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The Yamaguchi–Noshiro Type of Bi-Univalent Functions Connected with the Linear q-Convolution Operator

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Abstract: In the present paper, the authors introduce and investigate two new subclasses of the function class $\mathcal{B}$ of bi-univalent analytic functions in an open unit disk $\mathcal{U}$ connected with a linear q-convolution operator. The bounds on the coefficients $|c_2|$, $|c_3|$ and $|c_4|$ for the functions in these new subclasses of $\mathcal{B}$ are obtained. Relevant connections of the results presented here with those obtained in earlier work are also pointed out.

Keywords: univalent functions; fractional derivatives; convolution; analytic functions; bi-univalent functions; coefficient bounds

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1. Introduction

Let $\mathcal{A}$ be the class of analytical functions in an open unit disk

$$\mathcal{U} := \{\xi : \xi \in \mathbb{C} \text{ and } |\xi| < 1\}$$

and assume that Ω is a family of functions $\mathbb{F} \in \mathcal{A}$ satisfying the normalization conditions (see [1]):

$$\mathbb{F}(0) = \mathbb{F}'(0) - 1 = 0.$$

The functions in Ω are defined by

$$\mathbb{F}(\xi) = \xi + \sum_{r=2}^{\infty} c_r \xi^r \quad (\xi \in \mathcal{U}). \tag{1}$$

Assume that Γ denotes the class of all functions in Ω which are univalent in $\mathcal{U}$. For the functions $\mathbb{F}$, $\mathbb{H} \in \mathcal{A}$ defined by

$$\mathbb{F}(\xi) = \sum_{r=1}^{\infty} c_r \xi^r \quad \text{and} \quad \mathbb{H}(\xi) = \sum_{r=1}^{\infty} d_r \xi^r \quad (\xi \in \mathcal{U}),$$

the convolution of $\mathbb{F}$ and $\mathbb{H}$ denoted by $\mathbb{F} * \mathbb{H}$ is

$$(\mathbb{F} * \mathbb{H})(\xi) = \sum_{r=1}^{\infty} c_r d_r \xi^r = (\mathbb{H} * \mathbb{F})(\xi) \quad (\xi \in \mathcal{U}).$$

To start with, we recall the following differential and integral operators.

For $0 < q < 1$, El-Deeb et al. [2,3] defined the q-convolution operator (see also [4–7]) for $\mathbb{F} * \mathbb{H}$ by

$$\mathcal{D}_q(\mathbb{F} * \mathbb{H})(\xi) := \mathcal{D}_q\left(\xi + \sum_{r=2}^{\infty} c_r d_r \xi^r\right)$$
$$= \frac{(\mathbb{F} * \mathbb{H})(\xi) - (\mathbb{F} * \mathbb{H})(q\xi)}{\xi(1-q)} = 1 + \sum_{r=2}^{\infty} [r]_q c_r d_r \xi^{r-1}, \ \xi \in \mathcal{U},$$

where

$$[r]_q := \frac{1-q^r}{1-q} = 1 + \sum_{j=1}^{r-1} q^j, \qquad [0]_q := 0. \tag{2}$$

We used the linear operator $\mathcal{G}_{\mathbb{H}}^{\delta,q} : \mathcal{A} \to \mathcal{A}$ according to El-Deeb et al. [2] (see also [3]) for $\delta > -1$ and $0 < q < 1$. If

$$\mathcal{G}_{\mathbb{H}}^{\delta,q}\mathbb{F}(\xi) * \mathcal{I}_q^{\delta+1}(\xi) = \xi\mathcal{D}_q(\mathbb{F} * \mathbb{H})(\xi), \ \xi \in \mathcal{U},$$

where $\mathcal{I}_q^{\delta+1}$ is given by

$$\mathcal{I}_q^{\delta+1}(\xi) := \xi + \sum_{r=2}^{\infty} \frac{[\delta+1]_{q,r-1}}{[r-1]_q!}\xi^r, \ \xi \in \mathcal{U},$$

then

$$\mathcal{G}_{\mathbb{H}}^{\delta,q}\mathbb{F}(\xi) := \xi + \sum_{r=2}^{\infty} \frac{[r]_q!}{[\delta+1]_{q,r-1}} c_r d_r \, \xi^r \quad (\delta > -1, \ 0 < q < 1, \ \xi \in \mathcal{U}). \tag{3}$$

Using the operator $\mathcal{G}_{\mathbb{H}}^{\delta,q}$, we define a new operator as follows:

$$\begin{aligned}
\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,0}\mathbb{F}(\xi) &= \mathcal{G}_{\mathbb{H}}^{\delta,q}\mathbb{F}(\xi) \\
\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,1}\mathbb{F}(\xi) &= \mu\xi^3\left(\mathcal{G}_{\mathbb{H}}^{\delta,q}\mathbb{F}(\xi)\right)''' + (1+2\mu)\xi^2\left(\mathcal{G}_{\mathbb{H}}^{\delta,q}\mathbb{F}(\xi)\right)'' + \xi\left(\mathcal{G}_{\mathbb{H}}^{\delta,q}\mathbb{F}(\xi)\right)' \\
&\ \ . \\
&\ \ . \\
\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{F}(\xi) &= \mu\xi^3\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n-1}\mathbb{F}(\xi)\right)''' + (1+2\mu)\xi^2\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n-1}\mathbb{F}(\xi)\right)'' + \xi\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n-1}\mathbb{F}(\xi)\right)' \\
&= \xi + \sum_{r=2}^{\infty} r^{2n}(\mu(r-1)+1)^n \frac{[r]_q!}{[\delta+1]_{q,r-1}} c_r d_r \, \xi^r \\
&= \xi + \sum_{r=2}^{\infty} \Theta_r \, c_r \xi^r \\
(\delta &> -1, \ \mu > 0, \ 0 < q < 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \ \ \xi \in \mathcal{U}),
\end{aligned} \tag{4}$$

where

$$\Theta_r \ = r^{2n}(\mu(r-1)+1)^n \frac{[r]_q!}{[\delta+1]_{q,r-1}} \, d_r. \tag{5}$$

From the definition relation (3), we get

(i) $[\delta+1]_q \mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{F}(\xi) = [\delta]_q \mathcal{W}_{\mathbb{H},\mu}^{\delta+1,q,n}\mathbb{F}(\xi) + q^\delta \, \xi\mathcal{D}_q\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta+1,q,n}\mathbb{F}(\xi)\right), \ \varsigma \in \mathcal{U};$ (6)

(ii) $\mathcal{R}_{\mathbb{H},\mu}^{\delta,n}\mathbb{F}(\xi) := \lim_{q \to 1^-} \mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{F}(\xi)$

$$= \xi + \sum_{r=2}^{\infty} r^{2n} (\mu(r-1)+1)^n \tfrac{r!}{(\delta+1)_{r-1}} d_r\, c_r\, \xi^r, \;\; \xi \in \mathcal{U}. \tag{7}$$

Remark 1. *We find the following special cases for the operator* $\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}$ *by considering several particular cases for the coefficients* d_r *and* n *:*

(i) Putting $d_r = 1$ *and* $n = 0$ *into this operator, we obtain the operator* $QTRcalB_q^{\alpha}$ *defined by Srivastava et al. [8];*

(ii) Putting $d_r = \dfrac{(-1)^{r-1}\Gamma(\rho+1)}{4^{r-1}(r-1)!\Gamma(r+\rho)}$ $(\rho > 0)$ *and* $n = 0$ *in this operator, we obtain the operator* $\mathcal{N}_{\rho,q}^{\mu}$ *defined by El-Deeb and Bulboacă [9] and El-Deeb [10];*

(iii) Putting $d_r = \left(\dfrac{\tau+1}{\tau+r}\right)^j$ $(j > 0, \tau \geq 0)$ *and* $n = 0$ *in this operator, we obtain the operator* $\mathcal{M}_{\tau,q}^{\mu,j}$ *defined by El-Deeb and Bulboacă [11] and Srivastava and El-Deeb [12];*

(iv) Putting $d_r = \dfrac{\sigma^{r-1}}{(r-1)!}e^{-\sigma}$ $(\sigma > 0)$ *and* $n = 0$ *in this operator, we obtain the q-analogue of Poisson operator* $\mathcal{I}_q^{\mu,\sigma}$ *defined by El-Deeb et al. [2];*

(v) Putting $d_r = 1$ *in this operator, we obtain the operator* $QTRcalB_{\mu}^{\delta,q,n}$ *defined as follows:*

$$\mathcal{B}_{\delta,q}^{\alpha,n}\mathcal{F}(\varsigma) = \xi + \sum_{r=2}^{\infty} r^{2n}(\delta(r-1)+1)^n \tfrac{[r]_q!}{[\alpha+1]_{q,r-1}}\, c_r\, \xi^r\,; \tag{8}$$

(vi) Putting $d_r = \dfrac{(-1)^{r-1}\Gamma(\rho+1)}{4^{r-1}(r-1)!\Gamma(r+\rho)}$ $(\rho > 0)$ *in this operator, we obtain the operator* $\mathcal{N}_{\delta,\rho,q}^{\alpha,n}$ *defined as follows:*

$$\begin{aligned}\mathcal{N}_{\delta,\rho,q}^{\alpha,m}\mathcal{F}(\varsigma) &= \xi + \sum_{r=2}^{\infty} r^{2n}(\delta(r-1)+1)^n \tfrac{[r]_q!}{[\alpha+1]_{q,r-1}} \tfrac{(-1)^{r-1}\Gamma(\rho+1)}{4^{r-1}(r-1)!\Gamma(r+\rho)}\, c_r\, \xi^r \\ &= \xi + \sum_{r=2}^{\infty} \phi_r\, c_r\, \xi^r\,,\end{aligned} \tag{9}$$

where

$$\phi_r = r^{2n}(\delta(r-1)+1)^n \tfrac{[r]_q!}{[\alpha+1]_{q,r-1}} \tfrac{(-1)^{r-1}\Gamma(\rho+1)}{4^{r-1}(r-1)!\Gamma(r+\rho)}; \tag{10}$$

(vii) Putting $d_r = \left(\dfrac{\tau+1}{\tau+r}\right)^j$ $(j > 0, \tau \geq 0)$ *in this operator, we obtain the operator* $\mathcal{M}_{\delta,\tau,q}^{\alpha,n,j}$ *defined as follows:*

$$\mathcal{M}_{\delta,\tau,q}^{\alpha,n,j}\mathcal{F}(\varsigma) = \xi + \sum_{r=2}^{\infty} r^{2n}(\delta(r-1)+1)^n \left(\tfrac{\tau+1}{\tau+r}\right)^j \tfrac{[r]_q!}{[\alpha+1]_{q,r-1}}\, c_r\, \xi^r; \tag{11}$$

(viii) Putting $d_r = \dfrac{\sigma^{r-1}}{(r-1)!}e^{-\sigma}$ $(\sigma > 0)$ *in this operator, we obtain the q-analogue of Poisson operator* $\mathcal{I}_{\delta,\sigma,q}^{\alpha,m}$ *defined as follows:*

$$\mathcal{I}_{\delta,\sigma,q}^{\alpha,n}\mathcal{F}(\varsigma) = \xi + \sum_{r=2}^{\infty} r^{2n}(\delta(r-1)+1)^n \tfrac{[r]_q!}{[\alpha+1]_{q,r-1}} \tfrac{\sigma^{r-1}}{(r-1)!}e^{-\sigma}\, c_r\, \xi^r. \tag{12}$$

The well-known Koebe one-quarter theorem (see [1]) states that any univalent function $\mathbb{F} \in \Omega$ includes a disk with a radius of $\frac{1}{4}$ in its image of $\mathcal{U}$. For a result, the inverse of $\mathbb{F}$ is a univalent analytic function on the disk with the notation $\mathcal{U}_\rho := \{\xi : \xi \in \mathbb{C}$ and

$|\xi| < \rho;\ \rho \geq \frac{1}{4}\}$. As a result, there is an inverse function $\mathbb{F}^{-1}(\varpi)$ of $\mathbb{F}(\zeta)$ defined for each function $\mathbb{F}(\xi) = \varpi \in \sigma$

$$\mathbb{F}^{-1}(\mathbb{F}(\varsigma)) = \varsigma \quad (\varsigma \in \mathcal{U})$$

and

$$\mathbb{F}(\mathbb{F}^{-1}(\varpi)) = \varpi \quad (\varpi \in \mathcal{U}_\rho)$$

where

$$\mathbb{F}^{-1}(\varpi) = \varpi - c_2\varpi^2 + (2c_2^2 - c_3)\varpi^3 - (5c_2^3 - 5c_2c_3 + c_4)\varpi^4 + \ldots. \tag{13}$$

When both $\mathbb{F}$ and $\mathbb{F}^{-1}$ are univalent in $\mathcal{U}$, a function $\mathbb{F}$ is said to be bi-univalent in $\mathcal{U}$.

Let $\mathcal{B}$ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1). The concept of bi-univalent analytic functions was introduced by Lewin [13] in 1967 and he showed that $|c_2| < 1.51$. Subsequently, Brannan and Clunie [14] conjectured that $|c_2| \leq \sqrt{2}$. Netanyahu [15], on the other hand, showed that $\max_{\mathbb{F}\in\mathcal{B}} |c_2| = \frac{4}{3}$. The coefficient estimate problem for each of the following Taylor–Maclaurin coefficients:

$$|c_r| \quad (r \in \mathbb{N} \setminus \{1, 2\})$$

is presumably still an open problem.

In [16] (see also [2,10,17–26]), certain subclasses of the bi-univalent analytic functions class $\mathcal{B}$ were introduced and non-sharp estimates on the first two coefficients $|c_2|$ and $|c_3|$ were found. The object of the present paper is to introduce two new subclasses as in Definitions 1 and 2 of the function class $\mathcal{B}$ using the linear q-convolution operator and determine estimates of the coefficients $|c_2|$, $|c_3|$ and $|c_4|$ for the functions in these new subclasses of the function class $\mathcal{B}$.

Definition 1. *A function $\mathbb{F}(\xi)$ given by (1) is said to be in the class $\mathcal{N}_{\mathbb{H},\mu,m}^{\delta,q,n,\kappa}$ if the following conditions are satisfied:*

$$\mathbb{F} \in \mathcal{B} \text{ and } \left| \arg\left((1-m)\frac{\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{F}(\xi)}{\varsigma} + m\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{F}(\xi)\right)' \right) \right| < \frac{\kappa\pi}{2} \tag{14}$$

and

$$\left| \arg\left((1-m)\frac{\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{G}(\varpi)}{\varpi} + m\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{G}(\varpi)\right)' \right) \right| < \frac{\kappa\pi}{2} \tag{15}$$

where the function $\mathbb{G}$ is the inverse of $\mathbb{F}$ given in (13), where $0 < \kappa \leq 1, m \geq 1, \xi, \varpi \in \mathcal{U}$.

Definition 2. *A function $\mathbb{F}$ given by (1) is said to be in the class $\mathcal{M}_{\mathbb{H},\mu,m}^{\delta,q,n,\kappa}$ if the following conditions are satisfied:*

$$\mathbb{F} \in \mathcal{B} \text{ and } \Re\left[(1-m)\frac{\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{F}(\xi)}{\xi} + m\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{F}(\xi)\right)' \right] > \kappa \tag{16}$$

and

$$\Re\left[(1-m)\frac{\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{G}(\varpi)}{\varpi} + m\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{G}(\varpi)\right)' \right] > \kappa \tag{17}$$

where the function $\mathbb{G}$ is the inverse of $\mathbb{F}$ given in (13), where $0 < \kappa \leq 1, m \geq 1, \xi, \varpi \in \mathcal{U}$.

By fixing $m = 1$, we define a new subclass of $\mathcal{B}$ due to Noshiro [27].

Definition 3. *A function $\mathbb{F}$ given by (1) is said to be in the class $\mathcal{S}_{\mathbb{H},\mu}^{\delta,q,n,\kappa}$ if the following conditions are satisfied:*

$$\mathbb{F} \in \mathcal{B} \text{ and } \left| \arg \left(\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n} \mathbb{F}(\xi) \right)' \right) \right| < \frac{\kappa\pi}{2} \quad \text{and} \quad \left| \arg \left(\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n} \mathbb{G}(\varpi) \right)' \right) \right| < \frac{\kappa\pi}{2} \tag{18}$$

where the function $\mathbb{G}$ is the inverse of $\mathbb{F}$ given in (13), where $0 < \kappa \leq 1, m \geq 1, \xi, \varpi \in \mathcal{U}$.

Definition 4. *A function $\mathbb{F}$ given by (1) is said to be in the class $\mathcal{R}_{\mathbb{H},\mu}^{\delta,q,n,\kappa}$ if the following conditions are satisfied:*

$$\mathbb{F} \in \mathcal{B} \text{ and } \Re\left[\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n} \mathbb{F}(\xi) \right)' \right] > \kappa \quad \text{and} \quad \Re\left[\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n} \mathbb{G}(\varpi) \right)' \right] > \kappa \tag{19}$$

where the function $\mathbb{G}$ is the inverse of $\mathbb{F}$ given in (13), where $0 < \kappa \leq 1, m \geq 1, \xi, \varpi \in \mathcal{U}$.

By fixing $m = 0$, we define a new subclass of $\mathcal{B}$ due to Yamaguchi [28].

Definition 5. *A function $\mathbb{F}$ given by (1) is said to be in the class $\mathcal{Y}_{\mathbb{H},\mu}^{\delta,q,n,\kappa}$ if the following conditions are satisfied:*

$$\mathbb{F} \in \mathcal{B} \text{ and } \left| \arg \left(\frac{\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n} \mathbb{F}(\xi)}{\zeta} \right) \right| < \frac{\kappa\pi}{2} \quad \text{and} \quad \left| \arg \left(\frac{\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n} \mathbb{G}(\varpi)}{\varpi} \right) \right| < \frac{\kappa\pi}{2} \tag{20}$$

where the function $\mathbb{G}$ is the inverse of $\mathbb{F}$ given in (13), where $0 < \kappa \leq 1, m \geq 1, \xi, \varpi \in \mathcal{U}$.

Definition 6. *A function $\mathbb{F}$ given by (1) is said to be in the class $\mathcal{X}_{\mathbb{H},\mu}^{\delta,q,n,\kappa}$ if the following conditions are satisfied:*

$$\mathbb{F} \in \mathcal{B} \text{ and } \Re\left[\frac{\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n} \mathbb{F}(\xi)}{\xi} \right] > \kappa \quad \text{and} \quad \Re\left[\frac{\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n} \mathbb{G}(\varpi)}{\varpi} \right] > \kappa \tag{21}$$

where the function $\mathbb{G}$ is the inverse of $\mathbb{F}$ given in (13), where $0 < \kappa \leq 1, m \geq 1, \xi, \varpi \in \mathcal{U}$.

2. Coefficient Bounds

We state and prove our main results. We need the following lemma for our investigation.

Lemma 1 (see [1], p. 41)**.** *Let $\mathcal{P}$ be the class of all analytic functions $\psi(\xi)$ which has a form as follows*

$$\psi(\xi) = 1 + \sum_{r=1}^{\infty} b_r \xi^r$$

satisfying $\Re(\psi(\xi)) > 0 \;\; (\xi \in \mathcal{U})$ and $\psi(0) = 1$, then

$$|b_r| \leq 2 \;\; (r = 1, 2, 3, \ldots).$$

This inequality is sharp. In particular, this equality holds for all r for the function

$$\psi(\xi) = \frac{1+\xi}{1-\xi} = 1 + \sum_{r=1}^{\infty} 2\xi^r.$$

Theorem 1. *Let* $\mathbb{F}$ *given by (1) be in the class* $\mathcal{N}_{\mathbb{H},\mu,m}^{\delta,q,n,\kappa}$. *Then,*

$$|c_2| \leq \frac{2\kappa}{\sqrt{2\kappa(1+2m)\Theta_3 + (1-\kappa)(1+m)^2\Theta_2^2}}, \tag{22}$$

$$|c_3| \leq \frac{2\kappa}{(1+2m)\Theta_3}, \tag{23}$$

and

$$|c_4| \leq \frac{2\kappa}{(1+3m)\Theta_4}\left[1 + \frac{2(1-\kappa)(1+m)\Theta_2\{6\kappa(1+2m)\Theta_3 + (1-2\kappa)(1+m)^2\Theta_2^2\}}{3\{2\kappa(1+2m)\Theta_3 + (1-\kappa)(1+m)^2\Theta_2^2\}^{\frac{3}{2}}}\right], \tag{24}$$

where $\Theta_r \ (r = 2,3,4)$ *is given in (5).*

Proof. Let $\mathbb{F} \in \mathcal{N}_{\mathbb{H},\mu,m}^{\delta,q,n,\kappa}$. Hence, by Definition 1, there exist two functions $\varphi(\xi)$ and $\psi(\varpi) \in \mathcal{P}$ satisfying the conditions of Lemma 1 such that

$$(1-m)\frac{\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{F}(\xi)}{\xi} + m\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{F}(\xi)\right)' = [\varphi(\xi)\,]^{\kappa} \tag{25}$$

and

$$(1-m)\frac{\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{G}(\varpi)}{\varpi} + m\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{G}(\varpi)\right)' = [\psi(\varpi)]^{\kappa}. \tag{26}$$

Assume that

$$\varphi(\xi) \ = 1 + x_1\xi + x_2\xi^2 + x_3\xi^3 + \ldots \tag{27}$$

and

$$\psi(\varpi) = 1 + y_1\varpi + y_2\varpi^2 + y_3\varpi^3 + \ldots. \tag{28}$$

Equating the coefficients in (25) and (26), we get

$$(1+m)\Theta_2 c_2 = \kappa x_1 \tag{29}$$

$$(1+2m)\Theta_3 c_3 = \kappa x_2 + \frac{\kappa(\kappa-1)}{2}x_1^2 \tag{30}$$

$$(1+3m)\Theta_4 c_4 = \kappa x_3 + \kappa(\kappa-1)x_1x_2 + \frac{\kappa(\kappa-1)(\kappa-2)}{6}x_1^3 \tag{31}$$

and

$$-(1+m)\Theta_2 c_2 = \kappa y_1 \tag{32}$$

$$(1+2m)\Theta_3(2c_2^2 - c_3) = \kappa y_2 + \frac{\kappa(\kappa-1)}{2}y_1^2 \tag{33}$$

$$-(1+3m)\Theta_4(5c_2^3 - 5c_2c_3 + c_4) = \kappa y_3 + \kappa(\kappa-1)y_1y_2 + \frac{\kappa(\kappa-1)(\kappa-2)}{6}y_1^3. \tag{34}$$

From (29) and (32), we get

$$c_2 = \frac{\kappa x_1}{(1+m)\Theta_2} = -\frac{\kappa y_1}{(1+m)\Theta_2} \tag{35}$$

which implies

$$x_1 = -y_1.$$

Squaring and adding (29) and (32), we get

$$c_2^2 = \frac{\kappa^2}{(1+m)^2\Theta_2^2}(x_1^2 + y_1^2) \tag{36}$$

Adding (30) and (33), we obtain

$$2(1+2m)\Theta_3 c_2^2 = \kappa(x_2+y_2) + \frac{\kappa(\kappa-1)}{2}(x_1^2+y_1^2). \tag{37}$$

Substitute the value of c_2 from (35) in (37) and noting that $x_1^2 = y_1^2$, we observe that

$$x_1^2 = \frac{(1+m)^2\Theta_2^2(x_2+y_2)}{2\kappa(1+2m)\Theta_3 + (1-\kappa)(1+m)^2\Theta_2^2}. \tag{38}$$

By application of the triangle inequality and Lemma 1, we obtain

$$|x_1| \leq \frac{2(1+m)\Theta_2}{\sqrt{2\kappa(1+2m)\Theta_3 + (1-\kappa)(1+m)^2\Theta_2^2}}. \tag{39}$$

Then, (35) gives

$$|c_2| \leq \frac{2\kappa}{\sqrt{2\kappa(1+2m)\Theta_3 + (1-\kappa)(1+m)^2\Theta_2^2}}. \tag{40}$$

In order to find the bound on $|c_3|$, subtracting (33) from (30) with $x_1 = -y_1$ gives

$$\begin{aligned} 2(1+2m)\Theta_3 c_3 &= 2(1+2m)\Theta_3 c_2^2 + \kappa(x_2-y_2) \\ c_3 &= c_2^2 + \frac{\kappa(x_2-y_2)}{2(1+2m)\Theta_3}. \end{aligned} \tag{41}$$

Using (35) and (38) in (41), we have

$$\begin{aligned} 2(1+2m)\Theta_3 c_3 &= \frac{2\kappa^2\Theta_3(1+2m)}{2\xi(1+2m)\Theta_3 + (1+n)^2\Theta_2^2(1-\xi)}(x_2+y_2) + \kappa(x_2-y_2) \\ &= \left[\frac{2\kappa^2\Theta_3(1+2m)}{2\kappa(1+2m)\Theta_3 + (1+m)^2\Theta_2^2(1-\kappa)} + \kappa\right] x_2 \\ &+ \left[\frac{2\kappa^2\Theta_3(1+2m)}{2\kappa(1+2m)\Theta_3 + (1-\kappa)(1+m)^2\Theta_2^2} - \kappa\right] y_2 \\ &= \frac{\kappa\left[\left\{4\kappa(1+2m)\Theta_3 + (1-\kappa)(1+m)^2\Theta_2^2\right\}x_2 - (1-\kappa)(1+m)^2\Theta_2^2 y_2\right]}{2\kappa(1+2m)\Theta_3 + (1-\kappa)(1+m)^2\Theta_2^2}. \end{aligned} \tag{42}$$

Application of the triangle inequality to (42) gives

$$|c_3| \leq \frac{\kappa\left[\left\{4\kappa(1+2m)\Theta_3 + (1-\kappa)(1+m)^2\Theta_2^2\right\}x_2 - (1-\kappa)(1+m)^2\Theta_2^2 y_2\right]}{2(1+2m)\Theta_3\left\{2\kappa(1+2m)\Theta_3 + (1-\kappa)(1+m)^2\Theta_2^2\right\}}.$$

Applying Lemma 1 for the coefficients x_2 and y_2, we obtain

$$|c_3| \leq \frac{2\kappa}{(1+2m)\Theta_3}.$$

To determine the bound on $|c_4|$, by adding (31) and (34) with $x_1 = -y_1$, we obtain

$$-5(1+3m)\Theta_4 c_2^3 + 5(1+3m)\Theta_4 c_2 c_3 = \kappa(x_3+y_3) + \kappa(\kappa-1)x_1(x_2-y_2). \tag{43}$$

Substitute the values of c_2 and c_3 from (35) and (41) in (43) and simplify, then we obtain

$$x_1(x_2 - y_2) = \frac{2(1+2m)(1+m)\Theta_2\Theta_3}{5\kappa(1+3m)\Theta_4 + 2(1-\kappa)(1+2m)(1+m)\Theta_2\Theta_3}(x_3 + y_3), \tag{44}$$

subtracting (34) from (31) and using (38), (39), (43) and (44) in the result, we get

$$\begin{aligned} 2c_4(1+3m)\Theta_4 &= -5(1+3m)\Theta_4 c_2^3 + 5(1+3m)c_2c_3\Theta_4 + \kappa(x_3 - y_3) \\ &+ \kappa(\kappa-1)x_1(x_2+y_2) + \frac{\kappa(\kappa-1)(\kappa-2)}{3}x_1^3 \\ &= \kappa(x_3+y_3) + \kappa(\kappa-1)x_1(x_2-y_2) \\ &\quad +\kappa(x_3-y_3) + \kappa(\kappa-1)x_1(x_2+y_2) + \frac{\kappa(\kappa-1)(\kappa-2)}{3}x_1^3 \\ &= 2\kappa x_3 + \frac{2\kappa(\kappa-1)(1+2m)(1+m)\Theta_2\Theta_3}{5\kappa(1+3m)\Theta_4 + 2(1-\kappa)(1+2n)(1+n)\Theta_2\Theta_3}(x_3+y_3) \\ &+ \kappa(\kappa-1)x_1(x_2+y_2) + \frac{\kappa(\kappa-1)(\kappa-2)(1+m)^2\Theta_2^2}{3\{2\kappa(1+2m)\Theta_3 + (1-\kappa)(1+m)^2\Theta_2^2\}} \\ &= \frac{10\kappa^2(1+3m)\Theta_4 + 2\kappa(1-\kappa)(1+2m)(1+m)\Theta_2\Theta_3}{5\kappa(1+3m)\Theta_4 + 2(1-\kappa)(1+2m)(1+m)\Theta_2\Theta_3}x_3 \\ &- \frac{2\kappa(1-\kappa)(1+2m)(1+m)\Theta_2\Theta_3}{5(1+3m)\kappa\Theta_4 + 2(1-\kappa)(1+2m)(1+m)\Theta_2\Theta_3}y_3 \\ &\quad -\kappa(1-\kappa)\left[\frac{6\kappa(1+2m)\Theta_3 + (1-2\kappa)(1+m)^2\Theta_2^2}{6\kappa(1+2m)\Theta_3 + 3(1-\kappa)(1+m)^2\Theta_2^2}\right]x_1(x_2+y_2). \end{aligned} \tag{45}$$

Applying Lemma 1 with the triangle inequality in (45), we obtain

$$|c_4| \leq \frac{2\kappa}{(1+3m)\Theta_4}\left[1 + \frac{2(1-\kappa)(1+m)\Theta_2\{6\kappa(1+2m)\Theta_3 + (1-2\kappa)(1+m)^2\Theta_2^2\}}{3\{2\kappa(1+2m)\Theta_3 + (1-\kappa)(1+m)^2\Theta_2^2\}^{\frac{3}{2}}}\right],$$

this completes the proof of Theorem 1. □

Putting $q \to 1^-$, $\delta = 1$, $n = 0$ and $\mathbb{H}(\xi) = \frac{\xi}{1-\xi}$ in Theorem 1.

Example 1. *Let* $\mathbb{F}$ *given by (1) be in the class* $\lim\limits_{q\to 1^-} \mathcal{N}^{1,q,0,\kappa}_{\frac{\xi}{1-\xi},\mu,m}$, *then*

$$|c_2^*| \leq \frac{2\kappa}{\sqrt{2\kappa(1+2m) + (1-\kappa)(1+m)^2}},$$

$$|c_3^*| \leq \frac{2\kappa}{(1+2m)},$$

and

$$|c_4^*| \leq \frac{2\kappa}{(1+3m)}\left[1 + \frac{2(1-\kappa)(1+m)\{6\kappa(1+2m) + (1-2\kappa)(1+m)^2\}}{3\{2\kappa(1+2m) + (1-\kappa)(1+m)^2\}^{\frac{3}{2}}}\right].$$

Theorem 2. *Let* $\mathbb{F}$ *given by (1) be in the class* $\mathcal{M}^{\delta,q,n,\kappa}_{\mathbb{H},\mu,m}$. *Then,*

$$|c_2| \leq \sqrt{\frac{2(1-\kappa)}{(1+2m)\Theta_3}}, \tag{46}$$

$$|c_3| \leq \frac{2(1-\kappa)}{(1+2m)\Theta_3} \tag{47}$$

and

$$|c_4| \leq \frac{2(1-\kappa)}{(1+3m)\Theta_4}. \tag{48}$$

Proof. Let $\mathbb{F} \in \mathcal{M}_{\mathbb{H},\mu,m}^{\delta,q,n,\kappa}$, there are two functions $\varphi(\xi)$ and $\psi(\varpi) \in \mathcal{P}$ that satisfy the conditions of Lemma 1 such that

$$(1-m)\frac{\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{F}(\xi)}{\xi} + m\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{F}(\xi)\right)' = \kappa + (1-\kappa)\varphi(\xi) \tag{49}$$

and

$$(1-m)\frac{\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{G}(\varpi)}{\varpi} + m\left(\mathcal{W}_{\mathbb{H},\mu}^{\delta,q,n}\mathbb{G}(\varpi)\right)' = \kappa + (1-\kappa)\psi(\varpi), \tag{50}$$

where $\varphi(\xi)$ and $\psi(\varpi)$ have the form (27) and (28), respectively. Equating the coefficients in (49) and (50) gives

$$(1+m)\Theta_2 c_2 = (1-\kappa)x_1 \tag{51}$$

$$(1+2m)\Theta_3 c_3 = (1-\kappa)x_2 \tag{52}$$

$$(1+3m)\Theta_4 c_4 = (1-\kappa)x_3, \tag{53}$$

and

$$-(1+m)\Theta_2 c_2 = (1-\kappa)y_1 \tag{54}$$

$$(1+2m)\Theta_3(2c_2^2 - c_3) = (1-\kappa)y_2 \tag{55}$$

$$-(1+3m)\Theta_4(5c_2^3 - 5c_2c_3 + c_4) = (1-\kappa)y_3. \tag{56}$$

From (51) and (54), we obtain

$$c_2 = \frac{1-\kappa}{(1+m)\Theta_2}x_1 = -\frac{1-\kappa}{(1+m)\Theta_2}y_1 \tag{57}$$

which implies

$$x_1 = -y_1.$$

Adding (52) and (55), we obtain

$$2(1+2m)\Theta_3 c_2^2 = (1-\kappa)(x_2+y_2) \tag{58}$$

$$c_2^2 = \frac{(1-\kappa)}{2(1+2m)\Theta_3}(x_2+y_2). \tag{59}$$

Using (57) in (58), we have

$$x_1^2 = \frac{(1+m)^2\Theta_2^2}{2(1+2m)(1-\kappa)\Theta_3}(x_2+y_2). \tag{60}$$

Application of the triangle inequality and Lemma 1 in (60) yields

$$|x_1| \leq (1+m)\Theta_2\sqrt{\frac{2}{(1+2m)(1-\kappa)\Theta_3}}. \tag{61}$$

Using (61) in (57) gives

$$|c_2| \leq \sqrt{\frac{2(1-\kappa)}{(1+2m)\Theta_3}}. \tag{62}$$

Now, subtracting (55) from (52) and using (58), we obtain

$$|c_3| \leq \frac{2(1-\kappa)}{(1+2m)\Theta_3}, \tag{63}$$

which is the direct consequence of (52).

In order to obtain the bounds on $|c_4|$, we proceed as follows:

$$|c_4| = \left|\frac{(1-\kappa)x_3}{(1+3m)\Theta_4}\right| \leq \frac{2(1-\kappa)}{(1+3m)\Theta_4}. \tag{64}$$

On the other hand, subtracting (56) from (53) and using (57), we get

$$c_4 = \frac{1}{2(1+3m)\Theta_4}\left[\frac{-5(1+3m)(1-\kappa)^3\Theta_4}{(1+m)^3\Theta_2^3}x_1^3 + \frac{5(1+3m)(1-\kappa)\Theta_4}{(1+m)\Theta_2}c_3x_1 + (1-\kappa)(x_3-y_3)\right]. \tag{65}$$

Applying the triangle inequality in (65), we have

$$|c_4| \leq \frac{1}{2(1+3m)\Theta_4}\left[\frac{5(1+3m)(1-\kappa)^3\Theta_4}{(1+m)^3\Theta_2^3}|x_1|^3 + \frac{5(1+3m)(1-\kappa)\Theta_4}{(1+m)\Theta_2}|c_3||x_1| + (1-\kappa)(|x_3|+|y_3|)\right]. \tag{66}$$

Using (61), (63) and Lemma 1 in (66), and after simplification, yields

$$|c_4| \leq \frac{2(1-\kappa)}{(1+3m)\Theta_4}\left[1 + \frac{5(1+3m)}{(1+2m)\Theta_3}\sqrt{\frac{2(1-\kappa)}{(1+2m)\Theta_3}}\right]. \tag{67}$$

From (64) and (67), we observe that

$$\begin{aligned} |c_4| &\leq \min\left[\frac{2(1-\kappa)}{(1+3m)\Theta_4}, \frac{2(1-\kappa)}{(1+3m)\Theta_4}\left\{1 + \frac{5(1+3m)}{(1+2m)\Theta_3}\sqrt{\frac{2(1-\kappa)}{(1+2m)\Theta_3}}\right\}\right] \\ &= \frac{2(1-\kappa)}{(1+3m)\Theta_4}. \end{aligned}$$

This completes the proof of Theorem 2. □

3. Fekete–Szegö Inequalities

In this section, we obtain Fekete–Szegö inequalities results [29] (also see Zaprawa [30]) for $\mathbb{F} \in \mathcal{N}_{\mathbb{H},\mu,m}^{\delta,q,n,\kappa}$ and $\mathbb{F} \in \mathcal{M}_{\mathbb{H},\mu,m}^{\delta,q,n,\kappa}$,

Theorem 3. *For $\rho \in \mathbb{R}$, let $\mathbb{F}$ be given by (1) and $\mathbb{F} \in \mathcal{N}_{\mathbb{H},\mu,m}^{\delta,q,n,\kappa}$, then*

$$\left|c_3 - \rho c_2^2\right| \leq \begin{cases} \frac{2\kappa}{(1+2m)\Theta_3} & ; 0 \leq |T(\rho)| \leq \frac{\kappa}{2(1+2m)\Theta_3} \\ 4|T(\rho)| & ; |T(\rho)| \geq \frac{\kappa}{2(1+2m)\Theta_3} \end{cases}$$

where

$$T(\rho) = \frac{2\kappa^2(1-\rho)}{4\kappa(1+2m)\Theta_3 - (\kappa-1)(1+m)^2\Theta_2^2}.$$

Proof. From (41), we have

$$c_3 - \rho c_2^2 = \frac{\kappa(x_2-y_2)}{2(1+2m)\Theta_3} + (1-\rho)c_2^2 \tag{68}$$

$$= \frac{\kappa(x_2 - y_2)}{2(1+2m)\Theta_3} + \frac{2\kappa^2(1-\rho)(x_2+y_2)}{4\kappa(1+2m)\Theta_3 - (\kappa-1)(1+m)^2\Theta_2^2} \tag{69}$$

By simple computation, we have

$$c_3 - \rho c_2^2 = \left(T(\rho) + \frac{\kappa}{2(1+2m)\Theta_3}\right)x_2 + \left(T(\rho) - \frac{\kappa}{2(1+2m)\Theta_3}\right)y_2,$$

where

$$T(\rho) = \frac{2\kappa^2(1-\rho)}{4\kappa(1+2m)\Theta_3 - (\kappa-1)(1+m)^2\Theta_2^2}.$$

Thus, by taking the modulus of $c_3 - \rho c_2^2$, we get

$$\left|c_3 - \rho c_2^2\right| \leq \begin{cases} \frac{2\kappa}{(1+2m)\Theta_3} & ; 0 \leq |T(\rho)| \leq \frac{\kappa}{2(1+2m)\Theta_3}, \\ 4|T(\rho)| & ; |T(\rho)| \geq \frac{\kappa}{2(1+2m)\Theta_3} \end{cases}.$$

□

Theorem 4. *For $\rho \in \mathbb{R}$, let $\mathbb{F}$ be given by (1) and $\mathbb{F} \in \mathcal{M}_{\mathbb{H},\mu,m}^{\delta,q,n,\kappa}$, then*

$$\left|c_3 - \rho c_2^2\right| \leq \frac{(1-\kappa)|2-\rho|}{(1+2m)\Theta_3}\left[1 + \frac{\rho}{|2-\rho|}\right].$$

Proof. Subtracting (55) from (52), we obtain

$$c_3 = \frac{(1-\kappa)(x_2 - y_2)}{2(1+2m)\Theta_3} + c_2^2, \tag{70}$$

and using (59), we get

$$\begin{aligned} c_3 - \rho c_2^2 &= \frac{(1-\kappa)(x_2 - y_2)}{2(1+2m)\Theta_3} + (1-\rho)c_2^2 \\ &= \frac{(1-\kappa)(x_2 - y_2)}{2(1+2m)\Theta_3} + \frac{(1-\kappa)(1-\rho)}{2(1+2m)\Theta_3}(x_2 + y_2). \end{aligned} \tag{71}$$

Then,

$$\begin{aligned} c_3 - \rho c_2^2 &= \left(\frac{(1-\kappa)(1-\rho)}{2(1+2m)\Theta_3} + \frac{1-\kappa}{2(1+2m)\Theta_3}\right)x_2 + \left(\frac{(1-\kappa)(1-\rho)}{2(1+2m)\Theta_3} - \frac{1-\kappa}{2(1+2m)\Theta_3}\right)y_2, \\ &= \frac{(1-\kappa)}{2(1+2m)\Theta_3}((1-\rho)+1)x_2 + \frac{(1-\kappa)}{2(1+2m)\Theta_3}((1-\rho)-1)y_2, \\ &= \frac{(1-\kappa)}{2(1+2m)\Theta_3}[(2-\rho)x_2 - \rho y_2], \\ &= \frac{(1-\kappa)(2-\rho)}{2(1+2m)\Theta_3}\left[x_2 - \frac{\rho}{2-\rho}y_2\right]. \end{aligned} \tag{72}$$

By taking the modulus of (72), we have

$$\left|c_3 - \rho c_2^2\right| \leq \frac{(1-\kappa)|2-\rho|}{(1+2m)\Theta_3}\left[1 + \frac{\rho}{|2-\rho|}\right].$$

In particular, $\rho = 1$, then we obtain

$$\left|c_3 - c_2^2\right| \leq \frac{2(1-\kappa)}{(1+2m)\Theta_3}.$$

□

4. Conclusions

Geometric function theory is one of the most exciting areas of research in complex analysis. We investigated a unified subclass of bi-univalent functions of the Yamaguchi–Noshiro type combined with the linear q-convolution operator. For the functions in this new class, we obtained nonsharp bounds for the initial coefficients and the Fekete–Szegö inequalities. We also considered several interesting corollaries and applications of the results by suitably fixing the parameters, as illustrated in Remark 1.

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