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Inverse Problem for a Fourth-Order Hyperbolic Equation with a Complex-Valued Coefficient

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Abstract: This paper studies the existence and uniqueness of the classical solution of inverse problems for a fourth-order hyperbolic equation with a complex-valued coefficient with Dirichlet and Neumann boundary conditions. Using the method of separation of variables, formal solutions are obtained in the form of a Fourier series in terms of the eigenfunctions of a non-self-adjoint fourth-order ordinary differential operator. The proofs of the uniform convergence of the Fourier series are based on estimates of the norms of the derivatives of the eigenfunctions of a fourth-order ordinary differential operator and the uniform boundedness of the Riesz bases of the eigenfunctions.

Keywords: fourth-order hyperbolic equations; inverse problem; eigenfunction; the Riesz basis; the Fourier method

MSC: 35L35; 35N30; 35E99; 34L10

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1. Introduction

Various type of inverse problems for equations of mathematical physics have been investigated by many authors, see for instance [1,2] and references therein. It can also be noted that inverse problems for general parabolic equations were studied in [3–5].

Interest in problems for hyperbolic differential equations is due to their numerous applications. Various hyperbolic differential equations have been investigated in recent papers [6–17]. For example, in [8], inverse problems of finding the right-hand side and the initial conditions for a fourth-order hyperbolic equation

$$u_{tt}(x,t) + \alpha^2 \frac{\partial^4}{\partial x^4} u(x,t) = F(x,t)$$

are studied, and in [11], inverse problems for a fourth-order hyperbolic equation

$$u_{tt}(x,t) - \frac{\partial^4}{\partial x^4} u(x,t) + q(t)u(x,t) = f(x,t)$$

with a variable coefficient depending on t are considered. In works [18–20] (and references therein), the inverse problems for the fourth-order equations with a fractional differential operator are investigated.

However, in all the papers listed, equations with constant coefficients are studied. As for fourth-order hyperbolic equations with variable coefficients depending on x, inverse problems have not yet been studied.

Second-order differential equations with variable coefficients were considered in the works [12,21,22]. Direct problems for the heat equation and the wave equation with complex-valued coefficients were studied.

Note that the method of separation of variables gives an explicit form of the solution, but it imposes more stringent requirements on the original function. However, in this work, the problem of reducing the smoothness of the initial function was not posed.

The novelty of the work lies in the fact that the present paper is devoted to the study of the existence and uniqueness of the solution of inverse problems for a fourth-order hyperbolic equation with a complex-valued coefficient. Equations of this type describe the problems of the oscillations of rods, beams, and plates, the theory of the stability of running shafts and ship oscillations, etc. More detailed information on the development and applications of the fourth-order equations can be found in [6–11] and references therein. Let us consider an equation

$$u_{tt}(x,t) + \frac{\partial^4}{\partial x^4}u(x,t) + q(x)u(x,t) = f(x), \tag{1}$$

where $q(x) = q_1(x) + iq_2(x)$ is a known function, and $u(x,t)$ and $f(x)$ are unknown functions, in the open region $\Omega = \{-1 < x < 1,\ 0 < t < T\}$. The symbol $\bar{\Omega} = \{-1 \leq x \leq 1, 0 \leq t \leq T\}$ denotes the closed region. The space $C^{k,l}_{x,t}(\Omega)$ consists of all functions $u(x,t)$ that have continuous derivatives with respect to x and t of the order of k and l, respectively, in the domain Ω.

The purpose of this paper is to study the following inverse problems.

Inverse problem D. Find a pair of functions $u(x,t) \in C^{4,2}_{x,t}(\Omega) \bigcap C^{2,1}_{x,t}(\bar{\Omega})$ and $f(x) \in C[-1,1]$, satisfying Equation (1) in the domain Ω, Dirichlet boundary conditions

$$u(-1,t) = 0,\ u(1,t) = 0,\ u_{xx}(-1,t) = 0,\ u_{xx}(1,t) = 0,\ t \in [0,T], \tag{2}$$

and conditions

$$u(x,0) = \varphi(x),\ u_t(x,0) = 0,\ u(x,T) = \psi(x),\ x \in [-1,1], \tag{3}$$

where $\varphi(x)$ and $\psi(x)$ are known sufficiently smooth functions.

The third additional condition in Equation (3) is related to the statement of the inverse problem.

Inverse problem N. Find a pair of functions $u(x,t) \in C^{4,2}_{x,t}(\Omega) \bigcap C^{3,1}_{x,t}(\bar{\Omega})$ and $f(x) \in C[-1,1]$, satisfying Equation (1) in the domain Ω, Neumann boundary conditions

$$u_x(-1,t) = 0,\ u_x(1,t) = 0,\ u_{xxx}(-1,t) = 0,\ u_{xxx}(1,t) = 0,\ t \in [0,T], \tag{4}$$

and conditions satisfying Equation (3).

Definition 1. *A pair of functions $u(x,t) \in C^{4,2}_{x,t}(\Omega)$ and $f(x) \in C(-1,1)$ is called a regular solution of Equation* (1)*, if they satisfy this equation in region Ω.*

Definition 2. *The regular solution of Equation* (1) *is called a classical solution of inverse problem D, if the conditions of Equations* (2) *and* (3) *and $u(x,t) \in C^{4,2}_{x,t}(\Omega) \bigcap C^{2,1}_{x,t}(\bar{\Omega})$, $f(x) \in C[-1,1]$ are satisfied.*

Definition 3. *The regular solution of Equation* (1) *is called a classical solution of inverse problem N, if the conditions of Equations* (3) *and* (4) *and $u(x,t) \in C^{4,2}_{x,t}(\Omega) \bigcap C^{3,1}_{x,t}(\bar{\Omega})$, $f(x) \in C[-1,1]$ are satisfied.*

If the right side of Equation (1) has the form $f(x)g(t)$, where $f(x)$ is an unknown function and $g(t)$ is a known function, the existence and uniqueness of the solution to the inverse problem may depend on the third additional condition in Equation (3). For the case of a model problem when in Equation (1) $q(x) \equiv 0$, see for instance [8].

We thus seek a solution to inverse problem D in the form of a Fourier series [23]

$$u(x,t) = \sum_{k=0}^{\infty} C_k(t) X_k(x), \tag{5}$$

$$f(x) = \sum_{k=0}^{\infty} f_k X_k(x), \tag{6}$$

$$C_k(t) = \int_{-1}^{1} u(x,t)\bar{Z}_k(x)dx, \;\; f_k = \int_{-1}^{1} f(x)\bar{Z}_k(x)dx, \tag{7}$$

where $C_k(t)$ are unknown functions and f_k are unknown constants, by eigenfunctions of a non-self-conjugate boundary value problem

$$L_q X \equiv X^{IV}(x) + q(x)X(x) = \lambda X(x), \tag{8}$$

$$X(-1) = X(1) = X''(-1) = X''(1) = 0. \tag{9}$$

We write the problem conjugate to the boundary value problem of Equations (8) and (9) in the form

$$L_q^* Z \equiv Z^{IV}(x) + \bar{q}(x)Z(x) = \bar{\lambda} Z(x), \tag{10}$$

with boundary conditions in the form of Equation (9). The possibility of representing the solution of inverse problem D in the form of series Equations (5) and (6) depends on the properties of the eigenfunctions of the boundary value problem of Equations (8) and (9).

2. Properties of Eigenfunctions of Spectral Problems

It is more convenient to study the convergence of the expansions of continuous functions in terms of the eigenfunctions of the boundary value problem of Equations (8) and (9) if the system of its eigenfunctions $\{X_k(x)\}$ forms a Riesz basis in the class $L_2(-1,1)$. Therefore, in this section, we study the basis property of eigenfunctions $\{X_k(x)\}$.

Consider linear forms $U_i(y), \; i = 1,2,3,4,$

$$U_i(y) = a_{i1}y'''(-1) + a_{i2}y'''(1) + a_{i3}y''(-1) + a_{i4}y''(1) + a_{i5}y'(-1) + a_{i6}y'(1) +$$
$$+ a_{i7}y(-1) + a_{i8}y(1),$$

defining boundary conditions $U_i(y) = 0, \; i = 1,2,3,4,$ of a general form for the equation $L_q y(x) = \lambda y(x)$, where a_{ij} are given complex coefficients. Assume that the linear forms $U_1(y)$, $U_2(y)$, $U_3(y)$, $U_4(y)$ are linearly independent. The order of the highest derived form is called the order of the form. Then in the boundary conditions $U_i(y) = 0, \; i = 1,2,3,4,$ the maximum number of forms of order 3 is not higher than two. It is not difficult to transform them to the form

$$a_{11}y'''(-1) + a_{12}y'''(1) + a_{13}y''(-1) + a_{14}y''(1) + a_{15}y'(-1) + a_{16}y'(1) +$$
$$+ a_{17}y(-1) + a_{18}y(1) = 0,$$
$$a_{21}y'''(-1) + a_{22}y'''(1) + a_{23}y''(-1) + a_{24}y''(1) + a_{25}y'(-1) + a_{26}y'(1) +$$
$$+ a_{27}y(-1) + a_{28}y(1) = 0,$$
$$a_{33}y''(-1) + a_{34}y''(1) + a_{35}y'(-1) + a_{36}y'(1) + a_{37}y(-1) + a_{38}y(1) = 0,$$

$$a_{43}y''(-1)+a_{44}y''(1)+a_{45}y'(-1)+a_{46}y'(1)+a_{47}y(-1)+a_{48}y(1)=0, \tag{11}$$

called normalized boundary conditions [24]. For the sake of simplicity, we do not change the notation of the coefficients. We perform similar operations if the order of the highest derivative of the forms is less than 3.

Let us consider some well-known facts. Let $\lambda=\rho^4$. In the complex ρ-plane, consider a fixed region defined by the inequality $\frac{\nu\pi}{4}\le\arg\rho\le\frac{(\nu+1)\pi}{4}$. We enumerate different roots of the number $\sqrt[4]{-1}$ as $\omega_1,\omega_2,\omega_3,\omega_4$ so that for $\rho\in S_\nu$, $\mathrm{Re}(\rho\omega_1)\le\mathrm{Re}(\rho\omega_2)\le\mathrm{Re}(\rho\omega_3)\le\mathrm{Re}(\rho\omega_4)$.

It is well known that the normalized boundary conditions of Equation (11) are called regular (see, for example [24]) if the numbers θ_{-1}, θ_1 defined by the equality

$$\frac{\theta_{-1}}{s}+\theta_0+\theta_1 s=\begin{vmatrix} a_{11}\omega_1^3 & (a_{11}+sa_{12})\omega_2^3 & \left(a_{11}+\frac{1}{s}a_{12}\right)\omega_3^3 & a_{12}\omega_4^3 \\ a_{21}\omega_1^3 & (a_{21}+sa_{22})\omega_2^3 & \left(a_{21}+\frac{1}{s}a_{22}\right)\omega_3^3 & a_{22}\omega_4^3 \\ a_{33}\omega_1^2 & (a_{33}+sa_{34})\omega_2^2 & \left(a_{33}+\frac{1}{s}a_{34}\right)\omega_3^2 & a_{34}\omega_4^2 \\ a_{43}\omega_1^2 & (a_{43}+sa_{44})\omega_2^2 & \left(a_{43}+\frac{1}{s}a_{44}\right)\omega_3^2 & a_{44}\omega_4^2 \end{vmatrix}$$

are not equal to zero. Here the degree of the number ω_j is equal to the order of the highest derivative of the corresponding boundary condition. We proceed similarly if the order of the highest derivative of the forms is less than 3. If the additional condition $\theta_0^2-4\theta_{-1}\theta_1\neq 0$ is satisfied, then the boundary conditions of Equation (11) are called strongly regular. Note that the boundary value problem of Equation (8) with strongly regular boundary conditions can only have a finite number of multiple eigenvalues.

Papers [25,26] imply the following important theorem.

Theorem 1 ([25,26])**.** *If $q(x)\in L_1(-1,1)$, then the eigenfunctions and associated functions of the boundary value problem, Equation (8), with strongly regular boundary conditions, Equation (11), form the Riesz basis in the space $L_2(-1,1)$.*

It is easy to check that the boundary conditions Equations (2) and (4) are strongly regular, hence, the system of eigenfunctions $\{X_k(x)\}$ of the boundary value problem Equations (8) and (9) forms the Riesz basis in the space $L_2(-1,1)$. This is also valid for the system of eigenfunctions $\{Z_k(x)\}$ of the conjugate boundary value problem of Equations (9) and (10), where the systems of eigenfunctions $\{X_k(x)\}$ and $\{Z_k(x)\}$ satisfy the biorthogonality condition [24]

$$(X_k,Z_n)=\int_{-1}^{1}X_k(x)\bar{Z}_n(x)dx=\delta_{kn},$$

where δ_{kn} is the Kronecker symbol. Furthermore, we assume that all eigenvalues of the boundary value problem of Equations (8) and (9) are simple and zero is not an eigenvalue.

In the case of positive self-conjugate operators, the eigenvalues are real and positive. In the case of non-self-conjugate operators, the eigenvalues can be complex numbers. Therefore, it is necessary to study the condition of the non-negativity of their real parts.

Lemma 1. *Let $q(x)\in C[-1,1]$. Then the inequality $|\mathrm{Im}\ \lambda_k|\le\max|q_2(x)|$ holds for all eigenvalues λ_k of the boundary value problem of Equations (8) and (9). For the additional condition $\mathrm{Re}\ q(x)=q_1(x)\ge 0$ in the interval $-1\le x\le 1$, the inequality $\mathrm{Re}\ \lambda_k>0$ is valid for all eigenvalues λ_k of the boundary value problem of Equations (8) and (9).*

Proof. Let λ_k be eigenvalues of the boundary value problem of Equations (8) and (9) and $X_k(x)$ be the corresponding eigenfunctions. We multiply both parts of Equation (8) by the complex conjugate function $\bar{X}_k(x)$ and integrate the resulting equality twice by parts over the interval $(-1,1)$. We obtain the equality

$$\int_{-1}^{1} \left|X''_k(x)\right|^2 dx + \int_{-1}^{1} q(x)|X_k(x)|^2 dx = \lambda_k \int_{-1}^{1} |X_k(x)|^2 dx.$$

Writing out the imaginary and real parts of the last equality separately, we obtain the following two relations:

$$\int_{-1}^{1} q_2(x)|X_k(x)|^2 dx = \operatorname{Im}\lambda_k \int_{-1}^{1} |X_k(x)|^2 dx,$$

$$\int_{-1}^{1} \left|X''_k(x)\right|^2 dx + \int_{-1}^{1} q_1(x)|X_k(x)|^2 dx = \operatorname{Re}\lambda_k \int_{-1}^{1} |X_k(x)|^2 dx.$$

The first assertion of the lemma follows from the first equality.

To prove the second assertion of the lemma, we assume the contrary. Let there be a subsequence $\{\lambda_{n_k}\}$ satisfying the condition $\operatorname{Re}\lambda_{n_k} < 0$. Then from the second relation, we obtain the inequality

$$\int_{-1}^{1} \left|X''_k(x)\right|^2 dx + \int_{-1}^{1} q_1(x)|X_k(x)|^2 dx = \operatorname{Re}\lambda_k \int_{-1}^{1} |X_k(x)|^2 dx < 0.$$

It means that

$$\int_{-1}^{1} \left|X''_k(x)\right|^2 dx + \int_{-1}^{1} q_1(x)|X_k(x)|^2 dx < 0,$$

which contradicts the condition $q_1(x) \geq 0$. The lemma is proved. □

Note that this lemma is also valid for continuous $q(x) \in C[-1,1]$. In this case, $\operatorname{Re}\lambda_k > 0$, starting from some number k_0:$\operatorname{Re}\lambda_k \geq |\min q_1(x)|$ at $k \geq k_0$, if $\min q_1(x) < 0$.

Let the set $D(L_q)$ consist of functions $\phi(x) \in C^4(-1,1)\bigcap C^2[-1,1]$ satisfying the boundary conditions of Equation (2), and the set $N(L_q)$ consist of functions $\phi(x) \in C^4(-1,1)\bigcap C^3[-1,1]$ satisfying the boundary conditions of Equation (4).

Lemma 2. *For any function $\varphi \in D(L_q)$ $\left(\text{and } \varphi \in N(L_q)\right)$, each of the Fourier series*

$$\varphi(x) = \sum_{k=1}^{\infty} (\varphi, Z_k) X_k(x), \;\; \varphi(x) = \sum_{k=1}^{\infty} (\varphi, X_k) Z_k(x) \tag{12}$$

in eigenfunctions $\{X_k(x)\}$, $\{Z_k(x)\}$ converges uniformly for $-1 \leq x \leq 1$.

Proof. We rewrite Equation (8) in the form (the number $\lambda = 0$ is not an eigenvalue)

$$X_k(x) = \frac{X_k^{IV}(x) + q(x)X_k(x)}{\lambda_k}.$$

Then

$$(\varphi, X_k) = \int_{-1}^{1} \varphi(x)\bar{X}_k(x)dx = \int_{-1}^{1} \varphi(x)\frac{\bar{X}_k^{IV}(x) + \bar{q}(x)\bar{X}_k(x)}{\bar{\lambda}_k}dx$$

$$= \frac{1}{\overline{\lambda_k}} \int\limits_{-1}^{1} \left[\varphi^{IV}(x) + q(x)\varphi(x)\right] \bar{X}_k(x)dx = \frac{1}{\overline{\lambda_k}} \left(L_q\varphi, X_k\right),$$

where $L_q\varphi = \varphi^{IV}(x) + q(x)\varphi(x)$. Using this relation, the second row in Equation (12) can be written as

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{A_k}{\overline{\lambda_k}} Z_k(x), \tag{13}$$

where

$$A_k = \int\limits_{-1}^{1} \left[\varphi^{IV}(x) + q(x)\varphi(x)\right] \bar{X}_k(x)dx.$$

On the other hand, it is well known that the conjugate spectral problem is equivalent to the integral equation

$$Z_k(x) = \overline{\lambda_k} \int\limits_{-1}^{1} G^*(x,t)\bar{Z}_k(t)dt,$$

where $G^*(x,t)$ is Green's function of the conjugate boundary value problem for $\lambda = 0$. It is known [24] that the Green's function $G^*(x,t)$ is continuous for $x \in [-1,1]$ and $t \in [-1,1]$ and is therefore bounded. Let us denote $C_k(x) = \int\limits_{-1}^{1} G^*(x,t)\bar{Z}_k(t)dt$. Then equality Equation (13) takes the form

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{A_k}{\overline{\lambda_k}} Z_k(x) = \sum_{k=1}^{\infty} A_k C_k(x),$$

whence, using the inequality $ab \le \frac{1}{2}\left(a^2 + b^2\right)$, we obtain the estimate

$$\sum_{k=1}^{\infty} \left| \frac{A_k}{\overline{\lambda_k}} Z_k(x) \right| = \sum_{k=1}^{\infty} |A_k C_k(x)| \le \frac{1}{2} \sum_{k=1}^{\infty} |A_k|^2 + \frac{1}{2} \sum_{k=1}^{\infty} |C_k(x)|^2. \tag{14}$$

As A_k are the Fourier coefficients of the expansion in the Riesz basis $Z_k(x)$, $k = 1,2,3,\cdots$, and $C_k(x)$ are the Fourier coefficients of the expansion of the Green's function $G^*(x,t)$ in the Riesz basis $\{X_k(x)\}$, then due to the Bessel inequality for the Riesz bases, both series on the right side of inequality (14) converge and

$$\exists M_1 : \sum_{k=1}^{\infty} |C_k(x)|^2 \le M_1 \int\limits_{-1}^{1} |G^*(x,t)|^2 dt \le M_0, \ \forall x \in [-1,1],$$

where $M_0 = M_1 \max\limits_{-1 \le x \le 1} \int\limits_{-1}^{1} |G^*(x,t)|^2 dt$. This implies the absolute and uniform convergence of the second series Equation (12). The absolute and uniform convergence of the first series Equation (12) is proved similarly. The lemma is proved. □

3. Formal Solution to the Problem

In this section, we construct a formal solution to the inverse problem for Equation (1) with the Dirichlet boundary conditions of Equation (2) and the conditions of Equation (3). According to Theorem 1, the system of eigenfunctions $\{X_k(x)\}$ of the boundary value problem of Equations (8) and (9) forms the Riesz basis in the space $L_2(-1,1)$. Based on this fact, we can represent the formal solution of the inverse problem D in the form of series Equations (5) and (6). To determine unknown functions $C_k(t)$ and unknown constants f_k, we formally substitute functions (5) and (6) into Equation (1). After simple transformations, we obtain the equation

$$C''_k(t) + \lambda_k C_k(t) = f_k,$$

whose solution is written in the following form

$$C_k(t) = C_{k1}\cos\sqrt{\lambda_k}t + C_{k2}\sin\sqrt{\lambda_k}t + \frac{f_k}{\lambda_k}, \tag{15}$$

with unknown coefficients $C_{k1},\ C_{k2}$. From the conditions in Equation (3) and the first formula in Equation (7), we obtain

$$C_k(0) = \int_{-1}^{1} u(x,0)\bar{Z}_k(x)dx = \int_{-1}^{1} \varphi(x)\bar{Z}_k(x)dx = \varphi_k,$$

$$C_k(T) = \int_{-1}^{1} u(x,T)\bar{Z}_k(x)dx = \int_{-1}^{1} \psi(x)\bar{Z}_k(x)dx = \psi_k.$$

Using these relations, from Equation (15), we obtain the following system of equations

$$\begin{cases} C_{k1} + \dfrac{f_k}{\lambda_k} = \varphi_k, \\ C_{k1}\cos\sqrt{\lambda_k}T + C_{k2}\sin\sqrt{\lambda_k}T + \dfrac{f_k}{\lambda_k} = \psi_k, \\ C_{k2} = 0. \end{cases}$$

Solving this system of equations, we find the unknown coefficients

$$C_{k1} = \frac{\varphi_k - \psi_k}{1-\cos\sqrt{\lambda_k}T},\ f_k = \left(\varphi_k - \frac{\varphi_k - \psi_k}{1-\cos\sqrt{\lambda_k}T}\right)\lambda_k.$$

Substituting the found values of the unknown coefficients into Equations (5) and (6), we obtain the final form of the formal solution to the inverse problem D:

$$u(x,t) = \varphi(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1-\cos\sqrt{\lambda_k}T}\left[\cos\sqrt{\lambda_k}t - 1\right]X_k(x), \tag{16}$$

$$f(x) = L_q\varphi(x) - \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1-\cos\sqrt{\lambda_k}T}\lambda_k X_k(x). \tag{17}$$

Now we have to prove that functions (16) and (17) are the classical solution to inverse problem D.

4. Main Results

The main result of this work is the solvability of the inverse problem Equations (1) and (3) with the Dirichlet boundary conditions of Equation (2).

Theorem 2. *Let the following conditions be satisfied:*
(1) $q(x) \in C^4[-1,1],\ \ \varphi,\ \psi \in D(L_q),\ \ L_q\varphi,\ L_q\psi \in D(L_q)$;
(2) There is a positive number δ_0 *such that* $\left|1 - \cos\sqrt{\lambda_k}T\right| \geq \delta_0$.
Then the inverse problem in Equations (1)–(3) has a unique solution, which can be represented as a Fourier series, Equations (16) and (17).

Proof. We have to prove that the resulting formal solution in the form of series Equations (16) and (17) satisfies Equation (1) and the conditions in Equations (2) and (3). Let us prove the uniform convergence of the series Equations (16) and (17) in the open domain Ω and the uniform convergence of the formally differentiated series

$$u_t(x,t) = -\sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - \cos\sqrt{\lambda_k}T} \sqrt{\lambda_k} \sin\sqrt{\lambda_k} t X_k(x), \tag{18}$$

$$u_x(x,t) = \varphi'(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - \cos\sqrt{\lambda_k}T} \left[\cos\sqrt{\lambda_k}t - 1\right] X'_k(x), \tag{19}$$

$$u_{xx}(x,t) = \varphi''(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - \cos\sqrt{\lambda_k}T} \left[\cos\sqrt{\lambda_k}t - 1\right] X''_k(x), \tag{20}$$

$$u_{xxx}(x,t) = \varphi'''(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - \cos\sqrt{\lambda_k}T} \left[\cos\sqrt{\lambda_k}t - 1\right] X'''_k(x), \tag{21}$$

$$u_{xxxx}(x,t) = \varphi^{IV}(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - \cos\sqrt{\lambda_k}T} \left[\cos\sqrt{\lambda_k}t - 1\right] X_k^{IV}(x). \tag{22}$$

The uniform convergence of series Equation (16) follows from the obvious inequality

$$|u(x,t)| \le |\varphi(x)| + \left|\sum_{k=0}^{\infty} (\varphi, Z_k) X_k(x)\right| + \left|\sum_{k=0}^{\infty} (\psi, Z_k) X_k(x)\right|$$

and Lemma 2, taking into account Lemma 1 ($\operatorname{Im}\lambda_k \le const.$). To prove the uniform convergence of series Equation (17) in the expressions $\varphi_k = (\varphi, Z_k)$, $\psi_k = (\psi, Z_k)$, we replace the function $Z_k(x)$ by the conjugate Equation (10). Then

$$\lambda_k \varphi_k = \lambda_k (\varphi, Z_k) = \left(\varphi, L_q^* Z_k\right) = \left(L_q \varphi, Z_k\right), \ \lambda_k \psi_k = \left(L_q \psi, Z_k\right). \tag{23}$$

Substituting them into Equation (17), we obtain

$$|f(x)| \le \left|L_q \varphi(x)\right| + \left|\sum_{k=0}^{\infty} (L_q \varphi, Z_k) X_k(x)\right| + \left|\sum_{k=0}^{\infty} (L_q \psi, Z_k) X_k(x)\right|.$$

Due to the condition of the theorem $L_q\varphi,\, L_q\psi \in D\left(L_q\right)$, by virtue of Lemma 2, both series on the right-hand side of the last inequality converge uniformly. The uniform convergence of the series Equations (16) and (17) is proved. The uniform convergence of series Equation (18) is proved as well as the convergence of series Equation (17) taking into account the boundedness of trigonometric functions.

Let us prove the uniform convergence of series Equations (19)–(22). Applying Equation (23) to series Equation (19), we obtain the relation

$$|u_x(x,t)| \le \left|\varphi'(x)\right| + \left|\sum_{k=0}^{\infty} \frac{(L_q\varphi, Z_k) - (L_q\psi, Z_k)}{\lambda_k(1 - \cos\sqrt{\lambda_k}T)} \left(\cos\sqrt{\lambda_k}t - 1\right) X'_k(x)\right|.$$

In [27], it is shown that the estimates

$$\max\left|X_k^{(s)}(x)\right| \le \left(\sqrt[4]{|\lambda_k|}\right)^s \max|X_k(x)|, \ s = 1,2,3, \tag{24}$$

are valid for the eigenfunctions of the fourth-order differential operator. Using estimates Equation (24), from the last inequality, we obtain the estimate

$$|u_x(x,t)| \leq |\varphi'(x)| + c_1 \sum_{k=0}^{\infty} \left| \frac{(L_q\varphi, Z_k) - (L_q\psi, Z_k)}{\left(\sqrt[4]{\lambda_k}\right)^3} \right| \max|X_k(x)|.$$

It follows from [28] that only uniformly bounded systems of eigenfunctions of ordinary differential operators can be Riesz bases. Therefore, due to the conditions of the theorem $L_q\varphi, L_q\psi \in D(L_q)$, the Bessel inequality for the Riesz bases, the asymptotics of the eigenvalues [24], and the series on the right-hand side of the following inequality

$$|u_x(x,t)| \leq |\varphi'(x)| + c_1 \sum_{k=0}^{\infty} \left[|(L_q\varphi, Z_k)|^2 + |(L_q\psi, Z_k)|^2 + \frac{2}{|\sqrt{\lambda_k}|^3} \right]$$

converge. The uniform convergence of series Equation (19) is proved.

With the help of estimates Equation (24), the convergence of series Equations (20) and (21) in the open domain is similarly proved. Now, consider the uniform convergence of the series

$$u_{xxxx}(x,t) = \varphi^{IV}(x) + \sum_{k=0}^{\infty} \frac{(L_q\varphi, Z_k) - (L_q\psi, Z_k)}{\lambda_k(1 - \cos\sqrt{\lambda_k}T)} \left(\cos\sqrt{\lambda_k}t - 1\right) X_k^{IV}(x).$$

Replacing the fourth derivative with the help of Equation (8), we obtain the estimate

$$\begin{aligned} |u_{xxxx}(x,t)| \leq \left|\varphi^{IV}(x)\right| &+ \left| \sum_{k=0}^{\infty} \frac{q(x)}{\lambda_k} \left[(L_q\varphi, Z_k)X_k(x) - (L_q\psi, Z_k)X_k(x) \right] \right| \\ &+ \left| \sum_{k=0}^{\infty} \left[(L_q\varphi, Z_k)X_k(x) - (L_q\psi, Z_k)X_k(x) \right] \right|. \end{aligned} \tag{25}$$

The second series on the right-hand side of Equation (25) converges by virtue of the conditions of theorem $L_q\varphi,\ L_q\psi \in D(L_q)$ and Lemma 2. The convergence of the first series in Equation (25) is obvious due to the boundedness of the quantities $\frac{q(x)}{\lambda_k}$. This proves the uniform convergence of the series $u_{xxxx}(x,t)$ in the open region Ω. Thus, we show that series Equations (16) and (17) satisfy Equation (1).

Obviously, the formal solution of Equation (16) satisfies the boundary conditions of Equation (2) and the conditions of Equation (3):

$$\lim_{t\to 0+0} u(x,t) = \lim_{t\to 0+0} \left[\varphi(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - \cos\sqrt{\lambda_k}T} \left(\cos\sqrt{\lambda_k}t - 1\right) X_k(x) \right] = \varphi(x),$$

$$\lim_{t\to T-0} u(x,t) = \lim_{t\to T-0} \left[\varphi(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - \cos\sqrt{\lambda_k}T} \left(\cos\sqrt{\lambda_k}t - 1\right) X_k(x) \right] = \psi(x).$$

Let us prove the uniqueness of the solution. Assume that there are two sets of solutions, $\{u_1(x,t),\ f_1(x)\}$ and $\{u_2(x,t),\ f_2(x)\}$, of inverse problem D. Denote

$$u(x,t) = u_1(x,t) - u_2(x,t)$$

and

$$f(x) = f_1(x) - f_2(x).$$

Then the functions $u(x,t)$ and $f(x)$ satisfy Equation (1), the boundary conditions of Equation (2), and homogeneous conditions

$$u(x,0) = 0,\ u(x,T) = 0,\ u_t(x,0) = 0,\ \ x \in [-1,1]. \tag{26}$$

Consider the Fourier coefficients of the functions $u(x,t)$ and $f(x)$ in the Riesz basis $\{X_k(x)\}$:

$$u_k(t) = \int\limits_{-1}^{1} u(x,t)\bar{X}_k(x)dx,\ k \in N, \tag{27}$$

$$f_k = \int\limits_{-1}^{1} f(x)\bar{X}_k(x)dx,\ k \in N. \tag{28}$$

We thus search for an equation that satisfies the coefficient $u_k(t)$. By differentiating equality Equation (27), we obtain

$$u''_k(t) = \int\limits_{-1}^{1} u''_t(x,t)\bar{X}_k(x)dx = \int\limits_{-1}^{1} [-u_{xxxx}(x,t) - q(x)u(x,t) + f(x)]\bar{X}_k(x)dx.$$

After integration by parts four times, we have

$$u''_k(t) = \int\limits_{-1}^{1} \left[-\bar{X}_k^{IV}(x) - \bar{q}(x)\bar{X}_k(x)\right]u(x,t)dx + f_k,$$

or

$$u''_k(t) = -\overline{\lambda_k}\int\limits_{-1}^{1} \bar{X}_k(x)u(x,t)dx + f_k.$$

Then we obtain the equation

$$u''_k(t) + \overline{\lambda}_k u_k(t) = f_k,$$

for the coefficient $u_k(t)$ whose solution is written in the form of Equation (15). Due to Equations (26) and (27), and the unknown coefficients in Equation (15), $C_{k1} = 0$, $C_{k2} = 0$, and $f_k = 0$. Then, from Formula (15), we obtain $u_k(t) \equiv 0$. As the system $\{X_k(x)\}$ is a basis in $L_2(-1,1)$, then Equations (27) and (28) imply the equalities $f(x) \equiv 0$, $u(x,t) \equiv 0$, $(x,t) \in \Omega$. The uniqueness of the solution is proved. The theorem is completely proved. □

The assertion of the theorem is valid for the case of the problem of Equations (1), (3) and (4).

Theorem 3. *Let the following conditions be satisfied:*

(1) $q(x) \in C^4[-1,1]$, φ, $\psi \in N(L_q)$, $L_q\varphi$, $L_q\psi \in N(L_q)$;

(2) There is a positive number δ_0, such that $\left|1 - \cos\sqrt{\lambda_k}T\right| \geq \delta_0$.

Then the inverse problem of Equations (1), (4) and (30) has a unique solution, which can be represented as a Fourier series in Equations (16) and (17).

Remark 1. *We note the importance of the numerical discretization of the inverse problem and numerical experiments. However, this goal was not set in this work.*

Example 1. *Let us consider the inverse problem D for Equation (1), $u_{tt}(x,t) + \frac{\partial^4}{\partial x^4}u(x,t) = f(x)$, ($q(x) \equiv 0$), with the boundary conditions of Equation (2) and the conditions of Equation (3):*

$$u(x,0) = \sin(\pi x),\ u(x,T) = \cos\left(\frac{\pi}{2}x\right),\ u_t(x,0) = 0,\ x \in [-1,1].$$

According to the method of separation of variables for homogeneous equation $u_{tt}(x,t) + \frac{\partial^4}{\partial x^4}u(x,t) = 0$, *we find the eigenvalues*

$$\lambda_{k1} = (\pi k)^4, \ \lambda_{k2} = \left(k - \frac{1}{2}\right)^4 \pi^4, \ k \in N,$$

and the eigenfunctions

$$X_{k1}(x) = \sin(\pi k x), \ X_{k2}(x) = \cos\left(k - \frac{1}{2}\right)\pi x, \ k \in N,$$

of the spectral problem $X^{IV}(x) = \lambda X(x), \ X(-1) = X(1) = X''(-1) = X''(1) = 0$, *where* $\left\{X_{kj}\right\}$ *is the complete orthonormal system in space* $L_2(-1,1)$.

Then we can write the solution for Equations (16) and (17) of the inverse problem D as

$$\begin{aligned} u(x,t) &= \varphi(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - \cos\sqrt{\lambda_k}T}\left[\cos\sqrt{\lambda_k}t - 1\right]X_k(x) \\ &= \varphi(x) + \sum_{k=1}^{\infty} \frac{\varphi_{k1} - \psi_{k1}}{1 - \cos\sqrt{\lambda_{k1}}T}\left[\cos\sqrt{\lambda_{k1}}t - 1\right]\sin k\pi x \\ &+ \sum_{k=1}^{\infty} \frac{\varphi_{k2} - \psi_{k2}}{1 - \cos\sqrt{\lambda_{k2}}T}\left[\cos\sqrt{\lambda_{k2}}t - 1\right]\cos\left(k - \frac{1}{2}\right)\pi x. \end{aligned} \tag{29}$$

$$\begin{aligned} f(x) &= L_q\varphi(x) - \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - \cos\sqrt{\lambda_k}T}\lambda_k X_k(x) \\ &= \varphi^{IV}(x) - \sum_{k=1}^{\infty} \frac{\varphi_{k1} - \psi_{k1}}{1 - \cos\sqrt{\lambda_{k1}}T}\lambda_{k1}X_{k1}(x) - \sum_{k=1}^{\infty} \frac{\varphi_{k2} - \psi_{k2}}{1 - \cos\sqrt{\lambda_{k2}}T}\lambda_{k2}X_{k2}(x). \end{aligned} \tag{30}$$

Here $\varphi(x) = \sin(\pi x)$ *and* $\psi(x) = \cos\left(\frac{\pi}{2}x\right)$. *We have*

$$\varphi_{k1} = \int_{-1}^{1} \sin\pi x \sin(\pi k x)dx = \begin{cases} 1, & k = 1, \\ 0, & k \neq 1, \end{cases}$$

$$\varphi_{k2} = \int_{-1}^{1} \sin(\pi x)\cos\left(k - \frac{1}{2}\right)\pi x dx = 0, \ \forall k,$$

$$\psi_{k1} = \int_{-1}^{1} \cos\left(\frac{\pi}{2}\right)x \sin\pi k x dx = 0, \ \forall k,$$

$$\psi_{k2} = \int_{-1}^{1} \cos\left(\frac{\pi}{2}x\right)\cos\left(k - \frac{1}{2}\right)\pi x dx = \begin{cases} 1, & k = 1, \\ 0, & k \neq 1. \end{cases}$$

Thus, from Equations (29) and (30), we obtain the unique solution of inverse problem D

$$u(x,t) = \sin\pi x - \frac{1 - \cos\pi^2 t}{1 - \cos\pi^2 T}\sin(\pi x) + \frac{1 + \cos\frac{\pi^2}{4}t}{1 - \cos\frac{\pi^2}{4}T}\cos\left(\frac{\pi}{2}x\right),$$

$$f(x) = \pi^4\sin(\pi x) - \frac{\pi^4}{1 - \cos\pi^2 T}\sin(\pi x) + \frac{1}{16}\frac{\pi^4}{1 - \cos\frac{\pi^2}{4}T}\cos\left(\frac{\pi}{2}x\right),$$

for every T and fixed number m_0 *such that* $T \neq \frac{2m_0}{\pi}$, *since all conditions of Theorem 2 are satisfied.*

Example 2. *First, we solve the spectral problem of Equations (8) and (9), where* $q(x) = (1+i)e^x$*:*

$$X^{IV}(x) + (1+i)e^x X(x) = \lambda X(x), \tag{31}$$

$$X(-1) = X(1) = X''(-1) = X''(1) = 0. \tag{32}$$

We write the problem conjugate to the boundary value problem of Equations (31) and (32) in the form

$$Z^{IV}(x) + (1-i)e^x Z(x) = \bar{\lambda} Z(x), \tag{33}$$

with the boundary conditions in the form of Equation (32). It is easy to check that the boundary conditions of Equation (32) are strongly regular. Using Theorem 1, we conclude that the system of eigenfunctions $\{X_k(x)\} = \{X_{k1}(x),\ X_{k2}(x)\}$ *of the boundary value problem of Equations (31) and (32) forms the Riesz basis in the space* $L_2(-1,1)$. *This is also valid for the system of eigenfunctions* $\{Z_k(x)\} = \{Z_{k1}(x),\ Z_{k2}(x)\}$ *of the conjugate boundary value problem of Equations (32) and (33), where the systems of eigenfunctions* $\{X_k(x)\}$ *and* $\{Z_k(x)\}$ *satisfy the biorthogonality condition [24]*

$$(X_k, Z_n) = \int_{-1}^{1} X_k(x)\bar{Z}_n(x)dx = \delta_{kn},$$

where δ_{kn} *is the Kronecker symbol. The symbols* λ_{k1}, λ_{k2} *denote the eigenvalues [24] of Equations (31) and (32):* $\lambda_{k1} = (\pi k)^4\left(1 + O\left(\frac{1}{k}\right)\right)$, $\lambda_{k2} = \left(\pi k - \frac{\pi}{2}\right)^4\left(1 + O\left(\frac{1}{k}\right)\right)$, $k \in N$.

Next we consider the inverse problem D for Equation (1), $u_{tt}(x,t) + \frac{\partial^4}{\partial x^4}u(x,t) + (1+i)e^x u(x,t) = f(x)$, *with the boundary conditions of Equation (2) and the conditions of Equation (3):*

$$u(x,0) = X_{11}(x),\ u(x,T) = X_{12}(x),\ u_t(x,0) = 0,\ x \in [-1,1],$$

where $X_{11}(x)$ *and* $X_{12}(x)$ *are eigenfunctions of the problem Equations (31) and (32). Then we write the solution Equations (16) and (17) of the inverse problem D as*

$$\begin{aligned} u(x,t) &= \varphi(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - \cos\sqrt{\lambda_k}T}\left[\cos\sqrt{\lambda_k}t - 1\right]X_k(x) \\ &= \varphi(x) + \sum_{k=1}^{\infty} \frac{\varphi_{k1} - \psi_{k1}}{1 - \cos\sqrt{\lambda_{k1}}T}\left[\cos\sqrt{\lambda_{k1}}t - 1\right]X_{k1}(x) \\ &\quad + \sum_{k=1}^{\infty} \frac{\varphi_{k2} - \psi_{k2}}{1 - \cos\sqrt{\lambda_{k2}}T}\left[\cos\sqrt{\lambda_{k2}}t - 1\right]X_{k2}(x). \end{aligned} \tag{34}$$

$$\begin{aligned} f(x) &= L_q\varphi(x) - \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - \cos\sqrt{\lambda_k}T}\lambda_k X_k(x) \\ &= \varphi^{IV}(x) + (1+i)e^x\varphi(x) - \sum_{k=1}^{\infty} \frac{\varphi_{k1} - \psi_{k1}}{1 - \cos\sqrt{\lambda_{k1}}T}\lambda_{k1}\cdot X_{k1}(x) \\ &\quad - \sum_{k=1}^{\infty} \frac{\varphi_{k2} - \psi_{k2}}{1 - \cos\sqrt{\lambda_{k2}}T}\lambda_{k2}\cdot X_{k2}(x). \end{aligned} \tag{35}$$

Here $\varphi(x) = X_{11}(x)$, $\psi(x) = X_{12}(x)$. *From the biorthogonality conditions, we have*

$$\varphi_{k1} = \int_{-1}^{1} X_{11}(x)\bar{Z}_{k1}(x)dx = \begin{cases} 1, & k = 1, \\ 0, & k \neq 1, \end{cases}$$

$$\varphi_{k2} = \int_{-1}^{1} X_{11}(x)\bar{Z}_{k2}(x)dx = 0, \ \forall k,$$

$$\psi_{k1} = \int_{-1}^{1} X_{12}(x)\bar{Z}_{k1}(x)dx = 0, \ \forall k,$$

$$\psi_{k2} = \int_{-1}^{1} X_{12}(x)\bar{Z}_{k2}(x)dx = \begin{cases} 1, & k = 1, \\ 0, & k \neq 1. \end{cases}$$

Thus, from Equations (34) and (35), we obtain the unique solution of inverse problem D

$$u(x) = X_{11}(x) - \frac{1-\cos\sqrt{\lambda_{11}}t}{1-\cos\sqrt{\lambda_{11}}T}X_{11}(x) + \frac{1-\cos\sqrt{\lambda_{12}}t}{1-\cos\sqrt{\lambda_{12}}T}X_{12}(x),$$

$$f(x) = X_{11}^{IV}(x) + (1+i)e^{x}X_{11}(x) - \frac{\lambda_{11}}{1-\cos\sqrt{\lambda_{11}}T}X_{11}(x) + \frac{\lambda_{12}}{1-\cos\sqrt{\lambda_{12}}T}X_{12}(x),$$

for every T and fixed number m_0 such that $T \neq \frac{2\pi m_0}{\sqrt{\lambda_{11}}}$, since all conditions of Theorem 2 are satisfied.

5. Conclusions

In the present paper, we have, for the first time, studied the inverse problems for the fourth-order hyperbolic Equation (1) with a variable complex-valued coefficient. Therefore, in the future, it is of theoretical and practical interest to investigate other direct and inverse problems for Equation (1), a fourth-order hyperbolic equation with involution perturbation $u_{tt}(x,t) + \frac{\partial^4}{\partial x^4}u(x,t) - \alpha\frac{\partial^4}{\partial x^4}u(-x,t) + q(x)u(x,t) = f(x)$. As for the numerical discretization of the inverse problem and numerical experiments, we defer this challenging task to a possible future work.

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