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An Investigation of Linear Diophantine Fuzzy Nonlinear Fractional Programming Problems

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Abstract: The linear Diophantine fuzzy set notion is the main foundation of the interactive method of tackling nonlinear fractional programming problems that is presented in this research. When the decision maker (DM) defines the degree α of α level sets, the max-min problem is solved in this interactive technique using Zimmermann's min operator method. By using the updating technique of degree α, we can solve DM from the set of α-cut optimal solutions based on the membership function and non-membership function. Fuzzy numbers based on α-cut analysis bestowing the degree α given by DM can first be used to classify fuzzy Diophantine inside the coefficients. After this, a crisp multi-objective non-linear fractional programming problem (MONLFPP) is created from a Diophantine fuzzy nonlinear programming problem (DFNLFPP). Additionally, the MONLFPP can be reduced to a single-objective nonlinear programming problem (NLPP) using the idea of fuzzy mathematical programming, which can then be solved using any suitable NLPP algorithm. The suggested approach is demonstrated using a numerical example.

Keywords: nonlinear programming problems; fuzzy sets; linear Diophantine fuzzy sets; LDF-nonlinear programming problems

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1. Introduction

Decision-makers encounter numerous issues in everyday life while deciding between linear and nonlinear fractional programming problems (FPPs). The aims are typically conflicting, incommensurable, and fuzzy; therefore, many factors of uncertainty's ambiguous character should be taken into consideration when formulating the issue. For the objective functions and constraints, many fuzzy parameters have been used. With the help of numerous studies, fuzzy nonlinear fractional programming problems (FNLFPP) are divided into two categories: nonlinear fractional programming problems (NLFPP) with fuzzy goals and NLFPP with fuzzy coefficients. These fuzzy parameters are described as fuzzy numbers, introduced by Sakawa et al. [1–4].

The idea of fuzzy set was developed initially by Zadeh [5]. Bellman and Zadeh [6] also provided a definition for a fuzzy decision. According to the theory of fuzzy sets, an element's membership in a fuzzy set is represented by a single value between zero and one. However, because there may be some hesitation degree, it is not necessarily true that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the membership degree. Because it provides a generalization of fuzzy sets, the theory of intuitionistic fuzzy set (IFS) is anticipated to play a significant role in modern mathematics.

The intuitionistic fuzzy set was developed by Atanassov [7–9], who also expanded upon the idea of a fuzzy set. Since it includes the degree of belongingness, the degree of non-belongingness, and the hesitation margin introduced by Atanassov [10], the knowledge and semantic representation of IFS become more expressive, innovative, and appropriate. IFS are fascinating and helpful when describing an issue using a linguistic variable in terms of a membership function that appears too rough, according to Szmidt and Kacprzyk [11,12]

Hezibah [13] suggested a Taylor series method to solve the intuitionistic fuzzy multi-objective nonlinear programming problem (IFMONLPP). In an intuitionistic fuzzy setting, Singh and Yadav [14] presented a method for the handling of nonlinear programming issues. A method to resolve the intuitionistic fuzzy linear fractional programming problem was also devised by Singh and Yadav [15] (IFLFPP). The Sperm Motility algorithm was studied by Raouf et al. [16] as a solution to fractional programming problems under uncertainty (FPPU). An interactive intuitionistic fuzzy nonlinear fractional programming problem was introduced by Amer [17].

The idea of "α-cut optimality" is proposed in this study based on "α-level sets of fuzzy numbers" to address nonlinear fractional programming issues with fuzzy parameters classified by fuzzy numbers. Then, as a generalization of the findings in Sakawa et al. [1–4], an interactive decision-making method is described that may quickly determine the decision-maker's most satisfactory option from among a group of α-cut optimal solutions. Additionally, the objective function's coefficients are used to measure a linear Diophantine fuzzy nonlinear fractional programming problem (LDFNLFPP), and the constraints are a set of triangular linear Diophantine fuzzy numbers (LDFNs). A deterministic multi-objective nonlinear fractional programming problem (MONLFPP) is created from the given LDFNLFPP. Next, the MONLFPP is converted into a single-objective NLPP utilizing a fuzzy mathematical programming approach [18–20]. Finally, a numerical example is provided to show the effectiveness of this approach.

In [1,13,14,17,21–25], (multi-objective) nonlinear programming problems have been studied under the environment of fuzzy sets and intuitionistic fuzzy sets. However, here, we study the nonlinear fractional programming problems in terms of more generalized fuzzy sets called linear Diophantine fuzzy sets.

2. Preliminaries and Basic Definitions

This section is devoted to reviewing some fundamental ideas that are crucial in understanding the dominant model.

Definition 1 ([5])**.** *If X is a collection of objects denoted generically by X, then a fuzzy set $\aleph$ in X is a set of ordered pairs: $\{(\vartheta, \mu_{\aleph}(\vartheta))|\vartheta \in X\}$, $\mu_{\aleph}$ is called the membership function of $\aleph$, which maps X to $[0,1]$, and $\mu_{\aleph}(\vartheta)$ is called the membership degree of ϑ in $\aleph$.*

Definition 2 ([26])**.** *Let $\aleph$ be a fuzzy set on universal set X. Then, $\aleph$ is called convex FS if $\forall r, s \in X$ and $\lambda \in [0,1]$, and we have*

$$\mu_{\aleph}(\lambda r + (1-\lambda)s) \geq \min\{\mu_{\aleph}(r), \mu_{\aleph}(s)\}.$$

Definition 3 ([5])**.** *A fuzzy set $\aleph$ is said to be normalized if $h(\aleph) = 1$.*

Definition 4 ([26])**.** *An α-level set of an FS $\aleph$ is defined as*

$$\aleph^{\alpha} = \{\vartheta \in X : \mu_{\aleph}(\vartheta) \geq \alpha\} \text{ for each } \alpha \in (0,1].$$

Definition 5 ([26])**.** *A fuzzy subset $\aleph$ defined on a set $\mathbb{R}$ (of real numbers) is said to be a fuzzy number (FN) if $\aleph$ satisfies the following axioms:*

(a) $\aleph$ is continuous: $\mu_{\aleph}(t)$ is a continuous function from $\mathbb{R} \to [0,1]$.
(b) $\aleph$ is normalized: there exists $t \in \mathbb{R}$ such that $\mu_{\aleph}(t) = 1$.
(c) Convexity of $\aleph$, i.e., $\forall\ t, u, w \in \mathbb{R}$, if $t \leq u \leq w$ then $\mu_{\aleph}(u) \geq \min\{\mu_{\aleph}(t), \mu_{\aleph}(w)\}$.
(d) Boundedness of support, i.e., $\exists\ S \in \mathbb{R}$ and $\forall\ t \in \mathbb{R}$, if $|t| \geq S$ then $\mu_{\aleph}(t) = 0$.

We denote the set of all FNs by $F_{ns}(\mathbb{R})$.

Definition 6 ([27])**.** *Let X be the universe. An LDFS $\hbar_{\mathfrak{R}}$ on X is defined as follows*

$$\hbar_{\mathfrak{R}} = \{(\vartheta, \langle \aleph^{\tau}_{\mathfrak{R}}(\vartheta), \xi^{\nu}_{\mathfrak{R}}(\vartheta)\rangle, \langle \alpha(\vartheta), \beta(\vartheta)\rangle) : \vartheta \in X\}$$

where $\aleph^{\tau}_{\mathfrak{R}}(\vartheta), \xi^{\nu}_{\mathfrak{R}}(\vartheta), \alpha(\vartheta), \beta(\vartheta) \in [0,1]$ such that

$$0 \leq \alpha(\vartheta)\aleph^{\tau}_{\mathfrak{R}}(\vartheta) + \beta(\vartheta)\xi^{\nu}_{\mathfrak{R}}(\vartheta) \leq 1, \ \forall \vartheta \in X \tag{1}$$

$$0 \leq \alpha(\vartheta) + \beta(\vartheta) \leq 1. \tag{2}$$

and the hesitation part can be written as

$$\mathfrak{N}\pi_{\mathfrak{R}} = 1 - (\alpha(\vartheta)\aleph^{\tau}_{\mathfrak{R}}(\vartheta) + \beta(\vartheta)\xi^{\nu}_{\mathfrak{R}}(\vartheta)) \tag{3}$$

where $\mathfrak{N}$ is the reference parameter.

We write in short $\hbar_{\mathfrak{R}} = \left(\langle \aleph^{\tau}_{\mathfrak{R}}, \xi^{\nu}_{\mathfrak{R}}\rangle, \langle \alpha, \beta\rangle\right)$ or $\hbar_{\mathfrak{R}} = \left\langle \langle \aleph^{\tau}_{\mathfrak{R}}, \xi^{\nu}_{\mathfrak{R}}\rangle, \langle \alpha, \beta\rangle\right\rangle$ for

$$\hbar_{\mathfrak{R}} = \{(\vartheta, \langle \aleph^{\tau}_{\mathfrak{R}}(\vartheta), \xi^{\nu}_{\mathfrak{R}}(\vartheta)\rangle, \langle \alpha(\vartheta), \beta(\vartheta)\rangle) : \vartheta \in X\}.$$

Definition 7 ([27])**.** *An LDFS $\hbar_{\mathfrak{R}} = \left\{\left(\vartheta, \langle \aleph^{\tau}_{\mathfrak{R}}(\vartheta), \xi^{\nu}_{\mathfrak{R}}(\vartheta)\rangle, \langle \alpha(\vartheta), \beta(\vartheta)\rangle\right) : \vartheta \in X\right\}$ is called a linear Diophantine fuzzy number (LDFN) if the following hold:*

(i) There exists $m \in R$ such that $\aleph^{\tau}_{\mathfrak{R}}(\vartheta) = \alpha(\vartheta) = 1$ and $\xi^{\nu}_{\mathfrak{R}}(\vartheta) = \beta(\vartheta) = 0$, where m is the mean value of $\hbar_{\mathfrak{R}}$.

(ii) ($\aleph^{\tau}_{\mathfrak{R}}$ and α) and ($\xi^{\nu}_{\mathfrak{R}}$ and β) are piecewise continuous functions from R to the closed interval $[0,1]$ and $0 \leq \alpha(\vartheta)\aleph^{\tau}_{\mathfrak{R}}(\vartheta) + \beta(\vartheta)\xi^{\nu}_{\mathfrak{R}}(\vartheta) \leq 1, \ \forall \vartheta \in X$, where

$$\aleph^{\tau}_{\mathfrak{R}}(x) = \begin{cases} g_1(x) & m - \vartheta_1 \leq x < m \\ h_1(x) & \vartheta_3 \leq x \leq m + \vartheta_3 \\ 0 & \textit{otherwise} \end{cases}, \quad \xi^{\nu}_{\mathfrak{R}}(x) = \begin{cases} g_2(x) & \begin{array}{l} m - \vartheta_2 \leq x \leq m; \\ 0 \leq g_1(x) + g_2(x) \leq 1 \end{array} \\ h_2(x) & \begin{array}{l} m \leq x \leq m + \vartheta_3; \\ 0 \leq h_1(x) + h_2(x) \leq 1 \end{array} \\ 0 & \textit{otherwise}, \end{cases} \tag{4}$$

and

$$\alpha(x) = \begin{cases} g'_1(x) & m - \vartheta'_1 \leq x < m \\ h'_1(x) & \vartheta_3 \leq x \leq m + \vartheta_3 \\ 0 & \textit{otherwise} \end{cases}, \quad \beta(x) = \begin{cases} g'_2(x) & \begin{array}{l} m - \vartheta'_2 \leq x \leq m; \\ 0 \leq g'_1(x) + g'_2(x) \leq 1 \end{array} \\ h'_2(x) & \begin{array}{l} m \leq x \leq m + \vartheta'_3; \\ 0 \leq h'_1(x) + h'_2(x) \leq 1 \end{array} \\ 0 & \textit{otherwise}. \end{cases} \tag{5}$$

Definition 8 ([28])**.** *Let $\hbar_{\mathfrak{R}}$ be an LDFS on $\mathbb{R}$ with the following membership functions ($\aleph^{\tau}_{\mathfrak{R}}$ and α) and non-membership functions ($\xi^{\nu}_{\mathfrak{R}}$ and β)*

$$\aleph^{\tau}_{\mathfrak{R}}(x) = \begin{cases} \frac{x-\vartheta_1}{\vartheta_3-\vartheta_1} & \vartheta_1 \leq x \leq \vartheta_3 \\ \frac{\vartheta_5-x}{\vartheta_5-\vartheta_3} & \vartheta_3 \leq x \leq \vartheta_5 \\ 0 & \textit{otherwise} \end{cases}, \quad \xi^{\nu}_{\mathfrak{R}}(x) = \begin{cases} \frac{\vartheta_3-x}{\vartheta_3-\vartheta_2} & \vartheta_2 \leq x \leq \vartheta_3 \\ \frac{x-\vartheta_3}{\vartheta_4-\vartheta_3} & \vartheta_3 \leq x \leq \vartheta_4 \\ 0 & \textit{otherwise}, \end{cases} \tag{6}$$

and

$$\alpha(x) = \begin{cases} \frac{x-\vartheta'_2}{\vartheta'_3-\vartheta'_2} & \vartheta'_2 \leq x \leq \vartheta'_3 \\ \frac{\vartheta'_4-x}{\vartheta'_4-\vartheta'_3} & \vartheta'_3 \leq x \leq \vartheta'_4 \\ 0 & \textit{otherwise} \end{cases}, \quad \beta(x) = \begin{cases} \frac{\vartheta'_3-x}{\vartheta'_3-\vartheta'_1} & \vartheta'_1 \leq x \leq \vartheta'_3 \\ \frac{x-\vartheta'_3}{\vartheta'_5-\vartheta'_3} & \vartheta'_3 \leq x \leq \vartheta'_5 \\ 0 & \textit{otherwise}. \end{cases} \tag{7}$$

Throughout the paper, we consider only a triangular LDFN of type 1 and we refer to this type as triangular LDFN (TLDFN). This TLDFN is denoted by

$$\hbar_{\mathfrak{R}_{TLDFN}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{cases}.$$

Definition 9 ([28])**.** *A TLDFN* $\hbar_{\mathfrak{R}_{TLDFN}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{cases}$ *is said to be positive if and only if* $\vartheta_1 \geq 0$ *and* $\vartheta_1' \geq 0$.

Definition 10 ([28])**.** *Two TLDFNs* $\hbar_{\mathfrak{R}_{TLDFN}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{cases}$ *and* $\Im_{\mathfrak{R}_{TLDFN}} = \begin{cases} (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \\ (\delta_1', \delta_2', \delta_3, \delta_4', \delta_5') \end{cases}$ *are said to be equal if and only if* $\vartheta_1 = \delta_1$, $\vartheta_2 = \delta_2$, $\vartheta_3 = \delta_3$, $\vartheta_4 = \delta_4$, $\vartheta_5 = \delta_5$, $\vartheta_1' = \delta_1'$, $\vartheta_2' = \delta_2'$, $\vartheta_4' = \delta_4'$ *and* $\vartheta_5' = \delta_5'$.

We now define the arithmetic operations on TLDFNs using the concept of interval arithmetic.

Definition 11 ([28])**.** *Consider two positive TLDFNs* $\hbar_{\mathfrak{R}_{TLDFN}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{cases}$ *and* $\Im_{\mathfrak{R}_{TLDFN}} = \begin{cases} (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \\ (\delta_1', \delta_2', \delta_3, \delta_4', \delta_5') \end{cases}$, *then*

(i) $\hbar_{\mathfrak{R}_{TLDFN}} + \Im_{\mathfrak{R}_{TLDFN}} = \begin{cases} (\vartheta_1+\delta_1, \vartheta_2+\delta_2, \vartheta_3+\delta_3, \vartheta_4+\delta_4, \vartheta_5+\delta_5) \\ (\vartheta_1'+\delta_1', \vartheta_2'+\delta_2', \vartheta_3+\delta_3, \vartheta_4'+\delta_4', \vartheta_5'+\delta_5') \end{cases}$

(ii) $\hbar_{\mathfrak{R}_{TLDFN}} - \Im_{\mathfrak{R}_{TLDFN}} = \begin{cases} (\vartheta_1-\delta_5, \vartheta_2-\delta_4, \vartheta_3-\delta_3, \vartheta_4-\delta_2, \vartheta_5-\delta_1) \\ (\vartheta_1'-\delta_5', \vartheta_2'-\delta_4', \vartheta_3-\delta_3, \vartheta_4'-\delta_2', \vartheta_5'-\delta_1') \end{cases}$;

(iii) $\hbar_{\mathfrak{R}_{TLDFN}} \times \Im_{\mathfrak{R}_{TLDFN}} = \begin{cases} (\vartheta_1\delta_1, \vartheta_2\delta_2, \vartheta_3\delta_3, \vartheta_4\delta_4, \vartheta_5\delta_5) \\ (\vartheta_1'\delta_1', \vartheta_2'\delta_2', \vartheta_3\delta_3, \vartheta_4'\delta_4', \vartheta_5'\delta_5') \end{cases}$;

(iv) $\hbar_{\mathfrak{R}_{TLDFN}} \div \Im_{\mathfrak{R}_{TLDFN}} = \begin{cases} \left(\frac{\vartheta_1}{\delta_5}, \frac{\vartheta_2}{\delta_4}, \frac{\vartheta_3}{\delta_3}, \frac{\vartheta_4}{\delta_2}, \frac{\vartheta_5}{\delta_1}\right) \\ \left(\frac{\vartheta_1'}{\delta_5'}, \frac{\vartheta_2'}{\delta_4'}, \frac{\vartheta_3}{\delta_3}, \frac{\vartheta_4'}{\delta_2'}, \frac{\vartheta_5'}{\delta_1'}\right) \end{cases}$;

(v) $k \times \hbar_{\mathfrak{R}_{TLDFN}} = \begin{cases} \begin{cases} (k\vartheta_1, k\vartheta_2, k\vartheta_3, k\vartheta_4, k\vartheta_5) \\ (k\vartheta_1', k\vartheta_2', k\vartheta_3, k\vartheta_4', k\vartheta_5') \end{cases} & \text{if } k > 0 \\ \begin{cases} (k\vartheta_5, k\vartheta_4, k\vartheta_3, k\vartheta_2, k\vartheta_1) \\ (k\vartheta_5', k\vartheta_4', k\vartheta_3, k\vartheta_2', k\vartheta_1') \end{cases} & \text{if } k < 0. \end{cases}$

3. Problem Formulation and Solution Concepts

The general mathematical model of LDFNLFPP can be written as follows:

$$\begin{aligned} \text{Max } Z(\tilde{x}^L) \quad &= \frac{f(x, \tilde{A}^L)}{g(x, \tilde{B}^L)}, \\ \text{subject to} \quad &h_{j_1}(x, \tilde{C}^L) \leq \tilde{D}^L_{j_1}, \ j_1 = 1, 2, \ldots, m_1, \\ &h_{j_2}(x, \tilde{E}^L) \geq \tilde{j}^L_{j2}, \ j_2 = m_1 + 1, \ldots, m_2, \\ &h_{j_3}(x, \tilde{I}^L) = \tilde{L}^L_{j_3}, \ j_3 = m_2 + 1, \ldots, m, \\ &x \geq 0, \\ &g(x, \tilde{B}^L) \neq 0, \end{aligned} \tag{8}$$

where x is n-dimensional decision variable vector $x = (x_1, x_2, \ldots, x_n)$, $f(x, \tilde{A}^L)$ and $g(x, \tilde{B}^L) \neq 0, h_{j_1}(x, \tilde{C}^L), h_{j_2}(x, \tilde{E}^L)$ and $h_{j_3}(x, \tilde{I}^L)$, respectively, are supposed to be real valued continuous nonlinear functions with LDFNs. The parameters $\tilde{A}^L, \tilde{B}^L, \tilde{C}^L, \tilde{D}^L, \tilde{E}^L, \tilde{j}^L, \tilde{I}^L$ and $\tilde{L}^L$ are considered TLDFNs.

Similarly,

$$
\begin{array}{ll}
\text{Max } Z'(\widetilde{x}'^{L}) & = \frac{f'(x', \widetilde{A}'^{L})}{g'(x', \widetilde{B}'^{L})}, \\
\text{subject to} & h'_{j_1}(x', \widetilde{C}'^{L}) \leq \widetilde{D}'^{L}_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
& h'_{j_2}(x', \widetilde{E}'^{L}) \geq \widetilde{j}'^{L}_{j_2},\ j_2 = m_1 + 1, \ldots, m_2, \\
& h'_{j_3}(x', \widetilde{I}'^{L}) = \widetilde{L}'^{L}_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
& x' \geq 0, \\
& g(x', \widetilde{B}'^{L}) \neq 0,
\end{array} \tag{9}
$$

where x' is n-dimensional decision variable vector $x' = (x'_1, x'_2, \ldots, x'_n)$, $f'(x, \widetilde{A}'^{L})$ and $g'(x, \widetilde{B}'^{L}) \neq 0, h'_{j_1}(x, \widetilde{C}'^{L}), h'_{j_2}(x, \widetilde{E}'^{L})$ and $h'_{j_3}(x, \widetilde{I}'^{L})$, respectively, are supposed to be real valued continuous nonlinear functions with LDFNs. The parameters $\widetilde{A}'^{L}, \widetilde{B}'^{L}, \widetilde{C}'^{L}, \widetilde{D}'^{L}, \widetilde{E}'^{L}, \widetilde{j}'^{L}, \widetilde{I}'^{L}$ and $\widetilde{L}'^{L}$ are considered TLDFNs.

In this section, the methodology for the solution of an interactive LDFNLFPP is expanded where all coefficients are TLDFNs. This problem varies from the crisp problem via parametric values. The parameters are known precisely in crisp or non-fuzzy models. Consequently, for a precise degree of α, as in Definition 4, which is described by the DM, problems (8) and (9) can be redeveloped as the following linear Diophantine non-fuzzy α-nonlinear fractional programming problem (α-LDNLFPP) with linear Diophantine non-fuzzy numbers (α-LDFNs):

$$
\begin{array}{ll}
\text{Max } \theta(x) & = \frac{F(x, A)}{G(x, B)}, \\
\text{subject to} & H_{j_1}(x, C) \leq D_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
& H_{j_2}(x, E) \geq J_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
& H_{j_3}(x, I) = L_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
& x \geq 0, \\
& G(x, b) \neq 0, \quad \aleph_{\widetilde{J}^{L}}(J) \geq \alpha_J,
\end{array} \tag{10}
$$

where J is any coefficient and the parameters A, B, C, D, E, J, I and L, respectively, are assumed to be non-fuzzy numbers defined as $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$.

$$
\begin{array}{ll}
\text{Max } \theta'(x') & = \frac{F'(x', A')}{G'(x', B')}, \\
\text{subject to} & H'_{j_1}(x', C') \leq D'_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
& H'_{j_2}(x', E') \geq D'_{j_2},\ j_2 = m_1 + 1, \ldots, m_2, \\
& H'_{j_3}(x', I') = D'_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
& x' \geq 0, \\
& g(x', b') \neq 0, \quad \alpha_{\widetilde{J}'^{L}}(J) \geq \alpha_J,
\end{array} \tag{11}
$$

where J' is any coefficient and the parameters $A', B', C', D', E', J', I'$ and L', respectively, are assumed to be non-fuzzy numbers defined as $(\theta'_1, \theta'_2, \theta_3, \theta'_4, \theta'_5)$.

4. Solution Procedure for an Interactive LDFNLFPP

By using the division in Definition (12), problems (10) and (11) reduce to an equivalent linear Diophantine multi-objective nonlinear fractional programming problem (LD-MONLFPP) as follows:

$$\begin{array}{ll} \text{Max } \theta_1(x) & = \frac{F(x,\theta_1)}{G(x,\delta_5)}, \\ \text{Max } \theta_2(x) & = \frac{F(x,\theta_2)}{G(x,\delta_4)}, \\ \text{Max } \theta_3(x) & = \frac{F(x,\theta_3)}{G(x,\delta_3)}, \\ \text{Max } \theta_4(x) & = \frac{F(x,\theta_4)}{G(x,\delta_2)}, \\ \text{Max } \theta_5(x) & = \frac{F(x,\theta_5)}{G(x,\delta_1)} \\ \text{subject to} & H_{j_1}(x,C) \leq D_{j_1},\ j_1 = 1,2,\ldots,m_1, \\ & H_{j_2}(x,E) \geq J_{j_2},\ j_2 = m_1+1,\ldots,m_2, \\ & H_{j_3}(x,I) = L_{j_3},\ j_3 = m_2+1,\ldots,m, \\ & x \geq 0, \end{array} \tag{12}$$

and

$$\begin{array}{ll} \text{Max } \theta'_1(x') & = \frac{F'(x',\theta'_1)}{G'(x',\delta'_5)}, \\ \text{Max } \theta'_2(x') & = \frac{F'(x',\theta'_2)}{G'(x',\delta'_4)}, \\ \text{Max } \theta_3(x) & = \frac{F(x,\theta_3)}{G(x,\delta_3)}, \\ \text{Max } \theta'_4(x') & = \frac{F'(x',\theta'_4)}{G'(x',\delta'_2)}, \\ \text{Max } \theta'_5(x') & = \frac{F'(x',\theta'_5)}{G'(x',\delta'_1)}, \\ \text{subject to} & H'_{j_1}(x',C') \leq D'_{j_1},\ j_1 = 1,2,\ldots,m_1, \\ & H'_{j_2}(x',E') \geq D'_{j_2},\ j_2 = m_1+1,\ldots,m_2, \\ & H'_{j_3}(x',I') = D'_{j_3},\ j_3 = m_2+1,\ldots,m, \\ & x' \geq 0, \end{array} \tag{13}$$

Let us consider $\begin{cases} (\theta_1(x),\theta_2(x),\theta_3(x),\theta_4(x),\theta_5(x)) \\ (\theta'_1(x),\theta'_2(x),\theta_3(x),\theta'_4(x),\theta'_5(x)) \end{cases} \geq 0$ as feasible regions of problems (12) and (13). Hence, using Charnes and Cooper's transformation, the above model LDMONLFPP can be transformed into a linear Diophantine multi-objective nonlinear programming problem (LDMONLPP) by taking $y = tx, t > 0$, as follows:

$$\begin{array}{ll} \text{Max } \theta_1(y/t) & = F(y/t,\theta_1), \\ \text{Max } \theta_2(y/t) & = F(y/t,\theta_2), \\ \text{Max } \theta_3(y/t) & = F(y/t,\theta_3), \\ \text{Max } \theta_4(y/t) & = F(y/t,\theta_4), \\ \text{Max } \theta_5(y/t) & = F(y/t,\theta_5), \\ \text{s.t} & G(y/t,\delta_5) \leq 1, \\ & G(y/t,\delta_4) \leq 1, \\ & G(y/t,\delta_3) \leq 1, \\ & G(y/t,\delta_2) \leq 1, \\ & G(y/t,\delta_1) \leq 1, \\ & H_{j_1}(y/t,c_1) \leq (d_1)_{j_1},\ j_1 = 1,2,\ldots,m_1, \\ & H_{j_1}(y/t,c_2) \leq (d_2)_{j_1},\ j_1 = 1,2,\ldots,m_1, \\ & H_{j_1}(y/t,c_3) \leq (d_3)_{j_1},\ j_1 = 1,2,\ldots,m_1, \\ & H_{j_1}(y/t,c_4) \leq (d_4)_{j_1},\ j_1 = 1,2,\ldots,m_1, \\ & H_{j_1}(y/t,c_5) \leq (d_5)_{j_1},\ j_1 = 1,2,\ldots,m_1, \\ & H_{j_2}(y/t,e_1) \geq (J_1)_{j_2},\ j_2 = m_1+1,\ldots,m_2, \\ & H_{j_2}(y/t,e_2) \geq (J_2)_{j_2},\ j_2 = m_1+1,\ldots,m_2, \\ & H_{j_2}(y/t,e_3) \geq (J_3)_{j_2},\ j_2 = m_1+1,\ldots,m_2, \\ & H_{j_2}(y/t,e_4) \geq (J_4)_{j_2},\ j_2 = m_1+1,\ldots,m_2, \\ & H_{j_2}(y/t,e_5) \geq (J_5)_{j_2},\ j_2 = m_1+1,\ldots,m_2, \\ & H_{j_3}(y/t,i_1) = (L_1)_{j_3},\ j_3 = m_2+1,\ldots,m, \\ & H_{j_3}(y/t,i_2) = (L_2)_{j_3},\ j_3 = m_2+1,\ldots,m, \\ & H_{j_3}(y/t,i_3) = (L_3)_{j_3},\ j_3 = m_2+1,\ldots,m, \\ & H_{j_3}(y/t,i_4) = (L_4)_{j_3},\ j_3 = m_2+1,\ldots,m, \\ & H_{j_3}(y/t,i_5) = (L_5)_{j_3},\ j_3 = m_2+1,\ldots,m, \\ & y \geq 0, t > 0. \end{array} \tag{14}$$

Similarly,

$$
\begin{array}{ll}
\text{Max } \theta'_1(y'/t') & = F'(y'/t', \theta'_1), \\
\text{Max } \theta'_2(y'/t') & = F'(y'/t', \theta'_2), \\
\text{Max } \theta_3(y/t) & = F(y/t, \theta_3), \\
\text{Max } \theta'_4(y'/t') & = F'(y'/t', \theta'_4), \\
\text{Max } \theta'_5(y'/t') & = F'(y'/t', \theta'_5), \\
\text{s.t} & G'(y'/t', \delta'_5) \le 1, \\
& G'(y'/t', \delta'_4) \le 1, \\
& G(y/t, \delta_3) \le 1, \\
& G'(y'/t', \delta'_2) \le 1, \\
& G'(y'/t', \delta'_1) \le 1, \\
& H'_{j_1}(y'/t', c'_1) \le (d'_1)_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
& H'_{j_1}(y'/t', c'_2) \le (d'_2)_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
& H_{j_1}(y/t, c_3) \le (d_3)_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
& H'_{j_1}(y'/t', c'_4) \le (d'_4)_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
& H'_{j_1}(y'/t', c'_5) \le (d'_5)_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
& H'_{j_2}(y'/t', e'_1) \ge (J'_1)_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
& H'_{j_2}(y'/t', e'_2) \ge (J'_2)_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
& H_{j_2}(y/t, e_3) \ge (J_3)_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
& H'_{j_2}(y'/t', e'_4) \ge (J'_4)_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
& H'_{j_2}(y'/t', e'_5) \ge (J'_5)_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
& H'_{j_3}(y'/t', i'_1) = (L'_1)_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
& H'_{j_3}(y'/t', i'_2) = (L'_2)_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
& H_{j_3}(y/t, i_3) = (L_3)_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
& H'_{j_3}(y'/t', i'_4) = (L'_4)_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
& H'_{j_3}(y'/t', i'_5) = (L'_5)_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
& y' \ge 0, t' > 0.
\end{array}
\tag{15}
$$

Now, to solve problems (14) and (15), the following algorithm can be developed.

Step 1: Use the method proposed by Amer [17]; we expand this method to decompose problems (14) and (15) into nine sub-problems, MONLPPs, according to TLDFNs as follows:

$(\boldsymbol{P}_1)$:

$$
\begin{array}{ll}
\text{Max } \theta_1(y/t) & = F(y/t, \theta_1), \\
\text{Max } \theta_2(y/t) & = F(y/t, \theta_2), \\
\text{Max } \theta_3(y/t) & = F(y/t, \theta_3), \\
\text{Max } \theta_4(y/t) & = F(y/t, \theta_4), \\
\text{Max } \theta_5(y/t) & = F(y/t, \theta_5), \\
& G(y/t, \delta_5) \le 1, \\
& H_{j_1}(y/t, c_1) \le (d_1)_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
& H_{j_2}(y/t, e_1) \ge (J_1)_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
\text{subject to} & H_{j_3}(y/t, i_1) = (L_1)_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
& y \ge 0, t > 0,
\end{array}
$$

$(\boldsymbol{P}_2)$:

$$\begin{array}{ll}
\text{Max } \theta_1(y/t) & = F(y/t, \theta_1), \\
\text{Max } \theta_2(y/t) & = F(y/t, \theta_2), \\
\text{Max } \theta_3(y/t) & = F(y/t, \theta_3), \\
\text{Max } \theta_4(y/t) & = F(y/t, \theta_4), \\
\text{Max } \theta_5(y/t) & = F(y/t, \theta_5), \\
 & \quad G(y/t, \delta_4) \le 1, \\
 & \quad H_{j_1}(y/t, c_2) \le (d_2)_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
 & \quad H_{j_2}(y/t, e_2) \ge (J_2)_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
\text{subject to} & \quad H_{j_3}(y/t, i_2) = (L_2)_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
 & y \ge 0, t > 0,
\end{array}$$

$(\boldsymbol{P}_3)$:

$$\begin{array}{ll}
\text{Max } \theta_1(y/t) & = F(y/t, \theta_1), \\
\text{Max } \theta_2(y/t) & = F(y/t, \theta_2), \\
\text{Max } \theta_3(y/t) & = F(y/t, \theta_3), \\
\text{Max } \theta_4(y/t) & = F(y/t, \theta_4), \\
\text{Max } \theta_5(y/t) & = F(y/t, \theta_5), \\
 & \quad G(y/t, \delta_3) \le 1, \\
 & \quad H_{j_1}(y/t, c_3) \le (d_3)_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
 & \quad H_{j_2}(y/t, e_3) \ge (J_3)_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
\text{subject to} & \quad H_{j_3}(y/t, i_3) = (L_3)_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
 & y \ge 0, t > 0,
\end{array}$$

$(\boldsymbol{P}_4)$:

$$\begin{array}{ll}
\text{Max } \theta_1(y/t) & = F(y/t, \theta_1), \\
\text{Max } \theta_2(y/t) & = F(y/t, \theta_2), \\
\text{Max } \theta_3(y/t) & = F(y/t, \theta_3), \\
\text{Max } \theta_4(y/t) & = F(y/t, \theta_4), \\
\text{Max } \theta_5(y/t) & = F(y/t, \theta_5), \\
 & \quad G(y/t, \delta_2) \le 1, \\
 & \quad H_{j_1}(y/t, c_4) \le (d_4)_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
 & \quad H_{j_2}(y/t, e_4) \ge (J_4)_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
\text{subject to} & \quad H_{j_3}(y/t, i_4) = (L_4)_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
 & y \ge 0, t > 0,
\end{array}$$

$(\boldsymbol{P}_5)$:

$$\begin{array}{ll}
\text{Max } \theta_1(y/t) & = F(y/t, \theta_1), \\
\text{Max } \theta_2(y/t) & = F(y/t, \theta_2), \\
\text{Max } \theta_3(y/t) & = F(y/t, \theta_3), \\
\text{Max } \theta_4(y/t) & = F(y/t, \theta_4), \\
\text{Max } \theta_5(y/t) & = F(y/t, \theta_5), \\
 & \quad G(y/t, \delta_1) \le 1, \\
 & \quad H_{j_1}(y/t, c_5) \le (d_5)_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
 & \quad H_{j_2}(y/t, e_5) \ge (J_5)_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
\text{subject to} & \quad H_{j_3}(y/t, i_5) = (L_5)_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
 & y \ge 0, t > 0,
\end{array}$$

$(\boldsymbol{P}_6)$:

$$\begin{array}{ll}
\text{Max } \theta_1'(y'/t') & = F'(y'/t', \theta_1'), \\
\text{Max } \theta_2'(y'/t') & = F'(y'/t', \theta_2'), \\
\text{Max } \theta_3(y/t) & = F(y/t, \theta_3), \\
\text{Max } \theta_4'(y'/t') & = F'(y'/t', \theta_4'), \\
\text{Max } \theta_5'(y'/t') & = F'(y'/t', \theta_5'), \\
 & \quad G'(y'/t', \delta_5') \leq 1, \\
 & \quad H_{j_1}'(y'/t', c_1') \leq (d_1')_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
 & \quad H_{j_2}'(y'/t', e_1') \geq (J_1')_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
\text{subject to} & \quad H_{j_3}'(y'/t', i_1') = (L_1')_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
 & y' \geq 0, t' > 0,
\end{array}$$

$(\boldsymbol{P}_7)$:

$$\begin{array}{ll}
\text{Max } \theta_1'(y'/t') & = F'(y'/t', \theta_1'), \\
\text{Max } \theta_2'(y'/t') & = F'(y'/t', \theta_2'), \\
\text{Max } \theta_3(y/t) & = F(y/t, \theta_3), \\
\text{Max } \theta_4'(y'/t') & = F'(y'/t', \theta_4'), \\
\text{Max } \theta_5'(y'/t') & = F'(y'/t', \theta_5'), \\
 & \quad G'(y'/t', \delta_4') \leq 1, \\
 & \quad H_{j_1}'(y'/t', c_2') \leq (d_2')_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
 & \quad H_{j_2}'(y'/t', e_2') \geq (J_2')_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
\text{subject to} & \quad H_{j_3}'(y'/t', i_2') = (L_2')_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
 & y' \geq 0, t' > 0,
\end{array}$$

$(\boldsymbol{P}_8)$:

$$\begin{array}{ll}
\text{Max } \theta_1'(y'/t') & = F'(y'/t', \theta_1'), \\
\text{Max } \theta_2'(y'/t') & = F'(y'/t', \theta_2'), \\
\text{Max } \theta_3(y/t) & = F(y/t, \theta_3), \\
\text{Max } \theta_4'(y'/t') & = F'(y'/t', \theta_4'), \\
\text{Max } \theta_5'(y'/t') & = F'(y'/t', \theta_5'), \\
 & \quad G'(y'/t', \delta_2') \leq 1, \\
 & \quad H_{j_1}'(y'/t', c_4') \leq (d_4')_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
 & \quad H_{j_2}'(y'/t', e_4') \geq (J_4')_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
\text{subject to} & \quad H_{j_3}'(y'/t', i_4') = (L_4')_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
 & y' \geq 0, t' > 0,
\end{array}$$

and

$(\boldsymbol{P}_9)$:

$$\begin{array}{ll}
\text{Max } \theta_1'(y'/t') & = F'(y'/t', \theta_1'), \\
\text{Max } \theta_2'(y'/t') & = F'(y'/t', \theta_2'), \\
\text{Max } \theta_3(y/t) & = F(y/t, \theta_3), \\
\text{Max } \theta_4'(y'/t') & = F'(y'/t', \theta_4'), \\
\text{Max } \theta_5'(y'/t') & = F'(y'/t', \theta_5'), \\
 & \quad G'(y'/t', \delta_1') \leq 1, \\
 & \quad H_{j_1}'(y'/t', c_5') \leq (d_5')_{j_1},\ j_1 = 1, 2, \ldots, m_1, \\
 & \quad H_{j_2}'(y'/t', e_5') \geq (J_5')_{j2},\ j_2 = m_1 + 1, \ldots, m_2, \\
\text{subject to} & \quad H_{j_3}'(y'/t', i_5') = (L_5')_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\
 & y' \geq 0, t' > 0,
\end{array}$$

Step 2: Solve models $P_i, i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ as individual objective functions under the given constraints.

Step 3: Find the optimal points of all the sub-problems and let the total solution set be $X = \cup_{i=1}^{9} X_i$.

Step 4: Find the value of each objective function $\theta_i(y/t), i = 1, \ldots, 5$ and $\theta'_i(y/t)$, $i = 1, \ldots, 4$ at each point obtained in step 3.

Step 5: Find the upper and lower bounds U, U' and L, L', respectively, for objective functions

$$\begin{aligned} L &= \min\{\theta_i(y/t) : x \in X, i = 1, \ldots, 5\} \text{ and } U = \max\{\theta_i(y/t) : x \in X, i = 1, \ldots, 5\} \\ L' &= \min\{\theta'_i(y'/t') : x' \in X, i = 1, \ldots, 4\} \text{ and } U' = \max\{\theta'_i(y'/t') : x' \in X, i = 1, \ldots, 4\} \end{aligned}$$

Step 6: Then, IMONLFPPs (12) and (13) are equivalent to the following fuzzy model using Zimmermann's technique.

Find x

$$\text{such that} \quad \begin{array}{l} \theta(x) \geq U \\ H_{j_1}(x, C) \leq D_{j_1},\ j_1 = 1, \ldots, m_1, \\ H_{j_2}(x, E) \geq D_{j_2},\ j_2 = m_1 + 1, \ldots, m_2, \\ H_{j_3}(x, I) \approx D_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\ x \geq 0, \end{array} \tag{16}$$

and find x'

$$\text{such that} \quad \begin{array}{l} \theta'(x') \geq U' \\ H'_{j_1}(x', C') \leq D'_{j_1},\ j_1 = 1, \ldots, m_1, \\ H'_{j_2}(x', E') \geq D'_{j_2},\ j_2 = m_1 + 1, \ldots, m_2, \\ H'_{j_3}(x', I') \approx D'_{j_3},\ j_3 = m_2 + 1, \ldots, m, \\ x' \geq 0, \end{array} \tag{17}$$

where $\leq$, $\geq$ and $\approx$ are fuzzy inequality and fuzzy equality, respectively. Fuzzy in this context means that the DM's specified rigorous equality and inequality are subject to some tolerance.

Since the objective is to maximize it, the DM becomes more satisfied as the objective value approaches the upper bound. Let $\aleph_U$ and α_U stand for the degree of attainability of the upper bound U of the objective function $\theta(x)$ and L stand for the least sustainable level of the objective value by the DM.

Step 7: Take the membership functions as follows:

$$\aleph_U(\theta(x)) = \begin{cases} 0 & \text{if } \theta(x) < L, \\ \frac{(\theta(x))^t - L^t}{U^t - L^t} & \text{if } L \leq \theta(x) \leq U, \\ 1 & \text{if } \theta(x) > U. \end{cases} \tag{18}$$

Let $\aleph_D, \aleph_J$ and $\aleph_L$, respectively, represent the degree of achievability of the available constraints; they are defined by the following.

For $(J_1 = 1, 2, \ldots, m_1)$,

$$\aleph_D(H_{j_1}(x, C)) = \begin{cases} 0 & \text{if } H_{j_1}(x, C) < d_{j_1}, \\ \frac{(d^r_{j_1})^t - (H_{j_1}(x,C))^t}{(d^r_{j_1})^t - (d_{j_1})^t} & \text{if } d_{j_1} \leq H_{j_1}(x, C) \leq d^r_{j_1}, \\ 0 & \text{if } H_{j_1}(x, C) > d^r_{j_1}. \end{cases} \tag{19}$$

For $(J_2 = m_1 + 1, \ldots, m_2)$,

$$\aleph_J(H_{j_2}(x, E)) = \begin{cases} 0 & \text{if } H_{j_2}(x, E) < j_{j_2}, \\ \frac{(H_{j_2}(x,E))^t - (j^l_{j_2})^t}{(j_{j_2})^t - (j^L_{j_2})^t} & \text{if } j^L_{j_2} \leq H_{j_2}(x, E) \leq j_{j_2}, \\ 1 & \text{if } H_{j_2}(x, E) > j_{j_2}. \end{cases} \tag{20}$$

For $(J_3 = m_2 + 1, \ldots, m)$,

$$\aleph_J(H_{j_3}(x,I)) = \begin{cases} 0 & \text{if } H_{j_3}(x,I) < I_{j_3}, \\ \frac{(H_{j_3}(x,I))^t - (l^l_{j_3})^t}{(l_{j_3})^t - (l^l_{j_3})^t} & \text{if } l^l_{j_3} \leq H_{j_3}(x,I) \leq l_{j_3}, \\ \frac{(l^r_{j_3})^t - (H_{j_3}(x,I))^t}{(l^r_{j_3})^t - (l_{j_3})^t} & \text{if } l_{j_3} \leq H_{j_3}(x,I) \leq l^r_{j_3}, \\ 0 & \text{if } H_{j_3}(x,I) > l_{j_3} \end{cases} \tag{21}$$

where $t > 0$ is specified by the DM.

Similarly, for the membership fuction α,

$$\alpha_U(\theta'(x')) = \begin{cases} 0 & \text{if } \theta'(x) < L', \\ \frac{(\theta'(x'))^t - L'^t}{U'^t - L'^t} & \text{if } L' \leq \theta'(x) \leq U', \\ 1 & \text{if } \theta'(x) > U'. \end{cases} \tag{22}$$

Let α_D, α_J and α_L, respectively, represent the degree of achievability of the available constraints; they are defined by the following.

For $(J_1 = 1, 2, \ldots, m_1)$,

$$\alpha_D(H'_{j_1}(x',C')) = \begin{cases} 0 & \text{if } H'_{j_1}(x',C') < d'_{j_1}, \\ \frac{(d'^r_{j_1})^t - (H'_{j_1}(x',C'))^t}{(d'^r_{j_1})^t - (d'_{j_1})^t} & \text{if } d'_{j_1} \leq H'_{j_1}(x',C') \leq d'^r_{j_1}, \\ 0 & \text{if } H'_{j_1}(x',C') > d'^r_{j_1}. \end{cases} \tag{23}$$

For $(J_2 = m_1 + 1, \ldots, m_2)$,

$$\alpha_J(H'_{j_2}(x',E')) = \begin{cases} 0 & \text{if } H'_{j_2}(x',E') < j'_{j_2}, \\ \frac{(H_{j_2}(x',E'))^t - (j'^r_{j_2})^t}{(j'_{j_2})^t - (j'^L_{j_2})^t} & \text{if } j'^{L'}_{j_2} \leq H'_{j_2}(x',E') \leq j'_{j_2}, \\ 1 & \text{if } H'_{j_2}(x',E') > j'_{j_2}. \end{cases} \tag{24}$$

For $(J_3 = m_2 + 1, \ldots, m)$,

$$\alpha_J(H'_{j_3}(x',I')) = \begin{cases} 0 & \text{if } H'_{j_3}(x',I') < I'_{j_3}, \\ \frac{(H'_{j_3}(x',I'))^t - (l'^r_{j_3})^t}{(l'_{j_3})^t - (l'^l_{j_3})^t} & \text{if } l'^l_{j_3} \leq H'_{j_3}(x',I') \leq l'_{j_3}, \\ \frac{(l'^r_{j_3})^t - (H'_{j_3}(x',I'))^t}{(l'^r_{j_3})^t - (l'_{j_3})^t} & \text{if } l'_{j_3} \leq H'_{j_3}(x',I') \leq l'^r_{j_3}, \\ 0 & \text{if } H'_{j_3}(x',I') > l'_{j_3}. \end{cases} \tag{25}$$

The LDFNLFPP can be summarized as the question of how to formulate a sound strategy that will satisfy the DM to the greatest extent given a set of fuzzy objectives and fuzzy constraints. Between fuzzy objectives and fuzzy constraints, there should be the highest possible degree of balance.

Let

$$\lambda = \min \begin{cases} (\aleph_U(\theta(x), \aleph_D(H_{j_1}(x,C)),\ j_1 = 1, 2, \ldots, m_1, \\ \aleph_J(H_{j_2}(x,E)),\ j_2 = m_1 + 1, 2, \ldots, m_2, \\ \aleph_L(H_{j_3}(x,I)),\ j_3 = m_2 + 1, 2, \ldots, m, \end{cases}$$

and

$$\lambda' = \min \begin{cases} (\alpha_U(\theta(x), \alpha_{D'}(H'_{j_1}(x',C')),\ j_1 = 1, 2, \ldots, m_1, \\ \alpha_{J'}(H'_{j_2}(x',E')),\ j_2 = m_1 + 1, 2, \ldots, m_2, \\ \alpha_{L'}(H'_{j_3}(x',I')), j_3 = m_2 + 1, 2, \ldots, m, \end{cases}$$

where λ and λ' are the overall satisfaction levels for the DM.

Step 8: Ask the DM to select t; then, transform models (16) and (17) into the crisp model, which can easily be solved via suitable crisp NLPP methods as follows:

$$\begin{array}{ll} \text{Max} & \lambda \\ \text{subject to} & \begin{array}{l} \aleph_U(\theta(x) \geq \lambda \\ \aleph_D(H_{j_1}(x, C) \geq \lambda,\ j_1 = 1, \ldots, m_1, \\ \aleph_J(H_{j_2}(x, E) \geq \lambda,\ j_2 = m_1 + 1, \ldots, m_2, \\ \aleph_L(H_{j_3}(x, I) \geq \lambda,\ j_3 = m_2 + 1, \ldots, m, \end{array} \\ & x \geq 0. \end{array} \tag{26}$$

or

$$\begin{array}{ll} \text{Max} & \lambda \\ \text{subject to} & \begin{array}{l} (\theta(x))^t - L^t \geq \lambda\big(U^t - L^t\big) \\ (d^r_{j_1})^t - \ (H_{j_1}(x, C))^t \geq \lambda((d^r_{j_1})^t - (d_{j_1})^t),\ j_1 = 1, \ldots, m_1, \\ (H_{j_2}(x, E))^t - (j^l_{j_2})^t \geq \lambda((j_{j_2})^t - (j^L_{j_2})^t),\ j_2 = m_1 + 1, \ldots, m_2, \\ (H_{j_3}(x, I))^t - (l^l_{j_3})^t \geq \lambda((l_{j_3})^t - (l^l_{j_3})^t),\ j_3 = m_2 + 1, \ldots, m, \\ (l^r_{j_3})^t - (H_{j_3}(x, I))^t \geq \lambda((l^r_{j_3})^t - (l_{j_3})^t), j_3 = m_2 + 1, \ldots, m, \end{array} \\ & x \geq 0. \end{array} \tag{27}$$

Similarly,

$$\begin{array}{ll} \text{Max} & \lambda' \\ \text{subject to} & \begin{array}{l} \alpha_{U'}(\theta'(x') \geq \lambda' \\ \alpha_{D'}(H'_{j_1}(x', C') \geq \lambda',\ j_1 = 1, \ldots, m_1, \\ \alpha_{J'}(H'_{j_2}(x', E') \geq \lambda',\ j_2 = m_1 + 1, \ldots, m_2, \\ \alpha_{L'}(H'_{j_3}(x', I') \geq \lambda',\ j_3 = m_2 + 1, \ldots, m, \end{array} \\ & x' \geq 0. \end{array} \tag{28}$$

or

$$\begin{array}{ll} \text{Max} & \lambda' \\ \text{subject to} & \begin{array}{l} (\theta'(x'))^t - L'^t \geq \lambda'\big(U'^t - L'^t\big) \\ (d'^r_{j_1})^t - \ (H'_{j_1}(x', C'))^t \geq \lambda'((d'^r_{j_1})^t - (d'_{j_1})^t),\ j_1 = 1, \ldots, m_1, \\ (H'_{j_2}(x', E'))^t - (j'^l_{j_2})^t \geq \lambda((j'_{j_2})^t - (j'^L_{j_2})^t),\ j_2 = m_1 + 1, \ldots, m_2, \\ (H'_{j_3}(x', I'))^t - (l'^l_{j_3})^t \geq \lambda((l'_{j_3})^t - (l'^l_{j_3})^t),\ j_3 = m_2 + 1, \ldots, m, \\ (l'^r_{j_3})^t - (H_{j_3}(x', I'))^t \geq \lambda((l'^r_{j_3})^t - (l'_{j_3})^t), j_3 = m_2 + 1, \ldots, m, \end{array} \\ & x' \geq 0. \end{array} \tag{29}$$

5. Numerical Example

Let us consider the following LDFNLFPP :

$$\begin{array}{ll} \text{Max}\ \theta(\widetilde{x}^L) & = \frac{\widetilde{7}x_1^2 + \widetilde{6}x_2^2 + \widetilde{8}}{\widetilde{7}x_1^2 + \widetilde{8}x_2^2 + \widetilde{5}} \\ \text{subject to} & \begin{array}{l} \widetilde{9}x_1 + \widetilde{3}x_2 \leq \widetilde{28} \\ \widetilde{4}x_1 + \widetilde{3}x_2 \leq \widetilde{19} \end{array} \\ & x_1, x_2 \geq 0. \end{array} \tag{30}$$

where

$$\begin{aligned}
\widetilde{7} &= \begin{cases} (5,6,7,8,9) \\ (2,4,7,9,10) \end{cases}, \widetilde{6} = \begin{cases} (4,6,8,9,10) \\ (3,4,8,11,12) \end{cases}, \widetilde{8} = \begin{cases} (8,9,10,11,12) \\ (5,8,10,13,15) \end{cases}, \\
\widetilde{5} &= \begin{cases} (2,3,4,5,6) \\ (1,2,4,6,8) \end{cases}, \widetilde{9} = \begin{cases} (1,2,3,4,5) \\ (0,2,3,7,8) \end{cases}, \widetilde{3} = \begin{cases} (2,4,6,8,9) \\ (1,3,6,9,10) \end{cases}, \\
\widetilde{4} &= \begin{cases} (6,7,8,9,10) \\ (5,6,8,11,13) \end{cases}, \widetilde{26} = \begin{cases} (18,19,22,26,28) \\ (17,18,22,27,30) \end{cases}, \widetilde{19} = \begin{cases} (19,20,23,27,29) \\ (17,18,23,30,32) \end{cases}.
\end{aligned}$$

Taking

$$\begin{aligned}
\text{Max } \theta(x) &= \frac{\widetilde{7}x_1^2+\widetilde{6}x_2^2+\widetilde{8}}{\widetilde{7}x_1^2+\widetilde{8}x_2^2+\widetilde{5}} \\
\text{subject to } & \widetilde{9}x_1+\widetilde{3}x_2 \leq \widetilde{28} \\
& \widetilde{4}x_1+\widetilde{3}x_2 \leq \widetilde{19} \\
& x_1, x_2 \geq 0.
\end{aligned}$$

Suppose that the DM determines $\alpha = 0.5 \in [0,1]$. The membership function (6) is used to convert an LDFN of the above problem (30) into its linear Diophantine non-fuzzy numbers (α-LDFNs) referring to problem (10).

Let the LDFNs and α-LDFNs be given by the values listed in the Table 1 below.

Table 1. LDFNs and their corresponding α-LDFNs.

LDFNs	α-LDFNs
$\widetilde{7} = (5,6,7,8,9)$	$7 = (6,6.5,7,7.5,8)$
$\widetilde{6} = (4,6,8,9,10)$	$6 = (6,7,8,8.5,9)$
$\widetilde{8} = (8,9,10,11,12)$	$8 = (9,9.5,10,10.5,11)$
$\widetilde{5} = (2,3,4,5,6)$	$5 = (3,3.5,4,4.5,5)$
$\widetilde{9} = (1,2,3,4,5)$	$9 = (2,2.5,3,3.5,4)$
$\widetilde{3} = (2,4,6,8,9)$	$3 = (4,5,6,7,7.5)$
$\widetilde{4} = (6,7,8,9,10)$	$4 = (7,7.5,8,8.5,9)$
$\widetilde{26} = (18,19,22,26,28)$	$26 = (20,20.5,22,24,25)$
$\widetilde{19} = (19,20,23,27,29)$	$19 = (21,21.5,23,25,26)$

Problem (30) is equivalent to the following LDMONLFPP:

$$\begin{aligned}
\text{Max } \theta_1(x) &= \frac{6x_1^2+6x_2^2+9}{8x_1^2+11x_2^2+5}, \\
\text{Max } \theta_2(x) &= \frac{6.5x_1^2+7x_2^2+9.5}{7.5x_1^2+10.5x_2^2+4.5}, \\
\text{Max } \theta_3(x) &= \frac{7x_1^2+8x_2^2+10}{7x_1^2+10x_2^2+4}, \\
\text{Max } \theta_4(x) &= \frac{7.5x_1^2+8.5x_2^2+10.5}{6.5x_1^2+9.5x_2^2+3.5}, \\
\text{Max } \theta_5(x) &= \frac{8x_1^2+9x_2^2+11}{6x_1^2+9x_2^2+3}, \\
\text{subject to } & 2x_1+4x_2 \leq 20, \\
& 2.5x_1+5x_2 \leq 20.5, \\
& 3x_1+6x_2 \leq 22, \\
& 3.5x_1+7x_2 \leq 24, \\
& 4x_1+7.5x_2 \leq 25, \\
& 7x_1+4x_2 \leq 21, \\
& 7.5x_1+5x_2 \leq 21.5, \\
& 8x_1+6x_2 \leq 23, \\
& 8.5x_1+7x_2 \leq 25, \\
& 9x_1+7.5x_2 \leq 26, \\
& x_1, x_2 \geq 0.
\end{aligned} \tag{31}$$

Using the transformation of Charnes and Cooper, problem (31) is equivalent to the following LDMONLPP:

$$\begin{array}{ll}
\text{Max } \theta_1(y/t) & = 6y_1^2 + 6y_2^2 + 9t^2, \\
\text{Max } \theta_2(y/t) & = 6.5y_1^2 + 7y_2^2 + 9.5t^2, \\
\text{Max } \theta_3(y/t) & = 7y_1^2 + 8y_2^2 + 10t^2, \\
\text{Max } \theta_4(y/t) & = 7.5y_1^2 + 8.5y_2^2 + 10.5t^2, \\
\text{Max } \theta_5(y/t) & = 8y_1^2 + 9y_2^2 + 11t^2, \\
\text{subject to} & 8y_1^2 + 11y_2^2 + 5t^2 \leq 1, \\
& 7.5y_1^2 + 10.5y_2^2 + 4.5t^2 \leq 1, \\
& 7y_1^2 + 10y_2^2 + 4t^2 \leq 1, \\
& 6.5y_1^2 + 9.5y_2^2 + 3.5t^2 \leq 1, \\
& 6y_1^2 + 9y_2^2 + 3t^2 \leq 1, \\
& 2y_1 + 4y_2 - 20t \leq 0, \\
& 2.5y_1 + 5y_2 - 20.5t \leq 0, \\
& 3y_1 + 6y_2 - 22t \leq 0, \\
& 3.5y_1 + 7x_2 - 24t \leq 0, \\
& 4y_1 + 7.5y_2 - 25t \leq 0, \\
& 7y_1 + 4y_2 - 21t \leq 0 \\
& 7.5y_1 + 5y_2 - 21.5t \leq 0 \\
& 8y_1 + 6y_2 - 23t \leq 0 \\
& 8.5y_1 + 7y_2 - 25t \leq 0 \\
& 9y_1 + 7.5y_2 - 26t \leq 0 \\
& y_1, y_2 \geq 0,\ t > 0.
\end{array} \tag{32}$$

The above problem (32) can be transformed into the following five sub-problems of MONLPPs:

$\boldsymbol{P}_1$:

$$\begin{array}{ll}
\text{Max } \theta_1(y/t) & = 6y_1^2 + 6y_2^2 + 9t^2, \\
\text{Max } \theta_2(y/t) & = 6.5y_1^2 + 7y_2^2 + 9.5t^2, \\
\text{Max } \theta_3(y/t) & = 7y_1^2 + 8y_2^2 + 10t^2, \\
\text{Max } \theta_4(y/t) & = 7.5y_1^2 + 8.5y_2^2 + 10.5t^2, \\
\text{Max } \theta_5(y/t) & = 8y_1^2 + 9y_2^2 + 11t^2, \\
\text{subject to} & 8y_1^2 + 11y_2^2 + 5t^2 \leq 1, \\
& 2y_1 + 4y_2 - 20t \leq 0, \\
& 7y_1 + 4y_2 - 21t \leq 0 \\
& y_1, y_2 \geq 0,\ t > 0.
\end{array}$$

$\boldsymbol{P}_2$:

$$\begin{array}{ll}
\text{Max } \theta_1(y/t) & = 6y_1^2 + 6y_2^2 + 9t^2, \\
\text{Max } \theta_2(y/t) & = 6.5y_1^2 + 7y_2^2 + 9.5t^2, \\
\text{Max } \theta_3(y/t) & = 7y_1^2 + 8y_2^2 + 10t^2, \\
\text{Max } \theta_4(y/t) & = 7.5y_1^2 + 8.5y_2^2 + 10.5t^2, \\
\text{Max } \theta_5(y/t) & = 8y_1^2 + 9y_2^2 + 11t^2, \\
\text{subject to} & 7.5y_1^2 + 10.5y_2^2 + 4.5t^2 \leq 1, \\
& 2.5y_1 + 5y_2 - 20.5t \leq 0, \\
& 7.5y_1 + 5y_2 - 21.5t \leq 0, \\
& y_1, y_2 \geq 0,\ t > 0.
\end{array}$$

$\boldsymbol{P}_3$:

$$\begin{array}{ll} \text{Max } \theta_1(y/t) & = 6y_1^2 + 6y_2^2 + 9t^2, \\ \text{Max } \theta_2(y/t) & = 6.5y_1^2 + 7y_2^2 + 9.5t^2, \\ \text{Max } \theta_3(y/t) & = 7y_1^2 + 8y_2^2 + 10t^2, \\ \text{Max } \theta_4(y/t) & = 7.5y_1^2 + 8.5y_2^2 + 10.5t^2, \\ \text{Max } \theta_5(y/t) & = 8y_1^2 + 9y_2^2 + 11t^2, \\ \text{subject to} & 7y_1^2 + 10y_2^2 + 4t^2 \leq 1, \\ & 3y_1 + 6y_2 - 22t \leq 0, \\ & 8y_1 + 6y_2 - 23t \leq 0 \\ & y_1, y_2 \geq 0,\ t > 0. \end{array}$$

$\boldsymbol{P}_4$:

$$\begin{array}{ll} \text{Max } \theta_1(y/t) & = 6y_1^2 + 6y_2^2 + 9t^2, \\ \text{Max } \theta_2(y/t) & = 6.5y_1^2 + 7y_2^2 + 9.5t^2, \\ \text{Max } \theta_3(y/t) & = 7y_1^2 + 8y_2^2 + 10t^2, \\ \text{Max } \theta_4(y/t) & = 7.5y_1^2 + 8.5y_2^2 + 10.5t^2, \\ \text{Max } \theta_5(y/t) & = 8y_1^2 + 9y_2^2 + 11t^2, \\ \text{subject to} & 6.5y_1^2 + 9.5y_2^2 + 3.5t^2 \leq 1, \\ & 3.5y_1 + 7x_2 - 24t \leq 0, \\ & 8.5y_1 + 7y_2 - 25t \leq 0 \\ & y_1, y_2 \geq 0,\ t > 0. \end{array}$$

$\boldsymbol{P}_5$:

$$\begin{array}{ll} \text{Max } \theta_1(y/t) & = 6y_1^2 + 6y_2^2 + 9t^2, \\ \text{Max } \theta_2(y/t) & = 6.5y_1^2 + 7y_2^2 + 9.5t^2, \\ \text{Max } \theta_3(y/t) & = 7y_1^2 + 8y_2^2 + 10t^2, \\ \text{Max } \theta_4(y/t) & = 7.5y_1^2 + 8.5y_2^2 + 10.5t^2, \\ \text{Max } \theta_5(y/t) & = 8y_1^2 + 9y_2^2 + 11t^2, \\ \text{subject to} & 6y_1^2 + 9y_2^2 + 3t^2 \leq 1, \\ & 4y_1 + 7.5y_2 - 25t \leq 0 \\ & 9y_1 + 7.5y_2 - 26t \leq 0 \\ & y_1, y_2 \geq 0,\ t > 0. \end{array}$$

Solve models $P_i, i = 1,2,3,4,5$ as single-objective NLPPs. The lower and upper bounds L and U, respectively, for the objective functions are $L = 1.800000$ and $U = 3.666667$.

The LDMONLFPP (31) is equivalent to the following fuzzy model:

$$\begin{array}{ll} \text{Find} & x \\ \text{subject to} & \begin{array}{l} \frac{8x_1^2+9x_2^2+11}{6x_1^2+9x_2^2+3} \geq 3.666667 \\ 9x_1 + 3x_2 \leq 26 \\ 4x_1 + 3x_2 \leq 19 \end{array} \\ & x_1, x_2 \geq 0. \end{array} \tag{33}$$

Further, using the membership functions in (18)–(21), model (33) is equivalent to the following crisp model:

$$\begin{array}{ll} \text{Max} & \lambda \\ \text{subject to} & \begin{array}{l} (\frac{8x_1^2+9x_2^2+11}{6x_1^2+9x_2^2+3})^t - (1.800000)^t \geq (3.666667)^t - (1.800000)^t \\ 9x_1 + 3x_2 \leq 26 \\ 4x_1 + 3x_2 \leq 19 \end{array} \\ & x_1, x_2 \geq 0. \end{array}$$

Using LINGO, taking $t = 2$, the solution is $(x_1, x_2) = (1.234568, 1.234568)$, $\theta(x) = (0.803613, 0.941796, 1.098683, 1.251017, 1.427196)$ with satisfaction level $\lambda = 1$.

Now, suppose that the DM determines $\alpha = 0.5 \in [0, 1]$. The membership function (7) is used to convert an LDFN of the above problem (30) into its linear Diophantine non-fuzzy numbers (α-LDFNs) referring to problem (11).

Now,

$$\begin{array}{ll} \text{Max } \theta'(x) & = \frac{\widetilde{7}x_1'^2 + \widetilde{6}x_2'^2 + \widetilde{8}}{\widetilde{7}x_1'^2 + \widetilde{8}x_2'^2 + \widetilde{5}} \\ \text{subject to} & \widetilde{9}x_1' + \widetilde{3}x_2' \le \widetilde{28} \\ & \widetilde{4}x_1' + \widetilde{3}x_2' \le \widetilde{19} \\ & x_1', x_2' \ge 0. \end{array}$$

Let the LDFNs and α-LDFNs be given by the values listed in the Table 2 below.

Table 2. LDFNs and their corresponding α-LDFNs.

LDFNs	**α-LDFNs**
$\widetilde{7} = (2, 4, 7, 9, 10)$	$7 = (4.5, 5.5, 7, 8, 8.5)$
$\widetilde{6} = (3, 4, 8, 11, 12)$	$6 = (5.5, 6, 8, 9.5, 19)$
$\widetilde{8} = (5, 8, 10, 13, 15)$	$8 = (7.5, 9, 10, 11.5, 12.5)$
$\widetilde{5} = (1, 2, 4, 6, 8)$	$5 = (2.5, 3, 4, 5, 6)$
$\widetilde{9} = (0, 2, 3, 7, 8)$	$9 = (1.5, 2.5, 3, 5, 5.5)$
$\widetilde{3} = (1, 3, 6, 9, 10)$	$3 = (3.5, 4.5, 6, 7.5, 8)$
$\widetilde{4} = (5, 6, 8, 11, 13)$	$4 = (6.5, 7, 8, 9.5, 10.5)$
$\widetilde{26} = (17, 18, 22, 27, 30)$	$26 = (19.5, 20, 22, 24.5, 26)$
$\widetilde{19} = (17, 18, 23, 30, 32)$	$19 = (20, 20.5, 23, 26.5, 27.5)$

Problem (30) is equivalent to the following LDMONLFPP:

$$\begin{array}{ll} \text{Max } \theta_1'(x) & = \frac{4.5x_1'^2 + 5.5x_2'^2 + 7.5}{8.5x_1'^2 + 12.5x_2'^2 + 6}, \\ \text{Max } \theta_2'(x) & = \frac{5.5x_1'^2 + 6x_2'^2 + 9}{8x_1'^2 + 11.5x_2'^2 + 5} \\ \text{Max } \theta_3(x) & = \frac{7x_1^2 + 8x_2^2 + 10}{7x_1^2 + 10x_2^2 + 4}, \\ \text{Max } \theta_4'(x) & = \frac{8x_1'^2 + 9.5x_2'^2 + 11.5}{5.5x_1'^2 + 9x_2'^2 + 3}, \\ \text{Max } \theta_5'(x) & = \frac{8.5x_1'^2 + 10x_2'^2 + 12.5}{4.5x_1'^2 + 7.5x_2'^2 + 2.5}, \\ \text{subject to} & 1.5x_1' + 3.5x_2' \le 19.5, \\ & 2.5x_1' + 4.5x_2' \le 20, \\ & 3x_1 + 6x_2 \le 22, \\ & 5x_1' + 7.5x_2' \le 24.5, \\ & 5.5x_1' + 8x_2' \le 26, \\ & 6.5x_1' + 3.5x_2' \le 20, \\ & 7x_1' + 4.5x_2' \le 20.5, \\ & 8x_1 + 6x_2 \le 23, \\ & 9.5x_1' + 7.5x_2' \le 26.5, \\ & 10.5x_1' + 8x_2' \le 27.5, \\ & x_1', x_2' \ge 0. \end{array} \tag{34}$$

Using the transformation of Charnes and Cooper, problem (34) is equivalent to the following LDMONLPP:

$$\begin{array}{ll}
\text{Max } \theta_1'(y'/t') & = 4.5y_1'^2 + 5.5y_2'^2 + 7.5t'^2 \\
\text{Max } \theta_2'(y'/t') & = 5.5y_1'^2 + 6y_2'^2 + 9t'^2, \\
\text{Max } \theta_3(y/t) & = 7y_1^2 + 8y_2^2 + 10t^2, \\
\text{Max } \theta_4'(y'/t') & = 8y_1'^2 + 9.5y_2'^2 + 11.5t'^2, \\
\text{Max } \theta_5'(y'/t') & = 8.5y_1'^2 + 10y_2'^2 + 12.5t'^2, \\
\text{subject to} & 8.5y_1'^2 + 12.5y_2'^2 + 6t'^2 \leq 1, \\
& 8y_1'^2 + 11.5y_2'^2 + 5t'^2 \leq 1, \\
& 7y_1^2 + 10y_2^2 + 4t^2 \leq 1, \\
& 5.5y_1'^2 + 9y_2'^2 + 3t'^2 \leq 1 \\
& 4.5y_1'^2 + 7.5y_2'^2 + 2.5t'^2 \leq 1 \\
& 1.5y_1' + 3.5y_2' - 19.5t' \leq 0, \\
& 2.5y_1' + 4.5'y_2 - 20t' \leq 0, \\
& 3y_1 + 6y_2 - 22t \leq 0 \\
& 5y_1' + 7.5y_2' - 24.5t' \leq 0 \\
& 5.5y_1' + 8y_2' - 26t' \leq 0 \\
& 6.5y_1' + 3.5y_2' - 20t' \leq 0 \\
& 7y_1' + 4.5y_2' - 20.5t' \leq 0 \\
& 8y_1 + 6y_2 - 23t \leq 0 \\
& 9.5y_1' + 7.5y_2' - 25t' \leq 0 \\
& 10.5y_1' + 8y_2' - 27.5t' \leq 0 \\
& y_1', y_2' \geq 0,\ t' > 0.
\end{array} \tag{35}$$

The above problem (35) can be transformed into the following four sub-problems of MONLPPs:

$\boldsymbol{P_1'}$:

$$\begin{array}{ll}
\text{Max } \theta_1'(y'/t') & = 4.5y_1'^2 + 5.5y_2'^2 + 7.5t'^2 \\
\text{Max } \theta_2'(y'/t') & = 5.5y_1'^2 + 6y_2'^2 + 9t'^2, \\
\text{Max } \theta_3(y/t) & = 7y_1^2 + 8y_2^2 + 10t^2, \\
\text{Max } \theta_4'(y'/t') & = 8y_1'^2 + 9.5y_2'^2 + 11.5t'^2, \\
\text{Max } \theta_5'(y'/t') & = 8.5y_1'^2 + 10y_2'^2 + 12.5t'^2, \\
\text{subject to} & 8.5y_1'^2 + 12.5y_2'^2 + 6t'^2 \leq 1, \\
& 1.5y_1' + 3.5y_2' - 19.5t' \leq 0, \\
& 6.5y_1' + 3.5y_2' - 20t' \leq 0 \\
& y_1', y_2' \geq 0,\ t' > 0.
\end{array}$$

$\boldsymbol{P_2'}$:

$$\begin{array}{ll}
\text{Max } \theta_1'(y'/t') & = 4.5y_1'^2 + 5.5y_2'^2 + 7.5t'^2 \\
\text{Max } \theta_2'(y'/t') & = 5.5y_1'^2 + 6y_2'^2 + 9t'^2, \\
\text{Max } \theta_3(y/t) & = 7y_1^2 + 8y_2^2 + 10t^2, \\
\text{Max } \theta_4'(y'/t') & = 8y_1'^2 + 9.5y_2'^2 + 11.5t'^2, \\
\text{Max } \theta_5'(y'/t') & = 8.5y_1'^2 + 10y_2'^2 + 12.5t'^2, \\
\text{subject to} & 8y_1'^2 + 11.5y_2'^2 + 5t'^2 \leq 1, \\
& 2.5y_1' + 4.5'y_2 - 20t' \leq 0, \\
& 7y_1' + 4.5y_2' - 20.5t' \leq 0 \\
& y_1', y_2' \geq 0,\ t' > 0.
\end{array}$$

$\boldsymbol{P}_4'$:

$$\begin{array}{ll} \text{Max } \theta_1'(y'/t') & = 4.5y_1'^2 + 5.5y_2'^2 + 7.5t'^2 \\ \text{Max } \theta_2'(y'/t') & = 5.5y_1'^2 + 6y_2'^2 + 9t'^2, \\ \text{Max } \theta_3(y/t) & = 7y_1^2 + 8y_2^2 + 10t^2, \\ \text{Max } \theta_4'(y'/t') & = 8y_1'^2 + 9.5y_2'^2 + 11.5t'^2, \\ \text{Max } \theta_5'(y'/t') & = 8.5y_1'^2 + 10y_2'^2 + 12.5t'^2, \\ \text{subject to} & 5.5y_1'^2 + 9y_2'^2 + 3t'^2 \le 1, \\ & 5y_1' + 7.5y_2' - 24.5t' \le 0 \\ & 9.5y_1' + 7.5y_2' - 25t' \le 0, \\ & y_1', y_2' \ge 0,\ t' > 0. \end{array}$$

$\boldsymbol{P}_5'$:

$$\begin{array}{ll} \text{Max } \theta_1'(y'/t') & = 4.5y_1'^2 + 5.5y_2'^2 + 7.5t'^2 \\ \text{Max } \theta_2'(y'/t') & = 5.5y_1'^2 + 6y_2'^2 + 9t'^2, \\ \text{Max } \theta_3(y/t) & = 7y_1^2 + 8y_2^2 + 10t^2, \\ \text{Max } \theta_4'(y'/t') & = 8y_1'^2 + 9.5y_2'^2 + 11.5t'^2, \\ \text{Max } \theta_5'(y'/t') & = 8.5y_1'^2 + 10y_2'^2 + 12.5t'^2, \\ \text{subject to} & 4.5y_1'^2 + 7.5y_2'^2 + 2.5t^2 \le 1, \\ & 5.5y_1' + 8y_2' - 26t' \le 0, \\ & 10.5y_1' + 8y_2' - 27.5t' \le 0, \\ & y_1', y_2' \ge 0,\ t' > 0. \end{array}$$

Solve models $P_i', i' = 1,2,4,5$ as single-objective NLPPs. The lower and upper bounds L and U, respectively, for the objective functions are $L = 1.250000$ and $U = 5.000000$.

The LDMONLFPP (34) is equivalent to the following fuzzy model:

$$\begin{array}{ll} \text{Find} & x' \\ \text{subject to} & \frac{8.5x_1'^2+10x_2'^2+12.5}{4.5x_1'^2+7.5x_2'^2+2.5} \ge 5.000000 \\ & 9x_1' + 3x_2' \le 26 \\ & 4x_1' + 3x_2' \le 19 \\ & x_1', x_2' \ge 0. \end{array} \tag{36}$$

Further, using the membership functions in (21)–(24), model (36) is equivalent to the following crisp model:

$$\begin{array}{ll} \text{Max} & \lambda' \\ \text{subject to} & (\frac{8.5x_1'^2+10x_2'^2+12.5}{4.5x_1'^2+7.5x_2'^2+2.5})^t - (1.250000)^t \ge (5.000000)^t - (1.250000)^t \\ & 9x_1' + 3x_2' \le 26 \\ & 4x_1' + 3x_2' \le 19 \\ & x_1', x_2' \ge 0. \end{array}$$

Using LINGO, taking $t = 2$, the solution is $(x_1, x_2) = (1.234568, 1.234568)$, $\theta'(x) = (0.598347, 0.764026, 1.098683, 1.5208096, 1.957533)$ with satisfaction level $\lambda = 1$.

Hence, the optimal solution of the above TFLDFLP problem is

$$\left\{ \begin{array}{l} (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \\ (\theta_1', \theta_2', \theta_3, \theta_4', \theta_5') \end{array} \right. = \left\{ \begin{array}{l} (0.803613, 0.941796, 1.098683, 1.251017, 1.427196) \\ (0.598347, 0.764026, 1.098683, 1.5208096, 1.957533) \end{array} \right. .$$

The flow chart is given in Figure 1.

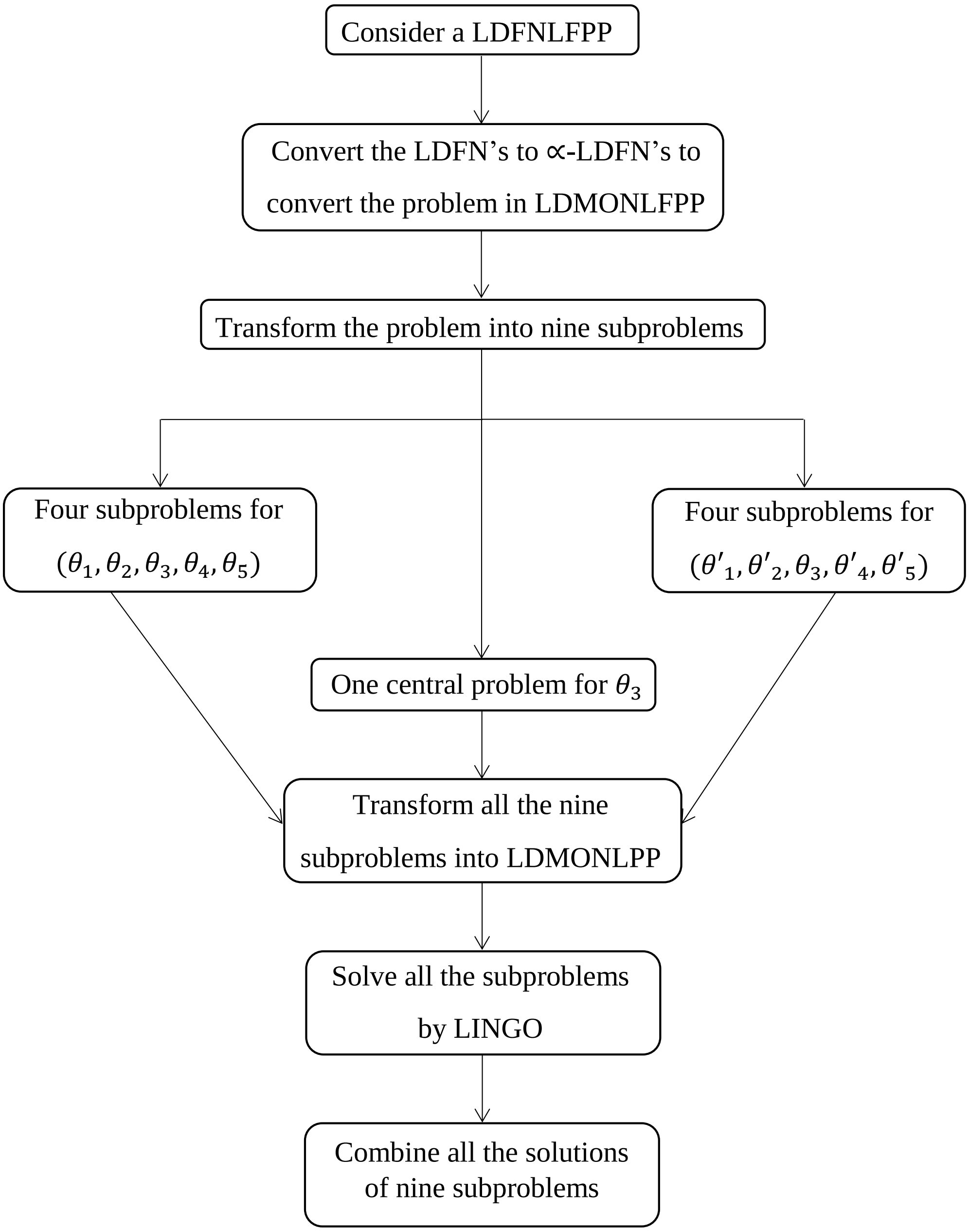

Figure 1. Problem-solving flow chart.

6. Conclusions

This study suggests an interactive technique to solve the LDFNLFPP in which the coefficients of the objective function and the constraints are taken as TLDFNs based on α-cut analysis defined by the DM. In the suggested methodology, the problem is transformed from an LDFNLFPP to an IMONLFPP using a fuzzy mathematical programming approach, and then the solution is transformed into an NLPP. For problems with uncertain and hesitant decision-making in manufacturing, planning, and scheduling systems, the suggested methodology will be highly beneficial. The approach can be modified in the future to address bi-level multi-objective nonlinear fractional programming problems using the goal linear Diophantine fuzzy method.

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