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# Properties of Multivariate Hermite Polynomials in Correlation with Frobenius–Euler Polynomials

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**Abstract:** A comprehensive framework has been developed to apply the monomiality principle from mathematical physics to various mathematical concepts from special functions. This paper presents research on a novel family of multivariate Hermite polynomials associated with Apostol-type Frobenius–Euler polynomials. The study derives the generating expression, operational rule, differential equation, and other defining characteristics for these polynomials. Additionally, the monomiality principle for these polynomials is verified. Moreover, the research establishes series representations, summation formulae, and operational and symmetric identities, as well as recurrence relations satisfied by these polynomials.

**Keywords:** multivariate special polynomials; monomiality principle; explicit form; operational connection; symmetric identities; summation formulae

**MSC:** 33E20; 33C45; 33B10; 33E30; 11T23



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## 1. Introduction and Preliminaries

A current field of study with practical applications involves investigating the convolution of multiple polynomials as a method for introducing innovative multivariate generalized polynomials. These polynomials hold immense importance due to their useful characteristics, which include recurring and explicit relations, functional and differential equations, summation formulae, symmetric and convolution identities, determinant forms, and more.

Multivariate hybrid special polynomials exhibit a wide range of features that show great promise for their utilization in various areas of pure and practical mathematics, such as number theory, combinatorics, classical and numerical analysis, theoretical physics, and approximation theory. The development of diverse new classes of hybrid polynomials is motivated by the desire to harness their utility and potential for application.

Sequences of polynomials hold significant relevance in various domains of applied mathematics, theoretical physics, approximation theory, and other branches of mathematics. Particularly, the Bernstein polynomials of degree  $n$  serve as a foundational basis for the space of polynomials with degrees less than or equal to  $n$ . Dattoli and collaborators utilized operational approaches to examine Bernstein polynomials [1], exploring the Appell sequences—a broad class encompassing several well-known polynomial sequences, including the Miller–Lee, Bernoulli, and Euler polynomials, among others.

The introduction and study of classes of hybrid special polynomials connected to the Appell sequences, as seen in references [2–7], play a significant role in engineering, biological, medical, and physical sciences. These hybrid polynomials are of paramount importance due to their key characteristics, such as differential equations, generating functions, series definitions, integral representations, and more. In numerous scientific and technical fields, problems are often expressed as differential equations, and their solutions typically manifest as special functions. Consequently, the challenges encountered in the development of scientific fields can be addressed by utilizing the differential equations satisfied by these hybrid special polynomials.

The multivariate special polynomials are extremely important in many areas of mathematics and have many uses. They are crucial in algebraic geometry, which examines the geometric properties of algebraic varieties. They are used to define and study significant geometric objects such as algebraic curves, surfaces, and higher-dimensional varieties. These polynomials describe the intersection of curves and surfaces, the singularities of algebraic varieties, and the properties of their coordinate rings. They may also be observed in many areas of theoretical physics, including quantum mechanics and quantum field theory. They show up as differential equation solutions in mathematical physics, especially when eigenvalue issues, boundary value issues, and symmetry analysis are involved. These polynomials have applications in quantum field theory, statistical mechanics, the study of integrable systems, etc. Due to such significance, several authors introduced multivariate Hermite and other special polynomials. Datolli et al. [8] introduced the generating function:

$$e^{u_1t+u_2t^2+u_3t^3} = \sum_{n=0}^{\infty} \mathfrak{H}_n(u_1, u_2, u_3) \frac{t^n}{n!}, \tag{1}$$

representing three-variable Hermite polynomials (3VHPs)  $\mathfrak{H}_n(u_1, u_2, u_3)$ .

Further, by taking  $u_3 = 0$ , 3VHPs reduce to the polynomials  $\mathfrak{H}_n(u_1, u_2)$  widely known as 2-v Hermite Kampé de Fériet polynomials (2VHKdFPs) [9] and on taking  $u_3 = 0, u_1 = 2u_1$  and  $u_2 = -1$  3VHPs become the classical Hermite polynomials  $\mathfrak{H}_n(u_1)$  [10] (Equation (5.1), p. 167).

At this point, it is noteworthy to mention that many semi-classical orthogonal polynomials, serving as generalizations of classical orthogonal polynomials such as Hermite, Laguerre, and Jacobi polynomials, have been extensively studied in recent years. Enthusiastic readers are encouraged to explore the works of [11,12] (and the references cited therein), along with the valuable insights presented in the book [13]. Furthermore, other interesting results concerning recurrence relations for generalized Appell polynomials and summation problems involving simplex lattice points or operators with a summing effect can be found in [14–16].

Recently, the polynomials represented by  $\mathcal{Y}_n^{[m]}(u_1, u_2, \dots, u_m)$ , known as multivariate Hermite polynomials (MHPs), were introduced in [17] and are given by generating relation:

$$\exp(u_1\zeta + u_2\zeta^2 + \dots + u_m\zeta^m) = \sum_{n=0}^{\infty} \mathcal{Y}_n^{[m]}(u_1, u_2, \dots, u_m) \frac{\zeta^n}{n!}, \tag{2}$$

with the operational rule:

$$\exp\left(u_2 \frac{\partial^2}{\partial u_1^2} + u_3 \frac{\partial^3}{\partial u_1^3} + \dots + u_m \frac{\partial^m}{\partial u_1^m}\right) u_1^n = \mathcal{Y}_n^{[m]}(u_1, u_2, \dots, u_m), \tag{3}$$

and series representation:

$$\mathcal{Y}_n^{[m]}(u_1, u_2, \dots, u_m) = n! \sum_{r=0}^{[n/m]} \frac{u_m^r \mathcal{Y}_{n-mr}^{[m]}(u_1, u_2, \dots, u_{m-1})}{r! (n - mr)!}. \tag{4}$$

Several mathematicians are keen to introduce different forms of various special polynomials. The unified forms of Apostol-type polynomials are introduced in the study of [18].

These polynomials are known as the Apostol-type Frobenius–Euler polynomials and they are represented mathematically by the symbol  $F_n(u_1; u)$  [19]. For  $\lambda = 1$ , these polynomials reduce to the Frobenius–Euler polynomials [20]. We now recall the generating expression of these Frobenius–Euler polynomials, which is as follows:

$$\left(\frac{1-u}{e^\zeta-u}\right)e^{u_1\zeta} = \sum_{n=0}^{\infty} F_n(u_1; u) \frac{\zeta^n}{n!}, \tag{5}$$

where  $u \in \mathbb{C}, u \neq 1$ .

Therefore, on taking  $u_1 = 0$ , expression (5) gives the Frobenius–Euler numbers (FENs)  $F_n(u)$ , defined by

$$\frac{1-u}{e^\zeta-u} = \sum_{n=0}^{\infty} F_n(u) \frac{\zeta^n}{n!}. \tag{6}$$

Further, on taking  $u = -1$ , the FEPs becomes Euler polynomials (EPs)  $A_n(u_1)$  [21].

Extensive research has been dedicated to the advancement and integration of the monomiality principle, operational rules, and other properties within the domain of hybrid special polynomials. This line of investigation traces its roots back to 1941 when Steffenson initially proposed the concept of poweroids as a means to understanding monomiality [22]. Building upon Steffenson’s work, Dattoli further refined the theory, offering valuable insights and refinements [2]. Their contributions have paved the way for a more comprehensive understanding of the monomiality principle and its application within the context of the so-called hybrid special polynomials. Therefore, on a combination of multivariate Hermite polynomials  $\mathcal{Y}_n^{[m]}(u_1, u_2, \dots, u_m)$  given by (2) and Frobenius–Euler polynomials [23,24] given by (5) by using the concept of the monomiality principle and operational rules, the convoluted new polynomial, namely, multivariate Hermite–Frobenius–Euler polynomials are given by the formal expression:

$$\left(\frac{1-u}{e^\zeta-u}\right)\exp(u_1\zeta + u_2\zeta^2 + \dots + u_m\zeta^m) := \sum_{n=0}^{\infty} \mathcal{Y}F_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\zeta^n}{n!}. \tag{7}$$

The rest of the article is as follows: The multivariate Hermite–Frobenius–Euler polynomials are introduced and studied in Section 2. Also, operational formulae for these polynomials are derived. In Section 3, the monomiality principle is verified and the differential equation is deduced. Further, several identities satisfied by these multivariate Hermite–Frobenius–Euler polynomials are established by using operational formalism. In Section 4, summation formulae and symmetric identities for these polynomials are established. Further, several special cases of these polynomials are taken and the corresponding results are deduced. Section 5 is devoted to some illustrative examples. Finally, Section 6 consists of concluding remarks.

### 2. Multivariate Hermite–Frobenius–Euler Polynomials

In this section, a novel and comprehensive method is introduced for determining the multivariate Hermite–Frobenius–Euler polynomials (MHFEPs)  $\mathcal{Y}F_n^{[m]}(u_1, u_2, \dots, u_m; u)$ . The approach presents an alternative viewpoint and methodology when compared to existing methods. By employing this innovative technique, our objective is to enrich the comprehension and investigation of these polynomial sequences, offering a new outlook on their properties and potential applications. As a result, we have introduced a fresh perspective to advance the understanding and utilization of these polynomials.

Now, we will use two different approaches to show that the representation series (7) is meaningful. Thus, MHFEPs are well-defined through the generating function method.

**Theorem 1.** The MHFEPs represented by  ${}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)$  satisfy the generating expression:

$$\left(\frac{1-u}{e^\xi-u}\right) \exp(u_1\xi + u_2\xi^2 + \dots + u_m\xi^m) = \sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\xi^n}{n!}. \tag{8}$$

**Proof.** We prove the result in two alternative ways:

- (i) Expanding the product of terms  $\left(\frac{1-u}{e^\xi-u}\right)$  and  $\exp(u_1\xi + u_2\xi^2 + \dots + u_m\xi^m)$  by Newton series and ordering the product of the developments of functions  $\left(\frac{1-u}{e^\xi-u}\right)$  and  $\exp(u_1\xi + u_2\xi^2 + \dots + u_m\xi^m)$  w.r.t. the powers of  $\xi$ , we obtain the polynomials  ${}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)$  expressed in (7) as coefficients of  $\frac{\xi^n}{n!}$ .
- (ii) Substituting the multiplicative operator  $\hat{M} = u_1 + 2u_2\partial_{u_1} + 3u_3\partial_{u_1}^2 + \dots + mu_m\partial_{u_1}^{m-1}$  of MHFEPs given in [17] in expression (5) in place of  $u_1$  on both sides, we find

$$\left(\frac{1-u}{e^\xi-u}\right) e^{(u_1+2u_2\partial_{u_1}+3u_3\partial_{u_1}^2+\dots+mu_m\partial_{u_1}^{m-1})\xi} = \sum_{n=0}^{\infty} \mathbf{F}_n(u_1 + 2u_2\partial_{u_1} + 3u_3\partial_{u_1}^2 + \dots + mu_m\partial_{u_1}^{m-1}; u) \frac{\xi^n}{n!} \tag{9}$$

In view of the identity given in [5], (Equation (7)) gives the l.h.s. of (8) and, denoting the r.h.s.  ${}_y\mathbf{F}_n(u_1 + 2u_2\partial_{u_1} + 3u_3\partial_{u_1}^2 + \dots + mu_m\partial_{u_1}^{m-1}; u)$  by  ${}_y\mathbf{F}_n(u_1, u_2, \dots, u_m; u)$ , assertion (8) is deduced.

□

The following result shows that the MHFEPs behave component-wise as Appell-type polynomial sequences.

**Theorem 2.** The multivariate Hermite–Frobenius–Euler polynomials  ${}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)$  satisfy the following differential relations:

$$\frac{\partial}{\partial u_j} [{}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)] = (n)_j {}_y\mathbf{F}_{n-j}^{[m]}(u_1, u_2, \dots, u_m; u), \quad 1 \leq j \leq m \leq n, \tag{10}$$

where  $(n)_j$  denotes the falling factorial, given by

$$(n)_j = \begin{cases} 1, & \text{if } j = 0, \\ \prod_{i=1}^j (n - i + 1), & \text{if } j \geq 1, \\ 0, & \text{if } j < 0. \end{cases}$$

**Proof.** By taking derivatives of expression (7) w.r.t.  $u_1$ , it follows that

$$\frac{\partial}{\partial u_1} \left[ \left(\frac{1-u}{e^\xi-u}\right) \exp(u_1\xi + u_2\xi^2 + \dots + u_m\xi^m) \right] = \xi \left[ \left(\frac{1-u}{e^\xi-u}\right) \exp(u_1\xi + u_2\xi^2 + \dots + u_m\xi^m) \right]. \tag{11}$$

Substituting the r.h.s. of (7) into (11), we find

$$\frac{\partial}{\partial u_1} \left[ \sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\xi^n}{n!} \right] = \sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\xi^{n+1}}{n!}, \tag{12}$$

By replacing  $n \rightarrow n - 1$  on the r.h.s. of the previous expression and then equating the coefficients of like exponents of  $\xi$ , the first expression of the system of expressions (10) is deduced.

Next, on taking derivatives of expression (7) w.r.t.  $u_2$ , it follows that

$$\frac{\partial}{\partial u_2} \left[ \left(\frac{1-u}{e^\xi-u}\right) \exp(u_1\xi + u_2\xi^2 + \dots + u_m\xi^m) \right] = \xi^2 \left[ \left(\frac{1-u}{e^\xi-u}\right) \exp(u_1\xi + u_2\xi^2 + \dots + u_m\xi^m) \right]. \tag{13}$$

Substituting the r.h.s. of expression (7) into (13), we find

$$\frac{\partial}{\partial u_2} \left[ \sum_{n=0}^{\infty} y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\zeta^n}{n!} \right] = \sum_{n=0}^{\infty} y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\zeta^{n+2}}{n!}, \quad (14)$$

by replacing  $n \rightarrow n - 2$  on the r.h.s. of the previous expression and then equating the coefficients of like exponents of  $\zeta$ , the second expression of the system of expressions (10) is deduced.

Similarly, continuing in the same fashion, we deduce other expressions of system (10).  $\square$

Concerning the operational formalism satisfied by the multivariate polynomials  $y\mathbf{F}_n(u_1, u_2, \dots, u_m; u)$ , we have the following:

**Theorem 3.** For MHFEPs  $y\mathbf{F}_n(u_1, u_2, \dots, u_m; u)$ , the operational rule:

$$\exp \left( u_2 \frac{\partial^2}{\partial u_1^2} + u_3 \frac{\partial^3}{\partial u_1^3} + \dots + u_m \frac{\partial^m}{\partial u_1^m} \right) \{ \mathbf{F}_n(u_1; u) \} = y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \quad (15)$$

holds true.

**Proof.** To prove result (15), we proceed by taking derivatives of expression (7) as:

$$\begin{aligned} \frac{\partial}{\partial u_1} [y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)] &= n y\mathbf{F}_{n-1}^{[m]}(u_1, u_2, \dots, u_m; u), \\ \frac{\partial^2}{\partial u_1^2} [y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)] &= n(n-1) y\mathbf{F}_{n-2}^{[m]}(u_1, u_2, \dots, u_m; u), \\ \frac{\partial^3}{\partial u_1^3} [y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)] &= n(n-1)(n-2) y\mathbf{F}_{n-3}^{[m]}(u_1, u_2, \dots, u_m; u), \\ &\vdots \\ \frac{\partial^m}{\partial u_1^m} [y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)] &= (n)_m y\mathbf{F}_{n-m}^{[m]}(u_1, u_2, \dots, u_m; u), \end{aligned} \quad (16)$$

and

$$\begin{aligned} \frac{\partial}{\partial u_2} [y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)] &= n(n-1) y\mathbf{F}_{n-2}^{[m]}(u_1, u_2, \dots, u_m; u), \\ \frac{\partial}{\partial u_3} [y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)] &= n(n-1)(n-2) y\mathbf{F}_{n-3}^{[m]}(u_1, u_2, \dots, u_m; u), \\ &\vdots \\ \frac{\partial}{\partial u_m} [y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)] &= (n)_m y\mathbf{F}_{n-m}^{[m]}(u_1, u_2, \dots, u_m; u). \end{aligned} \quad (17)$$

In consideration of the system of Equations (16) and (17), we find that the MHFEPs are solutions of the equations:

$$\begin{aligned} \frac{\partial}{\partial u_2} [{}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)] &= \frac{\partial^2}{\partial u_1^2} [{}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)], \\ \frac{\partial}{\partial u_3} [{}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)] &= \frac{\partial^3}{\partial u_1^3} [{}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)], \\ &\vdots \\ \frac{\partial}{\partial u_m} [{}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)] &= \frac{\partial^m}{\partial u_1^m} [{}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)], \end{aligned} \tag{18}$$

under the initial conditions:

$${}_y\mathbf{F}_n^{[m]}(u_1, 0, 0, \dots, 0; u) = \mathbf{F}_n(u_1; u). \tag{19}$$

Therefore, in cognizance of previous expressions (18) and (19), assertion (15) is obtained.  $\square$

Next, we will obtain the series representation of MHFEPs  ${}_y\mathbf{F}_n(u_1, u_2, \dots, u_m; u)$  by proving the succeeding results:

**Theorem 4.** For MHFEPs  ${}_y\mathbf{F}_n(u_1, u_2, \dots, u_m; u)$ , the succeeding series representations are demonstrated:

$${}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) = \sum_{s=0}^n \binom{n}{s} \mathbf{F}_s(u) \mathcal{Y}_{n-s}^{[m]}(u_1, u_2, \dots, u_m) \tag{20}$$

and

$${}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) = \sum_{s=0}^n \binom{n}{s} \mathbf{F}_s(u_1; u) \mathcal{Y}_{n-s}^{[m]}(u_2, u_3, \dots, u_m). \tag{21}$$

**Proof.** Inserting expressions (6) and (2) on the l.h.s. of (7), we find

$$\sum_{s=0}^{\infty} \mathbf{F}_s(u) \frac{\xi^s}{s!} \sum_{n=0}^{\infty} \mathcal{Y}_n^{[m]}(u_1, u_2, \dots, u_m) \frac{\xi^n}{n!} = \sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\xi^n}{n!}. \tag{22}$$

Interchanging the expressions and replacing  $n \rightarrow n - s$  in the resultant expression in view of the Cauchy product rule, it follows that

$$\sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\xi^n}{n!} = \sum_{n=0}^{\infty} \sum_{s=0}^n \mathbf{F}_s(u) \mathcal{Y}_n^{[m]}(u_1, u_2, \dots, u_m) \frac{\xi^n}{(n-s)! s!}. \tag{23}$$

Multiplying and dividing by  $n!$  on the r.h.s. of the previous expression and then equating the coefficients of the same exponents of  $\xi$  on both sides, assertion (20) is deduced.

In a similar fashion, inserting expressions (5) and (2) (with  $u_1 = 0$ ) on the l.h.s. of (7), we find

$$\sum_{s=0}^{\infty} \mathbf{F}_s(u_1; u) \frac{\xi^s}{s!} \sum_{n=0}^{\infty} \mathcal{Y}_n^{[m]}(u_2, u_3, \dots, u_m) \frac{\xi^n}{n!} = \sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\xi^n}{n!}. \tag{24}$$

Interchanging the expressions and replacing  $n \rightarrow n - s$  in the resultant expression in view of the Cauchy product rule, it follows that

$$\sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\xi^n}{n!} = \sum_{n=0}^{\infty} \sum_{s=0}^n \mathbf{F}_s(u_1; u) \mathcal{Y}_n^{[m]}(u_2, u_3, \dots, u_m) \frac{\xi^n}{(n-s)! s!}. \tag{25}$$

Multiplying and dividing by  $n!$  on the r.h.s. of the previous expression and then equating the coefficients of the same exponents of  $\xi$  on both sides, assertion (21) is deduced.  $\square$

### 3. Monomiality Principle

The development and incorporation of the monomiality principle, operational rules, and other properties in hybrid special polynomials have been extensively studied. The concept of monomiality was first introduced by Steffenson in 1941 through the notion of poweroids [22] and was further refined by Dattoli [2]. In this context, the  $\hat{\mathcal{M}}$  and  $\hat{\mathcal{D}}$  operators play a crucial role as multiplicative and derivative operators for a polynomial set  $b_k(u_1)_{k \in \mathbb{N}}$ . These operators satisfy the following expressions:

$$b_{k+1}(u_1) = \hat{\mathcal{M}}\{b_k(u_1)\} \tag{26}$$

and

$$k b_{k-1}(u_1) = \hat{\mathcal{D}}\{b_k(u_1)\}. \tag{27}$$

Subsequently, the polynomial set  $b_k(u_1)_{m \in \mathbb{N}}$  under the manipulation of multiplicative and derivative operators is known as a quasi-monomial. It is essential for this quasi-monomial to adhere to the following formula:

$$[\hat{\mathcal{D}}, \hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1}, \tag{28}$$

and, as a result, it shows a Weyl group structure.

The significance and usage of the operators  $\hat{\mathcal{M}}$  and  $\hat{\mathcal{D}}$  can be exploited to extract the significance of the set  $\{b_k(u_1)\}_{k \in \mathbb{N}}$ , provided it is quasi-monomial. Hence, the succeeding axioms hold:

- (i)  $b_k(u_1)$  gives the differential equation

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{b_k(u_1)\} = k b_k(u_1), \tag{29}$$

provided  $\hat{\mathcal{M}}$  and  $\hat{\mathcal{D}}$  exhibit differential traits.

- (ii) The expression

$$b_k(u_1) = \hat{\mathcal{M}}^k \{1\}, \tag{30}$$

gives the explicit form, with  $b_0(u_1) = 1$ .

- (iii) Further, the expression

$$e^{w\hat{\mathcal{M}}}\{1\} = \sum_{k=0}^{\infty} b_k(u_1) \frac{w^k}{k!}, \quad |w| < \infty, \tag{31}$$

behaves as a generating expression, which is derived by usage of identity (30).

Many branches of mathematical physics, quantum mechanics, and classical optics still employ these methods today. As a result, these methods offer strong and efficient research tools. We thus confirm the monomiality concept for MHFEPs by taking into account the importance of this method. Thus we verify the monomiality principle for MHFEPs  ${}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)$  in this section by demonstrating the succeeding results:

**Theorem 5.** *The MHFEPs  ${}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)$  satisfy the succeeding multiplicative and derivative operators:*

$$\hat{M}_{y\mathbf{F}} = u_1 + 2u_2\partial_{u_1} + 3u_3\partial_{u_1}^2 + \dots + mu_m\partial_{u_1}^{m-1} - \frac{e^{\partial_{u_1}}}{e^{\partial_{u_1} - u}} \tag{32}$$

and

$$\hat{D}_{y\mathbf{F}} = \partial_{u_1}, \tag{33}$$

where  $\partial_{u_1} = \frac{\partial}{\partial u_1}$ .

**Proof.** By differentiating expression (7) w.r.t.  $\zeta$  on both sides, we find

$$\begin{aligned} \left(u_1 + 2u_2\zeta + 3u_3\zeta^2 + \dots + mu_m\zeta^{m-1} - \frac{e^\zeta}{e^\zeta - u}\right) \left(\frac{1-u}{e^\zeta - u} \exp(u_1\zeta + u_2\zeta^2 + \dots + u_m\zeta^m)\right) \\ = \sum_{n=0}^{\infty} n \, {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\zeta^{n-1}}{n!}. \end{aligned} \tag{34}$$

which further can be written as follows:

$$\begin{aligned} \left(u_1 + 2u_2\zeta + 3u_3\zeta^2 + \dots + mu_m\zeta^{m-1} - \frac{e^\zeta}{e^\zeta - u}\right) \left(\sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\zeta^{n-1}}{n!}\right) \\ = \sum_{n=0}^{\infty} n \, {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\zeta^{n-1}}{n!}. \end{aligned} \tag{35}$$

Also, by taking a derivative of (7) w.r.t.  $u_1$ , we find the identity

$$\begin{aligned} \frac{\partial}{\partial u_1} \left(\frac{1-u}{e^\zeta - u} \exp(u_1\zeta + u_2\zeta^2 + \dots + u_m\zeta^m)\right) &= \zeta \left(\frac{1-u}{e^\zeta - u} \exp(u_1\zeta + u_2\zeta^2 + \dots + u_m\zeta^m)\right), \\ \frac{\partial}{\partial u_1} \left(\sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\zeta^{n-1}}{n!}\right) &= \zeta \left(\sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\zeta^{n-1}}{n!}\right). \end{aligned} \tag{36}$$

By replacing  $n \rightarrow n + 1$  on the r.h.s. of (35) and equating the coefficients of same exponents of  $\zeta$  in view of expressions (37) and (26) in the resultant expression, assertion (32) is demonstrated.

Moreover, the second part of expression (36) can be written as:

$$\frac{\partial}{\partial u_1} \left(\sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\zeta^{n-1}}{n!}\right) = \left(\sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) \frac{\zeta^{n+1}}{n!}\right). \tag{37}$$

By replacing  $n \rightarrow n - 1$  on the r.h.s. of (37) and equating the coefficients of the same exponents of  $\zeta$  in view of (27) in the resultant expression, assertion (33) is demonstrated.  $\square$

Next, we deduce the differential equation for MHFEPs  ${}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)$  by demonstrating the succeeding result:

**Theorem 6.** *The MHFEPs  ${}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)$  satisfy the differential equation:*

$$\left(u_1\partial_{u_1} + 2u_2\partial_{u_1}^2 + 3u_3\partial_{u_1}^3 + \dots + mu_m\partial_{u_1}^m - \frac{e^{\partial_{u_1}}}{e^{\partial_{u_1}} - u}\partial_{u_1} - n\right) {}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u) = 0. \tag{38}$$

**Proof.** Inserting expression (32) and (33) into the expression (29), assertion (38) is proved.  $\square$

The operational formalism developed in Theorem 6 can be applied to numerous identities related to the Frobenius–Euler polynomials, which are widely investigated to produce MHFEPs  ${}_y\mathbf{F}_n^{[m]}(u_1, u_2, \dots, u_m; u)$ . To do this, we carry out the subsequent action of operator ( $\mathcal{O}$ ) given by  $\exp\left(u_2\frac{\partial^2}{\partial u_1^2} + u_3\frac{\partial^3}{\partial u_1^3} + \dots + u_m\frac{\partial^m}{\partial u_1^m}\right)$  on the identities involving Frobenius–Euler polynomials  $\mathbf{F}_n(u_1; u)$  [25]:

$$u \mathbf{F}_n(u_1; u^{-1}) + \mathbf{F}_n(u_1; u) = (1 + u) \sum_{k=0}^n \binom{n}{k} \mathbf{F}_{n-k}(u^{-1}) \mathbf{F}_k(u_1; u), \tag{39}$$

$$\frac{1}{n+1} \mathbf{F}_k(u_1; u) + \mathbf{F}_{n-k}(u_1; u) = \sum_{k=0}^{n-1} \frac{\binom{n}{k}}{n-k+1} \sum_{l=k}^n ((-u)\mathbf{F}_{l-k}(u)\mathbf{F}_{n-l}(u) + 2u\mathbf{F}_{n-k}(u)) \mathbf{F}_k(u_1; u) \mathbf{F}_n(u_1; u), \tag{40}$$



$$F_n(u_1; u) = \sum_{k=0}^n \binom{n}{k} F_{n-k}(u) F_k(u_1; u), \quad n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \tag{41}$$

The MHFEPs  $yF_n^{[m]}(u_1, u_2, \dots, u_m; u)$  are obtained after operating  $(\mathcal{O})$  on both sides of (39)–(41):

$$\begin{aligned} u yF_n^{[m]}(u_1, u_2, \dots, u_m; u^{-1}; u) + yF_n^{[m]}(u_1, u_2, \dots, u_m; u) &= (1 + u) \sum_{k=0}^n \binom{n}{k} yF_{n-k}^{[m]}(u^{-1}) yF_k^{[m]}(u_1, u_2, \dots, u_m; u), \\ \frac{1}{n+1} yF_k^{[m]}(u_1, u_2, u_3, \dots, u_m; u) + yF_{n-k}^{[m]}(u_1, u_2, u_3, \dots, u_m; u) \\ &= \sum_{k=0}^{n-1} \frac{\binom{n}{k}}{n-k+1} \sum_{l=k}^n ((-u)F_{n-l}(u)F_{l-k}(u) + 2uF_{n-k}(u)) yF_k^{[m]}(u_1, u_2, u_3, \dots, u_m; u) yF_n^{[m]}(u_1, u_2, u_3, \dots, u_m; u), \\ yF_n^{[m]}(u_1, u_2, u_3, \dots, u_m; u) &= \sum_{k=0}^n \binom{n}{k} F_{n-k}(u) yF_k^{[m]}(u_1, u_2, u_3, \dots, u_m; u), \quad n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \end{aligned}$$

**4. Summation Formulae and Symmetric Identities**

To derive the summation formulae for the MHFEPs  $yF_n^{[m]}(u_1, u_2, u_3, \dots, u_m; u)$ , the succeeding results are demonstrated:

**Theorem 7.** For the MHFEPs  $yF_n^{[m]}(u_1, u_2, u_3, \dots, u_m; u)$ , the succeeding implicit summation formula holds true:

$$yF_n^{[m]}(u_1 + w, u_2, u_3, \dots, u_m; u) = \sum_{k=0}^n \binom{n}{k} yF_k^{[m]}(u_1, u_2, u_3, \dots, u_m; u) w^{n-k}. \tag{42}$$

**Proof.** On taking  $u_1 \rightarrow u_1 + w$  in expression (7), it follows that

$$\left( \frac{1-u}{e^{\zeta}-u} \right) \exp((u_1+w)\zeta + u_2\zeta^2 + \dots + u_m\zeta^m) = \sum_{n=0}^{\infty} yF_n^{[m]}(u_1+w, u_2, \dots, u_m; u) \frac{\zeta^n}{n!}$$

which further can be written as

$$\left( \frac{1-u}{e^{\zeta}-u} \right) \exp(u_1\zeta + u_2\zeta^2 + \dots + u_m\zeta^m) \exp(w\zeta) = \sum_{n=0}^{\infty} yF_n^{[m]}(u_1+w, u_2, \dots, u_m; u) \frac{\zeta^n}{n!},$$

By making use of the series expansion of  $\exp(w\zeta)$  on the l.h.s. of the previous expression, we have

$$\sum_{k=0}^{\infty} yF_n^{[m]}(u_1+w, u_2, \dots, u_m; u) w^n \frac{\zeta^{n+k}}{n!k!} = \sum_{n=0}^{\infty} yF_n^{[m]}(u_1+w, u_2, \dots, u_m; u) \frac{\zeta^n}{n!}. \tag{43}$$

This results in the deduction of assertion (42) by substituting  $n \rightarrow n - k$  into the r.h.s. of consequent expression and then equating the coefficients of the identical powers of  $\zeta$  in the resulting equation. □

**Corollary 1.** For  $w = 1$  in expression (42), we have

$$yF_n^{[m]}(u_1 + 1, u_2, u_3, \dots, u_m; u) = \sum_{k=0}^n \binom{n}{k} yF_k^{[m]}(u_1, u_2, u_3, \dots, u_m; u). \tag{44}$$

**Theorem 8.** For the MHFEPs  $yF_n^{[m]}(u_1, u_2, u_3, \dots, u_m; u)$ , the succeeding implicit summation formula holds true:

$${}_y\mathbf{F}_n^{[m]}(u_1 + x, u_2 + y, u_3 + z, \dots, u_m; u) = \sum_{k=0}^n \binom{n}{k} {}_y\mathbf{F}_{n-k}^{[m]}(u_1, u_2, u_3, \dots, u_m; u) \mathcal{Y}_k(x, y, z). \tag{45}$$

**Proof.** On taking  $u_1 \rightarrow u_1 + x, u_2 \rightarrow u_1 + y$  and  $u_3 \rightarrow u_3 + z$  in expression (7), it follows that

$$\left(\frac{1-u}{e^\xi - u}\right) \exp((u_1 + x)\xi + (u_2 + y)\xi^2 + (u_3 + z)\xi^3 + \dots + u_m \xi^m) = \sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1 + x, u_2 + y, u_3 + z, \dots, u_m; u) \frac{\xi^n}{n!} \tag{46}$$

which further can be written as

$$\left(\frac{1-u}{e^\xi - u}\right) \exp(u_1 \xi + u_2 \xi^2 + \dots + u_m \xi^m) \exp(x\xi + y\xi^2 + z\xi^3) = \sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1 + x, u_2 + y, u_3 + z, \dots, u_m; u) \frac{\xi^n}{n!}. \tag{47}$$

By making use of the series expansion of  $\exp(x\xi + y\xi^2 + z\xi^3)$  on the l.h.s. of the previous expression, we have

$$\sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1, u_2, u_3, \dots, u_m; u) \mathcal{Y}_k(x, y, z) \frac{\xi^{n+k}}{n!k!} = \sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(u_1 + x, u_2 + y, u_3 + z, \dots, u_m; u) \frac{\xi^n}{n!}. \tag{48}$$

This results in the deduction of assertion (45) by substituting  $n \rightarrow n - k$  on the l.h.s. of the consequent expression and then equating the coefficients of the identical powers of  $x$  in the resulting equation.  $\square$

**Corollary 2.** For  $z = 0$  in expression (45), we have

$${}_y\mathbf{F}_n^{[m]}(u_1 + x, u_2 + y, u_3, \dots, u_m; u) = \sum_{k=0}^n \binom{n}{k} {}_y\mathbf{F}_{n-k}^{[m]}(u_1, u_2, u_3, \dots, u_m; u) \mathcal{Y}_k(x, y). \tag{49}$$

**Theorem 9.** For the MHFEPs  ${}_y\mathbf{F}_n^{[m]}(u_1, u_2, u_3, \dots, u_m; u)$ , the succeeding implicit summation formula holds true:

$${}_y\mathbf{F}_{n+s}^{[m]}(q, u_2, u_3, \dots, u_m; u) = \sum_{l,m=0}^{n,s} \binom{n}{l} \binom{s}{m} (q - u_1)^{l+m} {}_y\mathbf{F}_{n+s-l-m}^{[m]}(u_1, u_2, u_3, \dots, u_m; u). \tag{50}$$

**Proof.** By replacing  $\xi \rightarrow \xi + \eta$  and in view of the expression:

$$\sum_{M=0}^{\infty} g(M) \frac{(u_1 + u_2)^M}{M!} = \sum_{l,m=0}^{\infty} g(l+m) \frac{u_1^l u_2^m}{l! m!} \tag{51}$$

in relation (7) and afterward simplifying the resultant expression, we have

$$e^{-u_1(\xi+\eta)} \sum_{n,s=0}^{\infty} {}_y\mathbf{F}_{n+s}^{[m]}(u_1, u_2, u_3, \dots, u_m; u) \frac{\xi^n \eta^s}{n! s!} = \left(\frac{1-u}{e^{\xi+\eta} - u}\right) \exp(u_2(\xi + \eta)^2 + \dots + u_m(\xi + \eta)^m). \tag{52}$$

Substituting  $u_1 \rightarrow q$  into (52) and comparing the resultant expression to the previous expression and further expanding the exponential function gives

$$\sum_{n,s=0}^{\infty} {}_y\mathbf{F}_{n+s}^{[m]}(u_1, u_2, u_3, \dots, u_m; u) \frac{\xi^n \eta^s}{n! s!} = \sum_{M=0}^{\infty} (q - u_1)^M \frac{(\xi + \eta)^M}{M!} \times \sum_{n,s=0}^{\infty} {}_y\mathbf{F}_{n+s}^{[m]}(u_1, u_2, u_3, \dots, u_m; u) \frac{\xi^n \eta^s}{n! s!}. \tag{53}$$

Thus, in view of expression (51) in expression (53) and then replacing  $n \rightarrow n - l$  and  $s \rightarrow s - m$  in the resultant expression, we find

$$\sum_{n,s=0}^{\infty} \mathcal{Y}\mathbf{F}_{n+s}^{[m]}(u_1, u_2, u_3, \dots, u_m; u) \frac{\xi^n \eta^s}{n! s!} = \sum_{n,s=0}^{\infty} \sum_{l,m=0}^{n,s} \frac{(q - u_1)^{l+m}}{l! m!} \times \mathcal{Y}\mathbf{F}_{n+s-l-m}^{[m]}(u_1, u_2, u_3, \dots, u_m; u) \frac{\xi^n \eta^s}{(n-l)! (s-m)!}. \tag{54}$$

On comparison of the coefficients of the like exponents of  $\xi$  and  $\eta$  on both sides of the previous expression, assertion (50) is established.  $\square$

**Corollary 3.** For  $n = 0$  in expression (50), we find

$$\mathcal{Y}\mathbf{F}_s^{[m]}(q, u_2, u_3, \dots, u_m; u) = \sum_{m=0}^s \binom{s}{m} (q - u_1)^m \mathcal{Y}\mathbf{F}_{s-m}^{[m]}(u_1, u_2, u_3, \dots, u_m; u)$$

**Corollary 4.** Substituting  $q \rightarrow q + u_1$  and taking  $m = 2$  in expression (50), we have

$$\mathcal{Y}\mathbf{F}_{n+s}^{[m]}(q + u_1, u_2, u_3, \dots, u_m; u) = \sum_{l,m=0}^{n,s} \binom{n}{l} \binom{s}{m} (q)^{l+m} \mathcal{Y}\mathbf{F}_{n+s-l-m}^{[m]}(u_1, u_2, u_3, \dots, u_m; u).$$

**Corollary 5.** Substituting  $q \rightarrow q + u_1$  and taking  $m = 1$  in expression (50), we have

$$\mathcal{Y}\mathbf{F}_{n+s}^{[m]}(q + u_1; u) = \sum_{l,m=0}^{n,s} \binom{n}{l} \binom{s}{m} (q)^{l+m} \mathcal{Y}\mathbf{F}_{n+s-l-m}^{[m]}(u_1; u).$$

**Corollary 6.** Substituting  $q = 0$  in expression (50), we have

$$\mathcal{Y}\mathbf{F}_{n+s}^{[m]}(u_2, u_3, \dots, u_m; u) = \sum_{l,m=0}^{n,s} \binom{n}{l} \binom{s}{m} (-u_1)^{l+m} \mathcal{Y}\mathbf{F}_{n+s-l-m}^{[m]}(u_1, u_2, u_3, \dots, u_m; u).$$

In physics and applied mathematics, it is common to encounter problems where finding a solution requires evaluating infinite sums that involve special functions. The applications of generalized special functions can be found in various fields, including electromagnetics and combinatorics. Several authors [23–34] established and examined different types of identities related to Apostol-type polynomials. These investigations serve as a motivation to establish symmetry identities for the MHFEPs. Let us now review the following definitions:

**Definition 1.** The generalized sum of integer powers  $\mathfrak{S}_k(n)$  is defined by the generating function shown below for:

$$\sum_{j=0}^{\infty} \mathfrak{S}_j(n) \frac{\xi^j}{j!} = \frac{e^{(n+1)\xi} - 1}{e^\xi - 1}. \tag{55}$$

**Definition 2.** The multiple power sums  $\mathfrak{S}_k^{(l)}(m)$  are defined by the generating function shown below:

$$\sum_{n=0}^{\infty} \left\{ \sum_{q=0}^n \binom{n}{q} (-l)^{n-q} \mathfrak{S}_k^{(l)}(m) \right\} \frac{\xi^n}{n!} = \left( \frac{1 - e^{m\xi}}{1 - e^\xi} \right)^l. \tag{56}$$

In order to derive the symmetry identities for the MHFEPs  $\mathcal{Y}\mathbf{F}_n^{[m]}(u_1, u_2, u_3, \dots, u_m; u)$ , we prove the following results:

**Theorem 10.** *The following symmetry connection between the MHFEPs and generalized integer power sums is valid for any integers with  $\mu, \eta > 0$  and  $n \geq 0, u \in \mathbb{C}$ :*

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} \mu^{n-k} {}_y\mathbf{F}_{n-k}^{[m]}(\eta u_1, \eta^2 u_2, \eta^3 u_3, \dots, \eta^m u_m; u) \sum_{l=0}^k \binom{k}{l} \eta^k \mathfrak{S}_l(\mu - 1; \frac{1}{u}) \\ & \qquad \qquad \qquad \times {}_y\mathbf{F}_{k-1}^{[m]}(\mu U_1, \mu^2 U_2, \mu^3 U_3, \dots, \mu^m U_m; u) \\ & = \sum_{k=0}^n \binom{n}{k} \eta^{n-k} {}_y\mathbf{F}_{n-k}^{[m]}(\mu u_1, \mu^2 u_2, \mu^3 u_3, \dots, \mu^m u_m; u) \sum_{l=0}^k \binom{k}{l} \mu^k \mathfrak{S}_l(\eta - 1; \frac{1}{u}) \\ & \qquad \qquad \qquad \times {}_y\mathbf{F}_{k-1}^{[m]}(\eta U_1, \eta^2 U_2, \eta^3 U_3, \dots, \eta^m U_m; u). \end{aligned} \tag{57}$$

**Proof.** Consider

$$\mathfrak{S}(\xi) := \frac{(1-u) e^{\mu u_1 \eta \xi + u_2 (\mu \eta \xi)^2 + u_3 (\mu \eta \xi)^3} (e^{\mu \eta \xi} - u) e^{\mu \eta \xi U_1 + U_2 (\mu \eta \xi)^2 + U_3 (\mu \eta \xi)^3}}{(e^{\mu \xi} - u) (e^{\eta \xi} - u)}, \tag{58}$$

which in consideration of the Cauchy product rule becomes

$$\begin{aligned} \mathfrak{S}(\xi) & = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \binom{n}{k} \mu^{n-k} {}_y\mathbf{F}_{n-k}^{[m]}(\eta u_1, \eta^2 u_2, \eta^3 u_3, \dots, \eta^m u_m; u) \sum_{l=0}^k \binom{k}{l} \eta^k \mathfrak{S}_l(\mu - 1; \frac{1}{u}) \right. \\ & \qquad \qquad \qquad \left. \times {}_y\mathbf{F}_{k-1}^{[m]}(\mu U_1, \mu^2 U_2, \mu^3 U_3, \dots, \mu^m U_m; u) \frac{\xi^n}{n!} \right). \end{aligned} \tag{59}$$

Continuing in a similar fashion, we find

$$\begin{aligned} \mathfrak{S}(\xi) & = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \binom{n}{k} \eta^{n-k} {}_y\mathbf{F}_{n-k}^{[m]}(\mu u_1, \mu^2 u_2, \mu^3 u_3, \dots, \mu^m u_m; u) \sum_{l=0}^k \binom{k}{l} \mu^k \mathfrak{S}_l(\eta - 1; \frac{1}{u}) \right. \\ & \qquad \qquad \qquad \left. \times {}_y\mathbf{F}_{k-1}^{[m]}(\eta U_1, \eta^2 U_2, \eta^3 U_3, \dots, \eta^m U_m; u) \frac{\xi^n}{n!} \right). \end{aligned} \tag{60}$$

On comparison of the coefficients of like exponents of  $\xi$  in expressions (59) and (60), assertion (57) is deduced.  $\square$

**Theorem 11.** *The following symmetry connection for the MHFEPs is valid for any integers with  $\mu, \eta > 0$  and  $n \geq 0, u \in \mathbb{C}$ :*

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{\mu-1} \sum_{j=0}^{\eta-1} u^{\mu+\eta-2} \left(\frac{1}{u}\right)^{i+j} \mu^{n-k} \eta^k {}_y\mathbf{F}_k^{[m]}(\mu U_1 + \frac{\mu}{\eta} j, \mu^2 U_2, \mu^3 U_3, \dots, \mu^m U_m; u) \times {}_y\mathbf{F}_{n-k}^{[m]}(\eta u_1 + \frac{\eta}{\mu} i, \eta^2 u_2, \eta^3 u_3, \dots, \eta^m u_m; u) \\ & = \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{\eta-1} \sum_{j=0}^{\mu-1} u^{\mu+\eta-2} \left(\frac{1}{u}\right)^{i+j} \eta^{n-k} \mu^k {}_y\mathbf{F}_k^{[m]}(\eta U_1 + \frac{\eta}{\mu} j, \mu^2 U_2, \mu^3 U_3, \dots, \mu^m U_m; u) \\ & \qquad \qquad \qquad \times {}_y\mathbf{F}_{n-k}^{[m]}(\mu u_1 + \frac{\mu}{\eta} i, \eta^2 u_2, \eta^3 u_3, \dots, \eta^m u_m; u). \end{aligned} \tag{61}$$

**Proof.** Consider

$$\mathfrak{H}(\xi) := (1-u)^2 e^{\mu \eta \xi u_1 + u_2 (\mu \eta \xi)^2 + u_3 (\mu \eta \xi)^3 + \dots + u_m (\mu \eta \xi)^m} \times \frac{(e^{\mu \eta \xi} - u^\mu) (e^{\mu \eta \xi} - u^\eta) e^{\mu \eta \xi U_1 + U_2 (\mu \eta \xi)^2 + U_3 (\mu \eta \xi)^3 + \dots + U_m (\mu \eta \xi)^m}}{(e^{\mu \xi} - u) (e^{\eta \xi} - u)}, \tag{62}$$

which in consideration of series representations of  $\frac{(e^{\mu \eta \xi} - u^\mu)}{(e^{\eta \xi} - u)}$  and  $\frac{(e^{\mu \eta \xi} - u^\eta)}{(e^{\mu \xi} - u)}$  in final expression gives

$$\mathfrak{H}(\xi) = \left(\frac{1-u}{e^{\mu\xi}-u}\right) e^{\eta u_1(\mu\xi) + \eta^2 u_2(\mu\xi)^2 + \eta^3 u_3(\mu\xi)^3 + \dots + \eta^m u_m(\mu\xi)^m} u^{\mu-1} \sum_{i=0}^{\mu-1} \left(\frac{1}{u}\right)^i e^{\eta\xi i} \\ \times \left(\frac{1-u}{e^{\eta\xi}-u}\right) e^{\mu U_1(\eta\xi) + \mu^2 U_2(\eta\xi)^2 + \mu^3 U_3(\eta\xi)^3 + \dots + \mu^m U_m(\eta\xi)^m} u^{\eta-1} \sum_{j=0}^{\eta-1} \left(\frac{1}{u}\right)^j e^{\mu\xi j}. \quad (63)$$

Thus, in view of (7) and the usage of the Cauchy product rule in the previous expression (63), we find

$$\mathfrak{H}(\xi) := \sum_{n=0}^{\infty} \left[ \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{\mu-1} \sum_{j=0}^{\eta-1} u^{\mu+\eta-2} \left(\frac{1}{u}\right)^{i+j} \mu^{n-k} \eta^k {}_y\mathbf{F}_k^{[m]}(\mu U_1 + \frac{\mu}{\eta} j, \mu^2 U_2, \mu^3 U_3, \dots, \mu^m U_m; u) \right. \\ \left. \times {}_y\mathbf{F}_{n-k}^{[m]}(\eta u_1 + \frac{\eta}{\mu} i, \eta^2 u_2, \eta^3 u_3, \dots, \eta^m u_m; u) \right]. \quad (64)$$

Continuing in a similar fashion, we find another identity

$$\mathfrak{H}(\xi) := \sum_{n=0}^{\infty} \left[ \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{\eta-1} \sum_{j=0}^{\mu-1} u^{\mu+\eta-2} \left(\frac{1}{u}\right)^{i+j} \eta^{n-k} \mu^k {}_y\mathbf{F}_k^{[m]}(\eta U_1 + \frac{\eta}{\mu} j, \mu^2 U_2, \mu^3 U_3, \dots, \mu^m U_m; u) \right. \\ \left. \times {}_y\mathbf{F}_{n-k}^{[m]}(\mu u_1 + \frac{\mu}{\eta} i, \eta^2 u_2, \eta^3 u_3, \dots, \eta^m u_m; u) \right]. \quad (65)$$

On comparison of the coefficients of like exponents of  $\xi$  in expressions (64) and (65), assertion (61) is deduced.  $\square$

**Theorem 12.** *The following symmetry connection for the MHFEPs is valid for any integers with  $\mu, \eta > 0$  and  $n \geq 0, u \in \mathbb{C}$ :*

$$\sum_{k=0}^{\eta-1} u^{\eta-1} \left(\frac{1}{u}\right)^k \sum_{i=0}^n \binom{n}{i} {}_y\mathbf{F}_{n-i}^{[m]}(\mu u_1, \mu^2 u_2, \mu^3 u_3, \dots, \mu^m u_m; u) \eta^{n-i} (\mu k)^i \\ = \sum_{k=0}^{\mu-1} u^{\mu-1} \left(\frac{1}{u}\right)^k \sum_{i=0}^n \binom{n}{i} {}_y\mathbf{F}_{n-i}^{[m]}(\eta u_1, \eta^2 u_2, \eta^3 u_3, \dots, \eta^m u_m; u) \mu^{n-i} (\eta k)^i. \quad (66)$$

**Proof.** Consider

$$\mathfrak{N}(\xi) := \frac{(1-u) e^{\mu\eta\xi u_1 + u_2(\mu\eta\xi)^2 + u_3(\mu\eta\xi)^3 + \dots + u_m(\mu\eta\xi)^m}}{(e^{\mu\eta\xi}-u^\eta)(e^{\mu\xi}-u)(e^{\eta\xi}-u)}.$$

By continuing in a similar fashion to that performed in Theorem 11, assertion (4) is deduced.  $\square$

**Theorem 13.** *The following symmetry connection between the MHFEPs and multiple power sums is valid for any integers with  $\mu, \eta > 0$  and  $n \geq 0, u \in \mathbb{C}$ :*

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} {}_y\mathbf{F}_{n-k}^{[m]}(\eta u_1, \eta^2 u_2, \eta^3 u_3, \dots, \eta^m u_m; u) u^\eta \sum_{l=0}^k \binom{k}{l} \sum_{r=0}^l \binom{l}{r} (-1)^{l-r} \mathcal{S}_k(\eta; \frac{1}{u}) \\ & \qquad \qquad \qquad \times {}_y\mathbf{F}_{k-l}^{[m+1]}(\mu U_1, \mu^2 U_2, \mu^3 U_3, \dots, \mu^m U_m; u) \mu^{n-k+l} \eta^{k-l} \\ & = \sum_{k=0}^n \binom{n}{k} {}_y\mathbf{F}_{n-k}^{[m]}(\mu u_1, \mu^2 u_2, \mu^3 u_3, \dots, \mu^m u_m; u) u^\mu \sum_{l=0}^k \binom{k}{l} \sum_{r=0}^l \binom{l}{r} (-1)^{l-r} \mathcal{S}_k(\mu; \frac{1}{u}) \\ & \qquad \qquad \qquad \times {}_y\mathbf{F}_{k-l}^{[m+1]}(\eta U_1, \eta^2 U_2, \eta^3 U_3, \dots, \eta^m U_m; u) \eta^{n-k+l} \mu^{k-l}. \end{aligned} \tag{67}$$

**Proof.** Consider

$$\begin{aligned} \mathfrak{F}(\xi) & := (1-u)^2 e^{\mu u_1(\eta\xi) + \mu^2 u_2(\eta\xi)^2 + \mu^3 u_3(\eta\xi)^3 + \dots + \mu^m u_m(\eta\xi)^m} \\ & \qquad \qquad \qquad \times \frac{(e^{\mu\eta\xi} - u^\eta) e^{\mu U_1(\eta\xi) + \mu^2 U_2(\eta\xi)^2 + \mu^3 U_3(\eta\xi)^3 + \dots + \mu^m U_m(\eta\xi)^m}}{(e^{\eta\xi} - u) (e^{\mu\xi} - u)}, \end{aligned} \tag{68}$$

which on simplifying the exponents and usage of expressions (7) and (56) in the final expression gives

$$\begin{aligned} \mathfrak{F}(\xi) & := \sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[m]}(\eta u_1, \eta^2 u_2, \eta^3 u_3, \dots, \eta^m u_m; u) \mu^n \frac{\xi^n}{n!} u^\eta \sum_{m=0}^{\infty} \sum_{r=0}^m \binom{m}{r} (-1)^{m-r} \mathcal{S}_k(\eta; \frac{1}{u}) \mu^m \frac{\xi^m}{m!} \\ & \qquad \qquad \qquad \times {}_y\mathbf{F}_{k-l}^{[m+1]}(\mu U_1, \mu^2 U_2, \mu^3 U_3, \dots, \mu^m U_m; u) \eta^l \frac{\xi^l}{l!}. \end{aligned} \tag{69}$$

Therefore, in view of the Cauchy product rule, we have

$$\begin{aligned} \mathfrak{F}(\xi) & := \sum_{n=0}^{\infty} \left[ \sum_{l=0}^n \binom{n}{l} {}_y\mathbf{F}_{n-l}^{[m]}(\eta u_1, \eta^2 u_2, \eta^3 u_3, \dots, \eta^m u_m; u) \mu^{n-l} u^\eta \sum_{m=0}^l \binom{l}{m} \sum_{r=0}^m \binom{m}{r} (-1)^{m-r} \mathcal{S}_k(\eta; \frac{1}{u}) \right. \\ & \qquad \qquad \qquad \left. \times {}_y\mathbf{F}_{l-m}^{[m+1]}(\mu U_1, \mu^2 U_2, \mu^3 U_3, \dots, \mu^m U_m; u) \mu^m \eta^{l-m} \right] \frac{\xi^n}{n!}. \end{aligned} \tag{70}$$

Continuing in a similar fashion, we have

$$\begin{aligned} \mathfrak{F}(\xi) & := \sum_{n=0}^{\infty} \left[ \sum_{l=0}^n \binom{n}{l} {}_y\mathbf{F}_{n-l}^{[m]}(\mu u_1, \mu^2 u_2, \mu^3 u_3, \dots, \mu^m u_m; u) \eta^{n-l} u^\mu \sum_{m=0}^l \binom{l}{m} \sum_{r=0}^m \binom{m}{r} (-1)^{m-r} \mathcal{S}_k(\mu; \frac{1}{u}) \right. \\ & \qquad \qquad \qquad \left. \times {}_y\mathbf{F}_{l-m}^{[m+1]}(\eta U_1, \eta^2 U_2, \eta^3 U_3, \dots, \eta^m U_m; u) \mu^m \mu^{l-m} \right] \frac{\xi^n}{n!}. \end{aligned} \tag{71}$$

On comparison of the coefficients of like exponents of  $\xi$  in expressions (70) and (71), assertion (67) is deduced.  $\square$

**Theorem 14.** The following symmetry connection between the MHFEPs and generalized integer power sums is valid for any integers with  $\mu, \eta > 0$  and  $n \geq 0, u \in \mathbb{C}$ :

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} {}_y\mathbf{F}_{n-m}^{[m]}(\eta u_1, \eta^2 u_2, \eta^3 u_3, \dots, \eta^m u_m; u) \mu^{n-m} u^\mu \sum_{r=0}^m \binom{m}{r} (-1)^{m-r} \mathcal{S}_k(\mu; \frac{1}{u}) \eta^m \\ & = \sum_{m=0}^n \binom{n}{k} {}_y\mathbf{F}_{n-m}^{[m]}(\mu u_1, \mu^2 u_2, \mu^3 u_3, \dots, \mu^m u_m; u) \eta^{n-m} u^\eta \sum_{r=0}^m \binom{m}{r} (-1)^{m-r} \mathcal{S}_k(\eta; \frac{1}{u}) \mu^m. \end{aligned} \tag{72}$$

**Proof.** Consider

$$\mathfrak{M}(\xi) := \frac{(1 - u) e^{\eta u_1(\mu\xi) + \eta^2 u_2(\mu\xi)^2 + \eta^3 u_3(\mu\xi)^3 + \dots + \eta^m u_m(\mu\xi)^m} (e^{\mu\eta\xi} - u^\mu)}{(e^{\eta\xi} - u) (e^{\mu\xi} - u)}. \tag{73}$$

By continuing in a similar fashion to that performed in the previous Theorem, assertion (72) is deduced.  $\square$

**5. Some Illustrative Examples**

Here, we give some specific examples of MHFEPs by taking their special cases:

For  $m = 3$ , the MHFEPs reduce to three-variable HFEPs  ${}_y\mathbf{F}_n^{[3]}(u_1, u_2, u_3; u)$  specified by the generating expression:

$$\left(\frac{1 - u}{e^\xi - u}\right) \exp(u_1\xi + u_2\xi^2 + u_3\xi^3) = \sum_{n=0}^{\infty} {}_y\mathbf{F}_n^{[3]}(u_1, u_2, u_3; u) \frac{\xi^n}{n!}, \tag{74}$$

operational rule:

$$\exp\left(u_2 \frac{\partial^2}{\partial u_1^2} + u_3 \frac{\partial^3}{\partial u_1^3}\right) \{ \mathbf{F}_n(u_1; u) \} = {}_y\mathbf{F}_n^{[3]}(u_1, u_2, u_3; u), \tag{75}$$

series representations:

$${}_y\mathbf{F}_n^{[3]}(u_1, u_2, u_3; u) = \sum_{s=0}^n \binom{n}{s} \mathbf{F}_s(u) \mathcal{Y}_{n-s}^{[3]}(u_1, u_2, u_3) \tag{76}$$

and

$${}_y\mathbf{F}_n^{[3]}(u_1, u_2, u_3; u) = \sum_{s=0}^n \binom{n}{s} \mathbf{F}_s(u_1; u) \mathcal{Y}_{n-s}^{[3]}(u_2, u_3). \tag{77}$$

For  $m = 2$ , the MHFEPs reduce to two-variable HFEPs  ${}_y\mathbf{F}_n(u_1, u_2, u_3; u)$  specified by the generating expression:

$$\left(\frac{1 - u}{e^\xi - u}\right) \exp(u_1\xi + u_2\xi^2) = \sum_{n=0}^{\infty} {}_y\mathbf{F}_n(u_1, u_2; u) \frac{\xi^n}{n!}, \tag{78}$$

operational rule:

$$\exp\left(u_2 \frac{\partial^2}{\partial u_1^2}\right) \{ \mathbf{F}_n(u_1; u) \} = {}_y\mathbf{F}_n(u_1, u_2; u), \tag{79}$$

series representations:

$${}_y\mathbf{F}_n(u_1, u_2; u) = \sum_{s=0}^n \binom{n}{s} \mathbf{F}_s(u) \mathcal{Y}_{n-s}(u_1, u_2) \tag{80}$$

and

$${}_y\mathbf{F}_n(u_1, u_2; u) = \sum_{s=0}^n \binom{n}{s} \mathbf{F}_s(u_1; u) \mathcal{Y}_{n-s}(u_2). \tag{81}$$

For  $m = 1$ , they reduce to Frobenius–Euler polynomials.

**6. Conclusions**

We develop the generation function and recurrence rules for the multivariate Hermite-type Frobenius–Euler polynomials in this context. We may investigate the polynomials’ characteristics and potential applications to physics and related fields using this approach. The generating function is derived and gives a compact representation of the polynomials,

which makes it simpler to analyze their algebraic and analytical properties. The recurrence relations also enable rapid computation and analysis of polynomial values through the use of recursive computing.

The multivariate Hermite-type Frobenius–Euler polynomials offer a strong foundation for further research. They provide opportunities to explore several algebraic and analytical characteristics, including differential equations, orthogonality, and others. Quantum mechanics, statistical physics, mathematical physics, engineering, and other areas of physics all make use of these polynomials. By developing the generating function and recurrence relations of extended hybrid-type polynomials, this technique is reinforced. These discoveries not only add to our understanding of multivariate Hermite-type Frobenius–Euler polynomials but also open up new avenues for investigation into their characteristics and potential applications in physics and related fields.

Operational techniques are effective in constructing new families of special functions and deriving features related to both common and generalized special functions. By employing these techniques, explicit solutions for families of partial differential equations, including those of the Heat and D’Alembert type, can be obtained. The approach described in this article, in conjunction with the monomiality principle, enables the analysis of solutions for a wide range of physical problems involving various types of partial differential equations.

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