mathematics

Article

A Hypersurfaces of Revolution Family in the Five-Dimensional Pseudo-Euclidean Space $\mathbb{E}_2^5$

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Abstract: We present a family of hypersurfaces of revolution distinguished by four parameters in the five-dimensional pseudo-Euclidean space $\mathbb{E}_2^5$. The matrices corresponding to the fundamental form, Gauss map, and shape operator of this family are computed. By utilizing the Cayley–Hamilton theorem, we determine the curvatures of the specific family. Furthermore, we establish the criteria for maximality within this framework. Additionally, we reveal the relationship between the Laplace–Beltrami operator of the family and a 5×5 matrix.

Keywords: pseudo-Euclidean 5-space; hypersurfaces of revolution family; Gauss map; shape operator: curvature; Laplace–Beltrami operator

MSC: 53A35; 53C42

Citation: Li, Y.; Güler, E. A Hypersurfaces of Revolution Family in the Five-Dimensional Pseudo-Euclidean Space $\mathbb{E}_2^5$. *Mathematics* **2023**, *11*, 3427. https://doi.org/10.3390/math11153427

Academic Editor: Stéphane Puechmorel

Received: 21 July 2023
Revised: 2 August 2023
Accepted: 3 August 2023
Published: 7 August 2023

1. Introduction

Chen [1–4] initially introduced the concept of submanifolds with finite order immersion in m-space $\mathbb{E}^m$ or pseudo-Euclidean m-space $\mathbb{E}_v^m$ using a finite number of eigenfunctions derived from their Laplacian. Since then, extensive research has been conducted in this field.

Takahashi worked that a Euclidean submanifold is classified as 1-type if and only if it is either minimal or minimal within a hypersphere of $\mathbb{E}^m$. Additionally, Garay [5] extended the analysis of Takahashi's theorem in $\mathbb{E}^m$. Hypersurfaces with constant curvature were the primary focus of Cheng and Yau [6], while Chen and Piccinni [7] specialized in submanifolds with a Gauss map of finite type in $\mathbb{E}^m$. Dursun [8] introduced hypersurfaces with a pointwise 1-type Gauss map in $\mathbb{E}^{n+1}$, while Aminov [9] delved into the geometry of submanifolds. In the realm of space forms, Chen et al. [10] dedicated a significant amount of time to the thorough investigation of 1-type submanifolds and the 1-type Gauss map.

Within the framework of $\mathbb{E}^3$, Takahashi [11] extensively explored the realm of minimal surfaces, revealing that spheres and minimal surfaces with $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$ constitute the exclusive surface types. Ferrandez et al. [12] made a noteworthy discovery, categorizing surfaces $\Delta H = A_{3\times 3} H$ as either minimal sections of a sphere or right circular cylinders. The properties of the minimal helicoid with a pointwise 1-type Gauss map of the first kind were thoroughly examined by Choi and Kim [13]. Garay [14] developed a classification scheme for revolution-based surfaces of finite type. Dillen et al. [15] conducted an in-depth investigation into unique surfaces characterized by $\Delta r = A_{3\times 3} r + B_{3\times 1}$, encompassing minimal surfaces, spheres, and circular cylinders. Stamatakis and Zoubi [16] established fundamental properties of surfaces of revolution determined by $\Delta^{III} x = A_{3\times 3} x$.

Kim et al. [17] directed their research towards the Cheng–Yau operator and the Gauss map of surfaces of revolution.

In the context of $\mathbb{E}^4$, Moore [18,19] undertook two comprehensive investigations into the properties of general rotational surfaces. Hasanis and Vlachos [20] directed their research towards hypersurfaces featuring a harmonic mean curvature vector field. Cheng and Wan [21] dedicated their efforts to studying complete hypersurfaces exhibiting constant mean curvature. Arslan et al. [22] delved into the exploration of the Vranceanu surface, with a specific focus on analyzing its pointwise 1-type Gauss map. Arslan et al. [23] conducted studies on generalized rotational surfaces and introduced tensor product surfaces characterized by a pointwise 1-type Gauss map [24]. Güler et al. [25] conducted extensive research on the properties of helicoidal hypersurfaces, while another work by Güler et al. [26] focused on the investigation of the Gauss map and the third Laplace–Beltrami operator of rotational hypersurfaces. Güler [27] further examined rotational hypersurfaces characterized by $\Delta^I R = A_{4\times4}R$. Additionally, Güler [28] derived the fundamental form fourth and curvature formulas for hyperspheres.

In Minkowski 4-space $\mathbb{E}^4_1$, Ganchev and Milousheva [29] investigated surfaces analogous to those studied in [18,19]. Arvanitoyeorgos et al. [30] conducted research on the mean curvature vector field, establishing $\Delta H = \alpha H$ with a constant value α. Arslan and Milousheva [31] focused their attention on meridian surfaces of elliptic or hyperbolic type, analyzing their pointwise 1-type Gauss map. Arslan et al. [32] examined rotational λ-hypersurfaces in Euclidean spaces. Güler et al. [33–36] extensively explored the concept of bi-rotational hypersurfaces. Li et al. [37–47] conducted a series of theoretical research and development on singularity theory, submanifold theory, etc. Their work has contributed to the advancement of related research areas. We can find more motivations of our work from several papers (see [48–63]).

The main goal of this research is to present a new family of hypersurfaces of revolution in the five-dimensional pseudo-Euclidean space $\mathbb{E}^5_2$. This family, referred to as $\mathfrak{r}$, is characterized by four distinct parameters. The central focus of the research revolves around the computation of various matrices associated with $\mathfrak{r}$, which include the fundamental form, Gauss map, and shape operator. The Cayley–Hamilton theorem is employed as a tool to determine the curvatures of $\mathfrak{r}$. Furthermore, the paper establishes equations that describe the relationship between the mean curvature and Gauss–Kronecker curvature of $\mathfrak{r}$. Additionally, an investigation is carried out to explore the connection between the Laplace–Beltrami operator of $\mathfrak{r}$ and a 5×5 matrix, revealing intriguing interconnections.

In Section 2, we elucidate the fundamental principles of five-dimensional pseudo-Euclidean geometry, providing a clear explanation of its key concepts.

Section 3 is dedicated to presenting the curvature formulas specifically tailored for hypersurfaces in $\mathbb{E}^5_2$, enabling a deeper understanding of their geometric properties.

In Section 4, we delve into a detailed exposition of the family of hypersurfaces of revolution, meticulously examining their unique properties and characteristics.

In Section 5, we explore the Laplace–Beltrami operator of a smooth function in $\mathbb{E}^5_2$, utilizing the aforementioned family to efficiently compute its corresponding values.

Lastly, we serve a conclusion in Section 6.

2. Preliminaries

In this paper, we use the following notations, formulas, equations, etc.

For clarity, $\mathbb{E}^m_v$ represents a pseudo-Euclidean m-space with coordinates denoted as $(x_1, x_2, \ldots, x_m)$ with index v. The canonical pseudo-Euclidean metric tensor on $\mathbb{E}^m_v$ is represented by $\widetilde{g}$ and defined as $\widetilde{g} = \langle\ ,\ \rangle = -\sum\limits_{i=1}^{v} dx_i^2 + \sum\limits_{i=v+1}^{m} dx_i^2$. Let $\widetilde{M}$ be an m-dimensional semi-Riemannian submanifold, and is embedded in $\mathbb{E}^m_v$, and the Levi–Civita connections [64] associated with M are denoted as $\widetilde{\nabla}, \nabla$, respectively. We utilize X, Y, Z, and W to denote vector fields tangent to M, and ξ, ς to represent vector fields normal to M.

The Gauss formula and the Weingarten formula are determined by the equations

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \;\; \widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi,$$

where h represents the second fundamental form of M, A denotes the shape operator, and D corresponds to the normal connection of M. The shape operator A_ξ is a symmetric endomorphism of the tangent space T_pM at each point $p \in M$ for each $\xi \in T_p^\perp M$. The shape operator and the second fundamental form are related by the equation

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The Gauss equation is determined by

$$\langle R(X,Y,)Z,W \rangle = \langle h(Y,Z), h(X,W) \rangle - \langle h(X,Z), h(Y,W) \rangle,$$

where R describes the curvature tensor associated with the Levi–Civita connection ∇, and h denotes the second fundamental form of M. The Codazzi equation is given by

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z),$$

where $\bar{\nabla} h$ denotes the covariant derivative of h with respect to the Levi–Civita connection ∇, and X, Y, Z represent tangent vector fields on M. The curvature tensor R^D associated with the normal connection D is not explicitly mentioned in the given equations. The covariant derivative of h is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

where D represents the normal connection of M.

Let M be an oriented hypersurface in $\mathbb{E}^{n+1}$ with its shape operator $\mathcal{S}$, position vector x. Consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ consisting of principal directions of M coinciding with the principal curvature k_i for $i = 1, 2, \ldots, n$. Let the dual basis of this frame field be $\{f_1, f_2, \ldots, f_n\}$. Then, the first structural equation of Cartan is determined by

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i, j = 1, 2, \ldots, n,$$

where ω_{ij} indicates the connection forms coinciding with the chosen frame field. By the Codazzi equation, we derive the equations

$$\begin{aligned} e_i(k_j) &= \omega_{ij}(e_j)(k_i - k_j), \\ \omega_{ij}(e_l)(k_i - k_j) &= \omega_{il}(e_j)(k_i - k_l) \end{aligned}$$

for different $i, j, l = 1, 2, \ldots, n$.

We let $s_j = \sigma_j(k_1, k_2, \ldots, k_n)$, where σ_j denotes the j-th elementary symmetric function defined by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} a_{i_1} a_{i_2} \ldots a_{i_j},$$

and consider the notation

$$r_i^j = \sigma_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_n).$$

According to the given definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. The function s_k is referred to as the k-th mean curvature of the oriented hypersurface M. The mean curvature $H = \frac{1}{n} s_1$ is also defined, and the Gauss–Kronecker curvature of M is $K = s_n$. If $s_j \equiv 0$, the hypersurface M is known as j-minimal.

In Euclidean $(n+1)$-space, to obtain the i-th curvature formulas $\mathcal{K}_i$ (see [65,66] for details), where $i = 0, \ldots, n$, we have the following characteristic polynomial equation $P_{\mathcal{S}}(\lambda) = 0$ of $\mathcal{S}$:

$$\sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k} = \det(\mathcal{S} - \lambda \mathcal{I}_n) = 0. \tag{1}$$

Here, $\mathcal{I}_n$ indicates the identity matrix. Hence, we reveal the curvature formulas as $\binom{n}{i}\mathcal{K}_i = s_i$.

Let $\mathfrak{x} = \mathfrak{x}(u, v, \alpha, \beta)$ be an immersion from $M^4 \subset \mathbb{E}^4$ to $\mathbb{E}^5_2$.

Definition 1. *An inner product of* $\mathbf{a}^1 = (a^1_1, a^1_2, \ldots, a^1_5), \ldots, \mathbf{a}^2 = (a^2_1, a^2_2, \ldots, a^2_5)$ *of* $\mathbb{E}^5_2$ *is determined by*

$$\left\langle \mathbf{a}^1, \mathbf{a}^2 \right\rangle = a^1_1 a^2_1 - a^1_2 a^2_2 + a^1_3 a^2_3 - a^1_4 a^2_4 + a^1_5 a^2_5.$$

Definition 2. *A quadruple vector product of* $\mathbf{a}^1 = (a^1_1, a^1_2, \ldots, a^1_5), \ldots, \mathbf{a}^4 = (a^4_1, a^4_2, \ldots, a^4_5)$ *of* $\mathbb{E}^5_2$ *is defined by*

$$\mathbf{a}^1 \times \mathbf{a}^2 \times \mathbf{a}^3 \times \mathbf{a}^4 = \det \begin{pmatrix} e_1 & -e_2 & e_3 & -e_4 & e_5 \\ a^1_1 & a^1_2 & a^1_3 & a^1_4 & a^1_5 \\ a^2_1 & a^2_2 & a^2_3 & a^2_4 & a^2_5 \\ a^3_1 & a^3_2 & a^3_3 & a^3_4 & a^3_5 \\ a^4_1 & a^4_2 & a^4_3 & a^4_4 & a^4_5 \end{pmatrix}.$$

Definition 3. *The matrix* $\left(\mathfrak{g}_{ij}\right)^{-1} \cdot \left(\mathfrak{h}_{ij}\right)$ *determines the shape operator matrix* $\mathcal{S}$ *of hypersurface* $\mathfrak{x}$ *in pseudo-Euclidean 5-space* $\mathbb{E}^5_2$*, where,* $\left(\mathfrak{g}_{ij}\right)_{4\times4}$ *and* $\left(\mathfrak{h}_{ij}\right)_{4\times4}$ *describe the first and the second fundamental form matrices, respectively, and* $\mathfrak{g}_{ij} = \langle \mathfrak{x}_i, \mathfrak{x}_j \rangle$, $\mathfrak{h}_{ij} = \langle \mathfrak{x}_{ij}, \mathcal{G} \rangle$, $i, j = 1, 2, \ldots, 4$, $\mathfrak{x}_u = \frac{\partial \mathfrak{x}}{\partial u}$ *when* $i = 1$, $\mathfrak{x}_{uv} = \frac{\partial^2 \mathfrak{x}}{\partial u \partial v}$ *when* $i = 1$ *and* $j = 2$, *etc.,* e_k *denotes the natural base elements of* $\mathbb{E}^5$*, and*

$$\mathcal{G} = \frac{\mathfrak{x}_u \times \mathfrak{x}_v \times \mathfrak{x}_\alpha \times \mathfrak{x}_\beta}{\left\| \mathfrak{x}_u \times \mathfrak{x}_v \times \mathfrak{x}_\alpha \times \mathfrak{x}_\beta \right\|} \tag{2}$$

determines the Gauss map of the hypersurface $\mathfrak{x}$.

3. Curvatures in $\mathbb{E}^5_2$

In this section, we reveal the curvature formulas of any hypersurface $\mathfrak{x} = \mathfrak{x}(u, v, \alpha, \beta)$ *in* $\mathbb{E}^5_2$.

Theorem 1. *A hypersurface* $\mathfrak{x}$ *in* $\mathbb{E}^5_2$ *has the following curvature formulas,* $\mathcal{K}_0 = 1$ *by definition,*

$$4\mathcal{K}_1 = -\frac{\mathfrak{c}_3}{\mathfrak{c}_4},\ 6\mathcal{K}_2 = \frac{\mathfrak{c}_2}{\mathfrak{c}_4},\ 4\mathcal{K}_3 = -\frac{\mathfrak{c}_1}{\mathfrak{c}_4},\ \mathcal{K}_4 = \frac{\mathfrak{c}_0}{\mathfrak{c}_4}, \tag{3}$$

where $\mathfrak{c}_4 \lambda^4 + \mathfrak{c}_3 \lambda^3 + \mathfrak{c}_2 \lambda^2 + \mathfrak{c}_1 \lambda + \mathfrak{c}_0 = 0$ *denotes the characteristic polynomial equation* $P_{\mathcal{S}}(\lambda) = 0$ *of the shape operator matrix* $\mathcal{S}$, $\mathfrak{c}_4 = \det\left(\mathfrak{g}_{ij}\right)$, $\mathfrak{c}_0 = \det\left(\mathfrak{h}_{ij}\right)$*, and* $\left(\mathfrak{g}_{ij}\right)$, $\left(\mathfrak{h}_{ij}\right)$ *are the first and the second fundamental form matrices, respectively.*

Proof. The solution matrix $\left(\mathfrak{g}_{ij}\right)^{-1} \cdot \left(\mathfrak{h}_{ij}\right)$ gives the shape operator matrix $\mathcal{S}$ of hypersurface $\mathfrak{x}$ in pseudo-Euclidean 5-space $\mathbb{E}^5_2$. We reveal the characteristic polynomial equation $\det(\mathcal{S} - \lambda \mathcal{I}_4) = 0$ of $\mathcal{S}$. Thus, we obtain the curvatures

$$\begin{aligned}\binom{4}{0}\mathcal{K}_0 &= 1,\\ \binom{4}{1}\mathcal{K}_1 &= k_1+k_2+k_3+k_4=-\frac{\mathfrak{c}_3}{\mathfrak{c}_4},\\ \binom{4}{2}\mathcal{K}_2 &= k_1k_2+k_1k_3+k_1k_4+k_2k_3+k_2k_4+k_3k_4=\frac{\mathfrak{c}_2}{\mathfrak{c}_4},\\ \binom{4}{3}\mathcal{K}_3 &= k_1k_2k_3+k_1k_2k_4+k_1k_3k_4+k_2k_3k_4=-\frac{\mathfrak{c}_1}{\mathfrak{c}_4},\\ \binom{4}{4}\mathcal{K}_4 &= k_1k_2k_3k_4=\frac{\mathfrak{c}_0}{\mathfrak{c}_4},\end{aligned}$$

□

Definition 4. *A space-like hypersurface $\mathfrak{x}$ is called j-maximal if $\mathcal{K}_j = 0$, where $j = 1, \ldots, 4$.*

Theorem 2. *A hypersurface $\mathfrak{x} = \mathfrak{x}(u, v, \alpha, \beta)$ in $\mathbb{E}^5_2$ has the following relation*

$$\mathcal{K}_0\mathbb{V} - 4\mathcal{K}_1\mathbb{IV} + 6\mathcal{K}_2\mathbb{III}\mathbb{I} - 4\mathcal{K}_3\mathbb{II} + \mathcal{K}_4\mathbb{I} = \mathcal{O}_4,$$

where $\mathbb{I}, \mathbb{II}, \ldots, \mathbb{V}$ determines the fundamental form matrices, and $\mathcal{O}_4$ represents the zero matrix having order 4 of the hypersurface.

Proof. Regarding $n = 4$ in (1), it runs. □

4. Hypersurfaces of Revolution Family in $\mathbb{E}^5_2$

In this section, we define the hypersurfaces of revolution family (*HRF*), then find its differential geometric properties in pseudo-Euclidean 5-space $\mathbb{E}^5_2$. The *HR* in Riemannian space forms were given in [67].

The *HRF M* of Euclidean $(n+1)$-space constructed by a surface γ around rotating axis ℓ does not meet γ is acquired by taking the orbit of ℓ under the orthogonal transformations of $(n+1)$-space.

To construct the *HRF*, we start with the generating surface given by $\gamma = \gamma(u, v) = (f, 0, g, 0, h)$ and apply the rotation matrix $\mathfrak{L} = \operatorname{diag}(\mathfrak{L}_\alpha, \mathfrak{L}_\beta, 1)$ with the elements given by $\mathfrak{L}_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, $t = \alpha, \beta$, respectively, and $\mathfrak{L}\cdot\ell = \ell$, $\det \mathfrak{L} = 1$. Then, we state the *HRF* given by $\mathfrak{x} = \mathfrak{L}\cdot\gamma^T$ when γ rotates about axis $\ell = (0, 0, 0, 0, 1)$. Finally, we present the following.

Definition 5. *The HRF is an immersion $\mathfrak{x} : M^4 \subset \mathbb{E}^4 \longrightarrow \mathbb{E}^5_2$ with rotating axis $(0, 0, 0, 0, 1)$, defined by*

$$\mathfrak{x}(u, v, \alpha, \beta) = (f \cosh\alpha, f \sinh\alpha, g \cosh\beta, g \sinh\beta, h), \tag{4}$$

where f, g, h denote the differentiable functions that depend on $u, v \in \mathbb{R}$, $0 \le \alpha, \beta < 2\pi$.

Taking the first derivatives of *HRF* given by Equation (4) with respect to u, v, α, β, respectively, we obtain the first fundamental form matrix

$$(\mathfrak{g}_{ij}) = \begin{pmatrix} f_u^2 + g_u^2 + h_u^2 & f_u f_v + g_u g_v + h_u h_v & 0 & 0 \\ f_u f_v + g_u g_v + h_u h_v & f_v^2 + g_v^2 + h_v^2 & 0 & 0 \\ 0 & 0 & -f^2 & 0 \\ 0 & 0 & 0 & -g^2 \end{pmatrix}, \tag{5}$$

and $f_u = \frac{\partial f}{\partial u}$, $f_v = \frac{\partial f}{\partial v}$, $f_u^2 = \frac{\partial^2 f}{\partial u^2}$, etc. Hence, $\mathbf{g} = \det(\mathfrak{g}_{ij}) = f^2 g^2 \mathcal{W}$, where $\mathcal{W} = (\mathcal{G}_1)^2 + (\mathcal{G}_2)^2 + (\mathcal{G}_3)^2$, and

$$\begin{aligned} \mathcal{G}_1 &= h_u g_v - h_v g_u, \\ \mathcal{G}_2 &= f_u h_v - f_v h_u, \\ \mathcal{G}_3 &= g_u f_v - f_u g_v. \end{aligned}$$

Since $\mathbf{g} > 0$, the *HRF* determined by Equation (4) is a space-like hypersurface.

Using (2), we obtain the following Gauss map of the *HRF* determined by Equation (4):

$$\mathcal{G} = \frac{1}{\mathcal{W}^{1/2}}(\mathcal{G}_1 \cosh\alpha, \mathcal{G}_1 \sinh\alpha, \mathcal{G}_2 \cosh\beta, \mathcal{G}_2 \sinh\beta, \mathcal{G}_3). \tag{6}$$

By taking the second derivatives with respect to u, v, α, β, of *HRF* described by Equation (4), and by using the Gauss map given by Equation (6), we find the second fundamental form matrix

$$(\mathfrak{h}_{ij}) = \frac{1}{\mathcal{W}}\begin{pmatrix} \mathcal{G}_1 f_{uu} + \mathcal{G}_2 g_{uu} + \mathcal{G}_3 h_{uu} & \mathcal{G}_1 f_{uv} + \mathcal{G}_2 g_{uv} + \mathcal{G}_3 h_{uv} & 0 & 0 \\ \mathcal{G}_1 f_{uv} + \mathcal{G}_2 g_{uv} + \mathcal{G}_3 h_{uv} & \mathcal{G}_1 f_{vv} + \mathcal{G}_2 g_{vv} + \mathcal{G}_3 h_{vv} & 0 & 0 \\ 0 & 0 & f\mathcal{G}_1 & 0 \\ 0 & 0 & 0 & g\mathcal{G}_2 \end{pmatrix}, \tag{7}$$

and $f_{uu} = \frac{\partial^2 f}{\partial u^2}$, $f_{uv} = \frac{\partial^2 f}{\partial u \partial v}$, ect. By using (5) and (7), we compute the following shape operator matrix of (4):

$$\mathcal{S} = \operatorname{diag}\left(\ (\mathfrak{s}_{kl})_{2\times 2}, \quad \mathfrak{s}_{33}, \quad \mathfrak{s}_{44} \ \right),$$

with the following components

$$\begin{aligned}
\mathfrak{s}_{11} &= \frac{(f_v^2 + g_v^2 + h_v^2)(\mathcal{G}_1 f_{uu} + \mathcal{G}_2 g_{uu} + \mathcal{G}_3 h_{uu}) - (f_u f_v + g_u g_v + h_u h_v)(\mathcal{G}_1 f_{uv} + \mathcal{G}_2 g_{uv} + \mathcal{G}_3 h_{uv})}{\mathcal{W}^{3/2}}, \\
\mathfrak{s}_{12} &= \frac{(f_v^2 + g_v^2 + h_v^2)(\mathcal{G}_1 f_{uv} + \mathcal{G}_2 g_{uv} + \mathcal{G}_3 h_{uv}) - (f_u f_v + g_u g_v + h_u h_v)(\mathcal{G}_1 f_{vv} + \mathcal{G}_2 g_{vv} + \mathcal{G}_3 h_{vv})}{\mathcal{W}^{3/2}}, \\
\mathfrak{s}_{21} &= \frac{(f_u^2 + g_u^2 + h_u^2)(\mathcal{G}_1 f_{uv} + \mathcal{G}_2 g_{uv} + \mathcal{G}_3 h_{uv}) - (f_u f_v + g_u g_v + h_u h_v)(\mathcal{G}_1 f_{uu} + \mathcal{G}_2 g_{uu} + \mathcal{G}_3 h_{uu})}{\mathcal{W}^{3/2}}, \\
\mathfrak{s}_{22} &= \frac{(f_u^2 + g_u^2 + h_u^2)(\mathcal{G}_1 f_{vv} + \mathcal{G}_2 g_{vv} + \mathcal{G}_3 h_{vv}) - (f_u f_v + g_u g_v + h_u h_v)(\mathcal{G}_1 f_{uv} + \mathcal{G}_2 g_{uv} + \mathcal{G}_3 h_{uv})}{\mathcal{W}^{3/2}}, \\
\mathfrak{s}_{33} &= -\frac{\mathcal{G}_1}{f\mathcal{W}^{1/2}}, \\
\mathfrak{s}_{44} &= -\frac{\mathcal{G}_2}{g\mathcal{W}^{1/2}}.
\end{aligned}$$

Finally, using (3), with (5) and (7), respectively, we find the curvatures of the *HRF* defined by Equation (4) as follows.

Theorem 3. *Let* $\mathfrak{x}$ *be the HRF determined by Equation* (4) *in* $\mathbb{E}_2^5$. $\mathfrak{x}$ *contains the following curvatures*

$$
\begin{aligned}
\mathcal{K}_0 &= 1, \\
4\mathcal{K}_1 &= \frac{\mathfrak{g}_{11}(\mathfrak{g}_{33}\mathfrak{g}_{44}\mathfrak{h}_{22}+\mathfrak{g}_{22}\mathfrak{g}_{44}\mathfrak{h}_{33}+\mathfrak{g}_{22}\mathfrak{g}_{33}\mathfrak{h}_{44})+\mathfrak{g}_{33}\mathfrak{g}_{44}(\mathfrak{g}_{22}\mathfrak{h}_{11}-2\mathfrak{g}_{12}\mathfrak{h}_{12})-\mathfrak{g}_{12}^2(\mathfrak{g}_{44}\mathfrak{h}_{33}+\mathfrak{g}_{33}\mathfrak{h}_{44})}{\mathfrak{g}_{33}\mathfrak{g}_{44}(\mathfrak{g}_{11}\mathfrak{g}_{22}-\mathfrak{g}_{12}^2)}, \\
6\mathcal{K}_2 &= \frac{\left\{\begin{array}{l} \mathfrak{g}_{11}(\mathfrak{g}_{22}\mathfrak{h}_{33}\mathfrak{h}_{44}+\mathfrak{g}_{33}\mathfrak{h}_{22}\mathfrak{h}_{44}+\mathfrak{g}_{44}\mathfrak{h}_{22}\mathfrak{h}_{33})-2\mathfrak{g}_{12}\mathfrak{h}_{12}(\mathfrak{g}_{44}\mathfrak{h}_{33}+\mathfrak{g}_{33}\mathfrak{h}_{44}) \\ +\mathfrak{h}_{11}(\mathfrak{g}_{33}\mathfrak{g}_{44}\mathfrak{h}_{22}+\mathfrak{g}_{22}\mathfrak{g}_{44}\mathfrak{h}_{33}+\mathfrak{g}_{22}\mathfrak{g}_{33}\mathfrak{h}_{44})-\mathfrak{h}_{12}^2\mathfrak{g}_{33}\mathfrak{g}_{44}-\mathfrak{g}_{12}^2\mathfrak{h}_{33}\mathfrak{h}_{44} \end{array}\right\}}{\mathfrak{g}_{33}\mathfrak{g}_{44}(\mathfrak{g}_{11}\mathfrak{g}_{22}-\mathfrak{g}_{12}^2)}, \\
4\mathcal{K}_3 &= \frac{((\mathfrak{g}_{11}\mathfrak{h}_{22}+2\mathfrak{h}_{12}\mathfrak{g}_{12}-\mathfrak{g}_{22}\mathfrak{h}_{11})\mathfrak{h}_{33}-\mathfrak{g}_{33}\mathfrak{h}_{11}\mathfrak{h}_{22})\mathfrak{h}_{44}-\mathfrak{g}_{44}\mathfrak{h}_{11}\mathfrak{h}_{22}\mathfrak{h}_{33}+\mathfrak{h}_{12}^2(\mathfrak{g}_{44}\mathfrak{h}_{33}+\mathfrak{g}_{33}\mathfrak{h}_{44})}{\mathfrak{g}_{33}\mathfrak{g}_{44}(\mathfrak{g}_{11}\mathfrak{g}_{22}-\mathfrak{g}_{12}^2)}, \\
\mathcal{K}_4 &= \frac{(\mathfrak{h}_{11}\mathfrak{h}_{22}-\mathfrak{h}_{12}^2)\mathfrak{h}_{33}\mathfrak{h}_{44}}{(\mathfrak{g}_{11}\mathfrak{g}_{22}-\mathfrak{g}_{12}^2)\mathfrak{g}_{33}\mathfrak{g}_{44}}.
\end{aligned}
$$

Here, $\mathcal{K}_1$ represents the mean curvature, and $\mathcal{K}_4$ denotes the Gauss–Kronecker curvature.

Proof. By using the Cayley–Hamilton theorem, we reveal the following characteristic polynomial equation $P_{\mathcal{S}}(\lambda) = 0$ of $\mathcal{S}$:

$$\mathcal{K}_0\lambda^4 - 4\mathcal{K}_1\lambda^3 + 6\mathcal{K}_2\lambda^2 - 4\mathcal{K}_3\lambda + \mathcal{K}_4 = 0.$$

The curvatures $\mathcal{K}_i$ of $\mathfrak{x}$ are obtained by the above equations, where $\mathfrak{g}_{33} \neq 0$, $\mathfrak{g}_{44} \neq 0$, $\mathfrak{g}_{11}\mathfrak{g}_{22} - \mathfrak{g}_{12}^2 \neq 0$. □

Theorem 4. *Let $\mathfrak{x}$ be the HRF described by Equation (4) in $\mathbb{E}_2^5$. $\mathfrak{x}$ has the following principal curvatures*

$$
\begin{aligned}
k_1 &= \frac{1}{2}\left(\mathfrak{s}_{11}+\mathfrak{s}_{22}-\sqrt{(\mathfrak{s}_{11}-\mathfrak{s}_{22})^2+4\mathfrak{s}_{12}\mathfrak{s}_{21}}\right), \\
k_2 &= \frac{1}{2}\left(\mathfrak{s}_{11}+\mathfrak{s}_{22}+\sqrt{(\mathfrak{s}_{11}-\mathfrak{s}_{22})^2+4\mathfrak{s}_{12}\mathfrak{s}_{21}}\right), \\
k_3 &= \mathfrak{s}_{33} = -\frac{\mathcal{G}_1}{f\mathcal{W}^{1/2}}, \\
k_4 &= \mathfrak{s}_{44} = -\frac{\mathcal{G}_2}{g\mathcal{W}^{1/2}}.
\end{aligned}
$$

Proof. By using equation $\det(\mathcal{S} - k\mathcal{I}_4) = 0$, the theorem is clear. □

Corollary 1. *Let $\mathfrak{x}$ be the HRF defined by Equation (4) in $\mathbb{E}_2^5$. $\mathfrak{x}$ is 1-maximal if and only if the following partial differential equation appears*

$$
\begin{aligned}
0 &= \mathfrak{g}_{11}(\mathfrak{g}_{33}\mathfrak{g}_{44}\mathfrak{h}_{22}+\mathfrak{g}_{22}\mathfrak{g}_{44}\mathfrak{h}_{33}+\mathfrak{g}_{22}\mathfrak{g}_{33}\mathfrak{h}_{44}) \\
&\quad +\mathfrak{g}_{33}\mathfrak{g}_{44}(\mathfrak{g}_{22}\mathfrak{h}_{11}-2\mathfrak{g}_{12}\mathfrak{h}_{12})-\mathfrak{g}_{12}^2(\mathfrak{g}_{44}\mathfrak{h}_{33}+\mathfrak{g}_{33}\mathfrak{h}_{44}).
\end{aligned}
$$

Corollary 2. *Let $\mathfrak{x}$ be the HRF determined by Equation (4) in $\mathbb{E}_2^5$. $\mathfrak{x}$ is 2-maximal if and only if the following partial differential equation occurs*

$$
\begin{aligned}
0 &= \mathfrak{g}_{11}(\mathfrak{g}_{22}\mathfrak{h}_{33}\mathfrak{h}_{44}+\mathfrak{g}_{33}\mathfrak{h}_{22}\mathfrak{h}_{44}+\mathfrak{g}_{44}\mathfrak{h}_{22}\mathfrak{h}_{33})-2\mathfrak{g}_{12}\mathfrak{h}_{12}(\mathfrak{g}_{44}\mathfrak{h}_{33}+\mathfrak{g}_{33}\mathfrak{h}_{44}) \\
&\quad +\mathfrak{h}_{11}(\mathfrak{g}_{33}\mathfrak{g}_{44}\mathfrak{h}_{22}+\mathfrak{g}_{22}\mathfrak{g}_{44}\mathfrak{h}_{33}+\mathfrak{g}_{22}\mathfrak{g}_{33}\mathfrak{h}_{44})-\mathfrak{h}_{12}^2\mathfrak{g}_{33}\mathfrak{g}_{44}-\mathfrak{g}_{12}^2\mathfrak{h}_{33}\mathfrak{h}_{44}.
\end{aligned}
$$

Corollary 3. *Let $\mathfrak{x}$ be the HRF given by Equation (4) in $\mathbb{E}^5_2$. $\mathfrak{x}$ is 3-maximal if and only if the following partial differential equation holds*

$$\begin{aligned} 0 &= ((\mathfrak{g}_{11}\mathfrak{h}_{22} + 2\mathfrak{h}_{12}\mathfrak{g}_{12} - \mathfrak{g}_{22}\mathfrak{h}_{11})\mathfrak{h}_{33} - \mathfrak{g}_{33}\mathfrak{h}_{11}\mathfrak{h}_{22})\mathfrak{h}_{44} \\ &\quad -\mathfrak{g}_{44}\mathfrak{h}_{11}\mathfrak{h}_{22}\mathfrak{h}_{33} + \mathfrak{h}_{12}^2(\mathfrak{g}_{44}\mathfrak{h}_{33} + \mathfrak{g}_{33}\mathfrak{h}_{44}). \end{aligned}$$

Corollary 4. *Let $\mathfrak{x}$ be the HRF described by Equation (4) in $\mathbb{E}^5_2$. $\mathfrak{x}$ is 4-maximal if and only if the following partial differential equation determines*

$$0 = \left(\mathfrak{h}_{11}\mathfrak{h}_{22} - \mathfrak{h}_{12}^2\right)\mathfrak{h}_{33}\mathfrak{h}_{44}.$$

Hence, we find the following.

Example 1. *Let $\mathfrak{x}$ be the HRF determined by Equation (4) in $\mathbb{E}^5_2$. When the profile hypersurface γ of $\mathfrak{x}$ is parameterized by the unit sphere: $f = \cos u \cos v$, $g = \sin u \cos v$, $h = \sin v$, then $\mathcal{S} = \mathcal{I}_4$ and the HRF has the following curvatures $\mathcal{K}_i = 1$, where $i = 0, 1, \ldots, 4$.*

Example 2. *Assume $\mathfrak{x}$ be the HRF denoted by Equation (4) in $\mathbb{E}^5_2$. While the profile hypersurface γ of $\mathfrak{x}$ is parameterized by the rational unit sphere: $f = \frac{1-u^2}{1+u^2}\frac{1-v^2}{1+v^2}$, $g = \frac{2u}{1+u^2}\frac{1-v^2}{1+v^2}$, $h = \frac{2v}{1+v^2}$, the HRF has the same results determined by Example 1.*

Example 3. *Let $\mathfrak{x}$ be the HRF defined by Equation (4) in $\mathbb{E}^5_2$. When the generating hypersurface γ of $\mathfrak{x}$ is parameterized by the Riemann sphere: $f = \frac{2u}{u^2+v^2+1}$, $g = \frac{2v}{u^2+v^2+1}$, $h = \frac{u^2+v^2-1}{u^2+v^2+1}$, the HRF has $\mathcal{S} = -\mathcal{I}_4$ and has the following curvatures $\mathcal{K}_i = (-1)^i$, where $i = 0, 1, \ldots, 4$.*

Example 4. *Considering the pseudo-hypersphere $\mathbb{S}^4_2(\rho) := \{\mathbf{q} \in \mathbb{E}^5_2 \mid \langle \mathbf{q}, \mathbf{q} \rangle = \rho^2\}$, radius $\rho > 0$, parameterized by*

$$\mathbf{q}(u, v, \alpha, \beta) = \begin{pmatrix} \rho \cos u \cos v \cosh \alpha \\ \rho \cos u \cos v \sinh \alpha \\ \rho \sin u \cos v \cosh \beta \\ \rho \sin u \cos v \sinh \beta \\ \rho \sin v \end{pmatrix}, \tag{8}$$

we compute $\mathcal{S} = \frac{1}{\rho}\mathcal{I}_4$. Hence, we find the following curvatures $\mathcal{K}_i = \frac{1}{\rho^i}$, where $i = 0, 1, \ldots, 4$. Then, the hypersurface $\mathbf{q}$ described by Equation (8) is an umbilical hypersphere (i.e., $(\mathcal{K}_1)^4 = \mathcal{K}_4$) of $\mathbb{E}^5_2$.

5. Hypersurfaces of Revolution Family with $\Delta\mathfrak{x} = \mathcal{A}\mathfrak{x}$ in $\mathbb{E}^5_2$

In this section, our focus is on the Laplace–Beltrami operator of a smooth function in $\mathbb{E}^5_2$. We will proceed to compute it utilizing the *HRF*, which is defined by Equation (4).

By using the inverse matrix of the first fundamental form matrix $(\mathfrak{g}_{ij})_{4\times 4}$, we have the following.

Definition 6. *The Laplace–Beltrami operator of a smooth function $\varphi = \varphi(x^1, x^2, x^3, x^4) \mid_{\mathcal{D}}$ ($\mathcal{D} \subset \mathbb{R}^4$) of class C^4 depends on the first fundamental form $(\mathfrak{g}_{ij})$ of a hypersurface $\mathfrak{x}$, is the operator defined by*

$$\Delta\varphi = \frac{1}{\mathbf{g}^{1/2}} \sum_{i,j=1}^{4} \frac{\partial}{\partial x^i}\left(\mathbf{g}^{1/2}\mathfrak{g}^{ij}\frac{\partial \varphi}{\partial x^j}\right), \tag{9}$$

where $(\mathfrak{g}^{ij}) = (\mathfrak{g}_{kl})^{-1}$ and $\mathbf{g} = \det(\mathfrak{g}_{ij})$.

Therefore, the Laplace–Beltrami operator of the *HRF* given by Equation (4) is determined by

$$\begin{aligned}\Delta\mathfrak{x} \;=\;& \frac{1}{\mathbf{g}^{1/2}}\left[\frac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathfrak{g}^{11}\frac{\partial\mathfrak{x}}{\partial u}\right)+\frac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathfrak{g}^{12}\frac{\partial\mathfrak{x}}{\partial v}\right)+\frac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathfrak{g}^{21}\frac{\partial\mathfrak{x}}{\partial u}\right)\right.\\ &\left.+\frac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathfrak{g}^{22}\frac{\partial\mathfrak{x}}{\partial v}\right)+\frac{\partial}{\partial \alpha}\left(\mathbf{g}^{1/2}\mathfrak{g}^{33}\frac{\partial\mathfrak{x}}{\partial \alpha}\right)+\frac{\partial}{\partial \beta}\left(\mathbf{g}^{1/2}\mathfrak{g}^{44}\frac{\partial\mathfrak{x}}{\partial \beta}\right)\right].\end{aligned} \tag{10}$$

By using the derivatives of the functions in (10), with respect to u, v, α, β, respectively, we obtain the following.

Theorem 5. *The Laplace–Beltrami operator of the HRF $\mathfrak{x}$ denoted by Equation (4) is given by $\Delta\mathfrak{x} = 4\mathcal{K}_1\mathcal{G}$, where $\mathcal{K}_1$ describes the mean curvature and $\mathcal{G}$ represents the Gauss map of $\mathfrak{x}$.*

Proof. By direct computing (10), we obtain $\Delta\mathfrak{x}$. □

Theorem 6. *Let $\mathfrak{x}$ be the HRF defined by Equation (4). $\Delta\mathfrak{x} = \mathcal{M}\mathfrak{x}$, where $\mathcal{M}$ represents the squared matrix of order 5 if and only if $\mathfrak{x}$ has $\mathcal{K}_1 = 0$, i.e., it is a 1-maximal hypersurface.*

Proof. We find $4\mathcal{K}_1\mathcal{G} = \mathcal{M}\mathfrak{x}$, and then we have

$$\begin{aligned}
& m_{11} f\cosh\alpha + m_{12} f\sinh\alpha + m_{13} g\cosh\beta + a_{14} g\sinh\beta + a_{15} h\\
=\;& \Phi f g (h_u g_v - h_v g_u)\cosh\alpha,\\
& m_{21} f\cosh\alpha + m_{22} f\sinh\alpha + m_{23} g\cosh\beta + a_{24} g\sinh\beta + a_{25} h\\
=\;& \Phi f g (h_u g_v - h_v g_u)\sinh\alpha,\\
& m_{31} f\cosh\alpha + m_{32} f\sinh\alpha + m_{33} g\cosh\beta + a_{34} g\sinh\beta + a_{35} h\\
=\;& \Phi f g (f_u h_v - f_v h_u)\cosh\beta,\\
& m_{41} f\cosh\alpha + m_{42} f\sinh\alpha + m_{43} g\cosh\beta + a_{44} g\sinh\beta + a_{45} h\\
=\;& \Phi f g (f_u h_v - f_v h_u)\sinh\beta,\\
& m_{51} f\cosh\alpha + m_{52} f\sinh\alpha + m_{53} g\cosh\beta + a_{54} g\sinh\beta + a_{55} h\\
=\;& \Phi f g (g_u f_v - f_u g_v).
\end{aligned}$$

Here, $\mathcal{M} = (m_{ij})_{5\times 5}$, and $\Phi = 4\mathcal{K}_1\mathbf{g}^{-1/2}$, where $\mathbf{g} = f^2g^2\mathcal{W}$. Derivativing above ODEs twice with respect to α, we obtain the following $m_{i5} = 0$, $\Phi = 0$, where $i = 1, 2, \ldots, 5$. Then, we obtain $(m_{i1}\cosh\alpha + m_{i2}\sinh\alpha)f = 0$, where $i = 1, 2, \ldots, 5$. The functions cosh and sinh are linear independent on α, then all the components of the matrix $\mathcal{M}$ are 0. Since $\Phi = 4\mathcal{K}_1\mathbf{g}^{-1/2}$, then $\mathcal{K}_1 = 0$. This means that $\mathfrak{x}$ is a 1-maximal *HRF*. □

Hence, we obtain the following.

Example 5. *Let $\mathfrak{x}$ be the HRF given by Equation (4). When the generating hypersurface γ of $\mathfrak{x}$ is parameterized by the unit sphere determined by Example 1, then, HRF $\mathfrak{x}$ supplies $\Delta\mathfrak{x} = \mathcal{A}\mathfrak{x}$, where $\mathcal{A} = -5\mathcal{I}_4$, and $\mathcal{I}_4$ denotes the identity matrix.*

Example 6. *Let $\mathfrak{x}$ be the HRF denoted by Equation (4). While the generating hypersurface γ of $\mathfrak{x}$ is parameterized by the Riemann sphere defined by Example 3, HRF $\mathfrak{x}$ has the same results denoted by Example 5.*

6. Conclusions

In conclusion, this research has introduced a novel and distinct family of hypersurfaces characterized by four parameters in the five-dimensional pseudo-Euclidean space $\mathbb{E}^5_2$.

By performing comprehensive computations, we have determined the fundamental form, Gauss map, and shape operator matrices associated with this family, enabling a comprehensive understanding of its geometric attributes. The application of the Cayley–Hamilton theorem has allowed us to ascertain the curvatures of these hypersurfaces, providing crucial information about their intrinsic geometry. Furthermore, we have identified the conditions for maximality within this specific context, shedding light on the behavior and potential optimality of these hypersurfaces. Notably, we have uncovered a captivating connection between the Laplace–Beltrami operator of this family and a 5×5 matrix, enhancing our comprehension of their intricate relationship.

The findings presented in this study contribute significantly to the field of five-dimensional pseudo-Euclidean geometry, expanding the existing knowledge and paving the way for further exploration and advancements in this area of research.

Author Contributions: E.G. gave the idea of the hypersurfaces of revolution family in 5-dimensional pseudo-Euclidean space. Conceptualization, Y.L. and E.G.; methodology, Y.L. and E.G.; software, Y.L. and E.G.; validation, Y.L. and E.G.; investigation, Y.L. and E.G.; resources, Y.L. and E.G.; data curation, Y.L. and E.G.; writing—original draft preparation, Y.L. and E.G.; writing—review and editing, Y.L. and E.G.; visualization, Y.L. and E.G.; supervision, Y.L. and E.G.; funding acquisition, Y.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data were created.

Acknowledgments: We gratefully acknowledge the constructive comments from the editor and the anonymous referees.

Conflicts of Interest: The authors declare no conflict of interest.

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