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Single-Machine Maintenance Activity Scheduling with Convex Resource Constraints and Learning Effects

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Abstract: In this paper, the single-machine scheduling problems under the common and slack due date assignments are studied, where the actual processing time of the job needs to consider some factors, such as convex resource allocation, maintenance activity, and learning effects. The goal of this study is to find the optimal sequence, maintenance activity location, resource allocation and common due date (flow allowance). The objective function is (1) to minimize the sum of scheduling cost (including the weighted sum of earliness, tardiness and common due date (flow allowance), where the weights are position-dependent weights) and resource consumption cost, and (2) to minimize the scheduling cost under the resource consumption cost which is bounded. We prove that these problems can be solved in polynomial time.

Keywords: scheduling; learning effect; maintenance activity; resource allocation; due date assignment

MSC: 90B35

Citation: Wei, Z.-J.; Wang, L.-Y.; Zhang, L.; Wang, J.-B.; Wang, E. Single-Machine Maintenance Activity Scheduling with Convex Resource Constraints and Learning Effects. *Mathematics* **2023**, *11*, 3536. https://doi.org/10.3390/math11163536

Academic Editor: Andrea Scozzari

Received: 1 July 2023
Revised: 7 August 2023
Accepted: 11 August 2023
Published: 16 August 2023

1. Introduction

In a realistic scheduling system, the job-processing time is often affected by many practical settings, such as learning effects, resource allocation, and maintenance activities. Learning effects appear such that, for example, workers continue to process the same jobs, experience increases, and the actual processing time of the work is gradually shortened (see Mosheiov [1]; Cheng et al. [2]; Wu et al. [3]; Azzouz et al. [4]; Sun et al. [5]; Zhao [6]; and Wang et al. [7]). The manager assigns a certain amount of additional resources to the job to reduce and control the processing time of the artifact. Common limited resources are the financial budget, energy, fuel, or manpower (see Vixkson [8]; Shabtay and Kasoi [9]; Wang and Cheng [10]; Shabtay and Steiner [11]; Zhang et al. [12]; Wang et al. [13]). In production, due to machine failure or the processing time being too long, wear will inevitably occur and reduce the working rate, so maintenance activities can be carried out to reduce the job processing time (see Lee and Leon [14]; Wang and Wang [15]; Mosheiov and Sidney [16]; Bai et al. [17]; Yin et al. [18]; and Strusevich and Rustogi [19]).

In order to strengthen global business competition and improve customer satisfaction, new production technologies such as "just-in-time" (JIT) production are adopted. The emergence of the concept of JIT production and the sequencing problems have attracted widespread attention; in a just-in-time system, jobs (work pieces) can neither be completed early nor late, otherwise penalty costs will be incurred (see Parwalker et al. [20], and Cheng et al. [21]). In addition, in actual production, we found that task-processing rates can be affected by several factors simultaneously. Therefore, in previous studies, Ji et al. [22] considered a single machine due date assignment scheduling problem with job-dependent aging effects and a deteriorating maintenance activity, where due dates are assigned using the SLK due date determination method. He et al. [23] studied the single-machine sequencing problem of resource allocation with general truncated learning effects. A convex resource allocation model under the condition of finite resource consumption cost is proposed, and various optimal algorithms are given for different cases of the problem. Liu and Jiang [24] delved into due date assignment scheduling problems with learning effects and resource allocation. Under common due date assignment and slack due date assignment rules, a bi-criteria analysis is provided. Zhao et al. [25] examined a single machine scheduling problem with slack due date assignment in which the actual processing time of a job is determined by its position in a sequence, its resource allocation function, and a rate-modifying activity simultaneously.

In previous studies, two different resource allocation functions were usually employed. One was a linear function setting for the amount of resources and actual processing time associated with each job (Janiak and Kovalyov [26]), and the other was a convex function setting for the amount of resources assigned to each job (Monma et al. [27]). In general, there are two ways to model learning effects: one is a location-dependent learning effect (Biskup [28]), and the other is a time-dependent (sum-of-processing time) learning effect (Azzouz et al. [4]). Wang et al. [29] considered location-dependent learning effects and convex resource allocation to build the model $p_{jr}^A(u_j) = \left(\frac{\theta_j r^\alpha}{u_j}\right)^\eta$, where $\eta > 0$ is a given constant, $\alpha \leq 0$ is the learning factor, θ_j is the normal processing time of job J_j, and u_j is the resource allocated to the job J_j. Zhu et al. [30] addressed the job processing time considering the rate modification activity, learning effect and convex resource allocation, described as follows: when the rate modification activity is not carried out, the job-processing time is $p_{jr}^A(u_j) = \left(\frac{\theta_j r^\alpha}{u_j}\right)^\eta$; otherwise, it is $p_{jr}^A(u_j) = \left(\frac{a_j \theta_j r^\alpha}{u_j}\right)^\eta$, where a_j is the modifying rate. For a comparison with other similar papers (see Table 1; the related symbols are given later), this article extends the results of Wang and Wang [15], Bai et al. [17], Ji et al. [22], Zhao et al. [25], Wang et al. [29], and Zhu et al. [30] by scrutinizing a more general scheduling model.

Table 1. Models studied.

References	Scheduling Problem	Time Complexity
Wang et al. [29]	$1\left\|P_{jr}^A(u_j) = \left(\frac{\theta_j r^\alpha}{u_j}\right)^\eta\right\|\delta_1 C_{\max} + \delta_2 TC + \delta_3 TADC + \delta_4 \sum_{j=1}^{\check{n}} v_j u_j$	$O(n\log(n))$
	$1\left\|P_{jr}^A(u_j) = \left(\frac{\theta_j r^\alpha}{u_j}\right)^\eta\right\|\delta_1 C_{\max} + \delta_2 TW + \delta_3 TADW + \delta_4 \sum_{j=1}^{\check{n}} v_j u_j$	$O(n\log(n))$
Zhu et al. [30]	$1\|MALE, RE\|\delta_1 C_{\max} + \delta_2 TC + \delta_3 TADC + \delta_4 \sum_{j=1}^{\check{n}} v_j u_j$	$O(n^2\log(n))$
	$1\|MALE, RE\|\delta_1 C_{\max} + \delta_2 TW + \delta_3 TADW + \delta_4 \sum_{j=1}^{\check{n}} v_j u_j$	$O(n^2\log(n))$
Wang and Wang [15]	$1\|MA, SLK, P_j = (\theta_j, \beta_j\theta_j)\|\sum_{j=1}^{n}(\delta E_j + \omega T_j + \gamma q_{opt})$	$O(n^4)$
Bai et al. [17]	$1\|MADE, P_j = (\theta_j + bs_j, \beta_j\theta_j + bs_j)\|\sum_{j=1}^{n}(\delta E_j + \omega T_j + \gamma q_{opt})$	$O(n^4)$
Ji et al. [22]	$1\left\|MADE, P_{jr}^A = \theta_j(r-l)^{a_j}\right\|\sum_{j=1}^{n}(\delta E_j + \omega T_j + \gamma q_{opt})$	$O(n^4)$
Zhao et al. [25]	$1\left\|MADE, CRE, \sum_{j=1}^{n} v_{[j]}u_{[j]} \leq U\right\|\sum_{j=1}^{n}(\delta E_j + \omega T_j + \gamma q_{opt})$	$O(n^4)$
	$1\left\|MADE, CRE, \sum_{j=1}^{n}(\delta E_j + \omega T_j + \gamma q_{opt}) \leq V\right\|\sum_{j=1}^{n} v_{[j]}u_{[j]}$	$O(n^4)$
This article	$1\|MALE, CRE\|\sum_{j=1}^{n}\left(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma d_{opt}\right) + \sum_{j=1}^{n} v_{[j]}u_{[j]}$	$O(n^4)$
	$1\|MALE, CRE\|\sum_{j=1}^{n}\left(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma q_{opt}\right) + \sum_{j=1}^{n} v_{[j]}u_{[j]}$	$O(n^4)$
	$1\left\|MALE, CRE, \sum_{j=1}^{n} v_{[j]}u_{[j]} \leq U\right\|\sum_{j=1}^{n}\left(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma d_{opt}\right)$	$O(n^4)$
	$1\left\|MALE, CRE, \sum_{j=1}^{n} v_{[j]}u_{[j]} \leq U\right\|\sum_{j=1}^{n}\left(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma q_{opt}\right)$	$O(n^4)$

MA means "maintenance activity"; $MADE$ means "maintenance activity and aging effect"; RE means "resource allocation"; b means common deterioration rate of all jobs; a_j means aging factor of job J_j; s_j means starting time of job J_j; δ, ω, δ_1, δ_2, δ_3 and δ_4 are given constants.

This paper's contributions and novelties are as follows:

- Single-machine maintenance scheduling with convex resource constraint and learning effect is modeled and studied;

- Four algorithms are provided for the following two objective functions: (1) minimize the sum of scheduling cost (including the weighted sum of earliness, tardiness and common due date (flow allowance), where the weight is the position-dependent weight) and resource consumption cost; and (2) the resource consumption cost has an upper bound, minimizing the dispatch cost.
- It is shown that these problems can be solved in polynomial time, and the effectiveness of the algorithms is presented by numerical study.

The paper is organized as follows. Section 2 introduces the model. Section 3 describes the optimal properties. Section 4 performs an optimal analysis of the objective function and proves that it can be solved in polynomial time. In Section 5, an example is calculated, and numerical experiments are carried out to verify the effectiveness of the algorithm. Section 6 concludes this paper.

2. Problem Description

In this article, the use of symbols is listed in Table 2, and the problem can be stated as follows: there are n independent and non-preemptive jobs $J = \{J_1, J_2, \ldots, J_n\}$ to be processed on a single machine. Each job J_j is available for processing at time zero. The machine can handle one job at a time. In addition, for any sequence $\xi = (J_{[1]}, J_{[2]}, \ldots, J_{[n]})$, there is a maintenance activity whose duration time is t and no jobs are processed during this execution.

Table 2. Notations.

Notation	Meaning
n	the number of jobs
J_j	the j-th job
$J_{[j]}$	the job scheduled in the j-th position
$\theta_{[j]}$ (resp. θ_j)	the normal processing time of job $J_{[j]}$ (resp. J_j)
$p^A_{[j]}$ (p^A_j)	the actual processing time of job $J_{[j]}$ (resp. J_j)
β_j	the modifying rate of job J_j
p^A_{jr}	the actual processing time of job J_j in position r
α	the learning factor
t	the maintenance duration
l	the location of the maintenance activity
$u_{[j]}$ (resp. u_j)	the resource allocated to job $J_{[j]}$ (resp. J_j)
$C_{[j]}$ (resp. C_j)	the completion time of job $J_{[j]}$ (resp. J_j)
$S_{[j]}$ (resp. S_j)	the start time of job $J_{[j]}$ (resp. J_j)
$E_{[j]}$ (resp. E_j)	(=max$\{0, d_j - C_j\}$) the earliness of job $J_{[j]}$ (resp. J_j)
$T_{[j]}$ (resp. T_j)	(=max$\{0, C_j - d_j\}$) the tardiness of job $J_{[j]}$ (resp. J_j)
v_j (resp. $v_{[j]}$)	the cost when allocating unit resource to job $J_{[j]}$ (resp. J_j)
δ_j, ω_j	the position-dependent (but job-independent) weight (cost) of the j-th job
γ (η, U)	the given constant

In this paper, the model considered is as follows

$$P^A_{jr} = \begin{cases} \left(\frac{\theta_j (r)^\alpha}{u_j}\right)^\eta, & r \le l \\ \left(\frac{\beta_j \theta_j (r)^\alpha}{u_j}\right)^\eta, & r > l \end{cases} \tag{1}$$

where $j = 1, 2, \ldots, n$, $\eta > 0$ is a constant, $\alpha \le 0$ is the non-negative learning index, β_j is the modifying rate (it satisfies $0 < \beta_j \le 1$) and l denotes the position of the job preceding the maintenance activity (i.e., position $l+1$ is the first position after the maintenance activity). In addition, in this article, we discuss two due date assignment models, including the common (CON) due date and slack (SLK) due date assignment. For the CON, $d_j = d_{opt}$, where d_{opt} is a decision variable. For the SLK, $d_j = p^A_j + q_{opt}$, where the common flow allowance q_{opt} is a decision variable. Given that the maintenance activity time is a fixed value, our goal is to determine the optimal sequence of jobs ξ^*, optimal amount of resource allocation u^*, optimal due date (d_{opt} or q_{opt}) and maintenance location l^*. The first problem of this paper is to minimize

$$Z(u,\xi,d,l) = \sum_{j=1}^{n}(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma d_{opt}) + \sum_{j=1}^{n} v_j u_j, \tag{2}$$

using three-field representation, the first problem ($P1$) can be expressed as

$$1|MALE, CRE|\sum_{j=1}^{n}(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma d_{opt}) + \sum_{j=1}^{n} v_j u_j, \tag{3}$$

where $MALE$ means "a maintenance activity and learning effect", and CRE means "convex resource allocation".

The second problem is to minimize

$$Z(u,\xi,q,l) = \sum_{j=1}^{n}(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma q_{opt}) + \sum_{j=1}^{n} v_j u_j, \tag{4}$$

using three-field representation, and the second problem ($P2$) can be expressed as

$$1|MALE, CRE|\sum_{j=1}^{n}(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma q_{opt}) + \sum_{j=1}^{n} v_j u_j. \tag{5}$$

The third problem is to minimize $\sum\limits_{j=1}^{n}(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma d_{opt})$ under a common due date, and the cost of the resource consumption cannot exceed a ceiling, i.e., $\sum\limits_{j=1}^{n} v_j u_j \leq U$, and this problem (denoted by $P3$) is

$$1|MALE, CRE, \sum_{j=1}^{n} v_j u_j \leq U|\sum_{j=1}^{n}(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma d_{opt}), \tag{6}$$

where $U > 0$ is an upper bound on $\sum\limits_{j=1}^{n} v_j u_j$. The last problem is to consider the slack due date assignment, i.e., the fourth problem (denoted by $P4$) is

$$1|MALE, CRE, \sum_{j=1}^{n} v_j u_j \leq U|\sum_{j=1}^{n}(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma q_{opt}). \tag{7}$$

3. Main Properties

In this section, we show some main properties of the problems. Based on the above notations and allowing to perform a maintenance activity in position l, the maintenance time is a fixed constant t, and the completion time of each job j ($j = 1, 2, \ldots, n$) can be presented in the following:

$$C_{[j]} = C_{[j-1]} + \left(\frac{\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right), \quad j = 1, 2, \ldots, l,$$

$$C_{[j]} = C_{[j-1]} + t + \left(\frac{\beta_{[j]}\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right), j = l+1,$$

$$C_{[j]} = C_{[j-1]} + \left(\frac{\beta_{[j]}\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right), \quad j = l+2, \ldots, n.$$

Under the optimal schedule, the first job starts processing at time 0 and there is no idle time between jobs.

Lemma 1. *For any given sequence $\xi = \left[J_{[1]}, J_{[2]}, \ldots, J_{[n]}\right]$, the optimal value of d_{opt} is determined by the completion time of the k-th job, i.e., $d_{opt} = C_{[k]}$. The optimal value of the common flow allowance q_{opt} is decided by the start time of the k-th job, i.e., $q_{opt} = S_{[k]}$.*

Proof. Now, we prove $d_{opt} = C_{[k]}$. Consider a schedule $\xi = \left[J_{[1]}, J_{[2]}, \ldots, J_{[n]}\right]$, d_{opt} with $C_{[k]} < d_{opt} < C_{[k+1]}$ and let Z be the corresponding objective value. Define $x = d_{opt} - C_{[k]}$ and $y = C_{[k+1]} - d_{opt}$. Let Z' and Z'' be the objective values for $d_{opt} = C_{[k]}$ and $d_{opt} = C_{[k+1]}$. Then

$$Z' = Z - x\sum_{j=1}^{k+1}\left(\delta_j + w_j + \gamma\right) + x\sum_{j=k+2}^{n}\left(\delta_j + w_j + \gamma\right)$$

$$= Z + x[\sum_{j=k+2}^{n}\left(\delta_j + w_j + \gamma\right) - \sum_{j=1}^{k+1}\left(\delta_j + w_j + \gamma\right)]$$

$$Z'' = Z + y\sum_{j=1}^{k+1}\left(\delta_j + w_j + \gamma\right) - y\sum_{j=k+2}^{n}\left(\delta_j + w_j + \gamma\right)$$

$$= Z - y[\sum_{j=k+2}^{n}\left(\delta_j + w_j + \gamma\right) - \sum_{j=1}^{k+1}\left(\delta_j + w_j + \gamma\right)]$$

Thus, $Z' \leq Z$ if $\sum\limits_{j=k+2}^{n}\left(\delta_j + w_j + \gamma\right) \leq \sum\limits_{j=1}^{k+1}\left(\delta_j + w_j + \gamma\right)$ and $Z'' \leq Z$ otherwise. This implies that an optimal solution exists in which d_{opt} is equal to the completion time of some job.

The proof for $q_{opt} = S_{[k]}$ is similar to that for $d_{opt} = C_{[k]}$. □

Lemma 2. *In the optimal sequence, $d_{opt} = C_{[k]}$, $q_{opt} = S_{[k]}$, where k satisfies*

$$\left(\sum_{j=1}^{k-1}\delta_j - \sum_{j=k}^{n}\omega_j + n\gamma\right) \leq 0 \text{ and } \left(\sum_{j=1}^{k}\delta_j - \sum_{j=k+1}^{n}\omega_j + n\gamma\right) \geq 0.$$

Proof. From Lemma 1, in the CON, $d_{opt} = C_{[k]}$, through the small disturbance technique, we move $d_{opt} = C_{[k]}$.

(1) If $d_{opt} = C_{[k]}$, the total cost is

$$Z = \sum_{j=1}^{k-1}\delta_j(C_{[k]} - C_{[j]}) + \sum_{j=k+1}^{n}\omega_j(C_{[j]} - C_{[k]}) + n\gamma C_{[k]} + \sum_{j=1}^{n}v_j u_j.$$

(2) If $d_{opt} = C_{\xi(k)} - x$, the total cost is

$$Z_1 = \sum_{j=1}^{k-1}\delta_j(C_{[k]} - x - C_{[j]}) + \sum_{j=k}^{n}\omega_j(C_{[j]} + x - C_{[k]}) + n\gamma(C_{[k]} - x) + \sum_{j=1}^{n}v_j u_j.$$

(3) If $d_{opt} = C_{\xi(k)} + y$, the total cost is

$$Z_2 = \sum_{j=1}^{k}\delta_j(C_{[k]} + y - C_{[j]}) + \sum_{j=k+1}^{n}\omega_j(C_{[j]} - y - C_{[k]}) + n\gamma(C_{[k]} + y) + \sum_{j=1}^{n}v_j u_j.$$

We have

$$Z - Z_1 = \sum_{j=1}^{k-1} \delta_j x - \sum_{j=k}^{n} \omega_j x + n\gamma x \leq 0,$$

$$Z - Z_2 = -\sum_{j=1}^{k} \delta_j y + \sum_{j=k+1}^{n} \omega_j y - n\gamma y \leq 0.$$

So k satisfies both

$$(\sum_{j=1}^{k-1} \delta_j - \sum_{j=k}^{n} \omega_j + n\gamma) \leq 0 \text{ and } (\sum_{j=1}^{k} \delta_j - \sum_{j=k+1}^{n} \omega_j + n\gamma) \geq 0. \quad \square$$

Remark 1. *For a given schedule, if k satisfies both the above inequalities, the optimal common due date can be determined by Lemma 2. But k may not meet both of the above inequalities, so we need to set $d_{opt} = q_{opt} = 0$, and this term can be minimized by the HLP rule (see Hardy et al. [31]).*

4. Optimal Analysis

In this section, we perform an optimal analysis of single-machine maintenance activity scheduling problems with convex resource constraints and learning effects. In each case of the above problems in Section 2, the decision consists of four parts: optimal sequence of jobs ξ^*, optimal amount of resource allocation u^*, optimal due date (d_{opt} or q_{opt}) and maintenance location l^*.

4.1. Results of P1

In this section, we provide an optimal solution to the $P1$ problem. For the $1|MALE, CRE| \sum_{j=1}^{n} (\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma d_{opt}) + \sum_{j=1}^{n} v_j u_j$, its objective function can be expressed as follows.

If $k \leq l$, where $d_{opt} = C_{[k]} = \sum_{j=1}^{k} P_{[j]}^A$, we have

$$\begin{aligned}
Z(u, \xi, d_{opt}, l) &= n\gamma \sum_{j=1}^{k} P_{[j]}^A + \sum_{j=1}^{k-1} \delta_j (C_{[k]} - C_{[j]}) + \sum_{j=k+1}^{n} \omega_j (C_{[j]} - C_{[k]}) + \sum_{j=1}^{n} v_{[j]} u_{[j]} \\
&= \sum_{j=1}^{k} P_{[j]}^A (n\gamma + \sum_{m=1}^{j-1} \delta_m) + \sum_{j=k+1}^{l} P_{[j]}^A (\sum_{m=j}^{l} \omega_m + \sum_{m=l+1}^{n} \omega_m) \\
&+ \sum_{j=l+1}^{n} P_{[j]}^A (\sum_{m=j}^{n} \omega_m) + \sum_{j=1}^{n} v_{[j]} u_{[j]} + \sum_{j=l+1}^{n} \omega_j t \\
&= \sum_{j=1}^{l} \lambda_j \left(\frac{\theta_{[j]} (j)^\alpha}{u_{[j]}} \right)^\eta + \sum_{j=l+1}^{n} \lambda_j \left(\frac{\beta_{[j]} \theta_{[j]} (j)^\alpha}{u_{[j]}} \right)^\eta + \sum_{j=1}^{n} v_{[j]} u_{[j]} + \sum_{j=l+1}^{n} \omega_j t,
\end{aligned} \tag{8}$$

where

$$\lambda_j = \begin{cases} n\gamma + \sum_{m=1}^{j-1} \delta_m, & j = 1, 2, \ldots, k, \\ \sum_{m=j}^{l} \omega_m + \sum_{m=l+1}^{n} \omega_m, & j = k+1, k+2, \ldots, l, \\ \sum_{m=j}^{n} \omega_m, & j = l+1, l+2, \ldots, n. \end{cases} \tag{9}$$

If $k > l$, where $d_{opt} = C_{[k]} = \sum_{j=1}^{l} P_{[j]}^A + t + \sum_{j=l+1}^{k} P_{[j]}^A$, we have

$$
\begin{aligned}
Z(u,\xi,d_{opt},l) &= n\gamma(\sum_{j=1}^{l} P_{[j]}^{A} + t + \sum_{j=l+1}^{k} P_{[j]}^{A}) + \sum_{j=1}^{k} \delta_j (C_{[k]} - C_{[j]}) + \sum_{j=k+1}^{n} \omega_j (C_{[j]} - C_{[k]}) \\
&+ \sum_{j=1}^{n} v_{[j]} u_{[j]} \\
&= \sum_{j=1}^{l} P_{[j]}^{A} (n\gamma + \sum_{m=1}^{j-1} \delta_m) + \sum_{j=l+1}^{k} P_{[j]}^{A} (n\gamma + \sum_{m=1}^{l} \delta_m + \sum_{m=l+1}^{j-1} \delta_m) \\
&+ \sum_{j=k+1}^{n} P_{[j]}^{A} \sum_{m=j}^{n} \omega_m + (n\gamma + \sum_{j=1}^{l} \delta_j) t + \sum_{j=1}^{n} v_{[j]} u_{[j]} \\
&= \sum_{j=1}^{l} \psi_j \left(\frac{\theta_{[j]} (j)^{\alpha}}{u_{[j]}} \right)^{\eta} + \sum_{j=l+1}^{n} \psi_j \left(\frac{\beta_{[j]} \theta_{[j]} (j)^{\alpha}}{u_{[j]}} \right)^{\eta} + \sum_{j=1}^{n} v_{[j]} u_{[j]} \\
&+ (n\gamma + \sum_{j=1}^{l} \delta_j) t,
\end{aligned}
\tag{10}
$$

where

$$
\psi_j = \begin{cases} n\gamma + \sum\limits_{m=1}^{j-1} \delta_m, & j = 1, 2, \ldots, l, \\ n\gamma + \sum\limits_{m=1}^{l} \delta_m + \sum\limits_{m=l+1}^{j-1} \delta_m, & j = l+1, l+2, \ldots, k, \\ \sum\limits_{m=j}^{n} \omega_m, & j = k+1, k+2, \ldots, n. \end{cases}
\tag{11}
$$

Lemma 3. *For a given sequence ξ, the optimal resources $u_{[j]}^*$ allocation of P1 is as follows.*

If $k \le l$,

$$
u_{[j]}^* = \begin{cases} \left(\frac{\eta \lambda_j}{v_{[j]}} \right)^{\frac{1}{\eta+1}} \left(\theta_{[j]} (j)^{\alpha} \right)^{\frac{\eta}{\eta+1}}, & j = 1, 2, \ldots, l, \\ \left(\frac{\eta \lambda_j}{v_{[j]}} \right)^{\frac{1}{\eta+1}} (\beta_{[j]} \theta_{[j]} (j)^{\alpha})^{\frac{\eta}{\eta+1}}, j = l+1, l+2, \ldots, n. \end{cases}
\tag{12}
$$

If $k > l$,

$$
u_{[j]}^* = \begin{cases} \left(\frac{\eta \psi_j}{v_{[j]}} \right)^{\frac{1}{\eta+1}} (\theta_{[j]} (j)^{\alpha})^{\frac{\eta}{\eta+1}}, & j = 1, 2, \ldots, l, \\ \left(\frac{\eta \psi_j}{v_{[j]}} \right)^{\frac{1}{\eta+1}} (\beta_{[j]} \theta_{[j]} (j)^{\alpha})^{\frac{\eta}{\eta+1}}, j = l+1, l+1, \ldots, n. \end{cases}
\tag{13}
$$

Proof. From (8) and (10), the objective function is a convex function of $u_{[j]}$. For the case of $k \le l$, deriving (8) with respect to $u_{[j]}$, we have $\frac{\partial Z}{\partial u_{[j]}} = \frac{\partial \left[\lambda_j \left(\frac{\theta_{[j]} (j)^{\alpha}}{u_{[j]}} \right)^{\eta} + v_{[j]} u_{[j]} \right]}{\partial u_{[j]}} = v_{[j]} - \eta \lambda_j (\theta_{[j]} (j)^{\alpha})^{\eta} \left(\frac{1}{u_{[j]}} \right)^{\eta+1}, j = 1, 2, \ldots, l.$

Let $\frac{\partial Z}{\partial u_{[j]}} = 0$, we have $u_{[j]}^* = \left(\frac{\eta \lambda_j}{v_{[j]}} \right)^{\frac{1}{\eta+1}} (\theta_{[j]} (j)^{\alpha})^{\frac{\eta}{\eta+1}}$.

Similarly, if $k > l$, the result (13) can be obtained. □

Bringing optimal u^* (i.e., Equations (12) and (13)) into the objective function (i.e., Equations (8) and (10)) yields

$$
\widehat{Z}([l]) = (\eta^{\frac{-\eta}{\eta+1}} + \eta^{\frac{1}{\eta+1}}) \widehat{Z}_1([l]) + f_1(t)
\tag{14}
$$

where

$$f_1(t) = \begin{cases} \sum\limits_{j=l+1}^{n} \omega_j t, k \le l, \\ (n\gamma + \sum\limits_{j=1}^{l} \delta_j)t, k > l. \end{cases} \tag{15}$$

$$\widehat{Z}_1([l]) = \sum_{j=1}^{l} \left(v_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}} \widehat{\mu}_j^{\frac{1}{\eta+1}} + \sum_{j=l+1}^{n} \left(v_{[j]}\beta_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}} \widehat{\mu}_j^{\frac{1}{\eta+1}} \tag{16}$$

$$\widehat{\mu}_j = \begin{cases} \lambda_j, k \le l, \\ \psi_j, k > l. \end{cases} \tag{17}$$

It is obvious that $f_1(t)$ is a constant (if the location l of the maintenance activity location is deterministic), and $\widehat{Z}([l])$ is only related to the the location l so that we find the minimal value of $\widehat{Z}([l])$ is actually the minimal value of $\widehat{Z}_1([l])$. Let

$$\chi_{jr} = \begin{cases} \left(v_j\theta_j\right)^{\frac{\eta}{\eta+1}} \left(\widehat{\mu}_r(r)^{\alpha\eta}\right)^{\frac{1}{\eta+1}}, 1 \le r \le l \\ \left(v_j\beta_j\theta_j\right)^{\frac{\eta}{\eta+1}} \left(\widehat{\mu}_r(r)^{\alpha\eta}\right)^{\frac{1}{\eta+1}}, l+1 \le r \le n \end{cases} \tag{18}$$

$\widehat{Z}_1([l])$ can be minimized by solving the next assignment problem:

$$\begin{aligned} &Min\ \ \widehat{Z}_1([l]) = \sum_{j=1}^{n}\sum_{r=1}^{n} \chi_{jr} M_{jr} \\ &s.t\ \ \sum_{j=1}^{n} M_{jr} = 1 \quad r = 1,2,\ldots,n \\ &\qquad \sum_{r=1}^{n} M_{jr} = 1 \quad j = 1,2,\ldots,n \\ &\qquad M_{jr} = 0\ or\ 1 \quad 1 \le j, r \le n \end{aligned} \tag{19}$$

where M_{jr} is a 0 or 1 variable, if the job J_j is at location r, $M_{jr} = 1$, otherwise $M_{jr} = 0$.

Based on the above analysis, it is obtained that the problem $P1$ can be solved by the next Algorithm 1.

Algorithm 1: Solution for problem $P1$.

Initialization: Let $\widehat{Z} = \infty$, $\xi^* = 0$, $d^*_{opt} = 0$, $u^* = 0$ and $l^* = 0$.
Step 1. For $l = 1 \to n$
Step 2. If $k \le l$, then
obtain the minimum value $\widehat{Z}_1([l])$ and the schedule ξ by using (14)–(19);
If $\widehat{Z}_1([l]) < \widehat{Z}$, then
let $\widehat{Z} = \widehat{Z}_1([l])$, $l^* = l$, $u^* = u$, $d^*_{opt} = d_{opt}$ and $\xi^* = \xi$;
If $k > l$, then
obtain the minimum value $\widehat{Z}_1([l])$ and the schedule ξ by using (14)–(19);
If $\widehat{Z}_1([l]) < \widehat{Z}$, then
let $\widehat{Z} = \widehat{Z}_1([l])$, $l^* = l$, $u^* = u$, $d^*_{opt} = d_{opt}$ and $\xi^* = \xi$.
Step 3. Choose the minimum value $\widehat{Z}^* = \min\{\widehat{Z}_1[l], l = 1,2,\ldots,n\}$, and obtain the corresponding schedule ξ^*, d^*_{opt}, u^* and l^*.

Theorem 1. *The problem P1 can be solved by Algorithm 1 in $O(n^4)$ time.*

Proof. The correctness of Algorithm 1 follows the above analysis. Steps 1 need $O(n)$ time; for each maintenance position l, the complexity of the assignment problem is $O(n^3)$. Hence, Algorithm 1 can be solved in $O(n^4)$ time. □

4.2. Results of P2

For the $1|MALE, CRE|\sum_{j=1}^{n}(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma q_{opt}) + \sum_{j=1}^{n} v_j u_j$, its objective function (4) can be expressed as follows.

If $k-1 \le l$, where $q_{opt} = S_{[k]} = \sum_{j=1}^{k-1} P_{[j]}^A$,

$$\begin{aligned} Z(u,\xi,q_{opt},l) &= n\gamma \sum_{j=1}^{k-1} P_{[j]}^A + \sum_{j=1}^{k-1} \delta_j (C_{[k-1]} - C_{[j]}) + \sum_{j=k}^{n} \omega_j (C_{[j]} - C_{[k-1]}) + \sum_{j=1}^{n} v_{[j]} u_{[j]} \\ &= \sum_{j=1}^{k-1} P_{[j]}^A (n\gamma + \sum_{m=1}^{j-1} \delta_m) + \sum_{j=k}^{l} P_{[j]}^A (\sum_{m=l+1}^{n} \omega_m + \sum_{m=j}^{l} \omega_m) \\ &+ \sum_{j=l+1}^{n} P_{[j]}^A \sum_{m=j}^{n} \omega_m + \sum_{j=1}^{n} v_{[j]} u_{[j]} + \sum_{j=l+1}^{n} \omega_j t \\ &= \sum_{j=1}^{l} \lambda_j' \left(\frac{\theta_{[j]} (j)^\alpha}{u_{[j]}} \right)^\eta + \sum_{j=l+1}^{n} \lambda_j' \left(\frac{\beta_{[j]} \theta_{[j]} (j)^\alpha}{u_{[j]}} \right)^\eta + \sum_{j=1}^{n} v_{[j]} u_{[j]} + \sum_{j=l+1}^{n} \omega_j t, \end{aligned} \tag{20}$$

where

$$\lambda_j' = \begin{cases} n\gamma + \sum_{m=1}^{j-1} \delta_m, & j = 1,2,\ldots,k-1, \\ \sum_{m=j}^{l} \omega_m + \sum_{m=l+1}^{n} \omega_m, & j = k, k+1, \ldots, l, \\ \sum_{m=j}^{n} \omega_m, & j = l+1, l+2, \ldots, n. \end{cases} \tag{21}$$

If $k-1 > l$, where $q_{opt} = S_{[k]} = \sum_{j=1}^{l} P_{[j]}^A + t + \sum_{j=l+1}^{k-1} P_{[j]}^A$

$$\begin{aligned} Z(u,\xi,q_{opt},l) &= n\gamma (\sum_{j=1}^{l} P_{[j]}^A + t + \sum_{j=l+1}^{k-1} P_{[j]}^A) + \sum_{j=1}^{k-1} \delta_j (C_{[k-1]} - C_{[j]}) + \sum_{j=k}^{n} \omega_j (C_{[j]} - C_{[k-1]}) \\ &+ \sum_{j=1}^{n} v_{[j]} u_{[j]} \\ &= \sum_{j=1}^{l} P_{[j]}^A (n\gamma + \sum_{m=1}^{j-1} \delta_m) + \sum_{j=l+1}^{k-1} P_{[j]}^A (n\gamma + \sum_{m=1}^{l} \delta_m + \sum_{m=l+1}^{j-1} \delta_m) \\ &+ \sum_{j=k}^{n} P_{[j]}^A \sum_{m=j}^{n} \omega_m + (n\gamma + \sum_{m=1}^{l} \delta_m) t + \sum_{j=1}^{n} v_{[j]} u_{[j]} \\ &= \sum_{j=1}^{l} \psi_j' \left(\frac{\theta_{[j]} (j)^\alpha}{u_[j]} \right)^\eta + \sum_{j=l+1}^{n} \psi_j' \left(\frac{\beta_{[j]} \theta_{[j]} (j)^\alpha}{u_{[j]}} \right)^\eta \\ &+ \sum_{j=1}^{n} v_{[j]} u_{[j]} + (n\gamma + \sum_{m=1}^{l} \delta_m) t, \end{aligned} \tag{22}$$

where

$$\psi_j' = \begin{cases} n\gamma + \sum_{m=1}^{j-1}\delta_m, & j = 1,2,\ldots,l, \\ n\gamma + \sum_{m=1}^{l}\delta_m + \sum_{m=l+1}^{j-1}\delta_m, & j = l+1, l+2, \ldots, k-1, \\ \sum_{m=j}^{n}\omega_m, & j = k, k+1, \ldots, n. \end{cases} \tag{23}$$

Lemma 4. *The optimal resource $u^*_{[j]}$ allocation of the problem P2 is as follows.*

If $k-1 \le l$

$$u^*_{[j]} = \begin{cases} \left(\frac{\eta\lambda_j'}{v_{[j]}}\right)^{\frac{1}{\eta+1}} (\theta_{[j]}(j)^{\alpha})^{\frac{\eta}{\eta+1}}, j = 1,2,\ldots,l, \\ \left(\frac{\eta\lambda_j'}{v_{[j]}}\right)^{\frac{1}{\eta+1}} (\beta_{[j]}\theta_{[j]}(j)^{\alpha})^{\frac{\eta}{\eta+1}}, j = l+1, l+2,\ldots,n. \end{cases} \tag{24}$$

If $k-1 > l$

$$u^*_{[j]} = \begin{cases} \left(\frac{\eta\psi_j'}{v_{[j]}}\right)^{\frac{1}{\eta+1}} (\theta_{[j]}(j)^{\alpha})^{\frac{\eta}{\eta+1}}, j = 1,2,\ldots,l, \\ \left(\frac{\eta\psi_j'}{v_{[j]}}\right)^{\frac{1}{\eta+1}} (\beta_{[j]}\theta_{[j]}(j)^{\alpha})^{\frac{\eta}{\eta+1}}, j = l+1, l+1,\ldots,n. \end{cases} \tag{25}$$

Proof. The proof process is similar to that of Lemma 3. Taking a partial derivative of Equation (20) with respect to $u_{[j]}$ and making it equal to 0, result Equation (25) can be obtained. □

Substituting Equations (24) and (25) into Equations (20) and (22), we have

$$Z'([l]) = (\eta^{\frac{-\eta}{\eta+1}} + \eta^{\frac{1}{\eta+1}})Z_2'([l]) + f_2(t) \tag{26}$$

where

$$f_2(t) = \begin{cases} \sum_{j=l+1}^{n}\omega_j t, k-1 \le l, \\ (n\gamma + \sum_{j=1}^{l}\delta_j)t, k-1 > l. \end{cases} \tag{27}$$

$$Z_2'([l]) = \sum_{j=1}^{l}\left(v_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}} \mu_j'^{\frac{1}{\eta+1}} + \sum_{j=l+1}^{n}\left(v_{[j]}\beta_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}} \mu_j'^{\frac{1}{\eta+1}}$$

$$\mu_j' = \begin{cases} \lambda_j', k-1 \le l, \\ \psi_j', k-1 > l, \end{cases} \tag{28}$$

and it is obvious that $f_2(t)$ is a constant (if the location l of the maintenance activity location is deterministic), and $Z'([l])$ is only related to the location l, so that we find the minimal value of $Z'([l])$ is actually the minimal value of $Z_2{}'([l])$. Let

$$\chi_{jr}' = \begin{cases} \left(v_j\theta_j\right)^{\frac{\eta}{\eta+1}} \left(\mu_r'(r)^{\alpha\eta}\right)^{\frac{1}{\eta+1}}, 1 \le r \le l \\ \left(v_j\beta_j\theta_j\right)^{\frac{\eta}{\eta+1}} \left(\mu_r'(r)^{\alpha\eta}\right)^{\frac{1}{\eta+1}}, l+1 \le r \le n \end{cases} \tag{29}$$

and $Z_2{}'([l])$ can be minimized by solving the next assignment problem:

$$
\begin{aligned}
&Min \;\; {Z_2}'([l]) = \sum_{j=1}^{n}\sum_{r=1}^{n} \chi_{jr}^{'} {M_{jr}}' \\
&s.t \;\; \sum_{j=1}^{n} M_{jr}^{'} = 1 \quad r = 1, 2, \ldots, n \\
&\qquad \sum_{r=1}^{n} M_{jr}^{'} = 1 \quad j = 1, 2, \ldots, n \\
&\qquad M_{jr}^{'} = 0 \; or \; 1 \quad 1 \le j, r \le n
\end{aligned} \tag{30}
$$

where $M_{jr}^{'}$ is a 0 or 1 variable, if the job J_j is at location r, $M_{jr}^{'} = 1$, otherwise $M_{jr}^{'} = 0$.

Based on the above analysis, the problem $P2$ can be solved by the next Algorithm 2:

Algorithm 2: Solution for problem $P2$.

Initialization: Let $Z^{'} = \infty$, $\xi^* = 0$, $q_{opt}^* = 0$, $u^* = 0$ and $l^* = 0$.
Step 1. For $l = 1 \to n$
Step 2. If $k - 1 \le l$, then
  obtain the minimum value $Z_2^{'}([l])$ and the schedule ξ by using (26)–(30);
   If $Z_2^{'}([l]) < Z^{'}$, then
   let $Z^{'} = Z_2^{'}([l])$, $l^* = l$, $u^* = u$, $q_{opt}^* = q_{opt}$ and $\xi^* = \xi$;
 If $k - 1 > l$, then
  obtain the minimum value $Z_2^{'}([l])$ and the schedule ξ by using (26)–(30);
   If $Z_2^{'}([l]) < Z^{'}$, then
   let $Z^{'} = Z_2^{'}([l])$, $l^* = l$, $u^* = u$, $q_{opt}^* = q_{opt}$ and $\xi^* = \xi$.
Step 3. Choose the minimum value $Z^{'*} = \min\{Z_2^{'}[l], l = 1, 2, \ldots, n\}$, and obtain the corresponding schedule ξ^*, q_{opt}^*, u^* and l^*.

Theorem 2. *The problem $P2$ can be solved by Algorithm 2 in $O(n^4)$ time.*

4.3. Results of $P3$

In this subsection, we consider the problem $P3$, i.e.,

$$1|MALE, CRE, \sum_{j=1}^{n} v_j u_j \le U| \sum_{j=1}^{n} (\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma d_{opt}).$$

If $k \le l$, similarly, we have

$$\sum_{j=1}^{n}(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma d_{opt}) = \sum_{j=1}^{l} \kappa_j \left(\frac{\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right)^{\eta} + \sum_{j=l+1}^{n} \kappa_j \left(\frac{\beta_{[j]}\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right)^{\eta} + \sum_{j=l+1}^{n} \omega_j t. \tag{31}$$

If $k > l$, similarly, we have

$$\sum_{j=1}^{n}(\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma d_{opt}) = \sum_{j=1}^{l} \sigma_j \left(\frac{\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right)^{\eta} + \sum_{j=l+1}^{n} \sigma_j \left(\frac{\beta_{[j]}\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right)^{\eta} + (n\gamma + \sum_{j=1}^{l} \delta_j)t, \tag{32}$$

where

$$\kappa_j = \begin{cases} n\gamma + \sum_{m=1}^{j-1} \delta_m, & j = 1, 2, \ldots, k, \\ \sum_{m=l+1}^{n} \omega_m + \sum_{m=j}^{l} \omega_m, & j = k+1, k+2, \ldots, l, \\ \sum_{m=j}^{n} \omega_m, & j = l+1, l+2, \ldots, n, \end{cases} \tag{33}$$

$$\sigma_j = \begin{cases} n\gamma + \sum_{m=1}^{j-1} \delta_m, & j = 1, 2, \ldots, l, \\ n\gamma + \sum_{m=1}^{l} \delta_m + \sum_{m=l+1}^{j-1} \delta_m, & j = l+1, l+2, \ldots, k, \\ \sum_{m=j}^{n} \omega_m, & j = k+1, k+2, \ldots, n. \end{cases} \tag{34}$$

Lemma 5. *For a given sequence ξ, the corresponding optimal resources $u^*_{[j]}$ allocation for P3 is as follows.*

If $k \le l$

$$u^*_{[j]} = \frac{U\left(\frac{1}{v_{[j]}}\right)^{\frac{1}{\eta+1}} \kappa_j^{\frac{1}{\eta+1}} \left[\theta_{[j]}(j)^{\alpha} + \beta_{[j]}\theta_{[j]}(j)^{\alpha}\right]^{\frac{\eta}{\eta+1}}}{\sum_{j=1}^{l} \kappa_j^{\frac{1}{\eta+1}} \left(v_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}} + \sum_{j=l+1}^{n} \kappa_j^{\frac{1}{\eta+1}} \left(v_{[j]}\beta_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}}}, j = 1, 2, \ldots, n. \tag{35}$$

If $k > l$

$$u^*_{[j]} = \frac{U\left(\frac{1}{v_{[j]}}\right)^{\frac{1}{\eta+1}} \sigma_j^{\frac{1}{\eta+1}} \left[\theta_{[j]}(j)^{\alpha} + \beta_{[j]}\theta_{[j]}(j)^{\alpha}\right]^{\frac{\eta}{\eta+1}}}{\sum_{j=1}^{l} \sigma_j^{\frac{1}{\eta+1}} \left(v_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}} + \sum_{j=l+1}^{n} \sigma_j^{\frac{1}{\eta+1}} \left(v_{[j]}\beta_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}}}, j = 1, 2, \ldots, n. \tag{36}$$

Proof. Obviously, when $\sum_{j=1}^{n} v_j u_j = U$, $\sum_{j=1}^{n} (\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma d_{opt})$ is the smallest. Thus, this problem becomes a conditional extreme problem of minimizing $\sum_{j=1}^{n} (\delta_j E_{[j]} + \omega_j T_{[j]} + \gamma d_{opt})$ subject to $\sum_{j=1}^{n} v_j u_j = U$. For this problem, we use the Lagrange multiplier method to solve it, and the Lagrange function is

$$L(u, \pi, l, \phi) = \sum_{j=1}^{l} \kappa_j \left(\frac{\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right)^{\eta} + \sum_{j=l+1}^{n} \kappa_j \left(\frac{\beta_{[j]}\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right)^{\eta} + \sum_{j=l+1}^{n} \omega_j t + \phi\left(\sum_{j=1}^{n} v_{[j]} u_{[j]} - U\right) \tag{37}$$

where $\phi (0 \le \phi)$ is the Lagrange multiplier.

Deriving (37) with respect to f $u_{[j]}$ and ϕ, we have

$$\frac{\partial L}{\partial u} = \phi v_{[j]} - \eta\kappa_j \frac{\left(\theta_{[j]}(j)^{\alpha}\right)^{\eta}}{\left(u_{[j]}\right)^{\eta+1}} - \eta\kappa_j \frac{\left(\beta_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\eta}}{\left(u_{[j]}\right)^{\eta+1}} = 0, j = 1, 2, ..., n. \tag{38}$$

$$\frac{\partial L}{\partial \phi} = \sum_{j=1}^{n} v_{[j]} u_{[j]} - U = 0, j = 1, 2, \ldots, n. \tag{39}$$

From (38), we have

$$u^*_{[j]} = \left(\frac{\eta\kappa_j}{\phi v_{[j]}}\right)^{\frac{1}{\eta+1}} \times \left(\theta_{[j]}(j)^\alpha + \beta_{[j]}\theta_{[j]}(j)^\alpha\right)^{\frac{\eta}{\eta+1}}, j = 1, 2, \ldots, n. \quad (40)$$

From (39) and (40), we have

$$\phi^{\frac{1}{\eta+1}} = \frac{\sum_{j=1}^{n} (\eta\kappa_j)^{\frac{1}{\eta+1}} \left[(\theta_{[j]}(j)^\alpha + \beta_{[j]}\theta_{[j]}(j)^\alpha)v_{[j]}\right]^{\frac{\eta}{\eta+1}}}{U}, j = 1, 2, \ldots, n. \quad (41)$$

From (40) and (41), we have

$$u^*_{[j]} = \frac{U\left(\frac{1}{v_{[j]}}\right)^{\frac{1}{\eta+1}} \kappa_j^{\frac{1}{\eta+1}} \left[\theta_{[j]}(j)^\alpha + \beta_{[j]}\theta_{[j]}(j)^\alpha\right]^{\frac{\eta}{\eta+1}}}{\sum_{j=1}^{l} \kappa_j^{\frac{1}{\eta+1}} \left(v_{[j]}\theta_{[j]}(j)^\alpha\right)^{\frac{\eta}{\eta+1}} + \sum_{j=l+1}^{n} \kappa_j^{\frac{1}{\eta+1}} \left(v_{[j]}\beta_{[j]}\theta_{[j]}(j)^\alpha\right)^{\frac{\eta}{\eta+1}}}, j = 1, 2, \ldots, n.$$

□

Bringing optimal u^* (i.e., Equations (35) and (36)) into the objective function (i.e., Equations (31) and (32)) yields

$$\dot{Z}([l]) = U^{-\eta}\dot{Z}_3^{\eta+1}([l]) + f_1(t) \quad (42)$$

where $f_1(t)$ is given by Equation (15) (is a constant), and

$$\dot{Z}_3([l]) = \sum_{j=1}^{l} \mu_j^{\frac{1}{\eta+1}} \left(v_{[j]}\theta_{[j]}(j)^\alpha\right)^{\frac{\eta}{\eta+1}} + \sum_{j=l+1}^{n} \mu_j^{\frac{1}{\eta+1}} \left(v_{[j]}\beta_{[j]}\theta_{[j]}(j)^\alpha\right)^{\frac{\eta}{\eta+1}} \quad (43)$$

$$\mu_j = \begin{cases} \kappa_j, k \le l, \\ \sigma_j, k > l, \end{cases} \quad (44)$$

it is obvious that the minimal value of $\dot{Z}([l])$ is actually the minimal value of $\dot{Z}_3([l])$. Let

$$\dot{B}_{jr} = \begin{cases} \left(v_j\theta_j\right)^{\frac{\eta}{\eta+1}} (\mu_r(r)^{\alpha\eta})^{\frac{1}{\eta+1}}, 1 \le r \le l \\ \left(v_j\beta_j\theta_j\right)^{\frac{\eta}{\eta+1}} (\mu_r(r)^{\alpha\eta})^{\frac{1}{\eta+1}}, l+1 \le r \le n \end{cases} \quad (45)$$

$\dot{Z}_3([l])$ can be minimized by solving the next assignment problem:

$$\begin{aligned} &Min \;\; \dot{Z}_3([l]) = \sum_{j=1}^{n}\sum_{r=1}^{n} \dot{B}_{jr}\dot{N}_{jr} \\ &s.t \;\; \sum_{j=1}^{n} \dot{N}_{jr} = 1 \quad r = 1, 2, \ldots, n \\ &\sum_{r=1}^{n} \dot{N}_{jr} = 1 \quad j = 1, 2, \ldots, n \\ &\dot{N}_{jr} = 0 \; or \; 1 \quad 1 \le j, r \le n \end{aligned} \quad (46)$$

where $\dot{N}_{jr}$ is a 0 or 1 variable, and if the job J_j is at location r, $\dot{N}_{jr} = 1$; otherwise, $\dot{N}_{jr} = 0$.

Based on the above analysis, the problem $P3$ can be solved by the next Algorithm 3:

Algorithm 3: Solution for problem $P3$.

Initialization: Let $\dot{Z}=\infty$, $\xi^*=0$, $d^*_{opt}=0$, $u^*=0$ and $l^*=0$.
Step 1. For $l=1\rightarrow n$
Step 2. If $k\leq l$, then
obtain the minimum value $\dot{Z}_3([l])$ and the schedule ξ by using (43)–(46);
If $\dot{Z}_3([l])<\dot{Z}$, then
let $\dot{Z}=\dot{Z}_3([l])$, $l^*=l$, $u^*=u$, $d^*_{opt}=d_{opt}$ and $\xi^*=\xi$;
If $k>l$, then
obtain the minimum value $\dot{Z}_3([l])$ and the schedule ξ by using (43)–(46);
If $\dot{Z}_3([l])<\dot{Z}$, then
let $\dot{Z}=\dot{Z}_3([l])$, $l^*=l$, $u^*=u$, $d^*_{opt}=d_{opt}$ and $\xi^*=\xi$.
Step 3. Choose the minimum value $\dot{Z}^*=\min\{\dot{Z}_3[l], l=1,2,\ldots,n\}$, and obtain the corresponding schedule ξ^*, d^*_{opt}, u^* and l^*.

Theorem 3. *The problem P3 can be solved by Algorithm 3 in $O(n^4)$ time.*

4.4. Results of P4

Similar to Section 4.3, the $1|MALE, CRE, \sum_{j=1}^{n} v_j u_j\leq U|\sum_{j=1}^{n}(\delta_j E_{[j]}+\omega_j T_{[j]}+\gamma q_{opt})$ problem can be expressed as follows.

If $k-1\leq l$

$$\sum_{j=1}^{n}(\delta_j E_{[j]}+\omega_j T_{[j]}+\gamma q_{opt})=\sum_{j=1}^{l}\kappa_j'\left(\frac{\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right)^{\eta}+\sum_{j=l+1}^{n}\kappa_j'\left(\frac{\beta_j\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right)^{\eta}+\sum_{j=l+1}^{n}\omega_j t. \quad (47)$$

If $k-1>l$

$$\sum_{j=1}^{n}(\delta_j E_{[j]}+\omega_j T_{[j]}+\gamma q_{opt})=\sum_{j=1}^{l}\sigma_j'\left(\frac{\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right)^{\eta}+\sum_{j=l+1}^{n}\sigma_j'\left(\frac{\beta_j\theta_{[j]}(j)^{\alpha}}{u_{[j]}}\right)^{\eta}+\sum_{j=1}^{l}\delta_j t. \quad (48)$$

where

$$\kappa_j'=\begin{cases} n\gamma+\sum_{m=1}^{j-1}\delta_m, & j=1,2,\ldots,k-1,\\ \sum_{m=j}^{l}\omega_m+\sum_{m=l+1}^{n}\omega_m, & j=k,k+1,\ldots,l,\\ \sum_{m=j}^{n}\omega_m, & j=l+1,l+2,\ldots,n.\end{cases} \quad (49)$$

$$\sigma_j'=\begin{cases} n\gamma+\sum_{m=1}^{j-1}\delta_m, & j=1,2,\ldots,l,\\ n\gamma+\sum_{m=1}^{l}\delta_m+\sum_{m=l+1}^{j-1}\delta_m, & j=l+1,l+2,\ldots,k-1,\\ \sum_{m=j}^{n}\omega_m, & j=k,k+1,\ldots,n.\end{cases} \quad (50)$$

Lemma 6. *The optimal resource allocation of the problem P4 is as follows.*

If $k-1\leq l$

$$u^*_{[j]}=\frac{U\left(\frac{1}{v_{[j]}}\right)^{\frac{1}{\eta+1}}{\kappa_j'}^{\frac{1}{\eta+1}}\left[\theta_{[j]}(j)^{\alpha}+\beta_{[j]}\theta_{[j]}(j)^{\alpha}\right]^{\frac{\eta}{\eta+1}}}{\sum_{j=1}^{l}{\kappa_j'}^{\frac{1}{\eta+1}}\left(v_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}}+\sum_{j=l+1}^{n}{\kappa_j'}^{\frac{1}{\eta+1}}\left(v_{[j]}\beta_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}}}, j=1,2,\ldots,n. \quad (51)$$

If $k-1>l$

$$u^*_{[j]}=\frac{U\left(\frac{1}{v_{[j]}}\right)^{\frac{1}{\eta+1}}\sigma_j'^{\frac{1}{\eta+1}}\left[\theta_{[j]}(j)^{\alpha}+\beta_{[j]}\theta_{[j]}(j)^{\alpha}\right]^{\frac{\eta}{\eta+1}}}{\sum_{j=1}^{l}\sigma_j'^{\frac{1}{\eta+1}}\left(v_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}}+\sum_{j=l+1}^{n}\sigma_j'^{\frac{1}{\eta+1}}\left(v_{[j]}\beta_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}}}, j=1,2,\ldots,n. \tag{52}$$

Proof. Similar to the proof of Lemma 5. □

Bringing optimal u^* (i.e., Equations (51) and (52)) into the objective function (i.e., Equations (47) and (48)) yields

$$\ddot{Z}([l])=U^{-\eta}\ddot{Z}_4^{\eta+1}([l])+f_2(t) \tag{53}$$

where $f_2(t)$ is given by Equation (27) (is a constant), and

$$\ddot{Z}_4([l])=\sum_{j=1}^{l}\ddot{\mu}_j^{\frac{1}{\eta+1}}\left(v_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}}+\sum_{j=l+1}^{n}\ddot{\mu}_j^{\frac{1}{\eta+1}}\left(v_{[j]}\beta_{[j]}\theta_{[j]}(j)^{\alpha}\right)^{\frac{\eta}{\eta+1}}$$

$$\ddot{\mu}_j=\begin{cases}\kappa_j', k-1\le l,\\ \sigma_j', k-1>l.\end{cases} \tag{54}$$

Let

$$\ddot{B}_{jr}=\begin{cases}\left(v_j\theta_j\right)^{\frac{\eta}{\eta+1}}(\ddot{\mu}_r(r)^{\alpha\eta})^{\frac{1}{\eta+1}}, 1\le r\le\ l\\ \left(v_j\beta_j\theta_j\right)^{\frac{\eta}{\eta+1}}(\ddot{\mu}_r(r)^{\alpha\eta})^{\frac{1}{\eta+1}}, l+1\le r\le n,\end{cases} \tag{55}$$

and $\ddot{Z}_4([l])$ can be minimized by solving the next assignment problem:

$$\begin{aligned}&Min\ \ \ddot{Z}_4([l])=\sum_{j=1}^{n}\sum_{r=1}^{n}\ddot{B}_{jr}\ddot{N}_{jr}\\ &s.t\ \ \sum_{j=1}^{n}\ddot{N}_{jr}=1\quad r=1,2,\ldots,n\\ &\sum_{r=1}^{n}\ddot{N}_{jr}=1\quad j=1,2,\ldots,n\\ &\ddot{N}_{jr}=0\ or\ 1\quad 1\le j,r\le n\end{aligned} \tag{56}$$

where $\ddot{N}_{jr}$ is a 0 or 1 variable, and if the job J_j is at location r, $\ddot{N}_{jr}=1$; otherwise, $\ddot{N}_{jr}=0$.

The problem $P4$ can be solved by the following Algorithm 4:

Algorithm 4: Solution for problem $P4$.

Initialization: Let $\ddot{Z} = \infty$, $\xi^* = 0$, $q^*_{opt} = 0$, $u^* = 0$ and $l^* = 0$.
Step 1. For $l = 1 \to n$
Step 2. If $k - 1 \le l$, then
obtain the minimum value $\ddot{Z}_4([l])$ and the schedule ξ by using (52)–(56);
If $\ddot{Z}_4([l]) < \ddot{Z}$, then
let $\ddot{Z} = \ddot{Z}_4([l])$, $l^* = l$, $u^* = u$, $q^*_{opt} = q_{opt}$ and $\xi^* = \xi$;
If $k - 1 > l$, then
obtain the minimum value $\ddot{Z}_4([l])$ and the schedule ξ by using (52)–(56);
If $\ddot{Z}_4([l]) < \ddot{Z}$, then
let $\ddot{Z} = \ddot{Z}_4([l])$, $l^* = l$, $u^* = u$, $q^*_{opt} = q_{opt}$ and $\xi^* = \xi$.
Step 3. Choose the minimum value $\ddot{Z}^* = \min\{\ddot{Z}_4[l], l = 1, 2, \ldots, n\}$, and obtain the corresponding schedule ξ^*, q^*_{opt}, u^* and l^*.

Theorem 4. *The problem P4 can be solved by Algorithm 4 in $O(n^4)$ time.*

5. An Example and Numerical Study

5.1. An Example

Since algorithms of $P1$ and $P2$ (resp. $P3$ and $P4$) are similar, we only consider the calculation steps of $P1$ (resp. $P3$).

Let $n = 6$, $t = 3$, $\alpha = -0.2$, $\eta = 1$, $\gamma = 4$, $U = 100$, and the remaining data are given in Table 3.

Table 3. The remaining data.

J_j	J_1	J_2	J_3	J_4	J_5	J_6
θ_j	9	11	4	13	22	6
β_j	0.4	0.1	0.6	0.5	0.2	0.8
δ_j	7	2	8	5	4	10
ω_j	5	4	2	12	6	9
v_j	22	16	15	20	7	9

Solution: According to Lemma 2, we can obtain $k = 2$.
Case 1. First, we consider the problem $P1$.
When $l = 3$, from Equations (17) and (18), we can obtain

$$\chi_{jr} = \begin{cases} (v_j\theta_j)^{\frac{1}{2}}\left[(n\gamma + \sum_{m=1}^{r-1}\delta_m)(r)^{-0.2}\right]^{\frac{1}{2}} & r = 1, 2 \\ (v_j\theta_j)^{\frac{1}{2}}\left[\sum_{m=r}^{n}\omega_m(r)^{-0.2}\right]^{\frac{1}{2}} & r = 3 \\ (v_j\beta_j\theta_j)^{\frac{1}{2}}\left[\sum_{m=r}^{n}\omega_m(r)^{-0.2}\right]^{\frac{1}{2}} & r = 4, 5, 6 \end{cases} \tag{57}$$

and it is easy to see that

$$\chi_{11} = (v_1\theta_1)^{\frac{1}{2}}\left[(n\gamma)(1)^{-0.2}\right]^{\frac{1}{2}} = (22 \times 9)^{\frac{1}{2}} \times (24)^{\frac{1}{2}} = 68.93475;$$
$$\chi_{12} = (v_1\theta_1)^{\frac{1}{2}}\left[(n\gamma + \delta_1)(2)^{-0.2}\right]^{\frac{1}{2}} = (22 \times 9)^{\frac{1}{2}} \times [(24 + 7) \times 2^{-0.2}]^{\frac{1}{2}} = 73.09883;$$
$$\chi_{13} = (v_1\theta_1)^{\frac{1}{2}}\left[(\omega_3 + \omega_4 + \omega_5 + \omega_6)(3)^{-0.2}\right]^{\frac{1}{2}} = (22 \times 9)^{\frac{1}{2}} \times [(2 + 12 + 6 + 9) \times 3^{-0.2}]^{\frac{1}{2}} = 67.89213;$$
$$\chi_{14} = (v_1\beta_1\theta_1)^{\frac{1}{2}}\left[(\omega_4 + \omega_5 + \omega_6)(4)^{-0.2}\right]^{\frac{1}{2}} = (22 \times 0.4 \times 9)^{\frac{1}{2}} \times [(12 + 6 + 9) \times 4^{-0.2}]^{\frac{1}{2}} = 40.25672;$$
$$\chi_{15} = (v_1\beta_1\theta_1)^{\frac{1}{2}}\left[(\omega_5 + \omega_6)(5)^{-0.2}\right]^{\frac{1}{2}} = (22 \times 0.4 \times 9)^{\frac{1}{2}} \times [(6 + 9) \times 5^{-0.2}]^{\frac{1}{2}} = 29.34345;$$

$\chi_{16} = (v_1\beta_1\theta_1)^{\frac{1}{2}}\left[\omega_6(6)^{-0.2}\right]^{\frac{1}{2}} = (22 \times 0.4 \times 9)^{\frac{1}{2}} \times [9 \times 6^{-0.2}]^{\frac{1}{2}} = 22.31869.$
Similarly, the rest of the values of χ_{jr} are given in Table 4.

Table 4. The values of χ_{jr}.

$j \backslash r$	**1**	**2**	**3**	**4**	**5**	**6**
1	68.93475	73.09883	**67.89213**	40.25672	29.34345	22.31869
2	64.9923	68.91824	64.00931	**18.9772**	13.83263	10.52113
3	**37.94733**	40.23958	37.37339	27.14108	19.78335	15.04725
4	78.99367	83.76537	77.79892	51.57599	37.59415	**28.59419**
5	60.79474	64.46711	59.87523	25.10448	**18.29885**	13.9184
6	36	**38.17462**	35.45551	29.73156	21.67157	16.48345

The bold values are the optimal solution.

We can obtain the optimal schedule $\xi = [J_3, J_4, J_1, J_6, J_5, J_2]$ by solving the assignment problem (19), and $\widehat{Z}_1([3]) = 67.89213 + 18.9772 + 37.94733 + 28.59419 + 18.29885 + 38.17462 = 209.88434$, $u = (2.5298, 4.18827, 3.086, 3.3035, 2.61412, 0.65757)$, $d_{opt} = p_3 + p_4 = 1.58115 + 2.7004 = 4.28155$, $f_1(t) = t(\omega_4 + \omega_5 + \omega_6) = 3 \times (12 + 6 + 9) = 81$, so $\widehat{Z}([3]) = (\eta^{\frac{-\eta}{\eta+1}} + \eta^{\frac{1}{\eta+1}})\widehat{Z}_1([3]) + f_1(t) = 2 \times 209.88434 + 81 = 500.76868$.

$l = 1$: Similarly, we can easily find the values of χ_{jr} from Equations (17) and (18), and solve the assignment problem (19) to obtain the optimal schedule $\xi = [J_5, J_2, J_3, J_6, J_4, J_1]$, and $\widehat{Z}_1([1]) = 169.78529$, $u = (8.68496, 1.36212, 1.93, 3.3035, 1.8797, 1.0145)$, $d_{opt} = p_5 + t + p_2 = 2.53311 + 3 + 0.70258 = 6.23569$, $f_1(t) = t(n\gamma + \delta_1) = 3 \times (24 + 7) = 93$, so $\widehat{Z}([1]) = (\eta^{\frac{-\eta}{\eta+1}} + \eta^{\frac{1}{\eta+1}})\widehat{Z}_1([1]) + f_1(t) = 2 \times 169.78529 + 93 = 432.57051$.

$l = 2$: With the above calculation, we can obtain the optimal schedule $\xi = [J_5, J_4, J_1, J_6, J_3, J_2]$, and $\widehat{Z}_1([1]) = 179.81382$, $u = (8.68496, 4.18827, 1.95176, 3.3035, 1.31889, 0.65757)$, $d_{opt} = p_5 + p_4 = 2.53311 + 2.7004 = 5.23351$, $f_1(t) = t(\omega_3 + \omega_4 + \omega_5 + \omega_6) = 3 \times (2 + 12 + 6 + 9) = 87$, so $\widehat{Z}([2]) = (\eta^{\frac{-\eta}{\eta+1}} + \eta^{\frac{1}{\eta+1}})\widehat{Z}_1([2]) + f_1(t) = 2 \times 179.81382 + 87 = 446.62764$.

$l = 4$: As above, we can obtain the optimal schedule $\xi = [J_4, J_5, J_1, J_6, J_3, J_2]$, and $\widehat{Z}_1([4]) = 242.07549$, $u = (3.94968, 9.20959, 3.086, 3.69343, 1.31889, 0.65757)$, $d_{opt} = p_4 + p_5 = 3.29141 + 2.07827 = 5.36968$, $f_1(t) = t(\omega_5 + \omega_6) = 3 \times (6 + 9) = 45$, so $\widehat{Z}([4]) = (\eta^{\frac{-\eta}{\eta+1}} + \eta^{\frac{1}{\eta+1}})\widehat{Z}_1([4]) + f_1(t) = 2 \times 242.07549 + 45 = 529.15098$.

$l = 5$: We can obtain the optimal schedule $\xi = [J_4, J_6, J_1, J_5, J_3, J_2]$, and $\widehat{Z}_1([5]) = 263.33595$, $u = (3.94968, 3.9395, 3.086, 8.0193, 1.70268, 0.65757)$, $d_{opt} = p_4 + p_6 = 3.29141 + 1.32504 = 4.61645$, $f_1(t) = t(\omega_6) = 3 \times (9) = 27$, so $\widehat{Z}([5]) = (\eta^{\frac{-\eta}{\eta+1}} + \eta^{\frac{1}{\eta+1}})\widehat{Z}_1([5]) + f_1(t) = 2 \times 263.33595 + 27 = 553.6719$.

$l = 6$: We can obtain the optimal schedule $\xi = [J_4, J_5, J_1, J_6, J_3, J_2]$, and $\widehat{Z}_1([6]) = 283.82959$, $u = (3.94968, 9.20959, 3.086, 2.69217, 1.70268, 2.07942)$, $d_{opt} = p_4 + p_5 = 3.29141 + 2.07827 = 5.36968$, $f_1(t) = 0$, so $\widehat{Z}([6]) = (\eta^{\frac{-\eta}{\eta+1}} + \eta^{\frac{1}{\eta+1}})\widehat{Z}_1([46]) + f_1(t) = 2 \times 283.82959 = 567.65918$.

After the above calculation, the results of Case 1 are shown in Table 5.

Table 5. The results of Case 1.

l	d_{opt}	u	$\widehat{Z}([l])$	ξ
1	6.23569	(8.68496, 1.36212, 1.93, 3.3035, 1.8797, 1.0145)	432.57051	$[J_5, J_2, J_3, J_6, J_4, J_1]$
2	5.23351	(8.68496, 4.18827, 1.95176, 3.3035, 1.31889, 0.65757)	446.62764	$[J_5, J_4, J_1, J_6, J_3, J_2]$
3	4.28155	(2.5298, 4.18827, 3.086, 3.3035, 2.61412, 0.65757)	500.76868	$[J_3, J_4, J_1, J_6, J_5, J_2]$
4	5.36968	(3.94968, 9.20959, 3.086, 3.69343, 1.31889, 0.65757)	529.15098	$[J_4, J_5, J_1, J_6, J_3, J_2]$
5	4.61645	(3.94968, 3.9395, 3.086, 8.0193, 1.70268, 0.65757)	553.6719	$[J_4, J_6, J_1, J_5, J_3, J_2]$
6	5.36968	(3.94968, 9.20959, 3.086, 2.69217, 1.70268, 2.07942)	567.65918	$[J_4, J_5, J_1, J_6, J_3, J_2]$

We can see that the optimal solution for $P1$ is $l = 1$, $\xi^* = [J_5, J_2, J_3, J_6, J_4, J_1]$, $d^*_{opt} = 6.23569$ and $u^* = (8.68496, 1.36212, 1.93, 3.3035, 1.8797, 1.0145)$.

Case 2. Next, we compute the problem $P3$.

When $l = 1$, from Equations (44) and (45), we can obtain

$$\dot{B}_{jr} = \begin{cases} (v_j\theta_j)^{\frac{1}{2}}\left[(n\gamma + \sum_{m=1}^{r-1}\delta_m)(r)^{-0.2}\right]^{\frac{1}{2}} & r = 1 \\ (v_j\beta_j\theta_j)^{\frac{1}{2}}\left[(n\gamma + \sum_{m=1}^{r-1}\delta_m)(r)^{-0.2}\right]^{\frac{1}{2}} & r = 2 \\ (v_j\beta_j\theta_j)^{\frac{1}{2}}\left[\sum_{m=r}^{n}\omega_m(r)^{-0.2}\right]^{\frac{1}{2}} & r = 3,4,5,6 \end{cases} \quad (58)$$

and it is easy to see that

$\dot{B}_{11} = (v_1\theta_1)^{\frac{1}{2}}\left[(n\gamma)(1)^{-0.2}\right]^{\frac{1}{2}} = (22\times 9)^{\frac{1}{2}} \times (24) = 337.70993$;

$\dot{B}_{12} = (v_1\beta_1\theta_1)^{\frac{1}{2}}\left[(n\gamma + \delta_1)(2)^{-0.2}\right]^{\frac{1}{2}} = (22\times 0.4\times 9)^{\frac{1}{2}} \times [(24+7)\times 2^{-0.2}]^{\frac{1}{2}} = 257.407552$;

$\dot{B}_{13} = (v_1\beta_1\theta_1)^{\frac{1}{2}}\left[(\omega_3+\omega_4+\omega_5+\omega_6)(3)^{-0.2}\right]^{\frac{1}{2}} = (22\times 0.4\times 9)^{\frac{1}{2}} \times [(2+12+6+9) \times 3^{-0.2}]^{\frac{1}{2}} = 231.23228$;

$\dot{B}_{14} = (v_1\beta_1\theta_1)^{\frac{1}{2}}\left[(\omega_4+\omega_5+\omega_6)(4)^{-0.2}\right]^{\frac{1}{2}} = (22\times 0.4\times 9)^{\frac{1}{2}} \times [(12+6+9)\times 4^{-0.2}]^{\frac{1}{2}} = 209.180094$;

$\dot{B}_{15} = (v_1\beta_1\theta_1)^{\frac{1}{2}}\left[(\omega_5+\omega_6)(5)^{-0.2}\right]^{\frac{1}{2}} = (22\times 0.4\times 9)^{\frac{1}{2}} \times [(6+9)\times 5^{-0.2}]^{\frac{1}{2}} = 113.64671$;

$\dot{B}_{16} = (v_1\beta_1\theta_1)^{\frac{1}{2}}\left[\omega_6(6)^{-0.2}\right]^{\frac{1}{2}} = (22\times 0.4\times 9)^{\frac{1}{2}} \times [9\times 6^{-0.2}]^{\frac{1}{2}} = 66.95607$.

Similarly, the rest of the values of $\dot{B}_{jr}$ are given in Table 6,

Table 6. The values of $\dot{B}_{jr}$.

$j\backslash r$	1	2	3	4	5	6
1	68.93475	46.23176	42.93875	40.25672	**29.34345**	22.31869
2	64.99230	**21.79386**	20.24152	18.9772	13.83264	10.52113
3	37.94733	31.16945	**28.94930**	27.14108	19.78335	15.04726
4	78.99367	59.23106	55.01214	51.57599	37.59415	**28.59419**
5	60.79474	28.83057	26.77702	**25.10448**	18.29886	13.91815
6	**36**	34.14442	31.7124	29.73156	21.67158	16.48345

The bold values are the optimal solution.

We can obtain the optimal schedule $\xi = [J_5, J_2, J_3, J_6, J_4, J_1]$ by solving the assignment problem (46), and $\dot{Z}_3([1]) = 169.78528$, $u = (5.60353, 2.65995, 1.22568, 2.92227, 1.91124, 1.11076)$, $d_{opt} = p_5 + p_2 = 4.33963$, $f_1(t) = t(n\gamma + \delta_1) = 3\times(24+7) = 93$, so $\dot{Z}([1]) = U^{-\eta}\dot{Z}_3^{\eta+1}([1]) + f_1(t) = 100^{-1}\times 169.78528^2 + 93 = 381.27041$.

$l = 2$: Similarly, we can easily find the values of $\dot{B}_{jr}$ from Equations (44) and (45), and solve the assignment problem (46) to obtain the optimal schedule $\xi = [J_5, J_4, J_1, J_6, J_3, J_2]$, and $\dot{Z}_3([2]) = 179.81382$, $u = (5.29097, 2.85518, 2.202719, 2.75966, 1.19380, 1.20519)$, $d_{opt} = p_5 + p_4 = 8.7116$, $f_1(t) = t(\omega_3+\omega_4+\omega_5+\omega_6) = 3\times(2+12+6+9) = 87$, so $\dot{Z}([2]) = U^{-\eta}\dot{Z}_3^{\eta+1}([2]) + f_1(t) = 100^{-1}\times 179.81382^2 + 87 = 404.331$.

$l = 3$: We can obtain the optimal schedule $\xi = [J_3, J_4, J_1, J_6, J_5, J_2]$, and $\dot{Z}_3([3]) = 209.88434$, $u = (3.02238, 2.44322, 1.73675, 2.36428, 3.04078, 1.03252)$, $d_{opt} = p_3 + p_4 = 6.64431$, $f_1(t) = t(\omega_4+\omega_5+\omega_6) = 3\times(12+6+9) = 81$, so $\dot{Z}([3]) = U^{-\eta}\dot{Z}_3^{\eta+1}([3]) + f_1(t) = 100^{-1}\times 209.88434^2 + 81 = 521.54136$.

$l = 4$: As given above, we can obtain the optimal schedule $\xi = [J_4, J_5, J_1, J_6, J_3, J_2]$, and $\dot{Z}_3([4])$

$= 242.07549$, $u = (1.99828, 4.16622, 1.5058, 2.04988, 0.88676, 0.89522)$, $d_{opt} = p_4 + p_5 = 11.78616$, $f_1(t) = t(\omega_5 + \omega_6) = 3 \times (6 + 9) = 45$, so $\dot{Z}([4]) = U^{-\eta}\dot{Z}_3^{\eta+1}([4]) + f_1(t) = 100^{-1} \times 242.07549^2 + 45 = 631.00543$.

$l = 5$: We can obtain the optimal schedule $\xi = [J_4, J_6, J_1, J_5, J_3, J_2]$, and $\dot{Z}_3([5]) = 263.33595$, $u = (1.83695, 2.16033, 1.38423, 3.34065, 0.81517, 0.82294)$, $d_{opt} = p_4 + p_6 = 9.85430$, $f_1(t) = t(\omega_6) = 3 \times (9) = 27$, so $\dot{Z}([5]) = U^{-\eta}\dot{Z}_3^{\eta+1}([5]) + f_1(t) = 100^{-1} \times 263.33595^2 + 27 = 720.45823$.

$l = 6$: We can obtain the optimal schedule $\xi = [J_4, J_5, J_1, J_6, J_3, J_2]$, and $\dot{Z}_3([6]) = 283.82959$, $u = (1.70432, 3.55333, 1.2843, 1.74832, 0.75632, 0.76352)$, $d_{opt} = p_4 + p_5 = 13.81905$, $f_1(t) = 0$, so $\dot{Z}([6]) = U^{-\eta}\dot{Z}_3^{\eta+1}([6]) + f_1(t) = 100^{-1} \times 283.82959^2 = 805.59236$.

After the above calculation, the results of Case 2 are shown in Table 7.

Table 7. The results of Case 2.

l	d_{opt}	u	$\dot{Z}([l])$	ξ
1	4.33963	(5.60353, 2.65995, 1.22568, 2.92227, 1.91124, 1.11076)	381.27041	$[J_5, J_2, J_3, J_6, J_4, J_1]$
2	8.71116	(5.29097, 2.85518, 2.202719, 2.75966, 1.19380, 1.20519)	404.331	$[J_5, J_4, J_1, J_6, J_3, J_2]$
3	6.64431	(3.02238, 2.44322, 1.73675, 2.36428, 3.04078, 1.03252)	521.54136	$[J_3, J_4, J_1, J_6, J_5, J_2]$
4	11.78616	(1.99828, 4.16622, 1.5058, 2.04988, 0.88676, 0.89522)	631.00543	$[J_4, J_5, J_1, J_6, J_3, J_2]$
5	9.85430	(1.83695, 2.16033, 1.38423, 3.34065, 0.81517, 0.82294)	720.45823	$[J_4, J_6, J_1, J_5, J_3, J_2]$
6	13.81905	(1.70432, 3.55333, 1.2843, 1.74832, 0.75632, 0.76352)	805.59236	$[J_4, J_5, J_1, J_6, J_3, J_2]$

We can see that the optimal solution for $P3$ is $l = 1$, $\xi^* = [J_5, J_2, J_3, J_6, J_4, J_1]$, $d^*_{opt} = 4.33963$ and $u^* = (5.60353, 2.65995, 1.22568, 2.92227, 1.91124, 1.11076)$.

5.2. Numerical Study

To test the validity of Algorithms 1–4, we randomly generated the instances. We implemented all the algorithms in Java language on JetBrains 2023, and coded on a PC with Intel(R) Core(TM) i5-10500 CPU @ 3.10 GHz, 8.00 GB of RAM on a Windows 10 OS. The features of the examples are listed below:

(1) $n = 35, 45, 55, 65, 75, 85, 95, 105, 115, 125, 135$, $t = 10$ and $\alpha = -0.3$;
(2) $\eta = 2$, $\gamma = 12$ and $U = 500$;
(3) θ_j $(j = 1, 2, \ldots, n)$ is drawn from a discrete uniform distribution in [1, 100] (i.e., $\theta_j \sim [1, 100]$);
(4) β_j $(j = 1, 2, \ldots, n) \sim [0.5, 1]$;
(5) δ_j, ω_j $(j = 1, 2, \ldots, n) \sim [1, 40]$;
(6) v_j $(j = 1, 2, \ldots, n) \sim [1, 50]$.

For each n, 20 instances are generated randomly. The computational tests for Algorithms 1–4 are given as follows. The average (mean) and maximum (max) CPU times (milliseconds (ms)) are shown in Table 8. From Table 8, we can see that Algorithms 1–4 are effective, and their CPU times increase moderately as n increases from 35 to 135, and the maximum CPU time is 139,834.75 ms for $n = 135$.

Table 8. CPU times (ms) of algorithms.

	Algorithm 1		**Algorithm 2**		**Algorithm 3**		**Algorithm 4**	
n	**mean**	**max**	**mean**	**max**	**mean**	**max**	**mean**	**max**
35	367.09	390.12	408.67	442.16	302.86	352.29	322.18	349.82
45	896.46	925.06	1250.46	1346.82	759.28	829.51	762.24	837.24
55	1864.52	1876.59	2386.35	2587.35	1562.37	1582.65	2118.32	2297.14
65	3624.21	3703.29	4103.54	4346.54	2993.52	3121.57	3314.17	3504.02
75	6735.58	6898.34	7827.50	8786.35	5615.58	5754.39	6849.91	6928.42
85	13,683.45	13,968.32	15,072.53	16,855.50	11,394.35	11,528.39	13,864.47	14,941.57
95	23,877.03	24,120.57	26,147.35	27,845.36	21,572.70	23,489.86	24,478.50	26,116.25
105	34,453.85	34,635.58	40,527.32	42,965.50	28,712.15	28,892.27	34,785.23	37,123.85

Table 8. *Cont.*

	Algorithm 1		Algorithm 2		Algorithm 3		Algorithm 4	
n	mean	max	mean	max	mean	max	mean	max
115	53,679.54	54,008.94	61,296.86	63,085.45	44,731.62	45,021.23	52,246.21	54,004.32
125	77,623.53	77,985.72	89,614.36	95,238.62	64,685.83	65,023.21	76,372.23	81,823.32
135	114,304.61	114,892.56	136,835.45	139,834.75	95,153.30	95,802.52	117,325.23	120,115.84

6. Conclusions

In previous studies, the influence of learning effects and resource allocation factors in single-machine scheduling was considered. In this paper, we extend this setting by allowing the execution of a maintenance activity. It is revealed that these four problems can be solved in $O(n^4)$ time (see Table 1). In future work, we will incorporate more realistic settings, such as multiple maintenance activities, or multiple machine (e.g., flow shop) conditions.

Author Contributions: Investigation, L.-Y.W.; Writing—original draft, Z.-J.W.; Writing—review and editing, Z.-J.W., L.-Y.W., L.Z., J.-B.W. and E.W. All authors have read and agreed to the published version of the manuscript.

Funding: This Work was supported by the Science Research Foundation of Educational Department of Liaoning Province (LJKMZ20220527). Ershen Wang was also supported by the SongShan Laboratory Foundation (YYJC062022017), the Open Fund of Key Laboratory of Flight Techniques and Flight Safety CAAC (FZ2021KF15, FZ2021ZZ06), the Applied Basic Research Programs of Liaoning Province (2022020502-JH2/1013), and the Special Funds program of Shenyang Science and technology (22-322-3-34).

Data Availability Statement: The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest: The authors declare no conflict of interest.

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