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Expansion Theory of Deng's Metric in $[0,1]$-Topology

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Abstract: The aim of this paper is to focus on a fuzzy metric called Deng's metric in $[0,1]$-topology. Firstly, we will extend the domain of this metric function from $M_0 \times M_0$ to $M \times M$, where M_0 and M are defined as the sets of all special fuzzy points and all standard fuzzy points, respectively. Secondly, we will further extend this metric to the completely distributive lattice L^X and, based on this extension result, we will compare this metric with the other two fuzzy metrics: Erceg's metric and Yang-Shi's metric, and then reveal some of its interesting properties, particularly including its quotient space. Thirdly, we will investigate the relationship between Deng's metric and Yang-Shi's metric and prove that a Deng's metric must be a Yang-Shi's metric on I^X, and consequently an Erceg's metric. Finally, we will show that a Deng's metric on I^X must be $Q - C_1$, and Deng's metric topology and its uniform structure are Erceg's metric topology and Hutton's uniform structure, respectively.

Keywords: Deng's pseudo-metric; expansion; M_0; M; metric topology; way below; $Q - C_1$

MSC: 54A40; 03E72; 54E35

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1. Introduction

In 1968, C.L. Chang [1] introduced the fuzzy set theory of Zadeh [2] into general topology [3] for the first time, which declared the birth of $[0,1]$-topology. Soon after that, J.A. Goguen [4] further generalized the L-fuzzy set to the proposed $[0,1]$-topology and his theory has been recognized as L-topology nowadays. From then on, this kind of lattice-valued topology formed another important branch of topology and thereafter many creative results and original thoughts have been presented (see [5–38], etc.).

Nevertheless, how to reasonably generalize the classical metric to the lattice-valued topology has always been a great challenge. So far, there are a significant number of fuzzy metrics introduced in the branch of learning (see [6,12,14,15,29–33,39–42], etc.). Considering that the codomain is either ordinary number or fuzzy number, these metrics are roughly divided into two types.

One type is composed of these metrics, each of which is defined by such a function whose distance between objects is fuzzy, while the objects themselves are crisp. Additionally, each of them always induces a fuzzifying topology. In recent years, these metrics have been promoted by many experts, such as I. Kramosil, J. Michalek, A. George, P. Veeramani, V. Gregori, S. Romaguera, J. Gutiérrez García, S. Morillas, F.G. Shi, etc. (see [17,18,32,33,40,43–49], etc.).

The other type consists of these metrics, each of which is defined by such a mapping $p : M \times M \to [0, +\infty)$, where M is the set of all standard fuzzy points of the underlying classical set X. In this case, every such fuzzy metric always induces a fuzzy topology (see [6,12–14,31,36], etc.).

Regarding the latter, there are roughly three kinds of fuzzy metrics in the history, with which the academic community has gradually become familiar. Regarding the three fuzzy metrics, we will list them below one by one.

The first is Erceg's metric, presented by M.A. Erceg [14] in 1979. Since then, many scholars have been engaged in its research and have obtained many compelling results on this fuzzy metric. Among them, a typical conclusion is the Urysohn's metrization theorem presented by J.H. Liang [24] in 1984: an L-topological space is Erceg-metrizable if it is T_1, regular and C_{II}. In 1985, M.K. Luo [26] listed an example of Erceg's metric on I^X whose metric topology has no σ-locally finite base. Therefore, the $[0,1]$-topological space of this example is not C_{II}, of course. Later on, based on Peng's simplification method [50], Erceg's metric was further simplified by P. Chen and F.G. Shi (see [9,10]) as seen below:

(I) An *Erceg's pseudo-metric* on L^X is a mapping $p : M \times M \to [0, +\infty)$ satisfying the following properties:

(A1) if $a \geq b$, then $p(a, b) = 0$;

(A2) $p(a, c) \leq p(a, b) + p(b, c)$;

(B1) $p(a, b) = \bigvee_{c \ll b} p(a, c)$;

(A3) $\forall a, b \in M$, $\exists x \not\leq a'$ s.t. $p(b, x) < r \Leftrightarrow \exists y \not\leq b'$ s.t. $p(a, y) < r$.

An Erceg's pseudo-metric p is called an *Erceg's metric* if it further satisfies the following property:

(A4) if $p(a, b) = 0$, then $a \geq b$.

where " $\ll$" is the way below relation in domain theory and L^X is a completely distributive lattice [51–53].

The second is Yang-Shi's metric (or p.q. metric), proposed by L.C. Yang [36] in 1988, where Yang also showed such a result: each topological molecular lattice with C_{II} property is p.q.-metrizable. After that, this kind of metric was studied in depth by F.G. Shi and P. Chen (see [9,10,29–31], etc.), whose definition is as follows:

(II) A *Yang-Shi's pseudo-metric* (resp., *Yang-Shi's metric*) on L^X is a mapping $p : M \times M \to [0, +\infty)$ satisfying (A1)–(A3) (resp., (A1)–(A4)) and the following property:

(B2) $p(a, b) = \bigwedge_{c \ll a} p(c, b)$.

The third is Deng's metric, supplied by Z.K. Deng [12] in 1982. Soon, Deng [13] proved that if a $[0,1]$-topological space is T_1, regular and C_{II}, then it is Deng-metrizable. Unfortunately, since Deng's research is only limited to this special lattice I^X and the family of special fuzzy points M_0 (see Definition 1), not many scholars later studied this metric. In this paper, we will extend the domain of Deng's pseudo-metric from I^X to L^X and its definition from M_0 to a class of standard fuzzy points M (see Definition 8 in this paper) as seen below:

(III) An *extended Deng's pseudo-metric* (resp., *extended Deng's metric*) on L^X is a mapping $p : M \times M \to [0, +\infty)$ satisfying (A1)–(A3) (resp., (A1)–(A4)) and the following condition:

(B3) $p(a, b) = \bigwedge_{b \ll c} p(a, c)$.

Therefore, based on this extension result, we will compare this metric with the other two fuzzy metrics, Erceg's metric and Yang-Shi's metric, and then reveal some of its interesting properties, particularly including its quotient space. Additionally, we will investigate the relationship between Deng's metric and Yang-Shi's metric and prove that a Deng's metric must be a Yang-Shi's metric on I^X, and consequently a Deng's metric also must be an Erceg's metric. Finally, we also will show that a Deng's metric on I^X must be $Q - C_1$, and Deng's metric topology and its uniform structure are Erceg's metric topology [14] and Hutton's uniform structure [22], respectively.

2. Preliminaries

All through this paper, $(L, \bigvee, \bigwedge, ')$ is a completely distributive lattice with an order-reversing involution " ′ " [51,52]. X is a nonempty set. L-fuzzy set in X is a mapping $A : X \to L$, and L^X is the set of all L-fuzzy sets. If $L = [0,1]$ and denote $[0,1]$ as I, then each element in I^X is claimed a fuzzy set in X [2]. A subfamily δ of I^X is called a $[0,1]$-*topology* if it satisfies the following three conditions: (O1) $\underline{1}, \underline{0} \in \delta$; (O2) if $A, B \in \delta$, then $A \wedge B \in \delta$; (O3) if $\{A_\lambda \mid \lambda \in \Lambda\} \subseteq \delta$, then $\bigvee_{\lambda\in\Lambda} A_\lambda \in \delta$. The pair (X, δ) is called a $[0,1]$-*topological space*. Two fuzzy sets A and B are quasi-coincidence if there is x such that $A(x) + B(x) > 1$ (see [53–55]). An open set A [12] is called an open neighborhood of x_λ if $\lambda < A(x)$. $X(x) \equiv 1$ and $X(x) \equiv 0$ are denoted by $\underline{1}$ and $\underline{0}$, respectively. And a is *way below* b, denoted by $a \ll b$, if and only if for every directed subset $D \subseteq L^X$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ ("$\leq$" refers to the following Definition 3). A family of fuzzy sets Ψ is called *locally finite* (resp., *discrete*) in a space (X, δ) if and only if each fuzzy point x_λ of the space has an open neighborhood which is quasi-coincidental with only finitely many members (resp., at most one member) of Ψ (see [52]). A family of fuzzy sets is called σ-*locally finite* (resp., σ-*discrete*) in a space (X, δ) if and only if it is the union of a countable number of locally finite (resp., discrete) subfamilies. A subfamily σ of I^X (resp., σ of δ) is called a (resp., an *open*) *cover* of a fuzzy set A in a space (X, δ) if for each $x_\alpha \in A$, there exists B belonging to σ such that $x_\alpha \in B$. Stipulate $\vee\emptyset = \underline{0}$, and $\wedge\emptyset = \underline{1}$.

In addition, the subsequent proofs also require some preliminary knowledge of definitions and theorems as follows:

Definition 1 ([12])**.** *A special fuzzy point x_λ in X is a fuzzy set with membership function $x_\lambda : X \to I$ defined by*

$$x_\lambda(y) = \begin{cases} \lambda, & y = x; \\ 0, & y \neq x, \end{cases}$$

where $\lambda \in (0,1)$. $x_\lambda(y)$ is usually written simply as x_λ. x, λ, and $x_{1-\lambda}$ are called support, value, and complementary point of x_λ, respectively, and the family of all special fuzzy points is denoted by M_0.

With the help of the above special fuzzy point, Deng [12] put forward a type of fuzzy metric as follows:

Definition 2 ([12])**.** *A Deng's pseudo-metric on I^X is a mapping $p : M_0 \times M_0 \to [0, +\infty)$ satisfying the following conditions:*

(A1) *if $\lambda_1 \geq \lambda_0$, then $p(x_{\lambda_1}, x_{\lambda_0}) = 0$;*

(A2) *$p(x_{\lambda_1}, z_{\lambda_3}) \leq p(x_{\lambda_1}, y_{\lambda_2}) + p(y_{\lambda_2}, z_{\lambda_3})$;*

(A3) *if $p(x_{\lambda_1}, y_{\lambda_2}) < r$, then $\exists \lambda' > \lambda_2$ such that $p(x_{\lambda_1}, y_{\lambda'}) < r$;*

(A4) *$p(x_{\lambda_1}, y_{\lambda_2}) = p(y_{1-\lambda_2}, x_{1-\lambda_1})$.*

A Deng's pseudo-metric p is called a Deng's metric if it further satisfies the following condition:

(A5) *if $p(x_{\lambda_1}, y_{\lambda_2}) = 0$, then $x = y, \lambda_1 \geq \lambda_2$.*

Definition 3 ([12])**.** *Let x_α, y_β belong to M and let A, B be fuzzy sets in X. Then,*

(1) $x_\alpha \leq A \Leftrightarrow \alpha \leq A(x)$;

(2) $x_\alpha \in A \Leftrightarrow \alpha < A(x)$;

(3) $x_\alpha \leq y_\beta \Leftrightarrow x = y, \alpha \leq \beta$;

(4) $A = B' \Leftrightarrow A(x) = 1 - B(x), \forall x \in X$.

Definition 4 ([12])**.** *Let p be a Deng's pseudo-metric on I^X and let $r \geq 0$ and $a \in M_0$. Define $U_r(a) = \bigvee\{b \in M_0 \mid p(a, b) < r\}$. Then, $U_r(a)$ is called an open sphere of p.*

Theorem 1 ([12]). *If p is a Deng's pseudo-metric on I^X, then the family of arbitrary unions of members of open spheres $\{U_r(a) \mid a \in M_0, r \in [0,+\infty)\}$ is a fuzzy topology denoted by ζ_p, and $\{U_r(a) \mid a \in M_0, r \in [0,+\infty)\}$ is a base for ζ_p.*

Therefore, the pair (I^X, ζ_p) and ζ_p are called Deng's pseudo-metric space and Deng's pseudo-metric topology, respectively.

Definition 5 ([12]). *The closure $\overline{A}$ of a fuzzy set A is the intersection of the members of the family of all fuzzy closed sets containing A.*

Definition 6 ([13]). *(I^X, δ) is T_1 if and only if for each $q \in M_0$, $q = \overline{q}$.*

Definition 7 ([52]). *(I^X, δ) is said the second axiom of countability denoted by C_{II} if and only if there is a countable base for δ.*

Pu and Liu [54] and Wang [52] have developed convincing theories about the Q-neighborhood and Remote-neighborhood, respectively. Therefore, corresponding with these theories, nowadays a standard fuzzy point on I^X has been accepted widely as follows:

Definition 8 ([52–54]). *$y_\alpha(x) \in I^X$ is called a standard fuzzy point if $y_\alpha(x)$ satisfies*

$$y_\alpha(x) = \begin{cases} \alpha, & y = x; \\ 0, & y \neq x, \end{cases}$$

where $\alpha \in (0,1]$. For convenience, $y_\alpha(x)$ is denoted by y_α. The set of all standard fuzzy points is denoted by M.

Definition 9. *For any $r \geq 0$ and $a \in M$, define $B_r(a) = \bigvee\{b \mid p(a,b) \leq r\}$, where p is a mapping from $M \times M$ to $[0,+\infty)$.*

Definition 10 ([24,25,54]). *Let (X, δ) be a [0, 1]-topological space. An open set B is called an open neighborhood of a fuzzy set A if $A < B$. An open set A is called a Q-neighborhood of x_λ if $\lambda + A(x) > 1$. If the family $Q(x_\lambda) = \{A \mid A$ is a Q-neighborhood of $x_\lambda\}$ is countable for each x_λ, then the space (X, δ) is called $Q - C_1$.*

Theorem 2 ([30]). *If p is a Yang-Shi's pseudo-metric on L^X, then it is $Q - C_I$.*

Theorem 3 ([12]). *If p is a Deng's pseudo-metric on I^X, then for any $A \in I^X$, $A^\circ = \bigvee\{a \mid \exists r > 0, a \in M_0, U_r(a) \leq A\}$.*

Theorem 4 ([12]). *Let v belong to I^X. Then, $v = \bigvee\{x_\alpha \in M_0 \mid x_\alpha \in v\}$=$\bigvee\{x_\lambda \in M_0 \mid x_\lambda \leq v\}$.*

Definition 11 ([12]). *A fuzzy point x_α is called a cluster point of a fuzzy set A if and only if each neighborhood of $x_{1-\alpha}$ is quasi-coincidence with A.*

Theorem 5 ([12,52]). *Let A be a fuzzy set. Then, $x_\alpha \leq \overline{A}$ if and only if x_α is a cluster point of A. Evidently, $\overline{A} = \bigvee\{x_\alpha \mid x_\alpha$ is a cluster point of $A\}$.*

Theorem 6 ([30]). *Let p be a Yang-Shi pseudo-metric on L^X and define $P_r(b) = \bigvee\{c \in M \mid p(c,b) \geq r\}$. Then, for $c, b \in M, c \leq P_r(b) \Leftrightarrow p(c,b) \geq r$.*

Theorem 7 ([10]). *Let p be a Erceg pseudo-metric on I^X. For any $a \in M_0$ and each $r \in [0,1)$ define $B_r(a) = \bigvee\{b \in M_0 \mid p(a,b) \leq r\}$. Then,*

1. *$\overline{B_r(a)} = B_r(a)$;*

2. $b \leq B_r(a) \Leftrightarrow p(a,b) \leq r$.

Theorem 8 ([10])**.** *If p is a Yang-Shi pseudo-metric on L^X, then it is an Erceg pseudo-metric. However, the converse is not true.*

3. Expansion Theorem of Deng's Metric

In this section, we will show that Deng's metric can be equivalently defined by using M_0 and M, and then its corresponding metric topology and uniform structure are Erceg's metric topology [14] and Hutton's uniform structure [22], respectively.

Definition 12. *An extended Deng's pseudo-metric metric on I^X is a mapping $p : M \times M \to [0, +\infty)$ satisfying the following conditions:*

(E1) *if $a \geq b$, then $p(a,b) = 0$;*

(E2) $p(a,c) \leq p(a,b) + p(b,c)$;

(E3) $p(a,b) = \bigwedge_{b<c} p(a,c)$;

(E4) $\forall a, b \in M$, $\exists x \not\leq b'$ *such that* $p(a,x) < r \Leftrightarrow \exists y \not\leq a'$ *such that* $p(b,y) < r$.

Theorem 9. *If p is a Deng's pseudo-metric on I^X, then p can be extended to $p^* : M \times M \to [0, +\infty)$ and p^* is an extended Deng's pseudo-metric.*

Proof. Based on the given conditions, we can construct a mapping $p^* : M \times M \to [0, +\infty)$ as follows:

(a) *if* $a, b \in M_0$, *then* $p^*(a,b) = p(a,b)$;

(b) *if* $a \in M_0, b = y_1$, *then* $p^*(a, y_1) = \bigvee_{e<1} p(a, y_e)$;

(c) *if* $a = x_1, b \in M_0$, *then* $p^*(x_1, b) = \bigwedge_{e<1} p(x_e, b)$;

(d) *if* $a = b = y_1$, *then* $p^*(y_1, y_1) = 0$;

(e) *if* $a = x_1, b = y_1, a \neq b$, *then* $p^*(x_1, y_1) = \bigwedge_{c<1} p^*(x_c, y_1) = \bigwedge_{c<1} \bigvee_{e<1} p(x_c, y_e)$.

Next, we will prove that p^* satisfies (E1)–(E4) and $p = p^* \mid M_0 \times M_0$.

(E1). Case 1. For any $y_1 \in M$, by (d) we can obtain $p^*(y_1, y_1) = 0$. Case 2. For any $y_\lambda \in M_0$, by (c) we can obtain $p^*(y_1, y_\lambda) = \bigwedge_{\alpha<1} p(y_\alpha, y_\lambda) = 0$. Therefore, p^* satisfies (E1).

(E2). Case 1. Let $x_1, z_1, b \in M$. Assume that $x_1 = z_1$. Then, it is evident that $p^*(x_1, y_\lambda) + p^*(y_\lambda, z_1) \geq p^*(x_1, z_1) = 0$. Assume that $x_1 \neq z_1$. Then, we can obtain the following situations:

(1) Let $b = y_\lambda \in M_0$. By definition, we have

$$p^*(x_1, z_1) \leq p^*(x_1, y_\lambda) + p^*(y_\lambda, z_1) \Leftrightarrow \bigwedge_{\alpha<1} \bigvee_{\gamma<1} p(x_\alpha, z_\gamma) \leq \bigwedge_{\alpha<1} p(x_\alpha, y_\lambda) + \bigvee_{\gamma<1} p(y_\lambda, z_\gamma).$$

Since $x_\alpha, z_\gamma, y_\lambda \in M_0$, it is true that $p(x_\alpha, z_\gamma) \leq p(x_\alpha, y_\lambda) + p(y_\lambda, z_\gamma)$. Therefore, we have

$$\bigvee_{\gamma<1} p(x_\alpha, z_\gamma) \leq p(x_\alpha, y_\lambda) + \bigvee_{\gamma<1} p(y_\lambda, z_\gamma),$$

and then $\bigwedge_{\alpha<1} \bigvee_{\gamma<1} p(x_\alpha, z_\gamma) \leq \bigwedge_{\alpha<1} p(x_\alpha, y_\lambda) + \bigvee_{\gamma<1} p(y_\lambda, z_\gamma)$.

(2) Let $b = y_1$. If $y_1 = x_1$ or $y_1 = z_1$, then

$$p^*(x_1, z_1) \leq p^*(x_1, y_1) + p^*(y_1, z_1) \Leftrightarrow p^*(x_1, z_1) \leq p^*(x_1, x_1) + p^*(x_1, z_1) = p^*(x_1, z_1)$$

or

$$p^*(x_1,z_1) \le p^*(x_1,y_1) + p^*(y_1,z_1) \Leftrightarrow p^*(x_1,z_1) \le p^*(x_1,z_1) + p^*(z_1,z_1) = p^*(x_1,z_1).$$

Therefore, p^* satisfies (E2).

Hence, let us assume that $y_1 \neq x_1$ and $y_1 \neq z_1$. In this case, we have the following formula:

$$p^*(x_1,z_1) \le p^*(x_1,y_1) + p^*(y_1,z_1) \Leftrightarrow \bigwedge_{\alpha<1}\bigvee_{\gamma<1} p(x_\alpha,z_\gamma) \le \bigwedge_{\alpha<1}\bigvee_{\beta<1} p(x_\alpha,y_\beta) + \bigwedge_{\beta<1}\bigvee_{\gamma<1} p(y_\beta,z_\gamma).$$

Since when $x_\alpha, z_\gamma, y_\beta \in M_0$, $p(x_\alpha,z_\gamma) \le p(x_\alpha,y_\beta) + p(y_\beta,z_\gamma)$, we have

$$z_\gamma: \quad \bigvee_{\gamma<1} p(x_\alpha,z_\gamma) \le p(x_\alpha,y_\beta) + \bigvee_{\gamma<1} p(y_\beta,z_\gamma);$$

$$y_\beta: \quad \bigvee_{\gamma<1} p(x_\alpha,z_\gamma) \le p(x_\alpha,y_\beta) + \bigwedge_{\beta<1}\bigvee_{\gamma<1} p(y_\beta,z_\gamma);$$

$$y_\beta: \quad \bigvee_{\gamma<1} p(x_\alpha,z_\gamma) \le \bigvee_{\beta<1} p(x_\alpha,y_\beta) + \bigwedge_{\beta<1}\bigvee_{\gamma<1} p(y_\beta,z_\gamma);$$

$$x_\alpha: \quad \bigwedge_{\alpha<1}\bigvee_{\gamma<1} p(x_\alpha,z_\gamma) \le \bigwedge_{\alpha<1}\bigvee_{\beta<1} p(x_\alpha,y_\beta) + \bigwedge_{\beta<1}\bigvee_{\gamma<1} p(y_\beta,z_\gamma).$$

Therefore, p^* still satisfies (E2).

Case 2. Let $x_1, z_\lambda \in M_0$ and let $b \in M$.

(1) if $b = y_\beta \in M_0$, then

$$p^*(x_1,z_\lambda) \le p^*(x_1,y_\beta) + p^*(y_\beta,z_\lambda) \Leftrightarrow \bigwedge_{\alpha<1} p(x_\alpha,z_\lambda) \le \bigwedge_{\alpha<1} p(x_\alpha,y_\beta) + p(y_\beta,z_\lambda).$$

In fact, since for $x_\alpha, z_\lambda, y_\beta \in M_0$, $p(x_\alpha,z_\lambda) \le p(x_\alpha,y_\beta) + p(y_\beta,z_\lambda)$, we have $\bigwedge_{\alpha<1} p(x_\alpha,z_\lambda) \le \bigwedge_{\alpha<1} p(x_\alpha,y_\beta) + p(y_\beta,z_\lambda)$. And so p^* satisfies (E2).

(2) Let $b = y_1$. If $y_1 = x_1$, then

$$p^*(x_1,z_\lambda) \le p^*(x_1,y_1) + p^*(y_1,z_\lambda) \Leftrightarrow p^*(x_1,z_\lambda) \le p^*(x_1,x_1) + p^*(x_1,z_\lambda) = p^*(x_1,z_\lambda).$$

If $y_1 \neq x_1$, then

$$p^*(x_1,z_\lambda) \le p^*(x_1,y_1) + p^*(y_1,z_\lambda) \Leftrightarrow \bigwedge_{\alpha<1} p(x_\alpha,z_\lambda) \le \bigwedge_{\alpha<1}\bigvee_{\beta<1} p(x_\alpha,y_\beta) + \bigwedge_{\beta<1} p(y_\beta,z_\lambda).$$

Due to any $x_\alpha, z_\lambda, y_\beta \in M_0$, $p(x_\alpha,z_\lambda) \le p(x_\alpha,y_\beta) + \bigwedge_{\beta<1} p(y_\beta,z_\lambda)$, we have the following formulas:

$$y_\beta: \quad p(x_\alpha,z_\lambda) \le \bigvee_{\beta<1} p(x_\alpha,y_\beta) + \bigwedge_{\beta<1} p(y_\beta,z_\lambda);$$

$$x_\alpha: \quad \bigwedge_{\alpha<1}(x_\alpha,z_\lambda) \le \bigwedge_{\alpha<1}\bigvee_{\beta<1} p(x_\alpha,y_\beta) + \bigwedge_{\beta<1} p(y_\beta,z_\lambda).$$

Therefore, p^* fulfills (E2).

Case 3. Let $x_\lambda \in M_0$ and let $z_1, b \in M$.

(1) Assume that $b \in M_0$. Then,

$p^*(x_\lambda,z_1) \le p^*(x_\lambda,b) + p^*(b,z_1) \Leftrightarrow \bigvee_{\alpha<1} p(x_\lambda,z_\alpha) \le p(x_\lambda,b) + \bigvee_{\alpha<1} p(b,z_\alpha)$.

For any $z_\alpha \in M_0$, we can obtain

$$p(x_\lambda, z_\alpha) \leq p(x_\lambda, b) + p(b, z_\alpha).$$

Furthermore, we have

$$z_\alpha : \qquad \bigvee_{\alpha<1} p(x_\lambda, z_\alpha) \leq p(x_\lambda, b) + \bigvee_{\alpha<1} p(b, z_\alpha).$$

Therefore, p^* satisfies (E2).
(2) Assume that $b = y_1$. Then, we have the following two cases:
If $y_1 = z_1$, then

$$p^*(x_\lambda, z_1) \leq p^*(x_\lambda, y_1) + p^*(y_1, z_1)$$
$$\Leftrightarrow p^*(x_\lambda, z_1) \leq p^*(x_\lambda, z_1) + p^*(z_1, z_1) = p^*(x_\lambda, z_1).$$

If $y_1 \neq z_1$, then

$$p^*(x_\lambda, z_1) \leq p^*(x_\lambda, y_1) + p^*(y_1, z_1)$$
$$\Leftrightarrow \bigvee_{\alpha<1} p(x_\lambda, z_\alpha) \leq \bigvee_{\beta<1} p(x_\lambda, y_\beta) + \bigwedge_{\gamma<1} \bigvee_{\delta<1} p(y_\gamma, z_\delta).$$

For $x_\lambda, z_\alpha, y_\beta \in M_0$, we have

$$p(x_\lambda, z_\alpha) \leq p(x_\lambda, y_\beta) + p(y_\beta, z_\alpha).$$

$$z_\alpha : \qquad p(x_\lambda, z_\alpha) \leq p(x_\lambda, y_\beta) + \bigvee_{\delta<1} p(y_\beta, z_\delta).$$

$$y_\beta : \qquad p(x_\lambda, z_\alpha) \leq p(x_\lambda, y_\beta) + \bigwedge_{\gamma<1} \bigvee_{\delta<1} p(y_\gamma, z_\delta),$$

$$y_\beta : \qquad p(x_\lambda, z_\alpha) \leq \bigvee_{\beta<1} p(x_\lambda, y_\beta) + \bigwedge_{\gamma<1} \bigvee_{\delta<1} p(y_\gamma, z_\delta),$$

$$z_\alpha : \qquad \bigvee_{\alpha<1} p(x_\lambda, z_\alpha) \leq \bigvee_{\beta<1} p(x_\lambda, y_\beta) + \bigwedge_{\gamma<1} \bigvee_{\delta<1} p(y_\gamma, z_\delta).$$

Therefore, in this case, p^* still satisfies (E2).
Case 4. Let x_α, $z_\gamma \in M_0$ and let $b \in M$.
(1) If $b = y_\beta \in M_0$, then

$$p^*(x_\alpha, z_\gamma) \leq p^*(x_\alpha, y_\beta) + p^*(y_\beta, z_\gamma) \Leftrightarrow p(x_\alpha, z_\gamma) \leq p(x_\alpha, y_\beta) + p(y_\beta, z_\gamma).$$

(2) If $b = y_1$, then

$$p^*(x_\alpha, z_\gamma) \leq p^*(x_\alpha, y_1) + p^*(y_1, z_\gamma) \Leftrightarrow p(x_\alpha, z_\gamma) \leq \bigvee_{\beta<1} p(x_\alpha, y_\beta) + \bigwedge_{\beta<1} p(y_\beta, z_\gamma).$$

For $x_\alpha, z_\gamma, y_\beta \in M_0$, we have

$$p(x_\alpha, z_\gamma) \leq p(x_\alpha, y_\beta) + p(y_\beta, z_\gamma).$$

Taking union and intersection for y_β, respectively, we can obtain

$$p(x_\alpha, z_\gamma) \leq \bigvee_{\beta<1} p(x_\alpha, y_\beta) + \bigwedge_{\beta<1} p(y_\beta, z_\gamma).$$

Hence, p^* fulfills (E2).
In summary, p^* satisfies (E2).

(E3). Case 1. Let $x_{\lambda_1}, y_{\lambda_2} \in M_0$. Since p^* satisfies (E1) and (E2), we have $p^*(x_{\lambda_1}, y_1) \geq p^*(x_{\lambda_1}, y_\lambda)$. Thus,

$$p^*(x_{\lambda_1}, y_{\lambda_2}) = \bigwedge_{1>s>\lambda_2} p^*(x_{\lambda_1}, y_s) \wedge p^*(x_{\lambda_1}, y_1) \Leftrightarrow p(x_{\lambda_1}, y_{\lambda_2}) = \bigwedge_{s>\lambda_2} p(x_{\lambda_1}, y_s).$$

Therefore, we have

$$p(x_{\lambda_1}, y_{\lambda_2}) = \bigwedge_{s>\lambda_2} p(x_{\lambda_1}, y_s).$$

Therefore, p^* satisfies (E3).
Case 2. Let $x_{\lambda_1} \in M_0$ and let $y_{\lambda_2} = y_1$.
Since p^* satisfies (E1) and (E2), we can obtain

$$p^*(x_{\lambda_1}, y_1) = \bigwedge_{s>1} p^*(x_{\lambda_1}, y_s).$$

Case 3. Let $x_{\lambda_1} = x_1$ and let $y_{\lambda_2} \in M_0$. Then, we have

$$p^*(x_1, y_{\lambda_2}) = \bigwedge_{\beta>\lambda_2} p^*(x_1, y_\beta) \Leftrightarrow \bigwedge_{\alpha<1} p(x_\alpha, y_{\lambda_2}) = \bigwedge_{\beta>\lambda_2} \bigwedge_{\alpha<1} p(x_\alpha, y_\beta).$$

Since p^* satisfies (E1) and (E2), it is true that $p^*(x_1, y_{\lambda_2}) \leq \bigwedge_{\beta>\lambda_2} p^*(x_1, y_\beta)$.
Conversely,

$$\forall e \leq \bigwedge_{\beta>\lambda_2} p^*(x_1, y_\beta) = \bigwedge_{\beta>\lambda_2} \bigwedge_{\alpha<1} p(x_\alpha, y_\beta)$$

$$\Rightarrow \forall \beta > \lambda_2, e \leq \bigwedge_{\alpha<1} p(x_\alpha, y_\beta) \Rightarrow \forall \beta > \lambda_2, \forall \alpha < 1, e \leq p(x_\alpha, y_\beta)$$

$$\Rightarrow \forall \alpha < 1, e \leq \bigwedge_{\beta>\lambda_2} p(x_\alpha, y_\beta) = p(x_\alpha, y_{\lambda_2})$$

$$\Rightarrow e \leq \bigwedge_{\alpha<1} p(x_\alpha, y_{\lambda_2}) = p^*(x_1, y_{\lambda_2})$$

$$\Rightarrow p^*(x_1, y_{\lambda_2}) \geq \bigwedge_{\beta>\lambda_2} p^*(x_1, y_\beta).$$

Case 4. Let $x_{\lambda_1} = x_1$ and let $y_{\lambda_2} = y_1$. This situation is meaningless and negligible.
In summary, p^* satisfies (E3).
(E4). Let $x_{\lambda_1}, y_{\lambda_2} \in M$.
Case 1. Let $x_{\lambda_1} = x_1$ and let $y_{\lambda_2} = y_\lambda \in M_0$.
Since (E4) $\Leftrightarrow \bigwedge_{s>1-\lambda_2} p^*(x_{\lambda_1}, y_s) = \bigwedge_{h>1-\lambda_1} p^*(y_{\lambda_2}, x_h)$, we need to testify

$$\bigwedge_{s>1-\lambda} p^*(x_1, y_s) = \bigwedge_{h>0} p^*(y_\lambda, x_h).$$

(1) Let $x_1 = y_1$. Owing to $\bigwedge_{s>1-\lambda} p^*(x_1, y_s) = \bigwedge_{1>s>1-\lambda} p^*(y_1, y_s) \wedge p^*(y_1, y_1) = 0$ and

$$\bigwedge_{h>0} p^*(y_\lambda, x_h) = \bigwedge_{1>h>0} p^*(x_\lambda, x_h) \wedge p^*(x_\lambda, x_1)$$

$$= \bigwedge_{h>0} p(x_\lambda, x_h) \wedge p^*(x_\lambda, x_1) = 0 \text{ (Because } h > 0 \Rightarrow \exists h = \lambda \Rightarrow p(x_\lambda, x_\lambda) = 0),$$

we can obtain $\bigwedge_{s>1-\lambda} p^*(x_1, y_s) = \bigwedge_{h>0} p^*(y_\lambda, x_h)$.

(2) Let $x_1 \neq y_1$. By (E1) and (E2), we have $p^*(x_1, y_s) \leq p^*(x_1, y_1)$ and $p^*(y_\lambda, x_h) \leq p^*(y_\lambda, x_1)$. Thus,

$$\bigwedge_{s>1-\lambda} p^*(x_1, y_s) = \bigwedge_{h>0} p^*(y_\lambda, x_h) \Leftrightarrow \bigwedge_{1>s>1-\lambda} p^*(x_1, y_s) \wedge p^*(x_1, y_1) =$$

$$\bigwedge_{s>1-\lambda} \bigwedge_{\alpha<1} p(x_\alpha, y_s) = \bigwedge_{1>h>0} p^*(y_\lambda, x_h) \wedge p^*(y_\lambda, x_1) = \bigwedge_{1>h>0} p(y_\lambda, x_h).$$

Therefore, we need to prove

$$\bigwedge_{s>1-\lambda} \bigwedge_{\alpha<1} p(x_\alpha, y_s) = \bigwedge_{1>h>0} p(y_\lambda, x_h).$$

In fact, by (A4) we have

$$\bigwedge_{s>1-\lambda} \bigwedge_{\alpha<1} p(x_\alpha, y_s) = \bigwedge_{s>1-\lambda} \bigwedge_{\alpha<1} p(y_{1-s}, x_{1-\alpha}) = \bigwedge_{\beta<\lambda} \bigwedge_{\gamma>0} p(y_\beta, x_\gamma).$$

Thus, we need to prove

$$\bigwedge_{1>h>0} p(y_\lambda, x_h) = \bigwedge_{\beta<\lambda} \bigwedge_{\gamma>0} p(y_\beta, x_\gamma).$$

This proof is as follows: for each $e \leq \bigwedge_{\beta<\lambda} \bigwedge_{\gamma>0} p(y_\beta, x_\gamma)$, we can obtain

$$e \leq \bigwedge_{\beta<\lambda} \bigwedge_{\gamma>0} p(y_\beta, x_\gamma) \Leftrightarrow \forall \beta < \lambda \text{ and } \forall \gamma > 0, e \leq p(y_\beta, x_\gamma) \Leftrightarrow \forall \gamma > 0 \text{ have } e \leq \bigwedge_{\beta<\lambda} p(y_\beta, x_\gamma)$$

$$\Leftrightarrow \forall \gamma > 0, e \leq \bigwedge_{\beta<\lambda} p(y_\beta, x_\gamma) = p(y_\lambda, x_\gamma) \Leftrightarrow e \leq \bigwedge_{\gamma>0} p(y_\lambda, x_\gamma) = \bigwedge_{h>0} p(y_\lambda, x_h).$$

Conversely, it is true for inequality similarly.

Case 2. Let $x_{\lambda_1} = x_\lambda \in M_0$ and let $y_{\lambda_2} = y_1$. By above Case 1 and (A4), we exchange x_1 and y_λ to fulfill. This proof is omitted.

Case 3. Let $x_{\lambda_1} = x_1$ and let $y_{\lambda_2} = y_1$.

Since

$$\bigwedge_{s>1-\lambda_2} p^*(x_{\lambda_1}, y_s) = \bigwedge_{h>1-\lambda_1} p^*(y_{\lambda_2}, x_h) \Leftrightarrow \bigwedge_{s>0} p^*(x_1, y_s) = \bigwedge_{h>0} p^*(y_1, x_h)$$

$$\Leftrightarrow \bigwedge_{s>0} (\bigwedge_{t<1} p(x_t, y_s)) = \bigwedge_{h>0} (\bigwedge_{r<1} p(y_r, x_h)) = \bigwedge_{h>0} (\bigwedge_{r<1} p(x_{1-h}, y_{1-r}))$$

$$= \bigwedge_{h>0} (\bigwedge_{v>0} p(x_{1-h}, y_v)) = \bigwedge_{u<1} (\bigwedge_{v>0} p(x_u, y_v)),$$

it is necessary to prove

$$\bigwedge_{s>0} (\bigwedge_{t<1} p(x_t, y_s)) = \bigwedge_{u<1} (\bigwedge_{v>0} p(x_u, y_v)).$$

This proof is based on the following equation:

$$w = \bigwedge_{s>0} (\bigwedge_{t<1} p(x_t, y_s)) \Rightarrow \forall s > 0, \forall t < 1, p(x_t, y_s) \geq w \Rightarrow \bigwedge_{u<1} (\bigwedge_{v>0} p(x_u, y_v)) \geq w.$$

Similarly, the inequality holds conversely.

In summary, p^* satisfies (E4).

Therefore, p^* is an extended Deng's pseudo-metric on I^X. Let $p = p^* \mid M_0 \times M_0$. Then, it is obvious that p is a Deng's pseudo-metric. □

Now, we analyze the relationship between the two topologies induced by p^* and p, respectively. For this purpose, we will need the following two lemmas:

Lemma 1. *Let $p : M \times M \to [0, +\infty)$ be a mapping and define $W_r(a) = \{b \in M \mid p(a, b) < r, r \in [0, +\infty)\}$. Then, p satisfies (A4) if and only if for each $b \in M$, $\bigvee_{b<a} W_r(b) = W_r(a)$.*

Proof. Because $W_r(b) \not\leq a' \Leftrightarrow$, there exists $x \not\leq a'$ such that $p(b, x) < r$, (E4) is equivalent to $W_r(a) \not\leq b' \Leftrightarrow W_r(b) \not\leq a'$ for any $a, b \in M$. Therefore,

$$W_r(a) \leq b' \Leftrightarrow W_r(b) \leq a' = \bigwedge_{x<a} x'$$

$$\Leftrightarrow x < a, W_r(b) \leq x' \Leftrightarrow x < a, W_r(x) \leq b' \Leftrightarrow \bigvee_{x<a} W_r(x) \leq b'.$$

Therefore, the proof is completed. □

Lemma 2. *Let p be an extended Deng's pseudo-metric on I^X. Then, the family $\{W_r(a) \mid a \in M, r \in [0, +\infty)\}$ is a base for a topology.*

Proof. We need to prove that the family τ_p of arbitrary unions of members of $\{W_r(a) \mid a \in M, r \in [0, +\infty)\}$ is a $[0, 1]$-topology, whose base is exactly the family $\{W_r(a) \mid a \in M, r \in [0, +\infty)\}$. Hence, we only need to prove that the intersection of any two elements of τ_p belongs to τ_p.

Let $A = W_s(a) \bigwedge W_t(b)$. If $s = 0$ or $t = 0$, then $A = \underline{0}$. Thus, we may as well suppose $s \neq 0$ and $t \neq 0$ and let $A \neq \underline{0}$. For any standard fuzzy point $c < A$ (here and in the proof, each "$<$ " is strictly smaller), we have $c < W_s(a)$ and $c < W_t(b)$, and then we have $p(a, c) < s$ and $p(b, c) < t$. Let $r_c = (s - p(a, c)) \wedge (t - p(b, c))$. Now, we come to prove $A = \bigvee_{c<A} W_{r_c}(c)$.

It is obvious that $A \leq \bigvee_{c<A} W_{r_c}(c)$. Conversely, let a standard fuzzy point $e < \bigvee_{c<A} W_{r_c}(c)$, then there exists $c < A$ such that $e < W_{r_c}(c)$, and then $p(c, e) < r_c$. Therefore, there are $p(c, e) < s - p(a, c)$ and $p(c, e) < t - p(b, c)$, which imply that $p(a, e) < s$ and $p(b, e) < t$ hold. Hence, we can obtain $e \leq W_s(a)$ and $e \leq W_t(b)$, and then $e \leq A$. Therefore, $A \geq \bigvee_{c<A} W_{r_c}(c)$. The proof is completed. □

Theorem 10. *Both p^* and p induce the same topology.*

Proof. By Theorem 1 and Lemma 2, $\{U_r(a) \mid a \in M_0, r \in [0, +\infty)\}$ and $\{W_r(b) \mid b \in M, r \in [0, +\infty)\}$ are a base for ζ_p and τ_{p^*}, respectively.

(i) let $b = x_\alpha \in M_0$.

Because $W_r(x_\alpha) = \vee\{y_\beta \in M \mid p^*(x_\alpha, y_\beta) < r\}$, we have

$$W_r(x_\alpha) = \bigvee\{y_\beta \in M_0 \mid p(x_\alpha, y_\beta) < r\} = U_r(x_\alpha)$$

for each $y_\beta \in M \Rightarrow y_\beta \in M_0$. Thus, in this case, $W_r(x_\alpha) = U_r(x_\alpha)$.

In the other case, besides $y_\beta \in M_0$, there exists index Γ with $\forall i \in \Gamma$ such that $y_{\beta_i} = y_1$ and $p^*(x_\alpha, y_1) < r$.

By (b) in definition of p^* (see Theorem 9), we can obtain $\bigvee_{\beta<1} p(x_\alpha, y_\beta) = p^*(x_\alpha, y_1) < r$.

Therefore, for each $i \in \Gamma$, we have $p(x_\alpha, y_\beta) < r$ if $p^*(x_\alpha, y_1) < r$, where $\beta < 1$. It follows that $y_\beta \in U_r(x_\alpha)$, and then $\vee y_\beta = y_1 \leq U_r(x_\alpha)$, which implies $W_r(x_\alpha) \leq U_r(x_\alpha)$.

Conversely, it is evident that $U_r(x_\alpha) \leq W_r(x_\alpha)$.

(ii) let $b = x_1$.

Since p^* is an extended Deng's pseudo-metric, by Lemma 1 and (i) we can obtain

$$W_r(x_1) = \bigvee_{\alpha<1} W_r(x_\alpha) = \bigvee_{\alpha<1} U_r(x_\alpha).$$

Therefore, for any $W_r(x_1)$, it is the union of some members of $\{U_r(b) \mid b \in M_0, r \in [0, +\infty)\}$. □

Corollary 1. *If p is a Deng's pseudo-metric, then ζ_p=τ_{p^*}.*

Proof. From Theorems 9 and 10, it is evident. □

Just because of Theorems 9 and 10, it is very natural for us to use M_0 to research Deng's pseudo-metric and its deduced topology. Therefore, it is no surprise that many scholars have achieved many excellent works by utilizing M_0 to investigate Deng's metric (for more details, see [12,13] etc.).

It is equivalent for us to use M_0 and M to characterize Deng's metric topology. Therefore, if we do not offer a special explanation, the subsequent discussions are based on M_0.

4. Quotient Space and the Further Extension of Deng's Metric

In this section, in order to discuss the properties of quotient space related to Deng's metrics, first of all, we define $T = \{p \mid p$ as an extended Deng's pseudo-metric on $I^X\}$ and $D = \{p_d \mid p_d$ is a Deng's pseudo-metric on $I^X\}$. Then, we can acquire the following result:

Theorem 11. *Define a mapping $f : T \to D$, where f is defined by $\forall p \in T$, let $f(p) = p_d = p \mid M_0 \times M_0$. Then,*

(i) *p_d is a Deng's pseudo-metric.*

(ii) *The mapping f is subjective.*

Proof. (i). By the definition of extended Deng's pseudo-metric, it is evident that (i) holds. (ii). By Theorem 9, we easily obtain that (ii) holds. □

According to Theorem 11, we can obtain a very interesting quotient space of the family of all extended Deng's pseudo-metrics. The details are as follows:

Take any $p_d \in D$ and let $B_{p_d} = f^{-1}(p_d)$. Then, B_{p_d} is the equivalence class of $D = \{p_d \mid p_d$ is a Deng's pseudo-metric on $I^X\}$. Define $\Omega = \{B_{p_d} \mid p_d \in D\}$. It is evident that Ω is a quotient space of T. The metric topology of each extended Deng's pseudo-metric in the equivalence class $f^{-1}(p_d)$ is the same topology induced by the expansion function p_d^* of p_d. It follows that there is a one-to-one mapping from D to Ω.

In addition, by Theorem 9, we can define an extended Deng's pseudo-metric on L^X, by using $M(L^X)$ as follows:

Definition 13. *A mapping $p : M(L^X) \times M(L^X) \longrightarrow [0, +\infty)$ is called a Deng's pseudo-metric on L^X if it satisfies the following conditions:*

(M1) *$\forall a, b \in M(L^X)$, if $a \geq b$, then $p(a, b) = 0$;*

(M2) *$\forall a, b, c \in M(L^X)$, $p(a, c) \leq p(a, b) + p(b, c)$;*

(M3) *$\forall a, b \in M(L^X)$, $p(a, b) = \bigwedge_{a \ll c} p(c, b)$;*

(M4) *$\forall a, b \in M(L^X)$, $\exists x \not\leq a'$ such that $p(b, x) < r \Leftrightarrow \exists y \not\leq b'$ such that $p(a, y) < r$.*

This is a type new metric on completely distributive lattice L^X, which is parallel to Erceg's metric [14] and Yang-Shi's metric [29]. So far, there almost is not any research about it on L^X. Maybe, this extended Deng's metric should be investigated.

5. The Relationship between Deng's Metric and Yang-Shi's Metric

In this section, we will show a commutative property of Deng's metric and investigate the relationship between Deng's metric and Yang-Shi's metric on I^X.

Theorem 12. *If a mapping* $p : M_0 \times M_0 \to [0, +\infty)$ *satisfies* (A1)–(A3) *and the following property:* (C4)* $\forall x_{\lambda_1}, y_{\lambda_2} \in M_0$, $p(x_{\lambda_1}, y_{\lambda_2}) = p(y_{\lambda_2}, x_{\lambda_1})$, *then* p *is a Deng's pseudo-metric.*

Proof. Case 1. Let $x_{\lambda_1}, y_{\lambda_2} \in M_0$ and let $y = x$. (i) if $\lambda_1 \geq \lambda_2$, then by (A1) $p(x_{\lambda_1}, x_{\lambda_2}) = 0$. In addition, since $1 - \lambda_1 < 1 - \lambda_2$, it is true that $p(x_{1-\lambda_2}, x_{1-\lambda_1}) = 0$. Therefore, we can obtain $p(x_{\lambda_1}, x_{\lambda_2}) = p(x_{1-\lambda_2}, x_{1-\lambda_1})$. (ii) when $\lambda_1 < \lambda_2$, by (C4)*, this conclusion is also valid.

Case 2. Let $x_{\lambda_1}, y_{\lambda_2} \in M_0$ and $x \neq y$. In this case, we will discuss it in two different situations.

Situation 1. Let $\lambda_1 \leq 1 - \lambda_1$. Under this condition, we still divide the discussion into two sub-situations (a) and (b) as follows:

(a) Assume that $\lambda_2 \leq 1 - \lambda_2$. Then,

$$\begin{aligned} p(y_{\lambda_2}, x_{\lambda_1}) = p(x_{\lambda_1}, y_{\lambda_2}) &\leq p(x_{\lambda_1}, y_{1-\lambda_2}) + p(y_{1-\lambda_2}, y_{\lambda_2}) \\ &= p(x_{\lambda_1}, y_{1-\lambda_2}) = p(y_{1-\lambda_2}, x_{\lambda_1}). \end{aligned}$$

Moreover, we can obtain the following equation:

$$p(y_{1-\lambda_2}, x_{\lambda_1}) \leq p(y_{1-\lambda_2}, y_{\lambda_2}) + p(y_{\lambda_2}, x_{\lambda_1}) = p(y_{\lambda_2}, x_{\lambda_1}).$$

Thus,

$$p(y_{\lambda_2}, x_{\lambda_1}) = p(y_{1-\lambda_2}, x_{\lambda_1}). \tag{1}$$

Similarly, we can obtain

$$p(y_{1-\lambda_2}, x_{\lambda_1}) \leq p(y_{1-\lambda_2}, x_{1-\lambda_1}) + p(x_{1-\lambda_1}, x_{\lambda_1}) \leq p(y_{1-\lambda_2}, x_{1-\lambda_1}).$$

In addition, we have

$$p(y_{1-\lambda_2}, x_{1-\lambda_1}) \leq p(y_{1-\lambda_2}, x_{\lambda_1}) + p(x_{\lambda_1}, x_{1-\lambda_1}) = p(y_{1-\lambda_2}, x_{\lambda_1}).$$

Thereby, we can assert

$$p(y_{1-\lambda_2}, x_{\lambda_1}) = p(y_{1-\lambda_2}, x_{1-\lambda_1}). \tag{2}$$

Furthermore, by (1) and (2), we know $p(y_{\lambda_2}, x_{\lambda_1}) = p(y_{1-\lambda_2}, x_{1-\lambda_1})$, that is to say, we have the following equation:

$$p(x_{\lambda_1}, y_{\lambda_2}) = p(y_{1-\lambda_2}, x_{1-\lambda_1}). \tag{3}$$

(b). Assume that $\lambda_2 > 1 - \lambda_2$. If $\beta = 1 - \lambda_2$, then $\lambda_2 = 1 - \beta$, and consequently, $1 - \beta = \lambda_2 > 1 - \lambda_2 = \beta$. Due to the fact that β satisfies (a), by (3) we have $p(x_{\lambda_1}, y_\beta) = p(y_{1-\beta}, x_{1-\lambda_1})$. Hence, let $p(x_{\lambda_1}, y_{1-\lambda_2})$ replace $p(y_{1-\lambda_2}, x_{\lambda_1})$. Then, in this way we can obtain

$$p(y_{1-\lambda_2}, x_{\lambda_1}) = p(y_{\lambda_2}, x_{1-\lambda_1}). \tag{4}$$

Moreover, by (4) we have the following formula:

$$\begin{aligned} p(y_{1-\lambda_2}, x_{1-\lambda_1}) = p(x_{1-\lambda_1}, y_{1-\lambda_2}) &\leq p(x_{1-\lambda_1}, y_{\lambda_2}) + p(y_{\lambda_2}, y_{1-\lambda_2}) \\ &= p(x_{1-\lambda_1}, y_{\lambda_2}) = p(y_{\lambda_2}, x_{1-\lambda_1}) = p(y_{1-\lambda_2}, x_{\lambda_1}). \end{aligned}$$

Again by $p(y_{1-\lambda_2}, x_{\lambda_1}) \le p(y_{1-\lambda_2}, x_{1-\lambda_1}) + p(x_{1-\lambda_1}, x_{\lambda_1}) = p(y_{1-\lambda_2}, x_{1-\lambda_1})$, we can obtain

$$p(y_{1-\lambda_2}, x_{\lambda_1}) = p(y_{1-\lambda_2}, x_{1-\lambda_1}), \tag{5}$$

According to (5), we need to prove

$$p(y_{\lambda_2}, x_{\lambda_1}) = p(x_{1-\lambda_1}, y_{1-\lambda_2}) = p(y_{1-\lambda_2}, x_{1-\lambda_1})$$

$$\Leftrightarrow p(y_{\lambda_2}, x_{\lambda_1}) = p(y_{1-\lambda_2}, x_{\lambda_1}) \Leftrightarrow p(y_{1-\beta}, x_{\lambda_1}) = p(y_{\beta}, x_{\lambda_1}).$$

This is exactly the case of (a). Thereby, it is true for $p(y_{\lambda_2}, x_{\lambda_1}) = p(y_{1-\lambda_2}, x_{1-\lambda_1})$, that is, it holds for

$$p(x_{\lambda_1}, y_{\lambda_2}) = p(y_{1-\lambda_2}, x_{1-\lambda_1}). \tag{6}$$

Situation 2. Let $\lambda_1 > 1-\lambda_1$. If $\alpha = 1-\lambda_1$, then $\lambda_1 = 1-\alpha$. Thus, $1-\alpha > \alpha$. By case 1, we can assert either $\lambda_2 \le 1-\lambda_2$ or $\lambda_2 > 1-\lambda_2$. Therefore, when $1-\alpha > \alpha$, we must have the following equation:

$$p(x_\alpha, y_{\lambda_2}) = p(y_{1-\lambda_2}, x_{1-\alpha}). \tag{7}$$

Namely

$$p(x_{1-\lambda_1}, y_{\lambda_2}) = p(y_{1-\lambda_2}, x_{\lambda_1}). \tag{8}$$

Similarly, by repeating the process from (4) to (6), we can obtain $p(x_{\lambda_1}, y_{\lambda_2}) = p(y_{1-\lambda_2}, x_{1-\lambda_1})$. In summary, this conclusion is true. Therefore, this proof is completed. □

Theorem 13. *If p is a Deng's pseudo-metric on I^X, then p is a Yang-Shi's pseudo-metric.*

Proof. For any two fuzzy points x_a and y_b, we only need to prove $p(x_a, y_b) = \bigwedge_{c<a} p(x_c, y_b)$. If $c < a$, then $p(x_a, y_b) \le p(x_c, y_b)$, and then $p(x_a, y_b) \le \bigwedge_{c<a} p(x_c, y_b)$. If $p(x_a, y_b) = r < \bigwedge_{c<a} p(x_c, y_b) = t$, then by (A4) we have $p(y_{1-b}, x_{1-a}) = r < t$, so that by (A3) there exists a number $s > 1-a$ such that $p(y_{1-b}, x_s) < t$, i.e., $p(x_{1-s}, y_b) < t$. But this contradicts $\bigwedge_{c<a} p(x_c, y_b) = t$. Consequently, $p(x_a, y_b) = \bigwedge_{c<a} p(x_c, y_b)$, as desired. □

Conversely, we have the following conclusion:

Theorem 14. *If p is a Yang-Shi's pseudo-metric and further satisfies the following condition: $(K3)^*$ $p(x_{\lambda_2}, y_{\lambda_1}) = \bigvee_{s>\lambda_2} p(x_s, y_{\lambda_1})$, then p is a Deng's pseudo-metric.*

To prove Theorem 14, we first need to prove the following two Lemmas.

Lemma 3. *Let p be a Yang-Shi pseudo-metric on I^X and for each $r \in [0,1)$ define $U_r(a) = \bigvee\{b \in I^X \mid p(a,b) < r\}$. Then, $U_r(y_\lambda) = \bigvee_{\alpha>1-\lambda} P_r(y_\alpha)'$.*

Proof. Let $x_\beta \in \bigvee_{\alpha>1-\lambda} P_r(y_\alpha)'$ and take γ such that $x_\beta < x_\gamma \le \bigvee_{\alpha>1-\lambda} P_r(y_\alpha)'$. Because $1-\gamma \ge \bigwedge_{\alpha>1-\lambda} P_r(y_\alpha)(x)$, there exists a number $\alpha > 1-\lambda$ such that $1-\gamma \ge P_r(y_\alpha)(x)$, and then for each $\delta > 1-\gamma$ we have $\delta > P_r(y_\alpha)(x)$. Therefore, by Theorem 6 we can obtain $p(x_\delta, y_\alpha) < r$. Again, by (A3) of (I) in Introduction ((A3) on the special case I^X of L^X is : for any $x_{\lambda_1}, y_{\lambda_2}$, $\exists t > 1-\lambda_1$ s.t. $p(y_{\lambda_2}, x_t) < r \Leftrightarrow \exists s > 1-\lambda_2$ s.t. $p(x_{\lambda_1}, y_s) < r$), there exists $x_\omega(x_\delta)$ (x_ω which has something to do with x_δ) with $\omega > 1-\delta$ such that $p(y_\lambda, x_\omega) < r$.

Let $x_q = \bigvee\{x_\omega(x_\delta) \mid \delta > 1-\gamma\}$. Then, $x_\delta \not\leq x_{1-q}$, i.e., $x_\delta > x_{1-q}$. This implies that as long as $x_\delta > x_{1-\gamma}$, it must hold that $x_\delta > x_{1-q}$. Thus, $x_\gamma \leq x_q$. Since $x_\beta < x_\gamma \leq x_q$, there exists $x_\omega(x_\delta)$ such that $x_\beta \leq x_\omega$, and so $p(y_\lambda, x_\beta) \leq p(y_\lambda, x_\omega) < r$. Hence, $x_\beta \leq U_r(y_\lambda)$. Because x_β is arbitrary, we have $\bigvee_{\alpha>1-\lambda} P_r(y_\alpha)' \leq U_r(y_\lambda)$.

Conversely, let $x_\alpha \in U_r(y_\lambda)$. Then, $p(y_\lambda, x_\alpha) < r$. For each $x_\beta > x_{1-\alpha}$, i.e., $\alpha > 1-\beta$, by (A3) there exists $\gamma > 1-\lambda$ such that $p(x_\beta, y_\gamma) < r$, and then by Theorem 6, $x_\beta \not\leq P_r(y_\gamma)$. Hence, $x_\beta \not\leq \bigwedge_{\gamma>1-\lambda} P_r(y_\gamma)$. That is to say, as long as $x_\beta > x_{1-\alpha}$, i.e., $x_\beta \not\leq x_{1-\alpha}$, it is true that $x_\beta \not\leq \bigwedge_{\gamma>1-\lambda} P_r(y_\gamma)$. Consequently, $\bigwedge_{\gamma>1-\lambda} P_r(y_\gamma)(x) \leq x_{1-\alpha}$, i.e., $x_\alpha \leq \bigvee_{\gamma>1-\lambda} P_r(y_\gamma)'$. Because x_α is arbitrary, we have $U_r(y_\lambda) \leq \bigvee_{\gamma>1-\lambda} P_r(y_\gamma)'$, as desired. □

Lemma 4. *If p is a Yang-Shi's pseudo-metric on I^X, then* $\bigvee_{\alpha>1-\lambda_1} p(x_\alpha, y_{\lambda_2}) = \bigvee_{\beta>1-\lambda_2} p(y_{\lambda_\beta}, x_{\lambda_1})$.

Proof. Denote $\bigvee_{\alpha>1-\lambda_1} p(x_\alpha, y_{\lambda_2}) = \bigvee_{\beta>1-\lambda_2} p(y_{\lambda_\beta}, x_{\lambda_1})$ as (H1). Then, it is easy to verify that (H1) is equivalent to the following property:

(H1)* $\exists\alpha > 1-\lambda_1$ s.t. $p(x_\alpha, y_{\lambda_2}) > r \Leftrightarrow \exists\beta > 1-\lambda_2$ s.t. $p(y_{\lambda_\beta}, x_{\lambda_1}) > r$.

Now, let us prove (H1)*.

Assume that there is α with $\alpha > 1-\lambda_1$ such that $p(x_\alpha, y_{\lambda_2}) > r$. Take a number s such that $p(x_\alpha, y_{\lambda_2}) > s > r$. By Theorems 7 and 8, we assert that $\lambda_2 > B_s(x_\alpha)(y)$. Therefore, by Lemma 3, we can obtain the following formula:

$$\lambda_2 > B_s(x_\alpha)(y) \geq U_s(x_\alpha)(y) = \bigvee_{\gamma>1-\alpha} P_s(x_\gamma)'(y).$$

Thus, for every $\gamma > 1-\alpha$ it is true that $\lambda_2 > P_s(x_\gamma)'(y)$. That is to say, as long as $\alpha > 1-\lambda_1$, i.e., $x_{\lambda_1} \not\leq x_{1-\alpha}$ such that $p(x_\alpha, y_{\lambda_2}) > r$, it is true that $\lambda_2 > P_s(x_{\lambda_1})'(y)$, i.e., $1-\lambda_2 < P_s(x_{\lambda_1})(y)$. Therefore, there exists y_ω such that $y_{1-\lambda_2} < y_\omega \leq P_s(x_{\lambda_1})$, and then $p(y_\omega, x_{\lambda_1}) \geq s > r$ by Theorem 7. Similarly, so is the reverse, as desired. □

Proof. The proof of Theorem 14 is as follows:

Let p be a Yang-Shi's pseudo-metric on I^X and it satisfies $p(x_{\lambda_2}, y_{\lambda_1}) = \bigvee_{s>\lambda_2} p(x_s, y_{\lambda_1})$. Then, we only need to prove that p satisfies (A3) and (A4).

(A4). Given any $x_{\lambda_1}, y_{\lambda_2} \in M_0$. According to Lemma 4, we have

$$\bigvee_{\alpha>1-\lambda_1} p(x_\alpha, y_{\lambda_2}) = \bigvee_{\beta>1-\lambda_2} p(y_{\lambda_\beta}, x_{\lambda_1}),$$

and then $p(x_{1-\lambda_1}, y_{\lambda_2}) = p(y_{1-\lambda_2}, x_{\lambda_1})$.

(A3). By (A1) and (A2), if $\lambda_3 > \lambda_1$, then $p(y_{\lambda_2}, x_{\lambda_1}) \leq p(y_{\lambda_2}, x_{\lambda_3})$. Thus, $p(y_{\lambda_2}, x_{\lambda_1}) \leq \bigwedge_{\lambda_3>\lambda_1} p(y_{\lambda_2}, x_{\lambda_3})$.

Conversely, take any r with $r \in (0, +\infty)$ such that $p(y_{\lambda_2}, x_{\lambda_1}) < r$. Then, by (A4) we have

$$p(y_{\lambda_2}, x_{\lambda_1}) = p(x_{1-\lambda_1}, y_{1-\lambda_2}) = \bigwedge_{h<1-\lambda_1} p(x_h, y_{1-\lambda_2}) < r.$$

Therefore, there at least exists h with $h < 1-\lambda_1$ such that $p(x_h, y_{1-\lambda_2}) < r$, i.e., $p(y_{\lambda_2}, x_{1-h}) < r$. Let $1-h = \lambda_3$. Then, $h < 1-\lambda_1 \Leftrightarrow \lambda_1 < 1-h = \lambda_3$ and $p(y_{\lambda_2}, x_{\lambda_3}) < r$. Consequently, $p(y_{\lambda_2}, x_{\lambda_1}) \geq \bigwedge_{\lambda_3>\lambda_1} p(y_{\lambda_2}, x_{\lambda_3})$, as desired. □

Example: Suppose that p_0 is distance function in usual sense on X. For any $b_\mu, a_\lambda \in M$, let $p(b_\mu, a_\lambda) = p_0(b, a) + max\{\lambda - \mu, 0\}$. Then (I^X, p) is a Deng's pseudo-metric.

Let us use Theorem 14 to verify this example. In fact, because $a_\lambda \not\leq b'_\mu$ implies $a = b$ and $\lambda > 1-\mu$, and $a_\lambda \ll b_\mu$ is equivalent to $a = b$ and $\lambda < \mu$, we need to verify that p satisfies the following conditions: (A1)–(A2), (B2), (A4) and (K3)* by $max\{\lambda-\mu, 0\} = \frac{1}{2}(\lambda-\mu+|\lambda-\mu|)$.

(A1). For any $a_\lambda, b_\mu \in M$ and $a_\lambda \leq b_\mu$, we can obtain $a = b$ and $\lambda \leq \mu$. Therefore, $p(b_\mu, a_\lambda) = 0$.

(A2). For any $a_\lambda, b_\mu, c_\nu \in M$, we have

$$\begin{aligned}
&p(b_\mu, a_\lambda) + p(c_\nu, b_\mu)\\
&= p_0(b,a) + \tfrac{1}{2}(\lambda-\mu+|\lambda-\mu|) + p_0(c,b) + \tfrac{1}{2}(\mu-\nu+|\mu-\nu|)\\
&= p_0(b,a) + p_0(c,b) + \tfrac{1}{2}(\lambda-\nu) + \tfrac{1}{2}(|\lambda-\mu|+|\mu-\nu|)\\
&\geq p_0(c,a) + \tfrac{1}{2}(\lambda-\nu+|\lambda-\mu|) = p(c_\nu, a_\lambda).
\end{aligned}$$

(B2). For any $a_\lambda, b_\mu \in M$, we have

$$\begin{aligned}
&\bigwedge_{c_\tau \ll b_\mu} p(c_\tau, a_\lambda) = \bigwedge_{c=b, \tau<\mu} [p_0(c,a) + \tfrac{1}{2}(\lambda-\tau+|\lambda-\tau|)]\\
&= p_0(b,a) + \bigwedge_{\tau<\mu} \tfrac{1}{2}(\nu-\mu+|\nu-\mu|)\\
&= p_0(b,a) + \tfrac{1}{2}(\lambda-\mu+|\lambda-\mu|) = p(b_\mu, a_\lambda).
\end{aligned}$$

(A4). To prove (B3), it only suffices to verify $\bigwedge_{x_\nu \not\leq a'_\lambda} p(b_\mu, x_\nu) = \bigwedge_{y_\tau \not\leq b'_\mu} p(a_\lambda, y_\tau)$. In fact, its proof is as follows:

$$\begin{aligned}
&\bigwedge_{x_\nu \not\leq a'_\lambda} p(b_\mu, x_\nu) = \bigwedge_{x=a, \nu>1-\lambda} [p_0(b,x) + \tfrac{1}{2}(\nu-\mu+|\nu-\mu|)]\\
&= p_0(b,a) + \bigwedge_{\nu>1-\lambda} \tfrac{1}{2}(\nu-\mu+|\nu-\mu|)\\
&= p_0(b,a) + \tfrac{1}{2}(1-\lambda-\mu+|1-\lambda-\mu|)] + \bigwedge_{\tau>1-\mu} \tfrac{1}{2}(\tau-\lambda+|\tau-\lambda|)\\
&= \bigwedge_{y_\tau \not\leq b'_\mu} p(a_\lambda, y_\tau).
\end{aligned}$$

(K3)*. For any $a_{1-\lambda}, b_\mu \in M$, we can verify the following equations:

$$\begin{aligned}
&\bigvee_{x_\nu \not\leq a'_\lambda} p(x_\nu, b_\mu) = \bigvee_{x=a, \nu>1-\lambda} [p_0(x,b) + \tfrac{1}{2}(\mu-\nu+|\mu-\nu|)]\\
&= p_0(a,b) + \bigvee_{\nu>1-\lambda} \tfrac{1}{2}(\mu-\nu+|\mu-\nu|)\\
&= p_0(a,b) + \tfrac{1}{2}(\lambda+\mu-1+|\lambda+\mu-1|)] = p(a_{1-\lambda}, b_\mu).
\end{aligned}$$

Corollary 2. *A Deng's pseudo-metric on* I^X *is* $Q - C_1$.

Proof. By Theorem 2 and Theorem 13, it is evident for the result to hold. □

According to Theorem 8, we have known that an Erceg's metric must be a Yang-Shi's metric. Again by Theorem 13, we can obtain that a Deng's metric must be an Erceg's metric. In addition, existing achievements (refer to [14,24,25]) have shown that Erceg's metric's uniform structure must be Hutton's uniform structure [22]. Therefore, we can assert that Deng's metric topology and its uniform structure are Erceg's metric topology and Hutton's uniform structure, respectively.

6. Conclusions

In this paper, firstly, we extend the domain of Deng's metric function from $M_0 \times M_0$ to $M \times M$. Secondly, we further extend this metric to L^X and, based on this extension result, we compare this metric with the other two kinds of familiar fuzzy metrics: Erceg's metric and Yang-Shi's metric, and then reveal some of its interesting properties, particularly including its quotient space. Thirdly, we prove that a Deng's metric must be a Yang-Shi's metric on I^X, and consequently an Erceg's metric. Finally, we will show that a Deng's metric must be $Q - C_1$, and Deng's metric topology and its uniform structure are Erceg's metric topology and Hutton's uniform structure, respectively.

In the future, we will continue to consider Deng's metric on L-topology. Additionally, we will further investigate Erceg's metric, Yang-Shi's metric and Deng's metric on L^X. Moreover, we will continue to conduct research on the kind of lattice-valued topological spaces, each of whose topologies has a σ-locally finite base. Beyond that, we also intend to inquire into the metrization problem in [0, 1]-topology.

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