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Existence of Best Proximity Point in *O*-CompleteMetric Spaces

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Abstract: In this work, we prove the existence of the best proximity point results for $\perp$-contraction (orthogonal-contraction) mappings on an *O*-complete metric space (orthogonal-complete metric space). Subsequently, these existence results are employed to establish the common best proximity point result. Finally, we provide suitable examples to demonstrate the validity of our results.

Keywords: best proximity point; *O*-complete metric space; *O*-closed set; P-property; weakly proximally $\perp$-preserving; $\perp$-continuous

MSC: 37C25

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1. Introduction and Preliminaries

Over the past 100 years, fixed point theory has been an active area of research, due to its significance in applications. Simultaneously, in the theory of functional analysis, the idea of proximity pairs for two sets was briefly discussed. Many researchers contributed their vision on when and where we can have the best proximity points for sets. Another group of researchers who were active on fixed point results wanted to analyze the case when we do not have an exact solution to the equation of the form $\mathcal{T}(x) = x$. Researchers such as Ky Fan, Segal, Singh, and Prolla [1–3] have provided a wealth of valuable results in best approximation theory. These findings shed light on situations where fixed points are absent, and under certain smooth conditions, we can obtain approximate solutions to the equations. Notably, Ky Fan [1] proved the existence of the best approximation for a continuous function on a compact convex subset of a normed space. In a subsequent study in 1989, Segal et al. [2] proved the existence of the best approximation for an approximately compact subset of a normed space. Furthermore, Prolla et al. [3] extended this concept to multifunctions. Around the end of the 1990s and the start of 2000, a group of researchers used the idea of the best proximity point for mappings, which unifies the fixed point and best approximation results [4–6]. Later, many generalizations were made by many researchers; refer to [7–11].

On the other hand, the Banach contraction principle is a significant mathematical discovery in fixed point theory. It has been expanded and applied to various types of metric spaces, such as semi-metrics, quasi-metrics, pseudo-metrics, fuzzy metric spaces, and partial metric spaces, among others (see [9,10,12–18]).

In that line, in 2017, Gordji et al. [19,20] introduced a new type of metric space called an orthogonal metric space and proved the fixed point results. They also demonstrated the application of these results in establishing the existence and uniqueness of solutions for first-order ordinary differential equations, where the Banach contraction mapping principle is not applicable.

Motivated by the aforementioned results [19,20], in this paper, we extend the results from the fixed point to the best proximity point for non-self-mappings in the context of an orthogonal set. Using these existence results, we prove a common best proximity point result. Finally, we provide suitable examples to demonstrate the validity of our results, which cannot be achieved through other best proximity point techniques. Furthermore, in the literature of fixed point theory, we have enormous results on the complete metric space and partially ordered metric space, but not many on the orthogonal metric space.

In [21], the existence of the best proximity points was provided for a map that is a continuous and proximal contraction, or it has to be a contraction map on an approximately compact set. In this paper, we provide the existence of the best proximity point for a weaker condition called $\perp$- continuity on an O-closed set.

Research on the concept of an orthogonal space is worth analyzing as it represents a more general space that cannot be compared with a partially ordered space. The upcoming examples will explain the necessity of having an Orthogonal space.

Example 1 ([20])**.** *Consider $M = \mathbb{R}^2$. Define $\perp$ as $u \perp v$ if $< u, v >= 0$ on M. Then, $(M, \perp)$ is an O-set, since $u = (0,0) \perp v$, for all $v \in M$. However, $(M, \perp)$ is not a partial order set. Choose $u = (1,0)$, $v = (0,1)$, $r = (-1,0)$; it is clear that $u \perp v$, $v \perp r$, but $u \not\perp r$.*

Example 2. *Consider $(M = \mathbb{R}, \leq)$. Then, M is a partially ordered set. but not an O-set with the $\leq$ relation, because we cannot find any $u \in M$ such that $u \leq p$ or $p \leq u$ for all $p \in \mathbb{R}$.*

Throughout this paper, the following notions are used:

Let A and B be any two nonempty subsets of a metric space X.

$$d(A, B) := \inf\{d(a, b) : a \in A \text{ and } b \in B\},$$
$$A_0 = \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\},$$
$$B_0 = \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}.$$

Definition 1. *Let A and B be any two nonempty subsets of a metric space X. Then, a point $p \in A$ is called a best proximity point of a mapping $T : A \to B$, if the following holds true:*

$$d(p, Tp) = d(A, B).$$

Definition 2 ([20])**.** *Let $M \neq \emptyset$, and let $\perp \subseteq M \times M$ be any binary relation. We call $(M, \perp)$ an O-set (orthogonal set) if $\perp$ satisfies the following condition:*

$$\exists u_0 \in M : (\forall v, v \perp u_0) \text{ or } (\forall v, u_0 \perp v).$$

We usually use $(M, \perp)$ to represent an O-set. Furthermore, note that this orthogonal relation is not a transitive relation.

Example 3 ([20])**.** *Take $M = [0, \infty)$, and if $uv \in \{u, v\}$, then $u \perp v$. It is clear to see that if $u_0 = 0$ or $u_0 = 1$, $(M, \perp)$ is an orthogonal set.*

Definition 3 ([20])**.** *Consider any O-set $(M, \perp)$. Let (u_n) be any sequence, then we say that (u_n) is an O-sequence if*

$$u_n \perp u_{n+1} \text{ or } u_{n+1} \perp u_n \text{ for all } n \in \mathbb{N}.$$

Example 4. *Let $M = \mathbb{R}$, and define $u \perp v$ by $uv \leq u$ or v. Take $u_n = 1/n$, then u_n is an O-sequence, since $\forall n$, $u_n \perp u_{n+1}$.*

Definition 4 ([20])**.** *Let $(X, \perp)$ be any O-set. Let A be any subset of X. Then, A is orthogonal closed set (O-closed set) if, when any O-sequence $x_n \to x$, then $x \in A$.*

Example 5. *Let $X = [0, \infty)$. Choose the usual order on X, then $(X, \leq)$ is an O-set. Consider $A = [0, 1]$, then A is an orthogonal closed set.*

Every closed set is an orthogonal closed set, but an orthogonal closed set need not be a closed set.

Example 6. *Let $X = [0,1]$ and $p \in (0,1)$, and define*

$$x \perp y \iff \begin{cases} x \leq y \leq p \\ x = 0 & \text{otherwise.} \end{cases}$$

Here, choose $A = [0,q)$ with $q \in (p,1)$. Then, A is an O-closed set. Furthermore, it is not a closed set.

Definition 5. *Let (A,B) be a pair of nonempty subsets of a metric space (X,d). The pair (A,B) satisfies the P-property if, whenever $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with,*

$$\left.\begin{aligned} d(a_1,b_1) = d(A,B) \\ d(a_2,b_2) = d(A,B) \end{aligned}\right\} \implies d(a_1,a_2) = d(b_1,b_2).$$

Definition 6 ([20])**.** *Let $(X,\perp,d)$ be an orthogonal metric space ($(X,\perp)$ is an O-set, and (X,d) is a metric space). Then, $T : X \to X$ is said to be orthogonally continuous (or $\perp$-continuous) in $a \in X$ if, for each O-sequence $\{a_n\}_{n\in N}$ in X with $a_n \to a$, we have $T(a_n) \to T(a)$. Furthermore, T is said to be $\perp$-continuous on X if T is $\perp$-continuous in each $a \in X$.*

Every continuous mapping is $\perp$-continuous, but the converse is not true.

Definition 7 ([20])**.** *Let $(X,\perp,d)$ be an orthogonal metric space and $0 < k < 1$. A mapping $T : X \to X$ is called an orthogonal-contraction (briefly, $\perp$-contraction) with Lipschitz constant k if, for all $x, y \in X$ with $x \perp y$,*

$$d(Tx,Ty) \leq kd(x,y).$$

Every contraction is a $\perp$-contraction, but the converse is not true.

2. Main Results

Now, we will prove the lemma that will be used to establish the existence of the best proximity point results.

Lemma 1. *Let A be an orthogonal closed subset of an O-complete metric space X, then A is an O-complete metric space.*

Proof. Let (x_n) be any O-Cauchy sequence in A. Then, $(x_n) \subseteq X$. Since X is an O-complete metric space, there exists $x \in X$ such that $x_n \to x$. Furthermore, (x_n) is an O-sequence, which converges to $x \in X$. Hence, $x \in A$. □

Definition 8. *Let A and B be any two nonempty subsets of a metric space (X,d). A map $T : A \to B$ is said to be proximally $\perp$-preserving if*

$$\left.\begin{aligned} d(a_1,Tb_1) = d(A,B) \\ d(a_2,Tb_2) = d(A,B) \end{aligned}\right\} \implies a_1 \perp a_2 \text{ if } b_1 \perp b_2,$$

for all a_1, a_2, b_1, $b_2 \in A$.

Theorem 1. *Let A and B be two nonempty O-closed subsets of an O-complete metric space $(X,\perp,d)$ such that $A_0 \neq \emptyset$. If (A,B) has the P-property and also $T : A \to B$ satisfies the following:*

1. *T is $\perp$-continuous and a $\perp$-contraction mapping;*

2. *$T(A_0) \subseteq B_0$;*
3. *T is proximally $\perp$-preserving;*
4. *A_0 is an O-set.*

Then, $d(u, Tu) = d(A, B)$, for some $u \in A$.

Proof. Since A_0 is an O-set, there exists $p \in A_0$ such that $u \perp p$, or $p \perp u$ for all $u \in A_0$. Without loss of generality, assume that $u \perp p$. From Condition 2, we have $Tp \in B_0$, and hence, there exists $u_1 \in A_0$ such that $d(u_1, Tp) = d(A, B)$. Furthermore, note that $Tu_1 \in B_0$, and hence, $d(u_2, Tu_1) = d(A, B)$. By the proximally $\perp$-preserving property of T, we obtain $u_1 \perp u_2$. Applying a similar argument, we construct an O-sequence $u_1 \perp u_2 \perp u_3 \perp \cdots \perp u_r \perp \cdots$ with $d(u_{r+1}, Tu_r) = d(A, B)$ for all $r \in \mathbb{N}$. Using the P-property of (A, B), we have $d(u_r, u_{r+1}) = d(Tu_{r-1}, Tu_r)$. Consider,

$$\begin{aligned} d(u_r, u_{r+1}) &= d(Tu_{r-1}, Tu_r) \\ &\leq kd(u_{r-1}, u_r) \\ &\vdots \\ &\leq k^r d(u_0, u_1). \end{aligned} \tag{1}$$

Since $k < 1$, $\lim_{r\to\infty} k^r = 0$. Hence, $\lim_{r\to\infty} d(u_r, u_{r+1}) = 0$. If $r, s \in \mathbb{N}$ and $s < r$, then

$$\begin{aligned} d(u_s, u_r) &\leq d(u_s, u_{s+1}) + d(u_{s+1}, u_{s+2}) \cdots + d(u_{r-1}, u_r) \\ &\leq k^s d(u_0, u_1) + k^{s+1} d(u_0, u_1) + \cdots + k^{r-1} d(u_0, u_1) \quad (by\ (1)) \\ &\leq k^s [1 + k + \cdots + k^{r-s-1}] d(u_0, u_1) \\ &\leq \frac{k^n}{1-k} d(u_0, u_1). \end{aligned}$$

As $s, r \to \infty$, $d(u_s, u_r) \to 0$, which means that (u_r) is an O-Cauchy sequence. Here, A is an O-closed subset of an O-complete metric space. By Lemma 1, A is an O-complete metric space $(X, \perp, d)$. Therefore, there exists $u^* \in A$ such that $\lim_{r\to\infty} u_r = u^*$. Since T is $\perp$-continuous, $\lim_{r\to\infty} Tu_{r-1} = Tu^*$, which implies $d(u_r, Tu_r) \to d(u^*, Tu^*)$ as $r \to \infty$. Hence, $d(u^*, Tu^*) = d(A, B)$. □

Theorem 2. *Let $(X, \perp, d)$ be any O-complete metric space. Let A and B be two nonempty subsets of X. Let $T : A \to B$ satisfy the following conditions:*

1. *T is $\perp$-continuous and a $\perp$-contraction;*
2. *$T(A_0) \subseteq B_0$ and (A, B) satisfy the P-property;*
3. *T is proximally $\perp$-preserving;*
4. *There exists u_0, $u_1 \in A_0$ such that $d(u_1, Tu_0) = d(A, B)$ and $u_0 \perp u_1$.*

Then, there exists an element $u \in A$ such that $d(u, Tu) = d(A, B)$.

Proof. By the hypothesis, there exists u_0 and u_1 in A_0 such that

$$d(u_1, Tu_0) = d(A, B) \text{ and } u_0 \perp u_1.$$

Since $u_1 \in A_0$, this implies $Tu_1 \in B_0$, and hence, there exists $u_2 \in A_0$ such that $d(u_2, Tu_1) = d(A, B)$, by the proximally $\perp$-preserving condition of T, we obtain $u_1 \perp u_2$. Proceeding like this, we obtain $u_1 \perp u_2 \perp \cdots \perp u_r \perp u_{r+1} \perp \cdots$. Then, (u_r) is an O-sequence with $d(u_{r+1}, Tu_r) = d(A, B)$ for all $r \in \mathbb{N}$. Since (A, B) has the P-property, we have

$$d(u_r, u_{r+1}) = d(Tu_{r-1}, Tu_r) \leq kd(u_{r-1}, u_r) \leq k^r d(u_0, u_1).$$

Since $k < 1$, $k^r \to 0$, $\lim_{r\to\infty} d(u_r, u_{r+1}) = 0$.

Claim: (u_r) is an O-Cauchy sequence. If $s, r \in \mathbb{N}$ and $r < s$, then

$$\begin{aligned} d(u_r, u_s) &\leq [d(u_r, u_{r+1}) + \cdots + d(u_{s-1}, u_s)] \\ &\leq k^r d(u_0, u_1) + \cdots + k^{s-1} d(u_0, u_1) \\ &\leq \frac{k^r}{1-k} d(u_0, u_1). \end{aligned}$$

Therefore, $d(u_r, u_r) \to 0$ as $s, r \to \infty$. Therefore, (u_r) is an O-Cauchy sequence. Hence, $\lim_{r\to\infty} u_r = u^*$. Since T is $\perp$-continuous, $\lim_{r\to\infty} Tu_{r-1} = Tu^*$, which implies $d(u_r, Tu_r) \to d(u^*, Tu^*)$. Therefore, u^* is a best proximity point. □

Example 7. *Consider $X := \mathbb{R}^2$ with $\perp$ defined as $u \perp v$ if $< u, v >= 0$. Now, define $T : \{0\} \times \mathbb{R} \to \{1\} \times \mathbb{R}$ by*

$$T(0, x) = \begin{cases} (1, x/2) : x \in \mathbb{Q} \cap \mathbb{R} \\ (1, 0) : x \in \mathbb{Q}^C \cap \mathbb{R}. \end{cases}$$

Here, observe that T is $\perp$-continuous and a $\perp$-contraction. It is easy to observe that $A_0 = A$ and $B_0 = B$; therefore, $T(A_0) \subseteq B_0$. Furthermore, (A, B) has the P-property. It is evident that the above map T satisfies all the conditions of Theorem 2. Clearly, $(0, 0)$ is the best proximity point for T.

Theorem 3. *Let $(X, \perp, d)$ be an O-complete metric space. Let A and B be two nonempty O-closed subsets of X such that $A_0 \neq \emptyset$. Furthermore, assume that (A, B) has the P-property. Let $T : A \to B$ satisfy the following conditions:*

1. *T is a $\perp$-contraction mapping and proximally $\perp$-preserving;*
2. *$T(A_0) \subseteq B_0$;*
3. *If (u_r) is any O-sequence with $u_r \to u$, then $u_r \perp u$ for all $r \in \mathbb{N}$;*
4. *A_0 is an O-set.*

Then, there exists $u \in A$ such that $d(u, Tu) = d(A, B)$.

Proof. By using the same technique as in Theorem 2, we can construct an O-Cauchy sequence (u_r) with $d(u_{r+1}, Tu_r) = d(A, B)$, and there exists $u \in A$, such that $u_r \to u$. Thus, for any $\epsilon/2 > 0$, there exists $N_1 \in \mathbb{N}$ such that $d(u_r, u) \leq \epsilon/2$, for all $r \geq N_1$. Similarly, for any $\epsilon/2k > 0$, there exists $N_2 \in \mathbb{N}$ such that $d(u_s, u) \leq \epsilon/2k$, where k is the contraction constant of T and for all $s \geq N_2$. Choosing, $N = \max\{N_1, N_2\}$, we obtain

$$\begin{aligned} d(u, Tu) &\leq d(u, u_N) + d(u_N, Tu_N) + d(Tu_N, Tu) \\ &\leq \epsilon/2 + d(A, B) + kd(u_N, u) \text{ (Since } u_N \perp u \text{ \& } T \text{ is } \perp - \text{contraction)} \\ &\leq \epsilon/2 + d(A, B) + \epsilon/2 \\ &\leq d(A, B) + \epsilon. \end{aligned}$$

Since,ϵ is arbitrary, we can conclude that $d(u, Tu) = d(A, B)$. □

Let us denote the new notion called weakly proximally $\perp$- preserving as follows.

Definition 9. *Two maps $T, S : A \to B$ are said to be weakly proximally $\perp$- preserving if:*

1. *For all $a \in A$, there exist $v_1, v_2 \in A$ with $d(v_1, Ta) = d(A, B)$, $d(v_2, Sv_1) = d(A, B)$ and $v_1 \perp v_2$.*
2. *For all $a \in A$, there exist $w_1, w_2 \in A$ with $d(w_1, Sa) = d(A, B)$, $d(w_2, Tw_1) = d(A, B)$ and $w_1 \perp w_2$.*

Theorem 4. *Let A and B be two nonempty O-closed subsets of an O-complete metric space $(X, \perp, d)$ with $A_0 \neq \emptyset$, and also, assume that (A, B) has the P-property. Let $T, S : A \to B$ be two non-self-mappings satisfying the following conditions:*

1. *(T, S) is weakly proximally $\perp$-preserving;*
2. *T or S is $\perp$-continuous;*
3. *For all u, v with $u \perp v$, $d(Tu, Sv) \leq kd(u, v)$ for some $k \in [0, 1)$,*
4. *If any O-sequence (u_n) converges, then $u_n \perp u$ for all n, where $u = \lim_{n\to\infty} u_n$.*

Then, there exists $u \in A$ such that $d(u, Tu) = d(u, Su) = d(A, B)$.

Proof. Since $A_0 \neq \emptyset$, choose any $u_0 \in A_0$. Applying T on u_0, then $Tu_0 \in B_0$. As (T, S) is weakly proximally $\perp$-preserving, we have $d(u_1, Tu_0) = d(A, B)$, $d(Tu_2, Su_1) = d(A, B)$, and $u_1 \perp u_2$. Continuing the same way using the weakly proximally $\perp$-preserving condition of (T, S), we can construct an O-sequence (u_r) with $d(u_{2r+1}, Tu_{2r}) = d(A, B)$, $d(u_{2r+2}, Su_{2r+1}) = d(A, B)$ and $u_{r+1} \perp u_{2r+2}$. Now, it is time for our usual technique of proving this (u_r) to be a Cauchy sequence. For that, observe

$$\begin{aligned} d(u_{2r+1}, u_{2r+2}) &= d(Tu_{2r}, Su_{2r+1}) \text{ (By P-Property)} \\ &\leq kd(u_{2r}, u_{2r+1}) \\ &= kd(Tu_{2r-1}, Su_{2r}) \\ &\leq k^2 d(u_{2r-1}, u_{2r}) \\ &\vdots \\ &\leq k^{2r+1} d(u_0, u_1). \end{aligned}$$

Since $k < 1$, $k^{2r+1} \to 0$, this implies $\lim_{r\to\infty} d(u_{2r+1}, u_{2r+2}) = 0$. Now, for $r, s \in \mathbb{N}$ with $s > r$, we have

$$\begin{aligned} d(u_r, u_s) &\leq d(u_r, u_{r+1}) + d(u_{r+1}, u_{r+2}) + \cdots + d(u_{s-1}, u_s) \\ &\leq k^r d(u_0, u_1) + k^{r+1} d(u_0, u_1) + \cdots k^{s-1} d(u_0, u_1) \\ &\leq k^r [1 + k + k^2 + \cdots + k^{s-r-1}] d(u_0, u_1). \end{aligned}$$

By the above inequality, it is evident that (u_r) is an O-Cauchy sequence. Since our space is O-complete, (u_r) converges, say u, which implies $u_r \perp u$ for all $r \in \mathbb{N}$. Without loss of generality, assume that T is $\perp$-continuous, then it is easy to conclude that $d(u_{2r+1}, Tu_{2r}) \to d(u, Tu)$. Furthermore, note that $d(u, Tu) = d(A, B)$. Thus, u is the best proximity point for T.

Next, our claim is to show that u is the best proximity point for S. By the convergence of (u_r), for $\epsilon/2 > 0$, there exists $N_1 \in \mathbb{N}$, such that $d(u_r, u) \leq \epsilon/2$ for all $r \geq N_1$; furthermore, for $\epsilon/2k > 0$, there exists $N_2 \in \mathbb{N}$, such that $d(u_r, u) \leq \epsilon/2$ for all $r \geq N_2$. By choosing $N = max\{N_1, N_2\}$, consider

$$\begin{aligned} d(u, Su) &\leq d(u, u_{2N+1}) + d(u_{2N+1}, Tu_{2N}) + d(Tu_{2N}, Su) \\ &\leq \epsilon/2 + d(u_{2N+1}, Tu_{2N}) + kd(u_{2N}, u) \\ &\leq \epsilon/2 + d(u_{2N+1}, Tu_{2N}) + \epsilon/2 \\ &\leq \epsilon + d(u_{2N+1}, Tu_{2N}). \end{aligned}$$

We obtain $d(u, Su) \leq d(A, B) + \epsilon$. It is easy to conclude that $d(u, Su) = d(A, B)$, since ϵ is arbitrary. Hence, $d(u, Tu) = d(u, Su) = d(A, B)$. □

Till now, in the literature on thew best proximity point, the existence of a common best proximity point in metric spaces or partially ordered metric spaces requires a stronger condition called the continuity of a map or the approximate compactness of a set. In the following example, one can easily observe that T is not a continuous map. Nevertheless, a common best proximity point exists.

Example 8. *Consider* $X = \mathbb{R}^2$ *with* $\perp$ *defined as* $(u_1, u_2) \perp (v_1, v_2)$, *if* $u_1 \le v_1$ *and* $u_2 \le v_2$. *Furthermore, choose* $d(u, v) = |u_1 - v_1| + |u_2 - v_2|$. *Then,* $(X, \perp, d)$ *is an O-complete metric space. Let us consider* $A := \{(0, a) : a \in \mathbb{R}\}$ *and* $B := \{(1, b) : b \in \mathbb{R}\}$. *Then,* $d(A, B) = 1$. *Now, define* $T : A \to B$ *by* $T(0, a) = \begin{cases} (1, -a/2) : a \in \mathbb{Q} \cap \mathbb{R} \\ (1, -a/4) : a \in \mathbb{Q}^C \cap \mathbb{R} \end{cases}$ *and* $S : A \to B$ *as* $S(0, b) = (1, -b/4)$. *We are now ready to verify the conditions of Theorem 4.*

Condition 1. *(T, S) is weakly proximally $\perp$-preserving:*

Let $u \in A$, *then* $u = (0, u_1)$, *where* $u_1 \in \mathbb{R}$.

Case (i): If $u_1 \in \mathbb{Q} \cap \mathbb{R}$, *then* $Tu = (1, -u_1/2)$. *It is easy to see that, if we take* $v = (0, -u_1/2)$ *and* $w = (0, -u_1/8)$, *then* $d(u, Tv) = d(A, B) = d(v, Sw)$ *and also* $v \perp w$.

Case (ii): If $u_1 \in \mathbb{Q}^C \cap \mathbb{R}$, *then* $Tu = (1, -u_1/4)$. *It is easy to see that, if we take* $v = (0, -u_1/4)$ *and* $w = (0, -u_1/16)$. *Then,* $d(u, Tv) = d(A, B) = d(v, Sw)$ *and also* $v \perp w$. *Similarly, for all* $u \in A$, *we can find* $w, w' \in A$ *with* $d(w, Su) = d(A, B)$, $d(w', Tw) = d(A, B)$, *which also implies* $w \perp w'$.

Condition 2. *T or S is $\perp$-continuous:*

Here, S is a continuous function, and hence, S is $\perp$-continuous. Furthermore, observe that T is not $\perp$-continuous, since O-sequence $x_n = (0, -1 - \sqrt{2}/n)$ *converges to* $x = (0, -1)$. *However,* $T(x_n) = \left(1, \dfrac{-(-1 - \sqrt{2}/n)}{4}\right)$ *converges to* $(1, 1/4)$, *which is not equal to* $Tx = (1, 1/2)$.

Condition 3. *If* $u \perp v$, *then* $d(Tu, Sv) \le kd(u, v)$ *for some* $k \in [0, 1)$. *Let* $u = (0, u_1)$, $v = (0, v_1) \in A$.

Case (i): If $u_1 \in \mathbb{Q}$, *then*

$$
\begin{aligned}
d(Tu, Sv) &= d((1, -u_1/2), (1, -v_1/4)) \\
&= | - u_1/2 + v_1/4| \\
&\le | - u_1/2 + v_1/2| \text{ (Since } u_1 \le v_1) \\
&\le \frac{1}{2} d(u, v).
\end{aligned}
$$

Case (ii): If $u_1 \in \mathbb{Q}^C$, *then*

$$
\begin{aligned}
d(Tu, Sv) &= d((1, -u_1/4), (1, -v_1/4)) \\
&= | - u_1/4 + v_1/4| \\
&\le \frac{1}{4} d(u, v) \\
&\le \frac{1}{2} d(u, v).
\end{aligned}
$$

By choosing $k = 1/2$, *it is evident that, for all* $x \perp y$, $d(Tu, Sv) \le d(u, v)$.

Condition 4. *If* (x_n) *is an O-sequence with* $x_n \to x$, *then* $x_n \perp x$ *for all n:*

Since (x_n) *is an O-sequence, we have* $x_n = (0, a_n) \le x_{n+1} = (0, a_{n+1})$, *which implies* $a_n \le a_{n+1}$. *Hence,* (x_n) *is a monotonically increasing sequence, which converges to the supremum, say* $x := (0, a)$. *It is clear that* $x_n \perp x$ *for all* $n \in \mathbb{N}$. *Furthermore, it is easy to observe that* (A, B) *has the P-property. Here,* $u^* = (0, 0)$ *satisfies* $d(u^*, Tu^*) = d(u^*, Su^*) = d(A, B)$.

Theorem 5. *Let A and B be two nonempty closed subsets of an O-complete metric space* $(X, \perp, d)$ *with* $A_0 \ne \emptyset$, *and also, assume that* (A, B) *has the P-property. Let* $T, S : A \to B$ *be two non-self-mappings satisfying the following conditions:*

1. *(T, S) is weakly proximally $\perp$-preserving;*
2. *T or S is $\perp$-continuous;*
3. *For all* u, v *with* $u \perp v$, $d(Tu, Sv) \le kd(u, v)$ *for some* $k \in [0, 1)$;

4. *If u is a best proximity point of either T or S, then $u \perp u$.*

Then, there exists $u \in A$ such that $d(u, Tu) = d(u, Su) = d(A, B)$.

Proof. Following the same technique that we used in Theorem 4, we can easily construct the O-Cauchy sequence (u_n) such that $d(u_{2n+1}, Tu_{2n}) = d(A, B)$, and $d(u_{2n+1}, Su_{2n+2}) = d(A, B)$. As usual, O-completeness provides the convergence of (u_n), that is there exists $u \in A$ such that $u_n \to u$. Without loss of generality, assume that S is $\perp$-continuous, then it is easy to conclude that $d(u_{2n+1}, Su_{2n+2}) \to d(u, Su)$. Furthermore, note that $d(u, Su) = d(A, B)$. Hence, u is the best proximity point for S; thus $u \perp u$. Consider

$$\begin{aligned} d(u, Tu) &\leq d(u, Su) + d(Su, Tu) \\ &\leq d(u, Su) + kd(u, u) \\ &\leq d(u, Su). \end{aligned}$$

Similarly, consider

$$\begin{aligned} d(u, Su) &\leq d(u, Tu) + d(Tu, Su) \\ &\leq d(u, Tu) + kd(u, u) \\ &\leq d(u, Tu). \end{aligned}$$

Hence, $d(u, Tu) = d(u, Su)$, which means that $d(u, Tu) = d(u, Su) = d(A, B)$. □

3. Conclusions

The fixed point and best proximity point results ensure the existence of solutions to many problems in non-linear analysis. In our paper, we have given the existence of the best proximity point and common best proximity point in a more general metric space called the O-metric space, which fails to satisfy the transitivity condition. Furthermore, we provided an example where our map fails to be continuous and fails to be a contraction; still, we can find the best proximity point and common best proximity points.

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