\ln\frac{R_s}{\rho} \tag{24}$$

As can be seen, this particular solution is singular at $\rho = 0$. However, the Macdonald function $K_0(\rho)$ has the same type of singularity and can be used to eliminate the singularity of the solution in Equation (24). We decompose the Bessel functions into the power series [5]:

$$I_0(\rho) = 1 + \frac{\rho^2}{4(1!)^2} + \frac{\rho^4}{4^2(2!)^2} + \cdots, \; K_0(\rho) = -\left[\gamma + \ln\frac{\rho}{2}\right]I_0(\rho) + \frac{\rho^2}{4(1!)^2} + \cdots \tag{25}$$

where $\gamma = 0.577$ is the Euler constant. Substituting series (25) in Equation (24) and putting $\rho \to 0$, we can conclude that the solution becomes regular if we take $C_2 = -P/(2\pi t)$. Finally, we arrive at:

$$w(\rho) = \frac{P}{2\pi t}\left[\ln\frac{R_s}{\rho} - K_0(\rho) + \frac{K_0(R_s)}{I_0(R_s)} I_0(\rho)\right]$$

At the membrane center, the deflection is finite, i.e.,:

$$w_0 = \frac{P}{2\pi t}\left[\ln R_s + 0.116 + \frac{K_0(R_s)}{I_0(R_s)}\right]$$

The derivative of the deflection,

$$w' = \frac{1}{s}\frac{dw}{d\rho} = \frac{P}{2\pi t s}\left[K_1(\rho) - \frac{1}{\rho} + \frac{K_0(R_s)}{I_0(R_s)} I_1(\rho)\right],$$

is zero at the membrane center because $K_1(\rho \to 0) \to 1/\rho$ and $I_1(0) = 0$.

We undertake the analysis of the obtained results. First, the singular classical solution in Equation (20) looks consistent since the right-hand side of Equation (19) includes the delta function, which is also singular. Thus, a singular action results in a singular solution. Being natural in mathematics, it is not acceptable in physics, in which the concentrated force and the membrane deflection are, in principle, different functions. A concentrated force does not exist in reality and is singular according to its definition, whereas a deflection is a physical variable that can be directly measured and cannot be infinitely high. Second, the consistency of the classical solution is proved in mathematics since it can be presented as a limit of a system of regular functions [6]. Thus, the mathematically consistent solution is not physically consistent. Finally, as follows from the foregoing derivation, the logarithmic-type singularity in the right-hand side of Equation (23) is compensated for by the same type of singularity in a fundamental solution of the corresponding homogeneous equation. If we formally change the type of singularity in the right-hand side of Equation (23) (e.g., introduce a power-type singularity), it will not be eliminated by the logarithmic fundamental solution. Thus, the proposed method works only for the equations that describe physical problems.

The obtained solution was verified experimentally. The experimental membrane was made of a polymeric film and had the following parameters: $h = 0.04$ mm, $R = 75$ mm, $E = 5.4$ GPa, $\nu = 0.4$, $t = 0.77$ N/mm, $P = 0.5$ N H. For these parameters, Equation (15) yields $s = 0.216$ mm. The calculated value of the maximum deflection—$w_0 = 0.593$mm is in good agreement with the experimental result—$w_0 = 0.6$ mm. The description of the experiment is presented in [7].

Dependence of the deflection on the radial coordinate is shown in Figure 7 with the solid line along with experimental results (dots). The line corresponds to both solutions (classical and nonlocal)—the difference between them can be seen only in the vicinity of the membrane center (Figure 8).

As follows from Figure 8, the difference between the curves is observed for $r \leq 1$ mm, whereas the membrane radius is $R = 75$ mm.

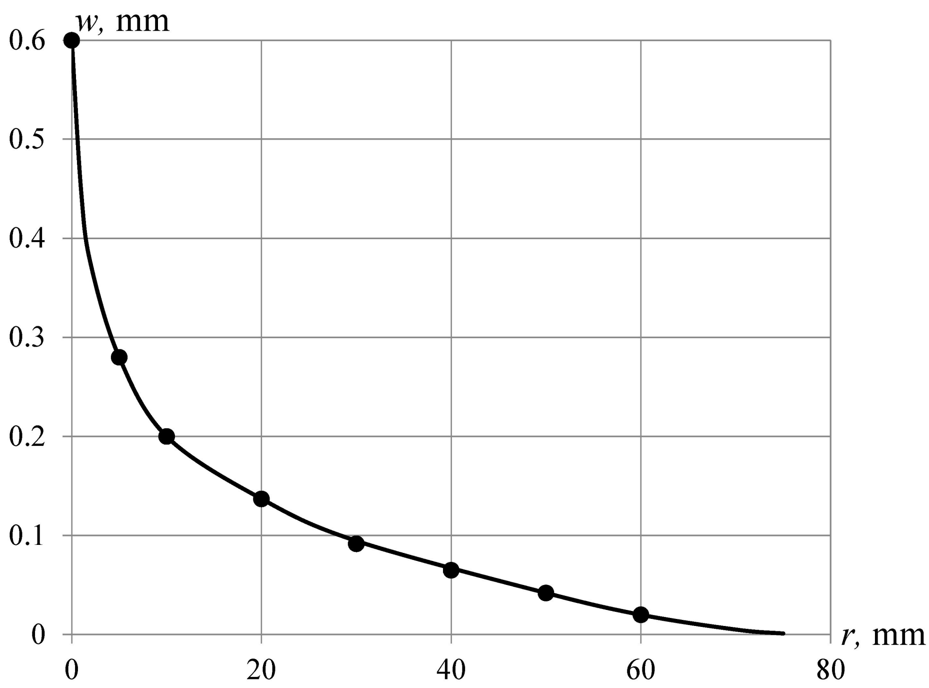

Figure 7. Dependences of the membrane deflection on the radial coordinate corresponding to the classical and nonlocal solutions (——) and experiment (•).

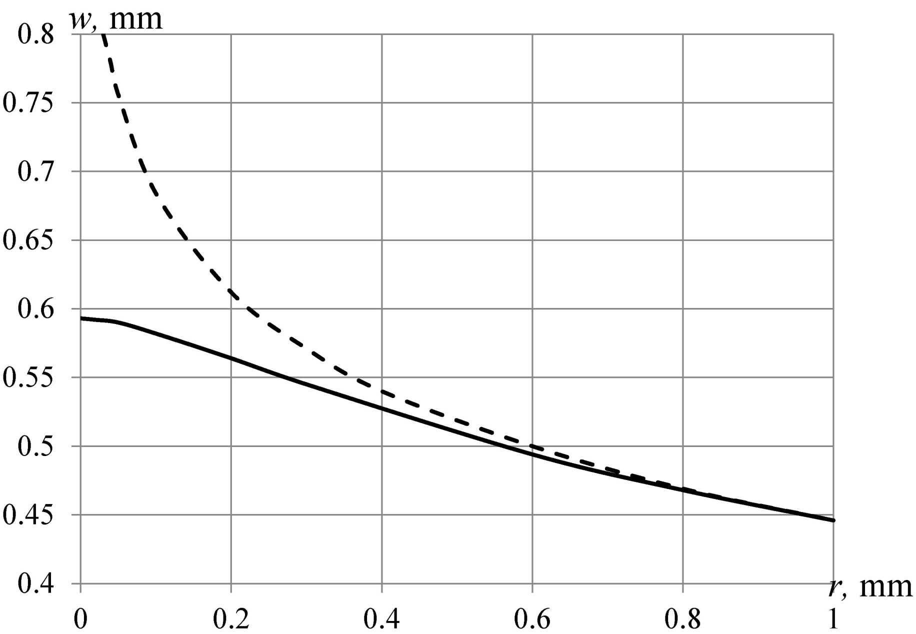

Figure 8. Membrane deflection corresponding to the classical solution (- - - - - -) and the nonlocal solution (——).

4. Conclusions

Regular solutions of the problems of mathematical physics for a string and a membrane following from physically consistent models simulating a string as a beam and a membrane as a circular plate cannot be obtained on the basis of the classical mathematical models of a string and a membrane as one- or two-dimensional manifolds, respectively, and can be found by applying the proposed nonlocal functions and derivatives.

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