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On the P_3-Coloring of Bipartite Graphs

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Abstract: The advancement in coloring schemes of graphs is expanding over time to solve emerging problems. Recently, a new form of coloring, namely P_3-coloring, was introduced. A simple graph is called a P_3-colorable graph if its vertices can be colored so that all the vertices in each P_3 path of the graph have different colors; this is called the P_3-coloring of the graph. The minimum number of colors required to form a P_3-coloring of a graph is called the P_3-chromatic number of the graph. The aim of this article is to determine the P_3-chromatic number of different well-known classes of bipartite graphs such as complete bipartite graphs, tree graphs, grid graphs, and some special types of bipartite graphs. Moreover, we have also presented some algorithms to produce a P_3-coloring of these classes with a minimum number of colors required.

Keywords: graph coloring; chromatic number; P_3-coloring; P_3-chromatic number; bipartite graphs

MSC: 05C05; 05C07; 05C15

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1. Introduction

Graph theory deals with the study of graphs, which are mathematical structures representing a set of vertices or objects connected by any set of lines; these lines are called edges. The study of graphs is a very important tool for the applications of different subjects, such as chemistry, biochemistry, computer science, communication networks, operations research, and coding theory (see [1]). The history of graph theory dates back to the 18th century when Leonhard Euler solved the famous seven bridges of Konigsberg problem (see [2]). Then, in the 19th century, graph theory was developed by mathematicians James Joseph Sylvester and Arthur Cayley (see [3]). In the 20th century, graph theory found significant importance in different fields. One of the important problems in graph theory is graph coloring, which involves assigning colors to the vertices of the graph such that no two vertices which are adjacent have the same color. Graph coloring has numerous applications, such as map coloring (see [4]), scheduling (see [5,6]), resource allocation, and register allocation (see [7]).

Graph coloring is a fundamental concept in graph theory, a branch of mathematics that deals with the study of networks or graphs. The history of graph coloring can be traced back to the 19th century when the four color theorem was first proposed by Francis Guthrie. This theorem states that any map on a plane can be colored with just four colors in such a way that no two adjacent regions have the same color. The proof of this theorem took several decades and involved significant mathematical developments, including the use of computers to verify thousands of cases. In 1880, Tait proved in [8] that the four color theorem is equivalent to the conjecture saying that every cubic map has a proper edge coloring with three colors. Haken et al. introduced a new type of coloring, face and map coloring [9]. The four color theorem sparked great interest in graph coloring and led to further research and the development of various coloring techniques. There are different types of graph coloring, each serving a specific purpose. The most well-known type is vertex coloring, where the goal is to assign colors to the vertices of a graph in such a way that no two adjacent vertices share the same color [10,11]. Another type is edge coloring, which focuses on coloring the edges of a graph so that no two adjacent edges have the same color [10,11]. In [12], Zhou discussed edge coloring and its applications.

A detailed review on vertex coloring was given in [13,14]. Baber described list coloring in [15]. A detailed review about list coloring and some properties and algorithms of list coloring are included in [16,17]. Jenson et al. explained path coloring in [18]. Total coloring is also a type of graph coloring, and a complete review of it was provided in [19]; the algorithm of total coloring was constructed by Isobe in [20]. These various types of graph coloring have contributed to a wide range of applications and continue to be studied and refined by mathematicians and computer scientists. Moreover, there are also different types of vertex coloring and edge coloring, such as Equitable vertex coloring [21–23], Circular vertex coloring [24–26], Acyclic vertex coloring [27,28], Star vertex coloring [28,29], Circular edge coloring [30], Acyclic edge coloring [31–33], Baerge Fulkerson coloring [34], and Fan Raspand coloring [35]. In 2023, Naeem et al. introduced (see [36]) a new form of graph coloring, "P_3-coloring", and they gave some general results about this coloring. In [36], the authors have also discussed P_3-coloring of some well-known families of graphs such as complete graphs, wheel graphs, star graphs, cycle graphs, prism graphs, ladder graphs, and path graphs.

Definition 1. *Let G be a simple graph and let $\ell : V(G) \to \{c_1, c_2, \dots, c_k\}$ be coloring of the vertices of G. If, for every P_3 path in G, the colors of its vertices are different, then ℓ is called P_3-coloring of G, that is, if uvw is a P_3 path on G, then $\ell(u) \neq \ell(v) \neq \ell(w) \neq \ell(u)$.*

Definition 2. *For a graph G, the minimum number of colors (or k in above definition) required to produce (or form) a P_3-coloring is called the P_3-chromatic number of G. It is denoted as $\chi_3(G)$. It is worth noticing that for all graphs G, we have $\chi_3(G) \geq 3$.*

The following results are useful to prove some of our main Theorems in this article.

Theorem 1 ([36], Corollary 3)**.** *Let S_n be a star graph on n vertices; then $\chi_3(S_n) = n$, for all $n \geq 3$.*

Theorem 2 ([36], Theorem 1)**.** *Let G be a graph and H be a subgraph of G; then $\chi_3(G) \geq \chi_3(H)$.*

The aim of this article is to discuss the P_3-coloring and P_3-chromatic number of bipartite graphs. Trees are one of the well-known types of bipartite graphs, and in Theorem 3, we have proved that χ_3 of a tree graph is $\Delta(T) + 1$, where $\Delta(T)$ is the maximum degree of the tree graph. Theorem 4 discusses the P_3-chromatic number of complete bipartite graphs. The mesh graph or the grid graphs are also bipartite graphs and the P_3-chromatic number of grid graphs is discussed in Theorem 5. Section 4 contains the main result of this article. In Theorem 6, we give the formula for the P_3-chromatic number of any bipartite graph having exactly one cycle. Moreover, we have also presented algorithms of these results, and using these algorithms, we can produce the P_3-coloring with a minimum number of colors.

2. P_3-Chromatic Number of Tree and Complete Bipartite Graphs

In graph theory, a tree is a simple graph in which any two vertices are connected by exactly one path, that is, a tree is a simple graph having no cycles. A tree graph is also a bipartite graph. A bipartite graph is a graph such that its vertices are partitioned into two sets of vertices in such a way that any edge of the graph connects only the vertices of one set to another. A complete bipartite graph is a special kind of bipartite graph such that $V(G) = V_1 \cup V_2$. In this graph, every vertex of set V_1 is connected with every vertex of set

V_2. It is denoted by $K_{m,n}$, where m and n are the number of vertices of the set V_1 and the set V_2, respectively.

In this section, we have determined the P_3-chromatic number of tree graphs and complete bipartite graphs. Let T be a tree graph and $\Delta(T)$ be the maximum degree of T. We have the following useful notions about the coloring of a graph and its elements:

- A P_3 path has different colors if all the vertices in P_3 are of different colors.
- We say that a vertex u of the graph G is P_3 colored if all the P_3 paths containing u have different colors.

Theorem 3. *Let T be a tree graph on n vertices; then $\chi_3(T) = \Delta(T) + 1$.*

Proof. Let T be a tree graph on $n \geq 3$ vertices and let $\Delta(T)$ be the maximum degree of T. Then, there exists a star subgraph S_m of T with $\Delta(T) + 1$ vertices. By Theorem 1, we have $\chi_3(S_m) = m = \Delta(T) + 1$, and by Theorem 2, $\chi_3(T) \geq \chi_3(S_m)$. So, $\chi_3(T) \geq \Delta(T) + 1$. For the converse, we draw the tree graph as shown in Figure 1, where we consider all the vertices with degree $\Delta(T)$ in the first layer.

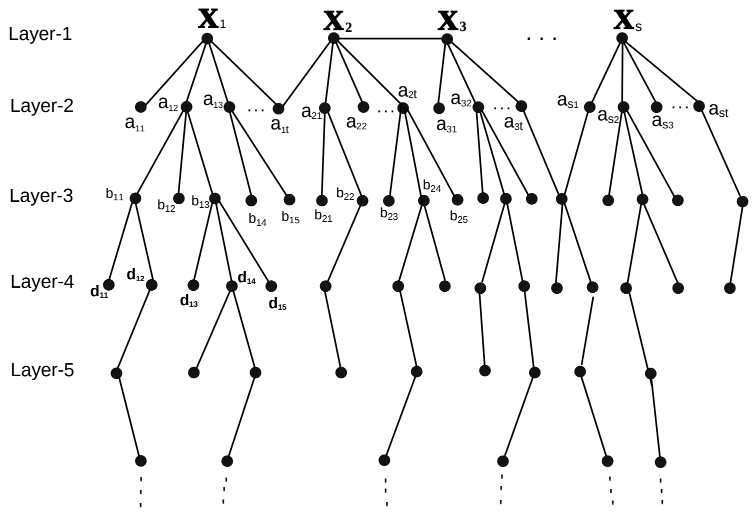

Figure 1. Tree graph.

As we move down in the layers by following any path, the degree of the vertices is decreasing and the degree of the vertices in the last layer is 1. In a tree graph, the path between all the vertices is unique, and if there is a P_3 path between any two vertices, then it is also unique. To show that $\chi_3(T) \leq \Delta(T) + 1$, we will show that $\Delta(T) + 1$ colors are enough to produce P_3-coloring of T. Let C be the set of colors and $|C| = \Delta(T) + 1$. We will produce a P_3 color function f from vertices of T to C. Notice that, in any coloring of a graph, if every vertex of the graph is P_3 colored, then such coloring is a P_3-coloring. Using this observation, firstly, we will show that x_1 is P_3 colored. So we start by assigning the color $f(x_1)$ to x_1 and the remaining $\Delta(T)$ colors are assigned to the neighboring vertices of x_1. In this way, the x_1 vertex is P_3 colored. To explain this claim, consider the vertex x_1, as shown in Figure 1. The degree of x_1 is $\Delta(T)$, and let $f(x_1)$ be the color of x_1. There exist three types of P_3 paths that contain x_1. The first type of P_3 path has x_1 as the middle vertex, the second type of P_3 path is the path whose one end point is x_1 and other is some x_i (if possible), and the third type of path is the path with one end as x_1 and the other as the vertex b_{1k}; such a path has some a_{1j} as the middle vertex. The first type of path whose middle vertex is x_1 is clearly of different colors under the assignment that we used. For the second type of path having x_1 as one end and the other as one of x_i (if it exists), we will have $\Delta(T) - 1$ choices of colors from C because there cannot exist any cycle in a tree graph, so any such x_i must be adjacent to exactly one neighbor of x_1. So, without any loss of generality, we can use the same color for such x_i as the color of any neighbor of x_1 that is not adjacent to x_i. We shall always prefer the fewest colors for the x_is (that is, if the set of colors has elements with increasing subscripts, then the first choice of color will be the color having the least subscripts). Now, for the third type of path (say $x_1a_{1j}b_{1k}$) having one end point as x_1 and the other as b_{1k}, we will assign different colors $f(b_{1k})$ from C to these vertices b_{1k}, such as the colors $f(x_1)$ and $f(a_{1j})$; see Figure 2. Because $d(a_{1j}) < d(x_1)$, we will have at least $\Delta(T) - 2$ choices for such a color scheme. So, this third type of path containing x_1 also has different colors. Thus, x_1 is a P_3 colored vertex.

Now, to show that the vertices in the second layer are also P_3-colored with the set C, we observe that if there is a vertex such as a_{11}, then this vertex is already P_3 colored by the above coloring scheme. For the other types of vertices, such as a_{12} in Figure 1, we proceed as follows. There are three possible P_3 paths that contain the vertex a_{12}. One path has x_1 as the middle vertex and a_{12} as the end vertex (such as $a_{11}x_1a_{12}$). The second type of path is that which starts from a_{12} and goes down to the descendant vertices (like $a_{12}b_{11}d_{11}$). The third type is the P_3 path that contains a_{12} as the middle vertex (such as $b_{11}a_{12}b_{12}$ or $x_1a_{12}b_{11}$). The first and third types of these P_3 paths already have different colors. For the second type of P_3 path, which has a middle vertex from b_{1k}s such as $a_{12}b_{11}d_{11}$, we assign different colors from C to the vertices d_{ik} so that none of these colors are equal to the assigned color of the middle vertex b_{1k} and $f(a_{12})$. Since $deg(b_{1k}) < deg(x_1)$, we have at least $\Delta(T) - 2$ choices of such colors. In this way, the third type of P_3 path has different colors. Thus, the vertex a_{12} is P_3 colored.

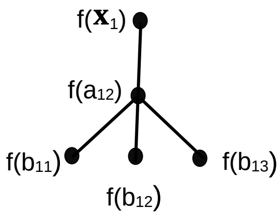

Figure 2. Assignment of colors.

So, we can use $\Delta(T) + 1$ or less colors for P_3-coloring of all the P_3 paths that contain a_{12}. Similarly, we can show that the vertex a_{12} is P_3 colored. Now, for the vertices in the third and all lower layers, we can use same scheme of coloring using at most $\Delta(T) + 1$ colors. We will apply the same scheme for the rest of the vertices of the graph T. This shows that all the vertices of T can be P_3 colorable with at most $\Delta(T) + 1$ colors. Therefore, by the definition of P_3-chromatic number, $\chi_3(T) \leq \Delta(T) + 1$. This concludes the proof. □

Algorithm to produce a P_3-coloring of tree graphs

Let T be a tree graph. Draw the tree graph shaped like a rooted tree such that all the vertices with maximum degree are in the first layer. Let C be a color class with colors $\{c_i | \, i = 1, 2, \ldots, \Delta(T) + 1\}$. So, $|C| = \Delta(T) + 1$. To understand the algorithm, we have labelled vertices of the k-th layer by $x^k_{\alpha_1\alpha_2\ldots\alpha_k}$. It represents a complete tracing of the vertices, that is, this is a vertex in the k-th layer which is connected to a vertex in the first layer by the path $x^1_{\alpha_1}x^2_{\alpha_1\alpha_2}\ldots x^k_{\alpha_1\alpha_2\ldots\alpha_k}$. For example, the vertex $x^2_{\alpha_i\alpha_3}$ shows that it is the third vertex of the second layer and it is adjacent to the i-th vertex $x^1_{\alpha_i}$ of the first layer. Let $f : V(T) \to C$ be the coloring function defined by the following steps. Fix $f(x^1_{\alpha_1}) = c_1$.

Step 1: If $x^1_{\alpha_i}, x^2_{\alpha_1\,\alpha_i} \in N(x^1_{\alpha_1})$, then $f(x^1_{\alpha_i}) = c_i$, for $i = 2, 3, \ldots, s$ and $f(x^2_{\alpha_1\,\alpha_i}) = c_j$, for $j = s + 1, s + 2, \ldots, \Delta(T) + 1$. We shall prefer the fewest colors for $x^1_{\alpha_i}$s while selecting the color of these vertices.

Step 2: Select a colored vertex, say $x^2_{\alpha_1\alpha_i}$, from the second layer having neighbor vertices in its lower layers and assign colors to these neighbor vertices in such way that the colors we are choosing are not assigned to $x^2_{\alpha_1\alpha_i}$ and to the neighbors of $x^2_{\alpha_1\alpha_i}$ in the upper layer.

Step 3: Apply "Step 2" to the vertices of lower layers having x_1^1 as the top vertex.

Step 4: Select a vertex, say $x_{\alpha_s}^1$, from first layer, which is already assigned a color, say c_j, then apply "Step 1" to $x_{\alpha_s}^1$ by setting $c_1 = c_j$. Moreover, apply "Step 3" to $x_{\alpha_s}^1$.

Step 5: Repeat "Step 4" until all the vertices have their colors.

Example 1. *For a better understanding of this algorithm, we provide an example. Consider a tree graph T as shown in Figure 3a.*

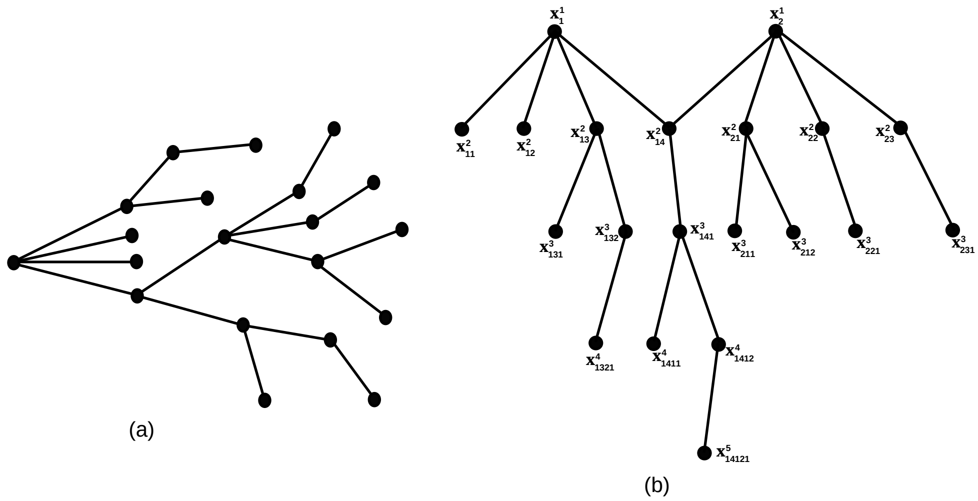

Figure 3. (**a**) A random tree graph. (**b**) Arrangement of the tree of part (**a**) according to the algorithm.

Arrange the graph in such a way that all the vertices with maximum degree are in the first layer (see Figure 3b), where we can see that $\Delta(T) = 4$. Consider the color class $C = \{c_1, c_2, c_3, c_4, c_5\}$. Fix $f(x_1^1) = c_1$.

Step 1: The $N(x_1^1) = \{x_{11}^2, x_{12}^2, x_{13}^2, x_{14}^2\}$. So, we set $f(x_{11}^2) = c_2,\ f(x_{12}^2) = c_3,\ f(x_{13}^2) = c_4$, $f(x_{14}^2) = c_5$.

Step 2: The assignment of colors to the neighbors of x_{13}^2 which are not yet assigned any color yet is $f(x_{131}^3) = c_2$, $f(x_{132}^3) = c_3$.
The assignment of colors to the neighbors of x_{14}^2 which are not yet assigned any color is $f(x_2^1) = c_2$, $f(x_{141}^3) = c_3$.

Step 3: There is only one vertex x_{1321}^4 in the neighbor of x_{132}^3 which is not assigned any color, so we put $f(x_{1321}^4) = c_1$.
The assignment of colors to the neighbors of x_{141}^3 which are not yet assigned any color is $f(x_{1411}^4) = c_1$, $f(x_{1412}^4) = c_2$.
There is only one vertex x_{14121}^5 in the neighbor of x_{1412}^4 which is not assigned any color, so we put $f(x_{14121}^5) = c_1$.

Step 4: The assignment of colors to the neighbors of x_2^1 which are not yet assigned any color is $f(x_{21}^2) = c_1$, $f(x_{22}^2) = c_3$, $f(x_{23}^2) = c_4$.
The assignment of colors to the neighbors of x_{21}^2 which are not yet assigned any color is $f(x_{211}^3) = c_3$, $f(x_{212}^3) = c_4$.
There is only one vertex x_{221}^3 in the neighbor of x_{22}^2 which is not assigned any color, so we put $f(x_{221}^3) = c_4$.
There is only one vertex x_{231}^3 in the neighbor of x_{23}^2 which is not assigned any color, so we put $f(x_{231}^3) = c_5$.

Step 5: All the vertices are already colored; see Figure 4. This completes the example.

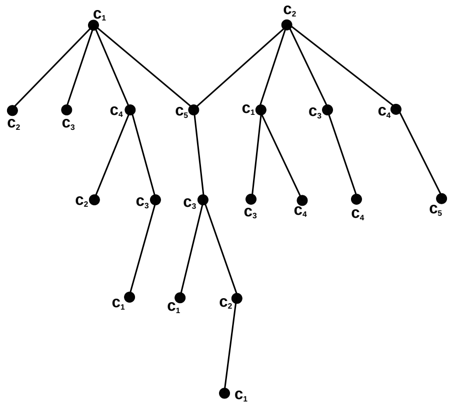

Figure 4. P_3-labelling of T.

The following Theorem 4 formulates the P_3-chromatic number of the complete bipartite graph.

Theorem 4. *Let $K_{m,n}$ be a complete bipartite graph; then $\chi_3(K_{m,n}) = m + n$.*

Proof. Let $K_{m,n}$ be a complete bipartite graph with two sets of vertices U and V, where U has m number of vertices and V has n number of vertices.

As the graph is complete bipartite, every vertex of set U is adjacent to each vertex of set V (see Figure 5). Now if we assign a color 0 to the vertex a_1, then for the vertex a_1 to be a P_3 colored vertex, we must assign n different colors to b_is. Now, select any vertex a_s different from a_1; then for this vertex to be a P_3 colored vertex, we cannot assign any color from the set $\{0, 1, \ldots n\}$. Because the path $a_1 b_t a_s$ is the P_3 path containing a_s for any arbitrary vertex b_t, we cannot assign the colors of b_t and a_1 to a_s. So, we must use a different color for every vertex of $K_{m,n}$ for P_3-labelling. Therefore, the number of colors must be equal to the number of vertices of $K_{m,n}$ and the number of vertices of $K_{m,n}$ is $m + n$. Thus, $\chi_3(K_{m,n}) = m + n$. □

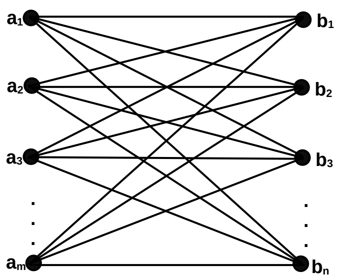

Figure 5. Complete bipartite graph $K_{m,n}$.

3. P_3-Chromatic Number of Grid Graph

In this section, we have computed the P_3 chromatic number of the grid graph. A grid graph is also one of the many well-known bipartite graphs. It is the Cartesian product $P_m \square P_n$ of path graphs with m and n vertices. The $m \times n$ grid graph is also denoted by $L(m, n)$. Grid graphs are also known as lattice graphs or rectangular graphs. In Theorem 5, the generalized form of the P_3-chromatic number of grid graph $(P_m \square P_n)$ is determined, where $m, n \geq 3$.

Theorem 5. *Let $P_m \square P_n$ be the grid graph; then $\chi_3(P_m \square P_n) = 5$ for all $m, n \geq 3$.*

Proof. Let $P_m \square P_n$ be a grid graph with $m, n \geq 3$. From the definition of grid graph and from Figure 6, we can see that the star graph S_5 is a subgraph of $P_m \square P_n$. Then, by Theorem 1 and Theorem 2, $P_m \square P_n \geq 5$. This means that we need at least 5 colors for the P_3-coloring of $P_m \square P_n$.

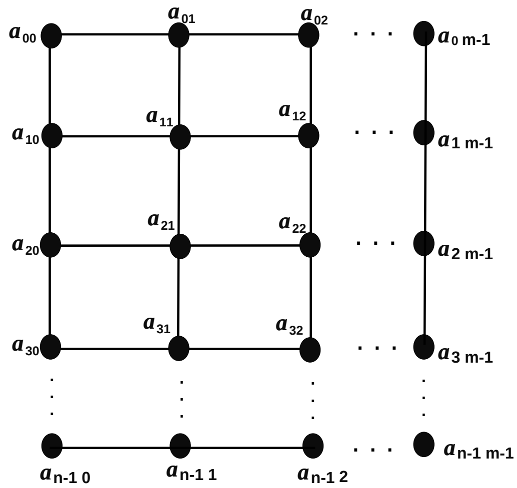

Figure 6. The grid graph $P_m \square P_n$.

To prove the converse, we will define a P_3-labeling $f : V(P_m \square P_n) \to \{0,1,2,3,4\}$, where the set $\{0,1,2,3,4\}$ is the set of colors. Let $j \in \{0,\ldots,m-1\}$; then we define f as follows.

$$f(a_{ij}) = 2i + j \pmod 5),\ 0 \leq i \leq n-1,\ 0 \leq j \leq m-1.$$

To show that f is indeed a P_3-coloring, we must show that each vertex of $P_m \square P_n$ is P_3 colored. As the graph is symmetric, it is sufficient to show that the vertices on P_3 paths in Figure 7 have this property. Because every vertex in $P_m \square P_n$ lies on one of these type of figures, if the vertices of Figure 7 are P_3 colored, then with the same scheme, we can say that it would be true for all vertices of the grid graph.

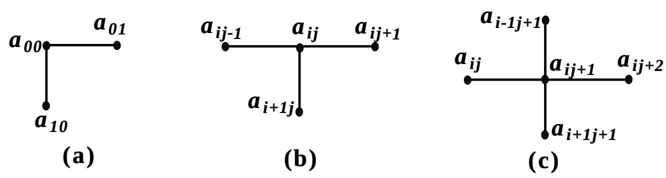

Figure 7. Sub-graphs of $P_m \square P_n$. (**a**) Represent the corner subgraphs. (**b**) Represent subgraphs from sides. (**c**) Represent subgraphs from inside of $P_m \square P_n$.

Figure 7a represents the four subgraphs on corners of the grid graph, Figure 7b represents such subgraphs on the borderline of the grid graph, and Figure 7c represents all such internal subgraphs of the grid graph. In Figure 7a, there are five possible P_3 paths containing the vertex a_{00}, where $0 \leq i \leq 2$ and $0 \leq j \leq 2$. The paths are

$$a_{00}a_{01}a_{02},\ a_{00}a_{01}a_{11},\ a_{00}a_{10}a_{20},\ a_{00}a_{10}a_{11},\ a_{01}a_{00}a_{10}.$$

We shall discuss only one path from the above five paths to show that they are colored. Similarly, the other paths can be shown to be colored. Let us consider the path $a_{01}a_{00}a_{10}$; then $f(a_{01}) = 1, f(a_{00}) = 0, f(a_{10}) = 2$.

Now, for $i = 0$ and $j = 1$, the sub-graph in Figure 7b shows that there are eight possible P_3 paths that contain the vertex a_{01}, and the list of such paths is

$$a_{00}a_{01}a_{02},\ a_{00}a_{01}a_{11},\ a_{02}a_{01}a_{11},\ a_{01}a_{02}a_{03},\ a_{01}a_{11}a_{12},\ a_{01}a_{11}a_{10},\ a_{01}a_{00}a_{10},\ a_{01}a_{02}a_{12}.$$

We shall discuss only one path from the above eight paths to show that they are colored. Similarly, the other paths can be shown to be colored. Let us consider the path $a_{00}a_{01}a_{02}$; then $f(a_{00}) = 0, f(a_{01}) = 1, f(a_{02}) = 2$.

For $i = 0$ and $1 < j < m-2$, the sub-graph in Figure 7b shows that there are nine possible P_3 paths that contain the vertex a_{ij}, and these P_3 path are

$$a_{i\,j-1}a_{ij}a_{i\,j+1},\ a_{i\,j-1}a_{ij}a_{i+1\,j},\ a_{i\,j+1}a_{ij}a_{i+1\,j},\ a_{ij}a_{i\,j+1}a_{i\,j+2},\ a_{ij}a_{i+1\,j}a_{i+1\,j+1},\ a_{ij}a_{i+1\,j}a_{i+1\,j-1},$$

$$a_{ij}a_{i\,j-1}a_{i\,j-2},\ a_{ij}a_{i\,j-1}a_{i+1\,j-1},\ a_{ij}a_{i\,j+1}a_{i+1\,j+1}.$$

Similarly, as above, we shall discuss only one path from the above nine paths to show that they are colored. The other paths can be shown to be colored by following a similar technique. Note that this case also proves that the result is true under the condition on the subscripts i and j as follows.

For $i = 0,\ n-1$ we have $1 < j < m-2$, and for $j = 0,\ m-1$ we have $1 < i < n-2$.

We shall discuss only one possibility here; the proofs for others will follow similarly. Let us consider the path $a_{i\,j-1}a_{ij}a_{i\,j+1}$, then $f(a_{i\,j-1}) = 2i + j - 1$, $f(a_{ij}) = 2i + j$, and $f(a_{i\,j+1}) = 2i + j + 1$.

Thus, all the vertices on the four sides of the grid are P_3 colored. Figure 8 represents the P_3-coloring and algorithm under the function f.

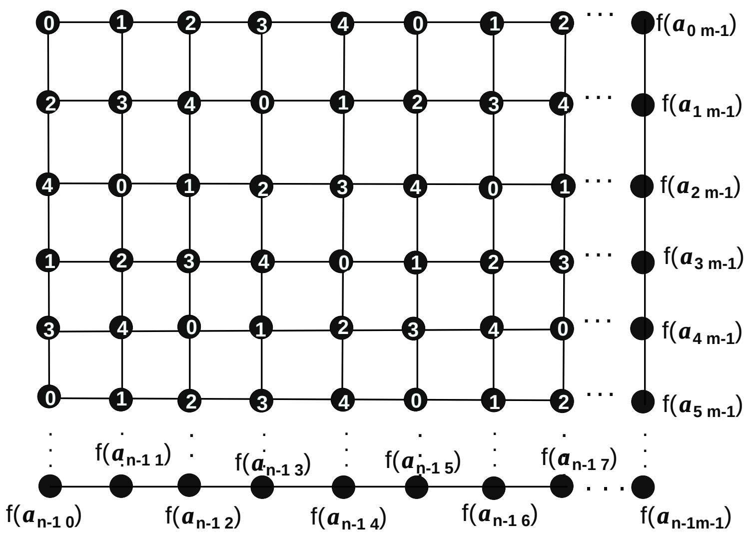

Figure 8. P_3-labelling of $P_m \square P_n$.

Now, we can see from Figure 7c that the vertex a_{ij}, where $1 \le i \le n-2$, $1 \le j \le m-2$, is contained in the following eighteen P_3 paths:

(i) $a_{i-1\,j}a_{ij}a_{i+1\,j}$, (ii) $a_{i\,j-1}a_{ij}a_{i\,j+1}$, (iii) $a_{i-1\,j}a_{ij}a_{i\,j-1}$, (iv) $a_{i-1\,j}a_{ij}a_{i\,j+1}$,
(v) $a_{i\,j-1}a_{ij}a_{i+1\,j}$, (vi) $a_{i\,j+1}a_{ij}a_{i+1\,j}$, (vii) $a_{ij}a_{i+1\,j}a_{i+2\,j}$, (viii) $a_{ij}a_{i-1\,j}a_{i-2\,j}$,
(ix) $a_{ij}a_{i\,j+1}a_{i\,j+2}$, (x) $a_{ij}a_{i\,j-1}a_{i\,j-2}$, (xi) $a_{ij}a_{i+1\,j}a_{i+1\,j-1}$, (xii) $a_{ij}a_{i+1\,j}a_{i+1\,j+1}$,
(xiii) $a_{ij}a_{i\,j+1}a_{i+1\,j+1}$, (xiv) $a_{ij}a_{i\,j-1}a_{i+1\,j-1}$, (xv) $a_{ij}a_{i\,j+1}a_{i-1\,j+1}$, (xvi) $a_{ij}a_{i\,j-1}a_{i-1\,j-1}$,
(xvii) $a_{ij}a_{i-1\,j}a_{i-1\,j+1}$, (xviii) $a_{ij}a_{i-1\,j}a_{i-1\,j-1}$.

We shall discuss one P_3 path from the above list. The computations for other paths will follow similarly. So, consider an arbitrary P_3 path from the above list, say $a_{ij}a_{i\,j+1}a_{i\,j+2}$; then

$$f(a_{ij}) = 2i + j,\ f(a_{i\,j+1}) = 2i + j + 1 \text{ and } f(a_{i\,j+2}) = 2i + j + 2.$$

This shows that all P_3 paths in this case are colored. Therefore, all the internal vertices of the grid graph are P_3 colored and this proves that f is a P_3-coloring. Similarly, every vertex is P_3 colored in all the cases. So, $\chi_3(P_m \square P_n) = 5$. □

4. P_3-Chromatic Number of Bipartite Graphs Having Exactly One Cycle.

In this section, we have discussed the P_3 chromatic number for a special class of bipartite graphs consisting of one or more cycles under different conditions. Theorem 6 provides the P_3-chromatic number of bipartite graphs, which contains exactly one cycle. We have constructed an algorithm for Theorem 6 after its proof.

Theorem 6. *Let G be a bipartite graph having exactly one cycle; then* $\chi_3(G) = \Delta(G) + 1$.

Proof. Let G be a bipartite graph having exactly one cycle, with U and V being a vertex partition of $V(G)$. For simplicity, let $\Delta(G) = \deg(u_1)$. Then, it contains a star subgraph S_m, such that $m = \Delta(G) + 1$. By Theorem 1, we have $\chi_3(S_m) = \Delta(G) + 1$, and from Theorem 2, we obtain $\chi_3(G) \geq \chi_3(S_m)$. So,

$$\chi_3(G) \geq \Delta(G) + 1$$

For the converse, we need to show that every vertex G can be P_3 colored with a color class $C = \{\alpha_i | \, i = 1, 2, 3, \ldots, \Delta(G) + 1\}$, that is, no vertices in any P_3 path have the same colors. For this, we will arrange(or draw) the graph in such a way that all the vertices having maximum degree are on the leftmost side of the graph and the degrees of the vertices are in decreasing order from left to right, as shown in Figure 9, where $deg(u_1) \geq deg(u_2) \geq deg(u_3) \geq \ldots \geq deg(u_m)$, and the same for the v_i's.

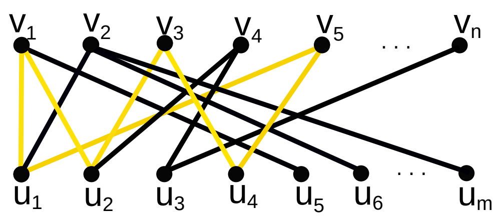

Figure 9. A bipartite graph G with one cycle. Cycle is in color.

Now, consider the vertex u_1 of the graph G. The vertex u_1 is adjacent to $\Delta(G)$ vertices of the set V. Therefore, we can assign $\Delta(G) + 1$ different colors to u_1 and to its $\Delta(G)$ neighbors for the production of a P_3-coloring, as shown in Figure 10.

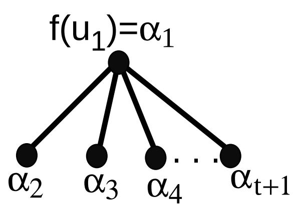

Figure 10. P_3-coloring at vertex u_1.

The graph G is a bipartite graph that contains a cycle, so there will be vertices which are connected by more than one P_3 path. Moreover, there are two types of P_3 paths containing each vertex $x \in V(G)$. In one P_3 path, the vertex x is a middle vertex (with end points from the opposite set of vertices U or V), and in the second type of P_3 path, the vertex x is one of the two end points (with the other end vertex from the same set U or V). So, for the vertex u_1, there also exist two types of P_3 paths which contain u_1. The first type of P_3

paths would be the paths whose middle vertex is u_1 and end points must be some v_is, for $i \in \{1, 2, 3, \ldots, \Delta(G)\}$; these types of P_3 paths are highlighted in Figure 11.

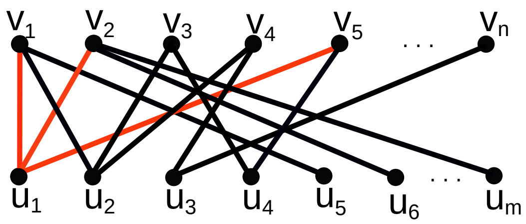

Figure 11. P_3 paths whose middle vertex is u_1.

The second type of P_3 path that contains u_1 starts from u_1 and must end at some other u_i, where the number of such u_is is at most $\Delta(G)$; such types of P_3 paths are shown in Figure 12.

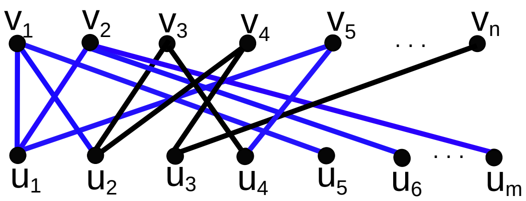

Figure 12. P_3 paths which start from u_1 and end at any other u_i

The first type of P_3 path which contains u_1 as the middle vertex is already colored because its end points are v_is, and such v_is are neighbors of u_1. From Figure 10, it is clear that we have already assigned colors to u_1 and its neighbor vertices using the C color class that has $\Delta(G) + 1$ colors. Now, for the second type of P_3 path containing u_1 that starts from u_1 and ends at some other u_i, the middle vertex of these types of P_3 paths must be some v_i. Such v_is are already assigned colors, as shown in Figure 10. So, now we need to assign color only to u_is, which are the end points of this second type of P_3 path.

To continue the procedure of producing a P_3-coloring, we will assign colors to these u_is (from the same color class C) which are not assigned to u_1 and to the neighbors of these u_is. As there exists only one cycle, in these types of paths, any u_i can be adjacent to at most two neighbors of u_1. Therefore, we can use the colors of v_js, say α_js, which are not adjacent to u_i, so in this way, we will have at least $\Delta(G) - 2$ and $\Delta(G) - 1$ choices of colors for each u_i when u_i is adjacent to two and one neighbors of u_1, respectively. Therefore, to produce a P_3-coloring at u_1 for the second type of P_3 paths, we can assign colors to such u_is from left to right in decreasing order with respect to subscripts of colors, as shown in Figure 13. Thus, u_1 is P_3 colored.

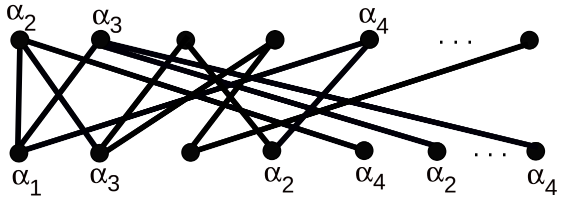

Figure 13. P_3-coloring of all P_3 paths containing u_1.

Now, consider any other u_i (if it exists) that is not colored yet, say u_ℓ. For example, in Figure 14, the vertex u_3 is such a vertex. We will show that u_ℓ is also P_3 colored. For this, firstly, we will assign α_1 color to u_ℓ, as shown in Figure 14; the vertex u_3 is assigned the color α_1.

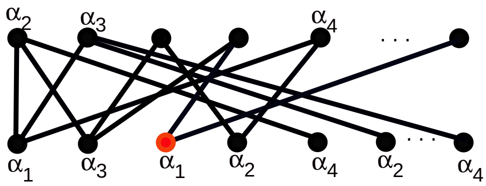

Figure 14. Assignment of color to u_ℓ. Here, it is u_3.

We can use the color assigned to u_1 for u_ℓ because u_ℓ is not adjacent to u_1 and also u_ℓ is not contained in any P_3 path that contains u_1. The vertex u_ℓ is also contained in two types of P_3 paths. The first type of path is the path having a middle vertex as u_ℓ. The degree of the vertex u_ℓ is less than or equal to $\Delta(G)$, so for its coloring, we can use the same color class C; see Figure 15. We will assign those colors to neighbors v_js of u_ℓ which are not used for u_ℓ, not used for the u_is, which are the end points of P_3 paths with u_ℓ as the second end point, and not used for the previous colored v_is, which are the end points of P_3 paths with v_js as the second end point (e.g., in Figure 15, we will not assign colors α_1, α_3, and α_2 to v_4). We will have choices of colors from C for v_js because there is only one cycle, and the degree of the vertices u_ℓ, u_is, and v_is (which are already assigned a color in some P_3 paths) is not more than $\Delta(G)$.

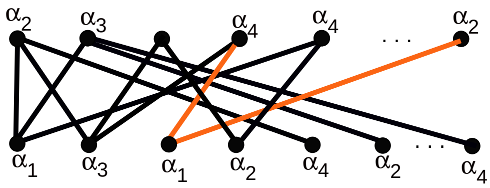

Figure 15. P_3 coloring of paths having u_3 as middle vertex.

Now, consider the second type of P_3 path having u_ℓ as one end point and u_i as the second end point (other than u_ℓ). Note that some vertices of these types of paths may already be colored, while some may not be. The ones that are not yet assigned colors, such as u_is, are not adjacent to any v_i that is adjacent to any u_k whose P_3-coloring is already completed. There will be choices (at least one) of colors for such vertices from color class C. For example, the second type of P_3 path for the vertex u_3 in Figure 15 is already colored. Similarly, for any u_i that is not assigned color, we can use color class C. That u_i must be contained in at most two types of P_3 paths with the same conditions. So, with the same technique, a P_3-coloring of all the P_3 paths which contain u_is can be produced by using the same color class C. We will select the fewest colors (with respect to the order in the subscripts) when selecting the colors of vertices u_is which are not used for vertices of any P_3 that contains the vertex u_i. We will apply this same scheme until all the u_is are colored. After assigning colors to vertices u_is, we shall have two cases:

(1) All v_ps are colored.
(2) Some v_ps are not colored. See Figure 15.

In the first case, the P_3-coloring of the graph is already completed. So, we will obtain our required result. But in the second case, we will consider the u_i vertex that is already colored, but some v_p that are adjacent to that u_i are not colored yet. That u_i must be contained in two types of P_3 paths. The first type of P_3 path has that u_i as its middle vertex (and v_p as one end point), and in the second type of P_3 path, that u_i would be one end point of the paths, and the other end point would be any other u_i (with v_p as the middle vertex). For the first type of path, we need to color only v_ps because u_i is already colored. As the degree of that u_i must be less than or equal to $\Delta(G)$, u_i must be connected with at most $\Delta(G)$ types of v_ps. So, we can use the colors from C that are not used for the neighbors of that v_p and also not used for the neighbors of the neighbors of v_ps. As the considered vertex u_i is already colored, but some of its neighbors are not colored, it shows that this u_i is the end point of any P_3 path whose other end point is some different u_p who is already P_3 colored and this u_i is assigned a color when we colored the second type of P_3 paths that contain the u_p vertex as their one end point. So, it means we can use the colors of that u_p, say α_p, for the v_p.

For example, the v_3 vertex is such a vertex in Figure 15, and it is assigned a color as shown in Figure 16, where $f(u_1)$ and $f(v_3)$ are the colors of vertices u_1 and v_3, respectively. By using this scheme, all the v_ps in this type of path would be colored by using same color class C. Then, we do not need to color the second type of P_3 path because that must already be colored, as here we assigned colors to v_ps and all the u_is are already labelled. So, by using this technique, we can color all the remaining vertices of set V by using at most $\Delta(G) + 1$ colors (see Figure 16). The process will end eventually since the graph is finite and the degree of vertices is non-increasing from left to right, with every vertex becoming P_3 colored in the process with at most $\Delta(G) + 1$ colors. Therefore, $\chi_3(G) \leq \Delta(G) + 1$. We have already proved that $\chi_3(G) \geq \Delta(G) + 1$; therefore,

$$\chi_3(G) = \Delta(G) + 1.$$

□

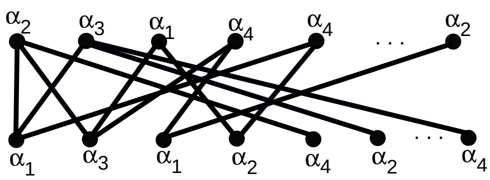

Figure 16. $f(u_1) = f(v_3)$

Algorithm to produce a P_3-coloring of graphs for Theorem 6

Let G be a bipartite graph having exactly one cycle and $V(G) = U \cup V$ be the vertex partition of G. Let $\Delta(G)$ be the maximum degree of G. Arrange the bipartite graph such that $d(u_1) \geq d(u_2) \geq d(u_3) \geq \ldots \geq d(u_m)$, $d(v_1) \geq d(v_2) \geq d(v_3) \geq \ldots \geq d(v_n)$ from left to right. We define a P_3-coloring $f : V(G) \to C$, where $C = \{\alpha_i | \, i = 1, 2, \ldots, \Delta(G) + 1\}$. Let $d(u_1) = \Delta(G)$ and fix $f(u_1) = \alpha_1$.

Step 1: If $v_j \in N(u_1)$ then $f(v_j) = \alpha_i, \; i = 2, 3, \ldots, \Delta(G) + 1$.

Step 2: Select the vertex $v_j \in N(u_1)$ that already has a color, say α_j, and assign different colors to neighbors of v_j. Choose colors that are not used for other neighbors of v_j.

Step 3: Select the immediate next vertex of set U which is not colored yet, say u_s, and put $f(u_s) = \alpha_1$. Then apply Step 1 and Step 2 to u_s.

Step 4: Repeat Step 3 until all the u_is are not colored.

Step 5: Select the vertex from set V which is not yet colored, say v_t. Assign the color α_t to the vertex v_t where α_t is the color that is not assigned to neighbors of v_t and neighbors of its neighbors.

Step 6: Repeat Step 5 until all the v_ts are not colored.

Example 2. Consider the bipartite graph G having exactly one cycle, as depicted in Figure 17.

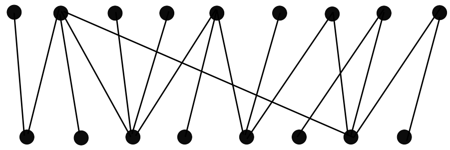

Figure 17. A bipartite graph having exactly one cycle

Arrange graph G such that the degrees of the vertices are in decreasing order from left to right, as shown in Figure 18. We can see $\Delta(G) = 4$. So, $C = \{\alpha_i | \, i = 1, 2, 3, 4, 5\}$. Fix $f(u_1) = \alpha_1$.

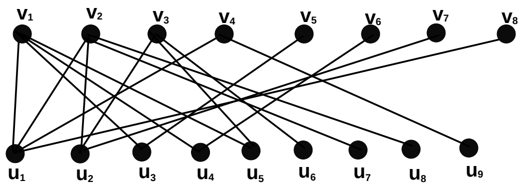

Figure 18. Arrangement of the vertices of G in non-increasing order from left to right.

Step 1: We have $N(u_1) = \{v_1, v_2, v_4, v_8\}$. We consider the assignment of colors to the neighbors of u_1 as $f(v_1) = \alpha_2$, $f(v_2) = \alpha_3$, $f(v_4) = \alpha_4$, $f(v_8) = \alpha_5$.

Step 2: The neighborhood of v_1 is $\{u_1, u_3, u_4, u_5\}$; therefore, we assign $f(u_3) = \alpha_3$, $f(u_4) = \alpha_4$, and $f(u_5) = \alpha_5$.
The neighborhood of v_2 is $\{u_1, u_2, u_7, u_8\}$; therefore, we assign $f(u_2) = \alpha_2$, $f(u_7) = \alpha_4$, and $f(u_8) = \alpha_5$.
The neighborhood of v_4 is $\{u_1, u_9\}$; therefore, we put $f(u_9) = \alpha_2$.

Step 3: Select the immediate next vertex u_6 of set U which is not colored yet and assign $f(u_6) = \alpha_1$. The vertex v_3 is the only neighbor of u_6, and we assign $f(v_3) = \alpha_4$.

Step 4: All the u_is are already colored. So we move to Step 5.

Step 5: The vertex v_5 from set V is not colored yet. We assign $f(v_5) = \alpha_1$.

Step 6: Now, select vertex v_6 and assign $f(v_6) = \alpha_1$. For the vertex v_7, we put $f(v_7) = \alpha_1$.

Thus, Figure 19 represents the final P_3-coloring of the graph.

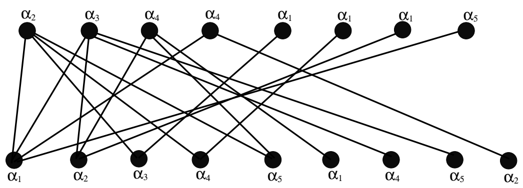

Figure 19. P_3-labelling of G.

5. Conclusions

In this article, the main interest of the authors was to study a recently introduced coloring of graphs called P_3-coloring. This coloring arises as a natural generalization of the coloring of a graph. In this respect, we have determined the P_3-chromatic number of different families of bipartite graphs. We have formalized the P_3-chromatic number of tree graphs, grid graphs, complete bipartite graphs, and the class of bipartite graphs that have only one cycle. Moreover, we have also presented algorithms to produce a P_3-coloring of tree graphs, grid graphs, and the bipartite graphs that have exactly one cycle with the minimum number of colors. In the future, the authors are interested in extending this study and making some more significant advancements.

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