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Fixed Point Results in Controlled Fuzzy Metric Spaces with an Application to the Transformation of Solar Energy to Electric Power

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Abstract: In this manuscript, we give sufficient conditions for a sequence to be Cauchy in the context of controlled fuzzy metric space. Furthermore, we generalize the concept of Banach's contraction principle by utilizing several new contraction conditions and prove several fixed point results. Furthermore, we provide a number of non-trivial examples to validate the superiority of main results in the existing literature. At the end, we discuss an important application to the transformation of solar energy to electric power by utilizing differential equations.

Keywords: fixed point theorems; fuzzy metric space (FMS); contraction principles; Green's function; differential equation

MSC: 47H10; 54H25

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1. Introduction and Preliminaries

The existence of a unique fixed point for self-mappings under suitable contraction conditions over complete metric spaces is guaranteed by Banach's fixed point theory (also known as "the contraction mapping theorem"), one of the most significant sources of existence and uniqueness theorems in numerous areas of analysis. New extensions and generalizations of fixed point results are important because they increase our understanding of mathematical systems, enable the solution of specific problems, extend current theorems, and lead to the development of new theories and applications. They are an important aspect of mathematical study and have far-reaching ramifications in a variety of fields.

The fuzzy logic was established by Zadeh [1]. Unlike the theory of traditional logic, some numbers are not contained within the set and fuzzy logic afflation of the numbers in the set defines an element within the interval $[0, 1]$. Uncertainty, the necessary section of real difficulty, has helped Zadeh to learn theories of fuzzy sets to bear the difficulty of indefinity. The theory is seen as a fixed point (FP) in the fuzzy metric space (FMS) for various processes, one of them utilizing a fuzzy logic (FL). Later on, following Zadeh's outcomes, Heilpern [2] established the fuzzy mapping (FM) notion and a theorem on an FP for fuzzy contraction mapping in linear MS, expressing a fuzzy general form of Banach's contraction theory. In the definition of FMSs provided by Kaleva and Saikkala [3], the imprecision is introduced if the distance between the elements is not a precise integer. After the first by Kramosil and Michalek [4] and further work by George and Veeramani [5], the notation of an FMS was introduced. Branga and Olaru [6] proved several fixed point results for self-mappings by utilizing generalized contractive conditions in the context of altered metric spaces. Al-Khaleel et al. [7] used cyclic contractive mappings of Kannan and Chatterjea type in generalized metric spaces. Czerwik [8] found the solution of the well-known Banach's fixed point theorem in the context of b-metric spaces (b-MS). Mlaiki [9] defined controlled MS as a generalization of b-MS by utilizing a control function $\alpha : \Xi \times \Xi \to [1, +\infty)$ of the other side of the b-triangular inequality. The relation between b-MS and FMS has been discussed by Hassanzadeh and Sedghi [10]. Li et al. [11] used Kaleva–Seikkala's type FbMSs and proved several fixed point results by using contraction mappings. Furthermore, Sedghi and Shobe [12,13] proved various common fixed point theorems for R-weakly commuting maps in the framework of FbMSs.

Sezen [14] introduced a controlled Fuzzy metric space (CFMS) as a generalization of FMS and FbMS by applying a control function $\alpha : \Xi \times \Xi \to [1,\ \infty)$ in a triangular inequality of the form:

$$M(b, \omega, t+s) \geq M\left(b, z, \frac{t}{\alpha(b,z)}\right) + M\left(z, \omega, \frac{s}{\alpha(z,\omega)}\right) \text{ for all } b, \omega, z \in \Xi \text{ and } s, t > 0.$$

If we take $\alpha(b,z) = (z,\omega) = 1$ then it is an FMS and for $\alpha(b,z) = (z,\omega) \geq s$ with $s \geq 1$ it is then an FbMS.

Ishtiaq et al. [15] established the theory of double-controlled intuitionistic fuzzy metric-like spaces by "considering the case where the self-distance is not zero"; if the metric's value is 0, afterward, it must be a self-distance and also an established FP theorem for contraction mappings. See [16] for triangular norm (TN), continuous triangular norm (CTN) [17], and TN of H-type [18,19]. In [20,21], authors worked on CFMSs by utilizing orthogonality and pentagonal CFMSs and proved several fixed point results for contraction mappings. Rakić [22] proved a fuzzy version of Banach's fixed point theorem by using Ciric-quasi-contraction in the context of FbMSs. Mehmood et al. [23] introduced the concept of extended fuzzy b-metric spaces and generalized the Banach contraction principle. Younis et al. [24] proved several fixed point results in the context of dislocated b-metric spaces and solved the turning circuit problem.

In this article,

- we prove that a sequence must be Cauchy in the CFMS under some conditions;
- we prove a fixed point result by using Ciric-quasi-contraction and generalize the Banach contraction principle by utilizing several new contraction conditions;
- we provide several non-trivial examples to show the validity of the main results;
- we discuss an application concerning the transformation of solar energy to electric power.

Now, we provide several definitions and results that are helpful to understand the main section.

Definition 1 ([16]). *A binary operation* $\Gamma : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ *is a CTN if it verifies the below conditions:*

(C1) Γ *is commutative and associative,*
(C2) Γ *is continuous,*
(C3) $\Gamma(\varkappa, 1) = \varkappa$ for all $\varkappa \in [0,\ 1]$,
(C4) $\Gamma(\varkappa, \rho) \leq \Gamma(c, d)$ *for* $\varkappa, \rho, c, d \in [0,\ 1]$ *such that* $\varkappa \leq c$ *and* $\rho \leq d$.
Examples of CTN are $T_p(a,b) = a.b,\ T_{\min}(a,b) = \min\{a,b\}$, *and* $T_L(a,b) = \max\{a+b-1, 0\}$.

Definition 2 ([18]). *Suppose that* Γ *is a TN and suppose that* $\Gamma_\tau : [0,1] \to [0,1],\ \tau \in \mathbb{N}$, *express the process given below:*

$$\Gamma_1(b) = \Gamma(b,b), \ \ \Gamma_{\tau+1}(b) = \Gamma(\Gamma_\tau(b), b), \ \tau \in \mathbb{N}, b \in [0,1].$$

Then, TN Γ is H-type if the family $\{\Gamma_\tau(b)\}_{\tau\in\mathbb{N}}$ is equicontinuous at $b = 1$.
A TN of H-type is $\Gamma_{\min}$ and see [18] for examples.
Each t-norm can be generalized in a different way to an n-ary process via associativity (see [17]), taking $(b_1, \ldots, b_n) \in [0]^n$ for the values

$$\Gamma_{i=1}^{1} b_i = b_1,\ \Gamma_{i=1}^{1} b_i = \Gamma\left(\Gamma_{i=1}^{1} b_i,\ b_\tau\right) = \Gamma(b_1, \ldots, b_\tau).$$

Example 1 ([19]). *An n-ary generalization of the TN Γ_{min}, Γ_L , and Γ_P are:*

$$\Gamma_{\min}(b_1, \ldots, b_\tau) = \min(b_1, \ldots, b_\tau),$$

$$\Gamma_L(b_1, \ldots, b_\tau) = \max\left(\sum_{i=1}^{\tau} b_i - (\tau - 1), 0\right),$$

$$\Gamma_P(b_1, \ldots, b_\tau) = \prod_{i=1}^{\tau} b_i.$$

A TN Γ (see [17]) can be extended to a countable infinite operation, for any sequence considering $(b_\tau)_{\tau\in\mathbb{N}}$ from $[0,1]$ the value

$$\Gamma_{i=1}^{\infty} = \lim_{\tau\to\infty} \Gamma_{i=1}^{\tau} b_i.$$

The sequence $\left(\Gamma_{i=1}^{\tau}\right)_{\tau\in\mathbb{N}}$ is non-increasing and bounded below and so the limit $\Gamma_{i=1}^{\infty} b_i$ exists.
In the FP theory (see [18,19]), it might be interesting to look at the category of TN Γ and sequence (b_τ) in the range $[0,1]$ such that $\lim_{\tau\to\infty} b_\tau = 1$ and

$$\lim_{\tau\to\infty} \Gamma_{i=1}^{\infty} b_i = \lim_{\tau\to\infty} \Gamma_{i=1}^{\infty} b_{\tau+i} = 1. \tag{1}$$

Proposition 1 ([18]). *Suppose $(b_\tau)_{\tau\in\mathbb{N}}$ is a series of numbers with the range $[0,1]$ such that $\lim_{\tau\to\infty} b_\tau = 1$ and assume Γ to be a TN of H-type. Then, $\lim_{\tau\to\infty} \Gamma_{i=\tau}^{\infty} b_{\tau+i} = 1$.*
Throughout this study, we utilize $\Xi^2 := \Xi \times \Xi$.

Definition 3 ([5]). *A 3-tuple (Ξ, M, Γ) is known as an FMS if Ξ is a random (nonempty) set, Γ is a CTN, and M is an FS on $\Xi^2 \times (0,\infty)$ and satisfies the following conditions for all $b, \omega, z \in \Xi$, and $t, s > 0$:*

(fm1) $M(b,\omega,t) > 0$,
(fm2) $M(b,\omega,t) = 1$ *iff* $b = \omega$,
(fm3) $M(b,\omega,t) = M(\omega,b,t)$,
(fm4) $\Gamma(M(b,\omega,t), M(\omega,z,s)) \leq M(b,z,t+s)$,
(fm5) $M(b,\omega,.) : (0,\infty) \to [0,1]$ *is continuous.*

Definition 4 ([12]). *A 3-tuple (Ξ, M, Γ) is called an FbMS if Ξ is a random (non-empty) set, Γ is a CTN, and M is an FS on $\Xi^2 \times (0,\infty)$ and satisfies the following conditions for all $b, \omega, z \in \Xi$, $t, s > 0$, and $\rho \geq 1$ as a real number:*

(b1) $M(b,\omega,t) > 0$,
(b2) $M(b,\omega,t) = 1$ *iff* $b = \omega$,
(b3) $M(b,\omega,t) = M(\omega,b,t)$,
(b4) $\Gamma(M(b,\omega,t), M(\omega,z,s)) \leq M(b,z,\rho(t+s))$,
(b5) $M(b,\omega,.) : (0,\infty) \to [0,1]$ *is continuous.*
Rakić [22] proved the following fixed point theorem by using Ciric-quasi-contraction in the context of FbMSs.

Theorem 1 ([22]). *Suppose that* $(\Xi, M, \Gamma_{\min})$ *is a complete FbMS, assume that* $f : \Xi \to \Xi$. *If for some* $\aleph \in (0,1)$, *such that*

$$M_\alpha(fb, f\omega, t) \geq \min\left\{ \begin{array}{c} M_\alpha\left(b, \omega, \frac{t}{\aleph}\right), M_\alpha\left(fb, b, \frac{t}{\aleph}\right), \\ M_\alpha\left(f\omega, \omega, \frac{t}{\aleph}\right), M_\alpha\left(fb, \omega, \frac{2t}{\aleph}\right), M_\alpha\left(b, f\omega, \frac{t}{\aleph}\right) \end{array} \right\}, \; b, \omega \in \Xi, t > 0.$$

Then, f *has a UFP in* Ξ.

Lemma 1 ([12,13]). *Let* $M(b, \omega, .)$ *be an FbMS. Then,* $M(b, \omega, t)$ *is b-non-decreasing with respect to* t *for all* $b, \omega \in \Xi$.

Definition 5 ([23]). *Let* Ξ *be a non-empty set,* $\alpha : \Xi \times \Xi \to [1, \infty)$, Γ *is a CTN, and* M_α *is an FS on* $\Xi^2 \times [0, \infty)$ *and satisfies the following conditions for all* $b, \omega, z \in \Xi$, s, *and* $t > 0$:

$(EM_\alpha 1) M_\alpha(b, \omega, 0) = 0,$
$(EM_\alpha 2) M_\alpha(b, \omega, t) = 1$ *iff* $b = \omega$,
$(EM_\alpha 3) M_\alpha(b, \omega, t) = M_\alpha(\omega, b, t),$
$(EM_\alpha 4) M_\alpha(b, z, \alpha(b, z)(t + s)) \geq \Gamma(M_\alpha(b, \omega, t), M_\alpha(\omega, z, s)),$
$(EM_\alpha 4)\, M(b, \omega, .) : (0, \infty) \to [0, 1]$ *is left continuous.*
Then, the triple (Ξ, M_α, Γ) *is said to be an extended fuzzy b-metric space and* M_α *is said to be controlled FM on* Ξ.

Theorem 2 ([23]). *Suppose that* (Ξ, M, Γ) *is a complete CFMS with* $\alpha : \Xi \times \Xi \to [1, \infty)$, *assume that*

$$\lim_{t \to \infty} M_\alpha(b, \omega, t) = 1,$$

for all $b, \omega \in \Xi$. *If* $f : \Xi \to \Xi$ *satisfies the following for some* $\aleph \in (0,1)$, *such that*

$$M_\alpha(fb, f\omega, t) \geq M_\alpha\left(b, \omega, \frac{t}{\aleph}\right), \qquad \text{for all } b, \omega \in \Xi, t > 0.$$

Also, suppose that for arbitrary $b_0 \in \Xi$ *and* $n, q \in \mathbb{N}$, *we have*

$$\alpha(b_n, b_{n+q}) \leq \frac{1}{\aleph},$$

where $b_n = f^n b_0$. *Then,* f *has a UFP in* Ξ.

Definition 6 ([14]). *Let* Ξ *be a non-empty set,* $\alpha : \Xi \times \Xi \to [1, \infty)$, Γ *is a CTN, and* M_α *is an FS on* $\Xi^2 \times [0, \infty)$ and *satisfies the following conditions for all* $b, \omega, z \in \Xi$, s, *and* $t > 0$:

$(FM_\alpha 1) M_\alpha(b, \omega, 0) = 0,$
$(FM_\alpha 2) M_\alpha(b, \omega, t) = 1$ *iff* $b = \omega$,
$(FM_\alpha 3) M_\alpha(b, \omega, t) = M_\alpha(\omega, b, t),$
$(FM_\alpha 4) M_\alpha(b, z, t + s) \geq \Gamma\left(M_\alpha\left(b, \omega, \frac{t}{\alpha(b,\omega)}\right), M_\alpha\left(\omega, z, \frac{s}{\alpha(\omega,z)}\right)\right),$
$(FM_\alpha 4)\, M(b, \omega, .) : (0, \infty) \to [0, 1]$ *is left continuous.*
Then, the triple (Ξ, M_α, Γ) *is said to be a CFMS and* M_α *is said to be a controlled FM on* Ξ.
Sezen [14] *proved the following Banach contraction principle in the context of CFMS.*

Theorem 3 ([14]). *Suppose that* (Ξ, M, Γ) *is a complete CFMS with* $\alpha : \Xi \times \Xi \to [1, \infty)$, *assume that*

$$\lim_{t \to \infty} M_\alpha(b, \omega, t) = 1,$$

for all $b, \omega \in \Xi$. *If* $f : \Xi \to \Xi$ *satisfies the following for some* $\aleph \in (0,1)$, *such that*

$$M_\alpha(fb, f\omega, t) \geq M_\alpha\left(b, \omega, \frac{t}{\aleph}\right), \text{ for all } b, \omega \in \Xi, t > 0.$$

Also, suppose that for all $b \in \Xi$*, we obtain* $\lim_{n\to\infty} M_\alpha(b_n, \omega)$ *and* $\lim_{n\to\infty} M_\alpha(\omega, b_n)$ *which exist and are finite. Then,* f *has a UFP in* Ξ*.*

Definition 7 ([14]). *Suppose* $M(b, \omega, t)$ *is a CFMS. For* $t > 0$*, the open ball* $B(b, l, t)$ *with center* $b \in \Xi$ *and radius* $0 < l < 1$ *is express as a sequence* $\{b_\tau\}$*:*
(a) G-Convergent to b if $M(b_\tau, b, t) \to 1$ *as* $\tau \to \infty$ *or for every* $t > 0$*. We write* $\lim_{\tau\to\infty} b_\tau = b$*;*
(b) is said to be Cauchy sequence (CS) if for all $0 < \varepsilon < 1$ *and* $t > 0$ *they exist satisfying* $\tau_0 \in \mathbb{N}$ *such that*
$M(b_\tau, b_m, t) > 1 - \varepsilon \; for \; all \; \tau, m \geq \tau_0$.
(c) The CFMS (Ξ, M, Γ) *is a G-complete if every CS is convergent in* Ξ*.*

2. Main Results

In this part, we discuss several new results in the context of CFMSs.

Lemma 2. *Suppose* $\{b_\tau\}$ *is a sequence in a CFMS* (Ξ, M_α, Γ)*. Let* $\aleph \in (0, 1)$ *exist such that*

$$M(b_\tau, b_{\tau+1}, t) \geq M\left(b_{\tau-1}, b_\tau, \frac{t}{\aleph}\right), \; \tau \in \mathbb{N}, t > 0, \tag{2}$$

and $b_0, b_1 \in \Xi$*, and* $v \in (0, 1)$ exist *such that*

$$\lim_{t\to\infty} M(b, \omega, t) = 1, \; t > 0. \tag{3}$$

Then, $\{b_\tau\}$ *is a CS.*

Proof. Suppose $\varkappa \in (\aleph, 1)$ and the total $\sum_{i=1}^{\infty} \varkappa^i$ is convergent, $\tau_0 \in \mathbb{N}$ exists such that $\sum_{i=1}^{\infty} \varkappa^i < 1$ for every $\tau > \tau_0$. Let $\tau > m > \tau_0$. Since M_α is b-non-decreasing, by $(FM_\alpha 4)$, for every $t > 0$, one can obtain

$$M_\alpha(b_\tau, b_{\tau+m}, t) \geq M_\alpha\left(b_\tau, b_{\tau+m}, \frac{\sum_{i=\tau}^{\tau+m-1} \varkappa^i}{\alpha(b_\tau, b_{\tau+m})}\right),$$

$$\geq \Gamma\left(M_\alpha\left(b_\tau, b_{\tau+1}, \frac{t\varkappa^\tau}{\alpha(b_\tau, b_{\tau+1})\alpha(b_\tau, b_{\tau+m})}\right), M_\alpha\left(b_{\tau+1}, b_{\tau+m}, \frac{t\sum\limits_{i=\tau+1}^{\tau+m-1} \varkappa^i}{\alpha(b_{\tau+1}, b_{\tau+m})\alpha(b_\tau, b_{\tau+m})}\right)\right)$$

$$\geq \Gamma\begin{pmatrix} M_\alpha\left(b_\tau, b_{\tau+1}, \frac{t\varkappa^\tau}{\alpha(b_\tau, b_{\tau+1})\alpha(b_\tau, b_{\tau+m})}\right), \\ M_\alpha\left(b_{\tau+1}, b_{\tau+2}, \frac{t\varkappa^{\tau+1}}{\alpha(b_{\tau+1}, b_{\tau+2})\alpha(b_{\tau+1}, b_{\tau+m})\alpha(b_\tau, b_{\tau+m})}\right), \\ M_\alpha\left(b_{\tau+2}, b_{\tau+m}, \frac{t\sum_{i=\tau+2}^{\tau+m-1} \varkappa^i}{\alpha(b_{\tau+2}, b_{\tau+m})\alpha(b_{\tau+1}, b_{\tau+m})\alpha(b_\tau, b_{\tau+m})}\right) \end{pmatrix}$$

$$\geq \Gamma\begin{pmatrix} M_\alpha\left(b_\tau, b_{\tau+1}, \frac{t\varkappa^\tau}{\alpha(b_\tau, b_{\tau+1})\alpha(b_\tau, b_{\tau+m})}\right), \\ M_\alpha\left(b_{\tau+1}, b_{\tau+2}, \frac{t\varkappa^{\tau+1}}{\alpha(b_{\tau+1}, b_{\tau+2})\alpha(b_{\tau+1}, b_{\tau+m})\alpha(b_\tau, b_{\tau+m})}\right), \\ M_\alpha\left(b_{\tau+2}, b_{\tau+3}, \frac{t\varkappa^{\tau+2}}{\alpha(b_{\tau+2}, b_{\tau+3})\alpha(b_{\tau+2}, b_{\tau+m})\alpha(b_{\tau+1}, b_{\tau+m})\alpha(b_\tau, b_{\tau+m})}\right), \\ M_\alpha\left(b_{\tau+3}, b_{\tau+4}, \frac{t\varkappa^{\tau+3}}{\alpha(b_{\tau+3}, b_{\tau+4})\alpha(b_{\tau+2}, b_{\tau+3})\alpha(b_{\tau+2}, b_{\tau+m})\alpha(b_{\tau+1}, b_{\tau+m})\alpha(b_\tau, b_{\tau+m})}\right), \\ \vdots \\ M_\alpha\left(b_{\tau+m-2}, b_{\tau+m-1}, \frac{t\varkappa^{\tau+m-2}}{\alpha(b_{\tau+m-2}, b_{\tau+m-1})\prod_{i=\tau}^{\tau+m-2} \alpha(b_i, b_{\tau+m})}\right), \\ M_\alpha\left(b_{\tau+m-1}, b_{\tau+m}, \frac{t\varkappa^{\tau+m-1}}{\prod_{i=\tau}^{\tau+m-1} \alpha(b_i, b_{\tau+m})}\right) \end{pmatrix}.$$

From inequality (2), we deduce

$$M_\alpha(b_\tau, b_{\tau+1}, t) \geq M_\alpha\left(b_0, b_1, \frac{t}{\aleph^\tau}\right), \ \tau \in \mathbb{N}, t > 0,$$

and since $\tau > m$ and $\alpha : \Xi \times \Xi \to [1, +\infty)$, one can obtain

$$M_\alpha(b_\tau, b_{\tau+m}, t) \geq \Gamma \begin{pmatrix} M_\alpha\left(b_0, b_1, \frac{t\varkappa^\tau}{\alpha(b_\tau, b_{\tau+1})\alpha(b_\tau, b_{\tau+m})\aleph^\tau}\right), \\ M_\alpha\left(b_0, b_1, \frac{t\varkappa^{\tau+1}}{\alpha(b_{\tau+1}, b_{\tau+2})\alpha(b_{\tau+1}, b_{\tau+m})\alpha(b_\tau, b_{\tau+m})\aleph^{\tau+1}}\right), \\ \vdots \\ M_\alpha\left(b_0, b_1, \frac{t\varkappa^{\tau+m-2}}{\alpha(b_{\tau+m-2}, b_{\tau+m-1})\prod_{i=\tau}^{\tau+m-2}\alpha(b_i, b_{\tau+m})\aleph^{\tau+m-1}}\right), \\ M_\alpha\left(b_0, b_1, \frac{t\varkappa^{\tau+m-1}}{\prod_{i=\tau}^{\tau+m-1}\alpha(b_i, b_{\tau+m})\aleph^{\tau+m-1}}\right) \end{pmatrix},$$

as $\tau \to +\infty$ and by utilizing (3), we obtain

$$\geq \Gamma(1, 1, \cdots, 1) = 1.$$

Hence, $\{b_\tau\}$ is a CS. □

Corollary 1. *Suppose $\{b_\tau\}$ is a sequence in CFMS (Ξ, M_α, Γ) and Γ is H-type. If $\aleph \in (0,1)$ exists such that*

$$M_\alpha(b_\tau, b_{\tau+1}, t) \geq M_\alpha\left(b_{\tau-1}, b_\tau, \frac{t}{\aleph}\right), \ \tau \in \mathbb{N}, t > 0, \tag{4}$$

Then, $\{b_\tau\}$ is a CS.

Lemma 3. *If for $b, \omega \in \Xi$, and some $\aleph \in (0,1)$,*

$$M(b, \omega, t) \geq M_\alpha\left(b, \omega, \frac{t}{\aleph}\right), \ t > 0, \tag{5}$$

Then, $b = \omega$.

Proof. An inequality (5) implies that

$$M_\alpha(b, \omega, t) \geq M_\alpha\left(b, \omega, \frac{t}{\aleph^\tau}\right), \ \tau \in \mathbb{N}, t > 0.$$

Now

$$M_\alpha(b, \omega, t) \geq \lim_{\tau\to\infty} M_\alpha\left(b, \omega, \frac{t}{\aleph^\tau}\right) = 1, t > 0,$$

and by $(FM_\alpha 2)$, it is easy to see that $b = \omega$. □

Theorem 4. *Suppose that (Ξ, M, Γ) is a complete CFMS and suppose that $f : \Xi \to \Xi$. Let them exist $\aleph \in (0, 1)$ such that*

$$M_\alpha(fb, f\omega, t) \geq \max\left\{\begin{array}{c} M_\alpha\left(b, \omega, \frac{t}{\aleph}\right), M_\alpha\left(b, f\omega, \frac{t}{\aleph}\right), M_\alpha\left(fb, \omega, \frac{t}{\aleph}\right), \\ \frac{M_\alpha\left(b, f\omega, \frac{t}{\aleph}\right) + M_\alpha\left(fb, \omega, \frac{t}{\aleph}\right)}{2}, \\ \frac{M_\alpha\left(b, f\omega, \frac{t}{\aleph}\right) . M_\alpha\left(fb, \omega, \frac{t}{\aleph}\right)}{1 + M_\alpha\left(b, \omega, \frac{t}{\aleph}\right)} \end{array}\right\}, \ b, \omega \in \Xi, \ t > 0, \tag{6}$$

and $b, \omega \in \Xi$ such that

$$\lim_{t\to\infty} M_\alpha(b, \omega, t) = 1, \ t > 0. \tag{7}$$

Then, f has a UFP in Ξ.

Proof. Suppose $b_0 \in \Xi$ and $b_{\tau+1} = fb_\tau,\ \tau \in \mathbb{N}$. Consider $b = b_\tau$ and $\omega = b_{\tau-1}$ in (6), then one can obtain

$$M_\alpha(b_\tau, b_{\tau+1},\ t) = M_\alpha(fb_{\tau-1}, fb_\tau,\ t)$$
$$\geq \max\left\{\begin{array}{c} M_\alpha\left(b_{\tau-1},\ b_\tau,\ \frac{t}{\aleph}\right), \Gamma\left(M_\alpha\left(b_{\tau-1},\ b_\tau,\ \frac{t}{\aleph}\right), M_\alpha\left(b_\tau,\ b_{\tau+1},\ \frac{t}{\aleph}\right)\right), \\ M_\alpha\left(b_\tau,\ b_\tau,\ \frac{t}{\aleph}\right), \frac{M_\alpha\left(b_{\tau-1},\ b_{\tau+1},\ \frac{t}{\aleph}\right)+M_\alpha\left(b_\tau,\ b_\tau,\ \frac{t}{\aleph}\right)}{2}, \\ \frac{M_\alpha\left(b_{\tau-1},\ b_{\tau+1},\ \frac{t}{\aleph}\right).M_\alpha\left(b_\tau,\ b_\tau,\ \frac{t}{\aleph}\right)}{1+M_\alpha\left(b_{\tau-1},\ b_\tau,\ \frac{t}{\aleph}\right)} \end{array}\right\},$$
$$\geq \max\left\{M_\alpha\left(b_{\tau-1}, b_\tau, \frac{t}{\aleph}\right), M_\alpha\left(b_\tau, b_{\tau+1}, \frac{t}{\aleph}\right)\right\}.$$

If

$$M_\alpha(b_\tau, b_{\tau+1}, t) \geq M_\alpha\left(b_\tau, b_{\tau+1}, \frac{t}{\aleph}\right),\ \tau \in \mathbb{N}, t > 0,$$

then by Lemma 3 such that $b_\tau = b_{\tau+1},\ \tau \in \mathbb{N}$, we have

$$M_\alpha(b_\tau, b_{\tau+1}, t) \geq M_\alpha\left(b_{\tau-1}, b_\tau, \frac{t}{\aleph}\right), \tau \in \mathbb{N}, t > 0,$$

and by Lemma 2 it obeys that $\{b_\tau\}$ is a CS. Since, (Ξ, M, Γ) is complete, $b \in \Xi$ exist such that $\lim_{\tau\to\infty} b_\tau = b$ and

$$\lim_{\tau\to\infty} M_\alpha(b, b_\tau, t) = 1,\ t > 0. \tag{8}$$

By utilizing (6) and $(FM_\alpha 4)$ it easy to see that b is a FP for f. Suppose $\varkappa_1 \in (\aleph, 1)$ and $\varkappa_2 = 1 - \varkappa_1$ by (6), one can obtain

$$M_\alpha(fb, b,\ t) \geq \Gamma\left(M_\alpha\left(fb,\ fb_{\tau-1},\ \frac{t\varkappa_1}{2\alpha(fb, b_\tau)}\right), M_\alpha\left(b_\tau,\ b,\ \frac{t\varkappa_2}{2\alpha(b_\tau, b)}\right)\right)$$
$$\geq \Gamma\left(\begin{array}{c} \max\left\{\begin{array}{c} M_\alpha\left(b, b_{\tau-1}, \frac{t\varkappa_1}{2\alpha(fb,b_\tau)\aleph}\right),\ M_\alpha\left(b, b_\tau, \frac{t\varkappa_1}{2\alpha(fb,b_\tau)\aleph}\right), \\ \Gamma\left(M_\alpha\left(fb, b, \frac{t\varkappa_1}{(2)^2\alpha(fb,b)\alpha(fb,b_\tau)\aleph}\right), M_\alpha\left(b, b_{\tau-1}, \frac{t\varkappa_1}{(2)^2\alpha(b,b_{\tau-1})\alpha(fb,b_\tau)\aleph}\right)\right) \\ , \frac{M_\alpha\left(b, b_\tau, \frac{t\varkappa_1}{2\alpha(fb,b_\tau)\aleph}\right)+\Gamma\left(\begin{array}{c} M_\alpha\left(fb, b, \frac{t\varkappa_1}{(2)^2\alpha(fb,b)\alpha(fb,b_\tau)\aleph}\right), \\ M_\alpha\left(b, b_{\tau-1}, \frac{t\varkappa_1}{(2)^2\alpha(b,b_{\tau-1})\alpha(fb,b_\tau)\aleph}\right)\end{array}\right)}{2}, \\ \frac{M_\alpha\left(b, b_\tau, \frac{t\varkappa_1}{2\alpha(fb,b_\tau)\aleph}\right).\Gamma\left(\begin{array}{c} M_\alpha\left(fb, b, \frac{t\varkappa_1}{(2)^2\alpha(fb,b)\alpha(fb,b_\tau)\aleph}\right), \\ M_\alpha\left(b, b_{\tau-1}, \frac{t\varkappa_1}{(2)^2\alpha(b,b_{\tau-1})\alpha(fb,b_\tau)\aleph}\right)\end{array}\right)}{1+M_\alpha\left(b, b_{\tau-1}, \frac{t}{2\alpha(fb,b_\tau)\aleph}\right)} \end{array}\right\}, \\ M_\alpha\left(b_\tau,\ b,\ \frac{t}{2\alpha(b_\tau, b)}\right) \end{array}\right),$$

for all $t > 0$. By (8) and as $\tau \to \infty$, we obtain

$$\geq \Gamma\left(\max\left\{\begin{array}{c} 1,\ 1, \\ \Gamma\left(M_\alpha\left(fb, b, \frac{t\varkappa_1}{(2)^2\alpha(fb,b)\alpha(fb,b_\tau)\aleph}\right), 1\right) \\ , \frac{1+\Gamma\left(\begin{array}{c} M_\alpha\left(fb, b, \frac{t\varkappa_1}{(2)^2\alpha(fb,b)\alpha(fb,b_\tau)\aleph}\right), \\ 1 \end{array}\right)}{2}, \\ \frac{1.\Gamma\left(\begin{array}{c} M_\alpha\left(fb, b, \frac{t\varkappa_1}{(2)^2\alpha(fb,b)\alpha(fb,b_\tau)\aleph}\right), \\ 1 \end{array}\right)}{\frac{1+1}{1}} \end{array}\right\}, \right) = 1.$$

Suppose that b and ω are two different FP for f. Then, by applying (6), one can obtainn

$$M_\alpha(b, \omega, t) = M_\alpha(fb, f\omega, t)$$

$$\geq \max\left\{\begin{array}{c} M_\alpha\left(b, \omega, \frac{t}{\aleph}\right),\ M_\alpha\left(b, f\omega, \frac{t}{\aleph}\right), M_\alpha\left(fb, \omega, \frac{t}{\aleph}\right) \\ , \frac{M_\alpha\left(b, f\omega, \frac{t}{\aleph}\right) + M_\alpha\left(fb, \omega, \frac{t}{\aleph}\right)}{2}, \\ \frac{M_\alpha\left(b, f\omega, \frac{t}{\aleph}\right).M_\alpha\left(fb, \omega, \frac{t}{\aleph}\right)}{1+M_\alpha\left(b, \omega, \frac{t}{\aleph}\right)} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} M_\alpha\left(b, \omega, \frac{t}{\aleph}\right),\ M_\alpha\left(b, \omega, \frac{t}{\aleph}\right), M_\alpha\left(b, \omega, \frac{t}{\aleph}\right) \\ , \frac{M_\alpha\left(b, \omega, \frac{t}{\aleph}\right) + M_\alpha\left(b, \omega, \frac{t}{\aleph}\right)}{2}, \\ \frac{M_\alpha\left(b, \omega, \frac{t}{\aleph}\right).M_\alpha\left(b, \omega, \frac{t}{\aleph}\right)}{1+M_\alpha\left(b, \omega, \frac{t}{\aleph}\right)} \end{array}\right\} = M_\alpha\left(b, \omega, \frac{t}{\aleph}\right),\ t > 0,$$

and by Lemma 3, it is easy to see that $b = \omega$. □

Remark 1. *If we take*

$$\max\left\{\begin{array}{c} M_\alpha\left(b, \omega, \frac{t}{\aleph}\right),\ M_\alpha\left(b, f\omega, \frac{t}{\aleph}\right), M_\alpha\left(fb, \omega, \frac{t}{\aleph}\right), \\ \frac{M_\alpha\left(b, f\omega, \frac{t}{\aleph}\right) + M_\alpha\left(fb, \omega, \frac{t}{\aleph}\right)}{2}, \\ \frac{M_\alpha\left(b, f\omega, \frac{t}{\aleph}\right).M_\alpha\left(fb, \omega, \frac{t}{\aleph}\right)}{1+M_\alpha\left(b, \omega, \frac{t}{\aleph}\right)} \end{array}\right\} = M_\alpha\left(b, \omega, \frac{t}{\aleph}\right),$$

in the above theorem we then obtain a fuzzy version of the Banach contraction principle in [14].

Example 2. *Let* $\Xi = \{0, 1, 3\}$, $M_\alpha(b, \omega, t) = e^{-\frac{(b-\omega)^2}{t}}$, *and* $\Gamma = \Gamma_p$. *Then,* (Ξ, M, Γ) *is a complete CFMS with* $\alpha(b, \omega) = b + \omega + 1$. *Define the function* $f : \Xi \to \Xi$ *such that* $f(0) = f(1) = 1$ *and* $f(3) = 0$. *Observe that if* $b = \omega$ or $\omega \in \{0, 1\}$, *then* $M_\alpha(fb, f\omega, t) = 1$ *and* $t > 0$ *and the condition (6) is fulfilled. Suppose* $b = 1$ *and* $\omega = 3$. *Then,* $\aleph \in \left(\frac{1}{9}, \frac{1}{4}\right)$ and *we obtain*

$$M_\alpha(fb, f\omega, t) = e^{-\frac{1}{t}} \geq \max\left\{e^{-\frac{4}{\aleph t}}, e^{-\frac{1}{\aleph t}},\ e^{-\frac{4}{\aleph t}}, \frac{e^{-\frac{1}{\aleph t}} + e^{-\frac{4}{\aleph t}}}{2}, \frac{e^{-\frac{1}{\aleph t}}.e^{-\frac{4}{\aleph t}}}{1 + e^{-\frac{4}{\aleph t}}}\right\}.$$

Now, suppose $b = 1$ *and* $\omega = 3$, *then, choosing* $\aleph$ *from* $\left(\frac{1}{9}, \frac{1}{4}\right)$, *we deduce*

$$M_\alpha(fb, f\omega, t) = e^{-\frac{1}{t}} \geq \max\left\{e^{-\frac{4}{\aleph t}}, e^{-\frac{1}{\aleph t}},\ e^{-\frac{4}{\aleph t}}, \frac{e^{-\frac{1}{\aleph t}} + e^{-\frac{4}{\aleph t}}}{2}, \frac{e^{-\frac{1}{\aleph t}}.e^{-\frac{4}{\aleph t}}}{1 + e^{-\frac{4}{\aleph t}}}\right\}.$$

Similarly, if $b = 3$ and $\omega = 1$ as well as $b = 3$ and $\omega = 1$, we establish that for $\aleph \in \left(\frac{1}{9}, \frac{1}{4}\right)$, condition (6) is fulfilled for all $b, \omega \in \Xi$, and $t > 0$. Hence, all the conditions of Theorem 4 are satisfied with a UFP $b = 1$.

Corollary 2. *Supposing that (Ξ, M, Γ) is a complete CFMS with $\alpha : \Xi \times \Xi \to [1, \infty)$, assume that*

$$\lim_{t\to\infty} M_\alpha(b, \omega, t) = 1,$$

For all $b, \omega \in \Xi$. If $f : \Xi \to \Xi$ satisfies the following, for some $\aleph \in (0, 1)$, such that

$$M_\alpha(fb, f\omega, t) \geq M_\alpha\left(b, \omega, \frac{t}{\aleph}\right), \text{ for all } b, \omega \in \Xi, t > 0.$$

Then, f has a UFP in Ξ.

Example 3. *$\Xi = A \cup B$, where $A = [0, 1]$, $B = \mathbb{N}\backslash\{1\}$, and $M_\alpha : \Xi \times \Xi \times (0, \infty) \to [0, 1]$, is defined by*

$$M_\alpha\ (b, \omega, t) = \begin{cases} 1 \text{ if } b = \omega \\ \frac{t}{t+\frac{1}{\omega}} \text{ if } b \in B \text{ and } \omega \in A \\ \frac{t}{t+\frac{1}{b}} \text{ if } b \in A \text{ and } \omega \in B \\ \frac{t}{t+\max\{b,\omega\}} \text{ otherwise.} \end{cases}$$

Then, (Ξ, M, Γ) is CFMS with $\Gamma(b, \omega) = b{\cdot}\omega$ and a controlled function $\alpha :\ \Xi \times \Xi \to [0, \infty)$ defined by

$$\alpha(b, \omega) = \begin{cases} 1 \text{ if } b = \omega \\ \max\{b, \omega\} \text{ otherwise.} \end{cases}$$

It is easy to see that all the conditions of Corollary 2 are satisfied.
Consider the triangular inequality $(EM_\alpha 4)$ of fuzzy extended b-metric space defined in Definition 5 as

$$M_\alpha(b, z, \alpha(b, z)(t + s)) \geq \Gamma(M_\alpha(b, \omega, t), M_\alpha(\omega, z, s)).$$

Let $b, z \in A$, and $\omega \in B$, the, n $\alpha(b, z) = 1$. Assume $t = s = 2$, $b = \frac{1}{2}$, $z = 1$, and $\omega = 40$. We have

$$\frac{t + s + \frac{1}{\max\{b,z\}}}{t + s + \frac{1}{\min\{b,z\}}} \geq \Gamma\left(\frac{t}{t + \frac{1}{\omega}}, \frac{s}{s + \frac{1}{\omega}}\right).$$

We obtain $0.8 > 0.975$ which is a contradiction. Hence, M_α is not an extended fuzzy b-metric space. Now, consider the triangular inequality $(b4)$ of FbMS defined in Definition 4 as

$$M_\alpha(b, z, \rho(t + s)) \geq \Gamma(M_\alpha(b, \omega, t), M_\alpha(\omega, z, s)).$$

We obtain

$$\frac{\rho(s + t) + \frac{1}{\max\{b,z\}}}{\rho(s + t) + \frac{1}{\min\{b,z\}}} \geq \Gamma\left(\frac{t}{t + \frac{1}{\omega}}, \frac{s}{s + \frac{1}{\omega}}\right).$$

For $\rho \in [1, 9]$, the above inequality not satisfied.

Theorem 5. *Supposing that* (Ξ, M_α, Γ) *is complete CFMS assuming that* $f : \Xi \to \Xi$, *then* $\aleph \in (0,1)$ *exists*

$$M_\alpha(fb, f\omega, t) \geq \min\left\{ M_\alpha\left(b, \omega, \frac{t}{\aleph}\right),\ M_\alpha\left(fb, b, \frac{t}{\aleph}\right),\ M_\alpha\left(f\omega, \omega, \frac{t}{\aleph}\right)\right\} \tag{9}$$

for all $b, \omega \in \Xi$, $t > 0$ *such that*

$$\lim_{\tau\to\infty} M_\alpha(b, \omega, t) = 1 \tag{10}$$

for all $t > 0$. *Then,* f *has a UFP in* Ξ.

Proof. Suppose $b_0 \in \Xi$, $b_{\tau+1} = fb_\tau$, *and* $\tau \in \mathbb{N}$ from (9) with $b = b_\tau$ *and* $\omega = b_{\tau-1}$, for every $\tau \in \mathbb{N}$ and $t > 0$, one can obtain

$$\begin{aligned} M_\alpha(b_{\tau+1}, b_\tau, t) &\geq \min\left\{M_\alpha\left(b_\tau, b_{\tau-1}, \tfrac{t}{\aleph}\right), M_\alpha\left(b_{\tau+1}, b_\tau, \tfrac{t}{\aleph}\right), M_\alpha\left(b_\tau, b_{\tau-1}, \tfrac{t}{\aleph}\right)\right\} \\ &\geq \min\left\{M_\alpha\left(b_\tau, b_{\tau-1}, \tfrac{t}{\aleph}\right), M_\alpha\left(b_{\tau+1}, b_\tau, \tfrac{t}{\aleph}\right)\right\}. \end{aligned}$$

If

$$M_\alpha(b_{\tau+1}, b_\tau, t) \geq M_\alpha\left(b_{\tau+1}, b_\tau, \frac{t}{\aleph}\right),\ \tau \in \mathbb{N},\ t > 0.$$

Then, Lemma 3 implies that $b_\tau = b_{\tau+1}$, $\tau \in \mathbb{N}$, such that

$$M_\alpha(b_{\tau+1}, b_\tau, t) \geq M_\alpha\left(b_\tau, b_{\tau-1}, \frac{t}{\aleph}\right),\ \tau \in \mathbb{N},\ t > 0.$$

Moreover, by Lemma 2 $\{b_\tau\}$ is a CS. Hence, $b \in \Xi$ exists such that $\lim_{\tau\to\infty} b_\tau = b$ and

$$\lim_{\tau\to\infty} M_\alpha(b, b_\tau, t) = 1,\ t > 0. \tag{11}$$

Now, we show that b is an FP for f. Letting $\varkappa_1 \in (\aleph, 1)$ and $\varkappa_2 = 1 - \varkappa_1$ by (9), one can obtain

$$M_\alpha(fb, b, t) \geq \Gamma\left(M_\alpha\left(fb, fb_\tau, \tfrac{t\varkappa_1}{\alpha(fb, fb_\tau)}\right),\ M_\alpha\left(b_{\tau+1}, fb_\tau, \tfrac{t\varkappa_2}{\alpha(b_{\tau+1}, fb_\tau)}\right)\right)$$

$$\geq \Gamma\left(\min\left\{\begin{array}{c} M_\alpha\left(b, b_\tau, \frac{t\varkappa_1}{\alpha(b, b_\tau)}\right), M_\alpha\left(b, fb, \frac{t\varkappa_1}{\alpha(b, fb)}\right), \\ M_\alpha\left(b_\tau, b_{\tau+1}, \frac{t\varkappa_1}{\alpha(b_\tau, b_{\tau+1})\aleph}\right) \end{array}\right\}, M_\alpha\left(b_{\tau+1}, b, \tfrac{t\varkappa_2}{\alpha(b_{\tau+1}, b_\tau)}\right)\right),$$

taking $\tau \to \infty$ and utilizing (11), we deduce

$$\begin{aligned} M_\alpha(fb, b, t) &\geq \Gamma\left(\min\left\{1, M_\alpha\left(b, fb, \tfrac{t\varkappa_1}{\alpha(b, fb)\aleph}\right), 1\right\}\right) \\ &= \Gamma\left(M_\alpha\left(b, fb, \tfrac{t\varkappa_1}{\alpha(b, fb)\aleph}\right), 1\right) = M_\alpha\left(b, fb, \tfrac{t}{v}\right),\ t > 0, \end{aligned}$$

where $v = \frac{\alpha(b, fb)\aleph}{\varkappa_1} \in (0,1)$, we have

$$M_\alpha(fb, b, t) \geq M_\alpha\left(fb, b, \frac{t}{v}\right),\ t > 0,$$

and by Lemma 3, we have $fb = b$. Suppose that b and ω are to different FP for f, that is, $fb = b$ and $f\omega = \omega$. By (9), we deduce

$$\begin{aligned} M_\alpha(fb, f\omega, t) &\geq \min\left\{M_\alpha\left(b, \omega, \tfrac{t}{\aleph}\right), M_\alpha\left(b, fb, \tfrac{t}{\aleph}\right), M_\alpha\left(\omega, f\omega, \tfrac{t}{\aleph}\right)\right\} \\ &= \min\left\{M_\alpha\left(b, \omega, \tfrac{t}{\aleph}\right), 1, 1\right\} = M_\alpha\left(b, \omega, \tfrac{t}{\aleph}\right) = M_\alpha\left(fb, f\omega, \tfrac{t}{\aleph}\right) \end{aligned}$$

for $t >$, and by utilizing the Lemma 3, we have $fb = f\omega$, which gives $b = \omega$. □

Remark 2. *If we take*

$$\min\left\{M_\alpha\left(b,\omega,\frac{t}{\aleph}\right),\ M_\alpha\left(fb,b,\frac{t}{\aleph}\right),\ M_\alpha\left(f\omega,\omega,\frac{t}{\aleph}\right)\right\} = M_\alpha\left(b,\omega,\frac{t}{\aleph}\right),$$

in the above theorem then we obtain a fuzzy version of the Banach contraction principle in [14].

Example 4. *Suppose* $\Xi = (0,\ 2), M_\alpha(b,\omega,t) = e^{-\frac{(b-\omega)^2}{t}}$, *and* $\Gamma = \Gamma_p$. *Then,* (Ξ, M, Γ) *is a complete CFMS with* $\alpha(b,\omega) = b + \omega + 2$. *Let*

$$f(b) = \begin{cases} 2-b, & b \in (0,1), \\ 1, & b \in [1,2). \end{cases}$$

part 1: *If* $b,\omega \in [1,2)$, *then* $M_\alpha(fb, f\omega, t) = 1,\ t > 0$ *and (9) are trivially verified.*
part 2: *If* $b \in [1,2)$ *and* $\omega \in (0,1)$, *such that* $\aleph \in \left(\frac{1}{4},\frac{1}{2}\right)$, *one can obtain*

$$M_\alpha(fb, f\omega, t) = e^{-\frac{(1-\omega)^2}{t}} \geq e^{-\frac{4\aleph(1-\omega)^2}{t}} = M_\alpha\left(f\omega,\omega,\frac{t}{\aleph}\right),\ t > 0.$$

part 3: *As in the preceding section, for* $\aleph \in \left(\frac{1}{4},\frac{1}{2}\right)$, *we obtain*

$$M_\alpha(fb, f\omega, t) \geq M_\alpha\left(fb,b,\frac{t}{\aleph}\right),\ b \in (0,1),\ \omega \in [1,2),\ t > 0.$$

part 4: *If* $b,\omega \in (0,1)$, *then for* $\aleph\left(\frac{1}{4},\frac{1}{2}\right)$ *we have*

$$M_\alpha(fb, f\omega, t) = e^{-\frac{(1-\omega)^2}{t}} \geq e^{-\frac{4\aleph(1-\omega)^2}{t}} = M_\alpha\left(f\omega,\omega,\frac{t}{\aleph}\right),\ b > \omega, t > 0,$$

and

$$M_\alpha(fb, f\omega, t) \geq M_\alpha\left(f\omega,\omega,\frac{t}{\aleph}\right),\ b < \omega, t > 0.$$

So, condition (9) is fulfilled $for\ all\ b,\omega \in, t > 0$ *and by Theorem 5 it follows that* $b = 1$ *is a UFP for* f.
We analyze a Ciric-quasi-contraction in the following theorem.

Theorem 6. *Supposing that* $(\Xi, M, \Gamma_{\min})$ *is a complete CFMS, assume that* $f : \Xi \to \Xi$. *If for some* $\aleph \in (0,1)$, *such that*

$$M_\alpha(fb, f\omega, t) \geq \min\left\{\begin{array}{c} M_\alpha\left(b,\omega,\frac{t}{\aleph}\right), M_\alpha\left(fb,b,\frac{t}{\aleph}\right), \\ M_\alpha\left(f\omega,\omega,\frac{t}{\aleph}\right), M_\alpha\left(fb,\omega,\frac{2t}{\aleph}\right), M_\alpha\left(b,f\omega,\frac{t}{\aleph}\right) \end{array}\right\},\ b,\omega \in \Xi, t > 0. \quad (12)$$

Then, f *has a UFP in* Ξ.

Proof. Suppose $b_0 \in \Xi$ and $b_{\tau+1} = fb_\tau,\ \tau \in \mathbb{N}$. By utilizing the condition (12) with $= b_\tau$, $\omega = b_{\tau-1}$, utilizing $(FM_\alpha 4)$

and the assumption that $\Gamma = \Gamma_{\min}$, one can obtain

$$M_\alpha(b_{\tau+1}, b_\tau, t) \geq \min\left\{\begin{array}{c} M_\alpha\left(b_\tau, b_{\tau-1}, \frac{t}{\aleph}\right), M_\alpha\left(b_{\tau+1}, b_\tau, \frac{t}{\aleph}\right), M_\alpha\left(b_\tau, b_{\tau-1}, \frac{t}{\aleph}\right), \\ \min\left\{M_\alpha\left(b_{\tau+1}, b_\tau, \frac{t}{\alpha(b_{\tau+1}, b_\tau)\aleph}\right), M_\alpha\left(b_\tau, b_{\tau-1}, \frac{t}{\alpha(b_\tau, b_{\tau-1})\aleph}\right)\right\}, \\ M_\alpha\left(b_\tau, b_\tau, \frac{t}{\aleph}\right), \end{array}\right\}$$

$$\geq \min\left\{\begin{array}{c} M_\alpha\left(b_\tau, b_{\tau-1}, \frac{t}{\alpha(b_\tau, b_{\tau-1})\aleph}\right), \\ M_\alpha\left(b_{\tau+1}, b_{\tau-1}, \frac{t}{\alpha(b_{\tau+1}, b_{\tau-1})\aleph}\right) \end{array}\right\}, \ \tau \in \mathbb{N}, \ t > 0.$$

By Lemma 3 and Corollary 2, we are able to demonstrate Theorem 5 such that

$$M_\alpha(b_{\tau+1}, b_\tau, t) \geq M_\alpha\left(b_\tau, b_{\tau-1}, \frac{t}{\alpha(b_\tau, b_{\tau-1})\aleph}\right), \ \tau \in \mathbb{N}, t > 0,$$

and $\{b_\tau\}$ is a CS. So, $b \in \Xi$ exists such that $\lim_{\tau\to\infty} b_\tau = b$ and

$$\lim_{\tau\to\infty} M_\alpha(b, b_\tau, t) = 1, \ t > 0. \tag{13}$$

Suppose $\varkappa_1 \in (\aleph, 1)$ and $\varkappa_2 = 1 - \varkappa_1$. By (12) and $(FM_\alpha 4)$, we deduce

$$M_\alpha(fb, b, t) \geq \min\left\{M_\alpha\left(fb, fb_\tau, \frac{t\varkappa_1}{\alpha(fb, fb_\tau)}\right), M_\alpha\left(fb_\tau, fb, \frac{t\varkappa_2}{\alpha(fb_\tau, fb)}\right)\right\}$$

$$\geq \min\left\{\begin{array}{c} \min\left\{\begin{array}{c} M_\alpha\left(b, b_\tau, \frac{t\varkappa_1}{\alpha(b, b_\tau)}\right), M_\alpha\left(b, fb, \frac{t\varkappa_1}{\alpha(b, fb)\aleph}\right), M_\alpha\left(b_\tau, b_{\tau+1}, \frac{t\varkappa_1}{\alpha(b_\tau, b_{\tau+1})\aleph}\right), \\ \min\left\{M_\alpha\left(fb, b, \frac{t\varkappa_1}{\alpha(b, fb)\alpha(fb, b_\tau)\aleph}\right), \left(b, b_\tau, \frac{t\varkappa_1}{\alpha(b, b_\tau)\alpha(fb, b_\tau)\aleph}\right)\right\}, \\ M_\alpha\left(b, b_{\tau+1}, \frac{t\varkappa_1}{\alpha(b, b_{\tau+1})\aleph}\right) \end{array}\right\}, \\ M_\alpha\left(b_{\tau+1}, b, \frac{t\varkappa_2}{\alpha(b_{\tau+1}, b)}\right) \end{array}\right\}$$

for all $\tau \in \mathbb{N}$ and $t > 0$. Taking $\tau \to \infty$ and utilizing (13), we obtain

$$M_\alpha(fb, b, t) \geq \min\left\{\min\left\{\begin{array}{c} 1, M_\alpha\left(b, fb, \frac{t\varkappa_1}{\alpha(b, fb)\aleph}\right), 1, \\ \min\left\{M_\alpha\left(fb, b, \frac{t\varkappa_1}{\alpha(b, fb)\alpha(fb, b_\tau)\aleph}\right), 1\right\}, 1 \end{array}\right\}, 1\right\}$$

$$= M_\alpha\left(fb, b, \frac{t\varkappa_1}{\alpha(b, fb)\alpha(fb, b_\tau)\aleph}\right), \ t > 0,$$

and by Lemma 3 with $v = \frac{\alpha(b, fb)\alpha(fb, b_\tau)\aleph}{\varkappa_1} \in (0, 1)$ such that $fb = b$. By condition (12), for two different FPs $b = fb$ and $\omega = f\omega$, one can obtain

$$M_\alpha(fb, f\omega, t) \geq \min\left\{\begin{array}{c} M_\alpha\left(b, \omega, \frac{t}{\aleph}\right), M_\alpha\left(fb, b, \frac{t}{\aleph}\right), M_\alpha\left(f\omega, \omega, \frac{t}{\aleph}\right), \\ \min\left\{M_\alpha\left(fb, b, \frac{t}{\alpha(fb, b)\aleph}\right), M_\alpha\left(b, \omega, \frac{t}{\alpha(b, \omega)\aleph}\right)\right\}, M_\alpha\left(b, f\omega, \frac{t}{\aleph}\right) \end{array}\right\}$$

$$= \min\left\{M_\alpha\left(b, \omega, \frac{t}{\aleph}\right), 1, 1, \min\left\{1, M_\alpha\left(b, \omega, \frac{t}{\alpha(b, \omega)\aleph}\right)\right\}, M_\alpha\left(b, \omega, \frac{t}{\aleph}\right)\right\}$$

$$= M_\alpha\left(b, \omega, \frac{t}{\alpha(b, \omega)\aleph}\right) = M_\alpha\left(fb, f\omega, \frac{t}{\alpha(b, \omega)\aleph}\right), t > 0,$$

and by Lemma 3, it follows that $b = \omega$. □

In the next theorem, we aim to establish a new contractive condition with the weaker TN.

Example 5. *Suppose* $\Xi = A \cup B$ *where* $A = [0,1], B = \mathbb{N}\backslash\{1\}$, *and* $M_\alpha : \Xi \times \Xi \times (0, \infty) \to [0,1]$ *is a fuzzy set defined by*

$$M(b, \omega, t) = \begin{cases} 1 \text{ if } b = \omega \\ e^{-\frac{1}{\omega t}} \text{ if } b \in B \text{ and } \omega \in A \\ e^{-\frac{1}{bt}} \text{ if } b \in A \text{ and } \omega \in B \\ e^{-\frac{\max\{b,\omega\}}{t}} \text{ otherwise.} \end{cases}$$

Then, (Ξ, M, Γ) *is a CFMS with* $\Gamma(b, \omega) = b{\cdot}\omega$ *and a controlled function* $\alpha : \Xi \times \Xi \to [0, \infty)$ *defined by*

$$\alpha(b, \omega) = \begin{cases} 1 \text{ if } b = \omega \\ \max\{b, \omega\} \text{ otherwise.} \end{cases}$$

It is easy to see that all the conditions of Theorem 6 are satisfied.
Let $b = \frac{1}{2}$, $\omega = 40$, $z = 1$, *and* $t = s = 1$. *Then, it does not satisfy the triangle inequality* $(EM_\alpha 4)$ *of Definition 5. Hence, it is not extended fuzzy b-metric space. Now, show that it is not an FbMS. Considering the triangle inequality (b4) of Definition 4, we have*

$$e^{-\frac{\max\{b,\omega\}}{\rho(t+s)}} \geq \Gamma\left(e^{-\frac{1}{\omega t}},\ e^{-\frac{1}{\omega s}}\right),$$
$$e^{-\frac{1}{2\rho}} \geq e^{-\frac{2}{20}} = e^{-\frac{1}{10}}.$$

It is clear that the above inequality is not satisfied for $\rho = 2$. *Hence, it is not FbMS.*

Theorem 7. *Supposing that* (Ξ, M, Γ), $\Gamma \geq \Gamma_p$ *is a complete CFMS, assume that* $f : \Xi \to \Xi$. For *some* $\aleph \in (0,\ 1)$, *let*

$$M_\alpha(fb, f\omega, t) \geq \min\left\{ \begin{array}{c} M_\alpha\left(b, \omega, \frac{t}{\aleph}\right), M_\alpha\left(fb, b, \frac{t}{\aleph}\right), M_\alpha\left(f\omega, \omega, \frac{t}{\aleph}\right), \\ \sqrt{M_\alpha\left(fb, \omega, \frac{2t}{\aleph}\right)}, M_\alpha\left(b, f\omega, \frac{t}{\aleph}\right) \end{array} \right\},\ b, \omega \in \Xi, t > 0, \quad (14)$$

and $b, \omega \in \Xi$ exists *such that*

$$\lim_{\tau \to \infty} M_\alpha(b, \omega, t) = 1,\ t > 0. \quad (15)$$

Then, f *has a UFP in* Ξ.

Proof. Let $b_0 \in \Xi$ and $b_{\tau+1} = fb_\tau$, $\tau \in \mathbb{N}$. Taking $b = b_\tau$ and $\omega = b_{\tau-1}$ in (14), by $(FM_\alpha 4)$ and $\Gamma \geq \Gamma_p$, one can obtain

$$M_\alpha(b_{\tau+1}, b_\tau, t) \geq \min\left\{ \begin{array}{c} M_\alpha\left(b_\tau, b_{\tau-1}, \frac{t}{\aleph}\right), M_\alpha\left(fb, b, \frac{t}{\aleph}\right), M_\alpha\left(b_{\tau+1}, b_\tau, \frac{t}{\aleph}\right), M_\alpha\left(b_\tau, b_{\tau-1}, \frac{t}{\aleph}\right), \\ \sqrt{M_\alpha\left(fb, \omega, \frac{2t}{\aleph}\right).M_\alpha\left(b_\tau, b_{\tau-1}, \frac{t}{\alpha(b_\tau, b_{\tau-1})\aleph}\right)}, M_\alpha\left(b_\tau, b_\tau, \frac{t}{\aleph}\right) \end{array} \right\},$$

for all $b, \omega \in \Xi, t > 0$. Since $M_\alpha(b, \omega, t)$ is a b-nondecreasing in t and $\sqrt{\varkappa.\rho} \geq \min\{\varkappa, \rho\}$, we deduce

$$M_\alpha(b_{\tau+1}, b_\tau, t) \geq \min\left\{ \begin{array}{c} M_\alpha\left(b_{\tau+1}, b_\tau, \frac{t}{\alpha(b_{\tau+1}, b_\tau)\aleph}\right), \\ M_\alpha\left(b_\tau, b_{\tau-1}, \frac{t}{\alpha(b_\tau, b_{\tau-1})\aleph}\right) \end{array} \right\}$$

for all $\tau \in \mathbb{N},\ t > 0$. By Lemmas 2 and 3 we have

$$M_\alpha(b_{\tau+1}, b_\tau, t) \geq M_\alpha\left(b_\tau, b_{\tau-1}, \frac{t}{\alpha(b_\tau, b_{\tau-1})\aleph}\right),\ \tau \in \mathbb{N},\ t > 0.$$

Hence, $\{b_\tau\}$ is a CS. Since (Ξ, M, Γ) is complete, $b \in \Xi$ exists such that

$$\lim_{\tau \to \infty} b_\tau = b \text{ and } \lim_{\tau \to \infty} M_\alpha(b, b_\tau, t) = 1,\ t > 0. \quad (16)$$

Supposing $\varkappa_1 \in (\aleph, 1)$ and $\varkappa_2 = 1 - \varkappa_1$, by (14) and $(FM_\alpha 4)$ one can obtain

$$\begin{aligned}
M_\alpha(fb, b, t) &\geq \Gamma\Big(M_\alpha\Big(fb, fb_\tau, \tfrac{t\varkappa_1}{\alpha(fb,b_{\tau+1})}\Big), M_\alpha\Big(fb_\tau, b, \tfrac{t\varkappa_2}{\alpha(b_{\tau+1},b)}\Big)\Big), \\
&\geq \Gamma\left(\begin{array}{c} \min\left\{\begin{array}{c} M_\alpha\Big(b, b_\tau, \frac{t\varkappa_1}{\alpha(b,b_\tau)\aleph}\Big), M_\alpha\Big(b, fb, \frac{t\varkappa_1}{\alpha(b,fb)\aleph}\Big), M_\alpha\Big(b_\tau, b_{\tau+1}, \frac{t\varkappa_1}{\alpha(b_\tau,b_{\tau+1})\aleph}\Big), \\ \sqrt{M_\alpha\Big(fb, b, \frac{t\varkappa_1}{\alpha(fb,b)\alpha(fb,b_\tau)\aleph}\Big). M_\alpha\Big(b, b_\tau, \frac{t\varkappa_1}{\alpha(fb,b_\tau)\alpha(b,b_\tau)\aleph}\Big)}, \\ M_\alpha\Big(b, b_{\tau+1}, \frac{t\varkappa_1}{\alpha(b,b_{\tau+1})\aleph}\Big), \end{array}\right\}, \\ M_\alpha\Big(b_{\tau+1}, b, \frac{t\varkappa_2}{\alpha(b_{\tau+1},b)}\Big) \end{array}\right) \\
&\geq \Gamma\left(\begin{array}{c} \min\left\{\begin{array}{c} M_\alpha\Big(b, b_\tau, \frac{t\varkappa_1}{\alpha(b,b_\tau)\aleph}\Big), M_\alpha\Big(b, fb, \frac{t\varkappa_1}{\alpha(b,fb)\aleph}\Big), M_\alpha\Big(b_\tau, b_{\tau+1}, \frac{t\varkappa_1}{\alpha(b_\tau,b_{\tau+1})\aleph}\Big), \\ \min\Big\{M_\alpha\Big(fb, b, \frac{t\varkappa_1}{\alpha(fb,b)\alpha(fb,b_\tau)\aleph}\Big), M_\alpha\Big(b, b_\tau, \frac{t\varkappa_1}{\alpha(fb,b_\tau)\alpha(b,b_\tau)\aleph}\Big)\Big\}, \\ M_\alpha\Big(b, b_{\tau+1}, \frac{t\varkappa_1}{\alpha(b,b_{\tau+1})\aleph}\Big), \end{array}\right\} \\ M_\alpha\Big(b_{\tau+1}, b, \frac{t\varkappa_2}{\alpha(b_{\tau+1},b)}\Big) \end{array}\right)
\end{aligned}$$

for all $\tau \in \mathbb{N}$ and $t > 0$. Taking $\tau \to \infty$ and utilizing (16), we have

$$\begin{aligned}
M_\alpha(fb, b, t) &\geq \Gamma\left(\min\left\{\begin{array}{c} 1, M\Big(b, fb, \frac{t\varkappa_1}{\alpha(b,fb)\aleph}\Big), 1, \\ \min\Big\{M\Big(fb, b, \frac{t\varkappa_1}{\alpha(fb,b)\alpha(fb,b_\tau)\aleph}\Big), 1\Big\}, 1 \end{array}\right\}\right) \\
&= M_\alpha\Big(fb, b, \frac{t\varkappa_1}{\alpha(fb,b)\alpha(fb,b_\tau)\aleph}\Big), \ t > 0
\end{aligned}$$

and by Lemma 3 with $v = \frac{\alpha(fb,b)\alpha(fb,b_\tau)\aleph}{\varkappa_1} \in (0,1)$ uch that $fb = b$.

Let b and ω are two different FPs for f. By (2.13), we obtain

$$\begin{aligned}
M_\alpha(fb, f\omega, t) &\geq \Gamma\left(\begin{array}{c} M_\alpha\big(b, \omega, \frac{t}{\aleph}\big), M_\alpha\big(fb, b, \frac{t}{\aleph}\big), M_\alpha\big(f\omega, \omega, \frac{t}{\aleph}\big), \\ \sqrt{M_\alpha\Big(fb, b, \frac{t}{\alpha(fb,b)\aleph}\Big).M_\alpha\Big(b, \omega, \frac{t}{\alpha(b,\omega)\aleph}\Big)}, M_\alpha\ \big(b, \omega, \frac{t}{\aleph}\big) \end{array}\right), \\
&\geq \Gamma\Big(M_\alpha\big(b, \omega, \tfrac{t}{\aleph}\big), 1, 1, \min\Big\{1, M_\alpha\Big(b, \omega, \tfrac{t}{\alpha(b,\omega)\aleph}\Big)\Big\}, M\big(b, \omega, \tfrac{t}{\aleph}\big)\Big), \\
&= M_\alpha\Big(b, \omega, \tfrac{t}{\alpha(b,\omega)\aleph}\Big) = M_\alpha\Big(fb, f\omega, \tfrac{t}{\alpha(fb,f\omega)\aleph}\Big), \ t > 0,
\end{aligned}$$

and thus, by Lemma 3, we have $b = \omega$. □

Example 6. *Suppose* $\Xi = \{0, 1, 3\}$, $M_\alpha(b, \omega, t) = e^{-\frac{(b-\omega)^2}{t}}$, *and* $\Gamma = \Gamma_p$. *Then,* (Ξ, M, Γ) *is a complete CFMS with* $\alpha(b, \omega) = b + \omega + 1$. *Define the function* $f : \Xi \to \Xi$ *such that* $f0 = f1 = 1$ and $f3 = 0$. *Observe that if* $b = \omega$ *or* $\omega \in \{0, 1\}$, *then,* $M_\alpha(fb, f\omega, t) = 1$, $t > 0$ *and (14) is fulfilled. Suppose* $b = 1$ *and* $\omega = 3$. *Then,* $\aleph \in \left(\frac{1}{9}, \frac{1}{4}\right)$, *we obtain*

$$M_\alpha(fb, f\omega, t) = e^{-\frac{1}{t}} \geq \min\Big\{e^{-\frac{9\aleph}{t}}, e^{-\frac{\aleph}{t}}, e^{-\frac{9\aleph}{t}}, e^{-\frac{\aleph}{t}}, 1\Big\}.$$

Suppose $b = 1$ *and* $\omega = 3$. *Then, by choosing* $\aleph$ *from* $\left(\frac{1}{9}, \frac{1}{4}\right)$, *we have*

$$M_\alpha(fb, f\omega, t) = e^{-\frac{1}{t}} \geq \min\Big\{e^{-\frac{4\aleph}{t}}, 1, e^{-\frac{9\aleph}{t}}, e^{-\frac{\aleph}{t}}, e^{-\frac{\aleph}{t}}\Big\}.$$

Similarly, if $b = 3$ *and* $\omega = 1$ *as well as* $b = 3$ *and* $\omega = 1$, *then for* $\aleph \in \left(\frac{1}{9}, \frac{1}{4}\right)$, *condition (14) is met for all* $b, \omega \in \Xi$, *and* $t > 0$. *As a result, Theorem 7 is satisfied with a UFP* $b = 1$.

3. An Application to the Transformation of Solar Energy to Electric Power

Sun-based boards are currently being distributed and shown widely to reduce people's reliance on petroleum derivatives which are less environmentally friendly. Nearly 19 trillion kilowatts of power were transported internationally in 2007. In comparison, the amount of daylight that enters the Earth's surface in a single hour is enough to illuminate the entire planet for a full year. The question is: how do those dazzling and warm beams of light obtain power? A numerical model of the electric flow in an RLC equal circuit, often known as a "tuning" circuit, can be presented with a basic understanding of how light is converted into power. In the fields of radio and communication engineering, this circuit has several uses. The version that is being presented can be used to calculate the production of electric power, provide tools to improve building performance, and can be used as a decision-making tool when designing a hybrid renewable electricity system based on solar power. Every aspect of this system is mathematically expressed as a differential equation in [24] using the following equation

$$\begin{cases} \frac{d^2\mathfrak{C}}{d\varpi^2} = \Omega(\varpi,\ \mathfrak{C}(\varpi)) - \frac{\mathfrak{R}}{\mathfrak{L}}\frac{d\mathfrak{C}}{d\varpi} \\ \mathfrak{C}(0) = 0,\ \mathfrak{C}'(0) = m, \end{cases} \tag{17}$$

where $\Omega : [0,\ 1] \times \mathcal{R}^+ \to \mathcal{R}$ is a continuous function that is condition (17) to the integral equation to which it is equivalent.

$$\varrho(\varpi) = \int_0^{\varpi} N(\varpi,\ l)\Omega(l,\ \mathfrak{C}(l))dl,\ \varpi \in [0,1], \tag{18}$$

where the Green's function $N(\varpi,\ b)$, it follows:

$$N(\varpi,\ l) = \begin{cases} (\varpi - l)e^{\Omega(M(b,\ \omega)(\varpi - l))}\ 0 \le l \le \varpi \le 1, \\ 0\ 0 \le \varpi \le l \le 1. \end{cases} \tag{19}$$

where $\Omega(M(b,\ \omega)) > 0$ is a constant, as determined by the values of $\mathfrak{R}$ and $\mathfrak{L}$, mentioned in (17).

Let $\Xi = C([0,\varpi],\ \mathcal{R}^+)$ be the set of all real continuous positive functions that are expressed on the set $[0,\ \varpi]$. Let Ξ be endowed with the CFMS given by the following

$$M(b,\ \omega,\ t) = \begin{cases} 0 \text{ if } t = 0 \\ \sup_{t\in[0,1]} \frac{\min\{b,\ \omega\}+t}{\max\{b,\ \omega\}+t} \text{ otherwise,} \end{cases} \text{for all } b,\ \omega \in \Xi. \tag{20}$$

One can verify that (Ξ, M, Γ) is a complete CFMS with a controlled function $\alpha :\ \Xi \times \Xi \to [0,\infty)$, defined by

$$\alpha(b,\ \omega) = b + \omega + 1.$$

It is obvious that b^* is a solution of integral Equation (18), and as a result, a solution of differential Equation (17) which governs the system of converting solar energy into electric power if and only if b^* is an FP of f. It is installed as a guarantee of the existence of FP of f.

Theorem 8. *Assume the following problem fulfills:*

(I) $f : [0,\varpi] \times [0,\varpi] \to \mathcal{R}^+$ *is a continuous function;*

(II) *there exists a continuous function* $N : [0,\varpi] \times [0,\varpi] \to \mathcal{R}^+$ *such that* $\sup_{\alpha\in[0,\varpi]} \int_0^{\varpi} N(\alpha,\ l) \ge 1;$

(III) $max\{f(\alpha,l,\ b(l),\ f(\alpha,l,\ \omega(l))\} \geq N(\alpha,b)max\{D(b(l),\ \omega(l))\}$ *and* $min\{f(\alpha,l,\ b(l),\ f(\alpha,l,\ \omega(l))\} \geq N(\alpha,b)min\{D(b(l),\ \omega(l))\}$ *for all* $\alpha,l,\ \in [0,1]$, $b,\omega \in \mathcal{R}^+$ *and* $\aleph \in (0,\ 1)$ *exists such that*

$$D(b(l),\ \omega(l)) = max\left\{\begin{array}{c} M_\alpha\left(b(l),\omega(l),\frac{t}{\aleph}\right),\ M_\alpha\left(b(l),f\omega(l),\frac{t}{\aleph}\right),M_\alpha\left(fb(l),\omega(l),\frac{t}{\aleph}\right) \\ ,\frac{M_\alpha\left(b(l),f\omega(l),\frac{t}{\aleph}\right)+M_\alpha\left(fb(l),\omega(l),\frac{t}{\aleph}\right)}{2}, \\ \frac{M_\alpha\left(b(l),f\omega(l),\frac{t}{\aleph}\right).M_\alpha\left(fb(l),\omega(l),\frac{t}{\aleph}\right)}{1+M_\alpha\left(b(l),\omega(l),\frac{t}{\aleph}\right)} \end{array}\right\}.$$

The differential Equation (17) that represents the solar energy problem has a solution as a result and the integral Equation (18) also has a solution.

Proof. For $b,\omega \in \Xi$, by use of assumptions (I) to (III), we have

$$M(fb,f\omega,t) = \sup_{t\in[0,1]} \frac{\min\left\{\int_0^{\varpi} N(\varpi,\ l)\Omega(l,\ b(l))dl,\ \int_0^{\varpi} N(\varpi,\ l)\Omega(l,\ \omega(l))dl,\right\}+t}{\max\left\{\int_0^{\varpi} N(\varpi,\ l)\Omega(l,\ b(l))dl,\ \int_0^{\varpi} N(\varpi,\ l)\Omega(l,\ \omega(l))dl,\right\}+t},$$
$$= \sup_{t\in[0,1]} \frac{\int_0^{\varpi} \min\{N(\varpi,\ l)\Omega(l,\ b(l))N(\varpi,\ l)\Omega(l,\ \omega(l))\}dl,+t}{\int_0^{\varpi} \max\{N(\varpi,\ l)\Omega(l,\ b(l))N(\varpi,\ l)\Omega(l,\ \omega(l))\}dl,+t},$$
$$= \sup_{t\in[0,1]} \frac{\int_0^{\varpi} N(\varpi,\ l)\min\{\Omega(l,\ b(l)),\Omega(l,\ \omega(l))\}\ dl,+t}{\int_0^{\varpi} N(\varpi,\ l)\max\{\Omega(l,\ b(l)),\Omega(l,\ \omega(l))\}\ dl,+t},$$
$$\geq \sup_{t\in[0,1]} \frac{\int_0^{\varpi} N(\varpi,\ l)\min\{D(b(l),\ \omega(l))\}\ dl,+t}{\int_0^{\varpi} N(\varpi,\ l)\max\{D(b(l),\ \omega(l))\}\ dl,+t},$$
$$= M(D(b,\omega,t)).$$

Thus, all conditions of Theorem 4 are fulfilled. That is, operator f has an FP which is the solution to differential Equation (17) regulating the conversion of solar energy to electrical power. □

Open Problems 1. *The following open problem is provided for further applications of the findings in this article:*

Optional appliance renewal is one of the most basic concerns in management science and engineering economics. A corporation periodically purchases a new appliance and sells the old one in order to operate the equipment permanently. If $\delta(t,z)$ *is the efficiency of the appliance at time period* $\mathcal{T}$ *and* $\delta(\mathcal{T})$ *is the cost at the purchasing time, then,*

$$e^{-\eta t}\delta(\ \mathcal{T}) = \int_{\mathcal{T}}^{\lambda^{-1}} e^{-\eta z}[\delta(\mathcal{T},z) - \delta(a(z),z)]du, -\infty < \mathcal{T} < +\infty,$$

where z is the usage time of machine and η *is the constant of industry wide discount rate.*
Can the results established in this note or their variants be applied to solve the aforementioned integral equation?
Can the results derived in this article be controlled in graphical fuzzy metric spaces?
Can we demonstrate the aforementioned findings for multivalued mappings?

4. Conclusions

In the perspective of controlled fuzzy metric spaces, this manuscript contains a number of fixed point theorems and a sufficient condition for a sequence to be Cauchy. As a result, we combined the well-known contraction requirements with controlled fuzzy metric spaces to simplify the proofs of several fixed point theorems. Furthermore, we discussed an application to transform solar energy to electric power. In the future, we will enhance these results in the framework of tripled controlled fuzzy metric spaces and pentagonal controlled fuzzy metrics spaces.

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