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On a General Formulation of the Riemann–Liouville Fractional Operator and Related Inequalities

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Abstract: In this paper, we present a general formulation of the Riemann–Liouville fractional operator with generalized kernels. Many of the known operators are shown to be particular cases of the one we present. In this new framework, we prove several known integral inequalities in the literature.

Keywords: generalized fractional Riemann–Liouville integral; fractional integral inequality; synchronous functions

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1. Introduction

Although the topic of inequalities has been dealt with since the Greeks, the consolidation of this discipline as a theoretical corpus is attributed to the classic text by Hardy, Littlewood, and Pólya [1], which was influential in increasing research on the study of different types of inequalities (such as the inequalities of Jensen, Gruss, Hermite–Hadamard, Fejer, which generalizes this list, Minkowski, and Polya–Szego, among others). Interested readers may refer to [2–6].

In the last 50 years, this area has become an object of attention for researchers from various disciplines (pure and applied) since the advent of what is now recognized as fractional calculus.

Throughout this work, we use the functions Γ (see [7–10]) and Γ_k (cf. defined by [11]):

$$\Gamma(z) = \int_0^\infty \tau^{z-1} e^{-\tau}\, d\tau,\ \mathrm{Re}(z) > 0, \tag{1}$$

$$\Gamma_k(z) = \int_0^\infty \tau^{z-1} e^{-\tau^k/k}\, d\tau,\ k > 0. \tag{2}$$

Evidently, as k approaches 1, we observe the convergence of $\Gamma_k(z)$ to $\Gamma(z)$, where $\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$. Additionally, we establish the k-beta function through the subsequent definition:

$$B_k(u,v) = \frac{1}{k}\int_0^1 \tau^{\frac{u}{k}-1}(1-\tau)^{\frac{v}{k}-1} d\tau,$$

noting that $B_k(u,v) = \frac{1}{k}B(\frac{u}{k},\frac{v}{k})$ and $B_k(u,v) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}$.

For the purpose of enhancing comprehension of the subject, we provide several definitions of fractional integrals, including some that are very recent (with $0 \le a_1 < \tau < a_2 \le \infty$). The first is the classic Riemann–Liouville fractional integrals.

Definition 1. *Let $f \in L_1[a_1, a_2]$; subsequently, the Riemann–Liouville fractional integrals of order $\alpha \in \mathbb{C}$, with $\mathrm{Re}(\alpha) > 0$, are given explicit definitions for the right and left cases as follows:*

$$\text{For } u > a_1: \quad I^{\alpha}_{a_1+} f(u) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{u} (u-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{3}$$

$$\text{For } u < a_2: \quad I^{\alpha}_{a_2-} f(u) = \frac{1}{\Gamma(\alpha)} \int_{u}^{a_2} (\tau-u)^{\alpha-1} f(\tau)\, d\tau. \tag{4}$$

Furthermore, the fractional derivatives, of any order n, corresponding to the aforementioned operators can be defined as follows:

Definition 2. *Given $f(u)$ belonging to $L_1(a_1, a_2)$, where α is a complex number with $\mathrm{Re}(\alpha) > 0$, and satisfying $n-1 < \alpha < n$, the left- and right-sided Riemann–Liouville fractional derivatives are characterized by*

$$(D^{\alpha}_{a_1^+} f)(u) = \frac{d^n}{du^n}\left(I^{n-\alpha}_{a_1^+} f\right)(u), \tag{5}$$

$$(D^{\alpha}_{a_2^-} f)(u) = (-1)^n \frac{d^n}{du^n}\left(I^{n-\alpha}_{a_2^-} f\right)(u). \tag{6}$$

The left- and right-sided Riemann–Liouville k-fractional integrals are given in [12].

Definition 3. *Now, suppose we have a function f belonging to the space $L_1[a_1, a_2]$. The expressions that define the Riemann–Liouville k-fractional integrals of order $\alpha \in \mathbb{C}$, where the real part of α is greater than 0 and $k > 0$ is a positive value, can be stated as follows:*

$${}^{\alpha}I^{k}_{a_1+} f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^{u} (u-\tau)^{\frac{\alpha}{k}-1} f(\tau)\, d\tau, \quad u > a_1, \tag{7}$$

$${}^{\alpha}I^{k}_{a_2-} f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{u}^{a_2} (\tau-u)^{\frac{\alpha}{k}-1} f(\tau)\, d\tau, \quad u < a_2. \tag{8}$$

A more general definition of the Riemann–Liouville fractional integrals is given in [13].

Definition 4. *Consider a function $f : [a_1, a_2] \to \mathbb{R}$ that is integrable. Additionally, let us suppose we have a function g defined on the interval $(a_1, a_2]$ which is both increasing and positive. This function g should have a continuous derivative g' within the interval (a_1, a_2). Now, turning our attention to the fractional integrals of function f with respect to another function g over the interval $[a_1, a_2]$ of a given order $\alpha \in \mathbb{C}$, where the real part of α is greater than 0, we can represent these integrals as follows:*

$${}^{\alpha}_{g}I_{a_1^+} f(u) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{u} (g(u)-g(\tau))^{\alpha-1} g'(\tau) f(\tau)\, d\tau, \quad u > a_1, \tag{9}$$

$${}^{\alpha}_{g}I_{a_2^-} f(u) = \frac{1}{\Gamma(\alpha)} \int_{u}^{a_2} (g(\tau)-g(u))^{\alpha-1} g'(\tau) f(\tau)\, d\tau, \quad u < a_2. \tag{10}$$

A k-fractional analogue of Definition 4 is given in the following (see [14–16]):

Definition 5. *Let $f : [a_1, a_2] \to \mathbb{R}$ be an integrable function. Let g be an increasing and positive function on $(a_1, a_2]$ with a continuous derivative g' on (a_1, a_2). The left- and right-sided k-fractional integrals of a function f with respect to another function g on $[a_1, a_2]$ of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ and $k > 0$ are expressed by*

$${}^{\alpha}I^{k}_{a_1^+} f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^{u} (g(u)-g(\tau))^{\frac{\alpha}{k}-1} g'(\tau) f(\tau)\, d\tau, \quad u > a_1, \tag{11}$$

$${}^{\alpha}I^{k}_{a_2^-} f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{u}^{a_2} (g(\tau)-g(u))^{\frac{\alpha}{k}-1} g'(\tau) f(\tau)\, d\tau, \quad u < a_2. \tag{12}$$

On the other hand, in 2011, Katugampola defined a new integral operator as a generalization of the n-integral, as follows:

Definition 6. *Let $f : [a_1, a_2] \to \mathbb{R}$ be an integrable function. The general Katugampola fractional integrals of a function f of order $\alpha \in \mathbb{R}$, with $\mathrm{Re}(\alpha) > 0$ and $s \neq -1$, is expressed by*

$$ {}^{s}_{a_1}I^{\alpha}_{u} f(u) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_{a_1}^{u} (u^{\alpha+1} - \tau^{\alpha+1})^{\alpha-1} \tau^{s} f(\tau)\, d\tau. \tag{13} $$

We are now in a position to define the k-generalized fractional Riemann–Liouville integral.

Definition 7. *The k-generalized fractional Riemann–Liouville integral of order α with $\mathrm{Re}(\alpha) > 0$ and the $s \neq -1$ of an integrable and non-negative function $f(u)$ on $[0, \infty)$ are given as follows:*

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1} f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^{u} \frac{F(\tau,s) f(\tau) d\tau}{[\mathbb{F}(u,\tau)]^{1-\frac{\alpha}{k}}}, \tag{14} $$

with $F(\tau, s)$, an integrable and non-negative function on $[0, +\infty)$, and $F(\tau, 0) = 1$, with $\mathbb{F}(u, \tau) = \int_{\tau}^{u} F(\theta, s) d\theta$ and $\mathbb{F}(\tau, u) = \int_{u}^{\tau} F(\theta, s) d\theta$.

In the subsequent remark, we will establish relationships between our generalized operator and some of the operators introduced in the earlier definitions.

Remark 1. *Let us consider the kernel $F(\tau, s) = \tau^s$; then, we will have, successively,*

$$ \mathbb{F}(u,\tau) = \int_{\tau}^{u} \theta^{s} d\theta = \frac{u^{s+1} - \tau^{s+1}}{s+1}, \tag{15} $$

$$ (\mathbb{F}(u,\tau))^{1-\frac{\alpha}{k}} = \left[\frac{u^{s+1} - \tau^{s+1}}{s+1}\right]^{1-\frac{\alpha}{k}}. \tag{16} $$

The (k,s)-Riemann–Liouville fractional integral is defined in Definition 2.1 of [17], and from here, we have the integral of Definition 6 with $k \equiv 1$.

Similarly, when we set $s \equiv 0$ and $k \equiv 1$, we arrive at the well-known Riemann–Liouville operator in its traditional form.

Remark 2. *Following Definition 2, it is not difficult to define the generalized derivative of the Riemann–Liouville type, following this formalism. So, we have*

$$ \left({}^{s}D^{\frac{\alpha}{k}}_{F,a_1} f\right)(u) = \frac{d^n}{du^n} \left({}^{s}J^{n-\frac{\alpha}{k}}_{F,a_1} f\right)(u), \tag{17} $$

with $n - 1 < \alpha < n$.

One of the fundamental inquiries regarding a new integral operator is its boundedness.

Theorem 1. *Let $f : [a_1, a_2] \to \mathbb{R}$ be a continuous function, with $\alpha, k > 0$ and $s \neq -1$. Then, ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1} f(u)$ exists for all $u \in [a_1, a_2]$.*

Proof. For all $f \in C[a_1, a_2]$, and $u \in [a_1, a_2]$, we have

$$\begin{aligned} |{}^{s}J^{\frac{\alpha}{k}}_{F,a_1} f(u)| &\leq \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^{u} F(\tau,s)[\mathbb{F}(u,\tau)]^{\frac{\alpha}{k}-1}|f(\tau)|d\tau \\ &\leq \frac{\|f\|}{k\Gamma_k(\alpha)} \int_{a_1}^{u} F(\tau,s)[\mathbb{F}(u,\tau)]^{\frac{\alpha}{k}-1}d\tau \\ &= \frac{\|f\|}{\Gamma_k(\alpha+k)}[\mathbb{F}(u,a_1)]^{\frac{\alpha}{k}}. \end{aligned}$$

□

Subsequently, we derive a targeted characteristic of the aforementioned integral operator: the commutativity and the semigroup property of the operator, as presented in Definition 7. We have the following:

Theorem 2. *Let f be a continuous function on $[a_1, a_2]$, $k > 0$ and $s \neq -1$. Then, we have*

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\left({}^{s}J^{\frac{\beta}{k}}_{F,a_1} f(u)\right) = {}^{s}J^{\frac{\alpha+\beta}{k}}_{F,a_1} f(u) = {}^{s}J^{\frac{\beta}{k}}_{F,a_1}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1} f(u)\right), \tag{18}$$

for all $\alpha > 0$, $\beta > 0$, $u \in [a_1, a_2]$.

Proof. Taking into account Definition 7 and the Dirichlet formula, we have

$$\begin{aligned} &{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\left({}^{s}J^{\frac{\beta}{k}}_{F,a_1} f(u)\right) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^{u} [\mathbb{F}(u,\tau)]^{\frac{\alpha}{k}-1}F(\tau,s)\left({}^{s}J^{\frac{\beta}{k}}_{F,a_1} f(\tau)\right)d\tau \\ &= \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^{u} [\mathbb{F}(u,\tau)]^{\frac{\alpha}{k}-1}F(\tau,s)\left(\frac{1}{k\Gamma_k(\beta)} \int_{a_1}^{\tau} [\mathbb{F}(\tau,w)]^{\frac{\beta}{k}-1}F(w,s)f(w)dw\right)d\tau \\ &= \frac{1}{k\Gamma_k(\alpha)} \frac{1}{k\Gamma_k(\beta)} \int_{a_1}^{u} F(w,s)f(w)\left(\int_{w}^{u} [\mathbb{F}(u,\tau)]^{\frac{\alpha}{k}-1}[\mathbb{F}(\tau,w)]^{\frac{\beta}{k}-1}F(\tau,s)d\tau\right)dw. \end{aligned}$$

Making $\rho = \frac{\mathbb{F}(\tau,w)}{\mathbb{F}(u,w)}$, we have

$$\begin{aligned} &\frac{1}{k\Gamma_k(\alpha)} \frac{1}{k\Gamma_k(\beta)} \int_{a_1}^{u} F(w,s)f(w)\left(\int_{w}^{u} [\mathbb{F}(u,\tau)]^{\frac{\alpha}{k}-1}[\mathbb{F}(\tau,w)]^{\frac{\beta}{k}-1}F(\tau,s)d\tau\right)dw \\ &= \frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)} \int_{a_1}^{u} F(w,s)[\mathbb{F}(u,w)]^{\frac{\alpha+\beta}{k}-1}f(w)\left(\int_{0}^{1} (1-\rho)^{\frac{\alpha}{k}-1}\rho^{\frac{\beta}{k}-1}d\rho\right)dw \\ &= \frac{kB_k(\alpha,\beta)}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)} \int_{a_1}^{u} F(w,s)[\mathbb{F}(u,w)]^{\frac{\alpha+\beta}{k}-1}f(w)dw \\ &= \frac{1}{k\Gamma_k(\alpha+\beta)} \int_{a_1}^{u} F(w,s)[\mathbb{F}(u,w)]^{\frac{\alpha+\beta}{k}-1}f(w)dw \\ &= {}^{s}J^{\frac{\alpha+\beta}{k}}_{F,a_1} f(u). \end{aligned}$$

The second part of equality of (18) is trivial. This completes the proof. □

Example 1. *If we take $a_1 = 0$ and $F \equiv 1$, then we have $\mathbb{F}(u,\tau) = u - \tau$, so we have the k-Riemann–Liouville fractional integral of order α [12]. In terms of this operator, we can formulate equality (18):*

Theorem 3. *Let $f \in Q(I), a_1, a_2 \in I$ with $1 < a < b$ and $f \in L_1[a_1, a_2]$. Then, for $\in (0,1]$, the following equality for the k-Riemann–Liouville fractional integral of order α holds:*

$$I_k^{\alpha}\left(I_k^{\beta} f\right)(u) = I_k^{\alpha+\beta} f(u) = I_k^{\beta}(I_k^{\alpha} f)(u) \tag{19}$$

This is the theorem presented in [18], p. 2.

If we take $k = 1$, $a_1 = 1$, and $F(\tau, s) = \tau^{-1}$, the above is still valid for the Hadamark fractional integral (see [19,20]).

Theorem 4. *Let $\alpha, \beta > 0$, $k > 0$ and $s \neq -1$. Then, we obtain*

$$^{s}J_{F,a_1}^{\frac{\alpha}{k}}[\mathbb{F}(u,a_1)^{\frac{\beta}{k}-1}] = \frac{\Gamma_k(\beta)\mathbb{F}(u,a_1)^{\frac{\alpha+\beta}{k}-1}}{\Gamma_k(\alpha+\beta)}. \tag{20}$$

Proof. Here,

$$^{s}J_{F,a_1}^{\frac{\alpha}{k}}[\mathbb{F}(u,a_1)^{\frac{\beta}{k}-1}] = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^{u} F(\tau,s)\mathbb{F}(u,\tau)^{\frac{\alpha}{k}-1}\mathbb{F}(\tau,a_1)^{\frac{\beta}{k}-1} d\tau.$$

Using the change in variable $\rho = \frac{\mathbb{F}(\tau,a_1)}{\mathbb{F}(u,a_1)}$, we obtain

$$\begin{aligned}
&= \frac{\mathbb{F}(u,a_1)^{\frac{\alpha+\beta}{k}-1}}{k\Gamma_k(\alpha)} \int_0^1 (1-\rho)^{\frac{\alpha}{k}-1}\rho^{\frac{\beta}{k}-1} d\rho \\
&= \frac{\mathbb{F}(u,a_1)^{\frac{\alpha+\beta}{k}-1} B_k(\alpha,\beta)}{\Gamma_k(\alpha)} \\
&= \frac{\Gamma_k(\beta)\mathbb{F}(u,a_1)^{\frac{\alpha+\beta}{k}-1}}{\Gamma_k(\alpha+\beta)}.
\end{aligned}$$

Thus, the proof is complete. □

An integral inequality that holds significant recognition is Chebyshev's inequality. When applied to the k-generalized Riemann–Liouville fractional integral operator defined in Definition 7, Chebyshev's inequality can be expressed as follows:

Theorem 5. *Suppose we have two synchronous functions, denoted as f and g, over the interval $[0,\infty)$. Under these conditions, for any values of τ, a_1, α, and λ satisfying $\tau > a_1 \geq 0$, $\alpha > 0$, and $\lambda > 0$, the ensuing set of inequalities holds:*

$$^{s}J_{F,a_1}^{\frac{\alpha}{k}}(fg)(\tau) \geq \frac{1}{^{s}J_{F,a_1}^{\frac{\alpha}{k}}(1)}\, ^{s}J_{F,a_1}^{\frac{\alpha}{k}} f(\tau)\, ^{s}J_{F,a_1}^{\frac{\alpha}{k}} g(\tau), \tag{21}$$

and

$$^{s}J_{F,a_1}^{\frac{\alpha}{k}}(fg)(\tau)\, ^{s}J_{F,a_1}^{\frac{\lambda}{k}}(1) + {}^{s}J_{F,a_1}^{\frac{\lambda}{k}}(fg)(\tau)\, ^{s}J_{F,a_1}^{\frac{\alpha}{k}}(1) \geq {}^{s}J_{F,a_1}^{\frac{\alpha}{k}} f(\tau)\, ^{s}J_{F,a_1}^{\frac{\lambda}{k}} g(\tau) + {}^{s}J_{F,a_1}^{\frac{\alpha}{k}} g(\tau)\, ^{s}J_{F,a_1}^{\frac{\lambda}{k}} f(\tau). \tag{22}$$

Proof. We say that the pair f and g are synchronous on $[0,\infty)$, if for all $u, v \geq 0$, we have

$$(f(u) - f(v))(g(u) - g(v)) \geq 0. \tag{23}$$

Therefore,

$$f(u)g(u) + f(v)g(v) \geq f(u)g(v) + f(v)g(u). \tag{24}$$

Then, multiplying both sides of (24) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, we integrate the resulting inequality with respect to u over (a_1,τ). It holds that

$$\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{f(u)g(u)F(u,s)du}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}+\frac{f(v)g(v)}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(u,s)du}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}\geq\frac{g(v)}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{f(u)F(u,s)du}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}+\frac{f(v)}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{g(u)F(u,s)du}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}},$$

and thus,

$${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau)+f(v)g(v)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(1)\geq g(v)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)+f(v)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g(\tau). \tag{25}$$

Multiplying both sides of (25) by $\frac{F(v,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,v)]^{1-\frac{\alpha}{k}}}$, and then integrating the resulting inequality with respect to v over (a_1,τ), we obtain

$${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau)\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(v,s)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\alpha}{k}}}+{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(1)\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(v,s)f(v)g(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\alpha}{k}}}\geq{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(v,s)g(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\alpha}{k}}}+{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g(\tau)\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(v,s)f(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\alpha}{k}}},$$

that is,

$$2[\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(1)]\geq 2[\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g(\tau)].$$

Thus, we obtain the first inequality. Now, multiplying both sides of (25) by $\frac{F(v,s)}{k\Gamma_k(\lambda)[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}}$, we integrate the resulting inequality with respect to v over (a_1,τ). It holds that

$${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau)\frac{1}{k\Gamma_k(\lambda)}\int_{a_1}^{\tau}\frac{F(v,s)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}}+{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(1)\frac{1}{k\Gamma_k(\lambda)}\int_{a_1}^{\tau}\frac{F(v,s)f(v)g(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}}\geq{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\frac{1}{k\Gamma_k(\lambda)}\int_{a_1}^{\tau}\frac{F(v,s)g(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}}+{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g(\tau)\frac{1}{k\Gamma_k(\lambda)}\int_{a_1}^{\tau}\frac{F(v,s)f(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}},$$

that is,

$${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(1)+{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(fg)(\tau)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(1)\geq{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}g(\tau)+{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}f(\tau).$$

This completes the proof. □

Remark 3. *If in Theorem 5 we take the kernel $F(u,s)=u^s$, we obtain Theorem 3.1 of [17]. In the case that $F(u,s)=\frac{1}{u}$ and $k=1$ from the previous result, Theorem 3.1 of [21] is obtained.*

The preceding outcome can be expanded by taking into account a specific positive "weight" function, denoted as h.

Theorem 6. *Considering two functions, f and g, synchronous over the interval $[0,\infty)$, with h being non-negative, we derive the subsequent inequality for all $\tau>a_1\geq 0,\alpha>0,\lambda>0$:*

$$
\begin{aligned}
&{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fgh)(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(1) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(1)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(fgh)(\tau) \\
&\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fh)(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}g(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(gh)(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}f(\tau) \\
&- {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(fg)(\tau) - {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau) \\
&{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}h(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(gh)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(fh)(\tau).
\end{aligned}
$$

Proof. Given that $h \geq 0$ and the functions f and g are synchronous over the interval $[0, \infty)$, we can deduce the following inequality for all $u, v \geq 0$:

$$(h(v) + h(u))(f(v) - f(u))(g(v) - g(u)) \geq 0.$$

Consequently, we have

$$
\begin{aligned}
f(u)g(u)h(u) + f(v)g(v)h(v) \geq{}& f(u)g(v)h(u) + f(v)g(u)h(u) - f(v)g(v)h(u) \\
&- f(u)g(u)h(v) + f(u)g(v)h(v) + f(v)g(u)h(v).
\end{aligned} \tag{26}
$$

Multiplying both sides of (26) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, we integrate the resulting inequality with respect to u over (a_1, τ). We have

$$
\begin{aligned}
&\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(u,s)f(u)g(u)h(u)du}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}} + \frac{f(v)g(v)h(v)}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(u,s)du}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}} \\
&\geq \frac{g(v)}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(u,s)f(u)h(u)du}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}} + \frac{f(v)}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(u,s)g(u)h(u)du}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}} \\
&- \frac{f(v)g(v)}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(u,s)h(u)du}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}} - \frac{h(v)}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(u,s)f(u)g(u)du}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}} \\
&+ \frac{g(v)h(v)}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(u,s)f(u)du}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}} + \frac{f(v)h(v)}{k\Gamma_k(\alpha)}\int_{a_1}^{\tau}\frac{F(u,s)g(u)du}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}.
\end{aligned}
$$

that is,

$$
\begin{aligned}
&{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fgh)(\tau) + f(v)g(v)h(v)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(1) \geq g(v)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fh)(\tau) + f(v)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(gh)(\tau) \\
&- f(v)g(v)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}h(\tau) - h(v)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau) + g(v)h(v)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau) + f(v)h(v)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g(\tau).
\end{aligned} \tag{27}
$$

Now, multiplying both sides of (27) by $\frac{F(v,s)}{k\Gamma_k(\lambda)[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}}$, we integrate the resulting inequality with respect to v over (a_1, τ). We obtain

$$
\begin{aligned}
&{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fgh)(\tau)\frac{1}{k\Gamma_k(\lambda)}\int_{a_1}^{\tau}\frac{F(v,s)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}} \\
&+ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(1)\frac{1}{k\Gamma_k(\lambda)}\int_{a_1}^{\tau}\frac{F(v,s)f(v)g(v)h(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}} \geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fh)(\tau)\frac{1}{k\Gamma_k(\lambda)}\int_{a_1}^{\tau}\frac{F(v,s)g(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}} \\
&+ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(gh)(\tau)\frac{1}{k\Gamma_k(\lambda)}\int_{a_1}^{\tau}\frac{F(v,s)f(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}} - {}^{s}J^{\frac{\alpha}{k}}_{F,a}h(\tau)\frac{1}{k\Gamma_k(\lambda)}\int_{a_1}^{\tau}\frac{F(v,s)f(v)g(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}} \\
&- {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau)\frac{1}{k\Gamma_k(\lambda)}\int_{a_1}^{\tau}\frac{F(v,s)h(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}} + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\frac{1}{k\Gamma_k(\lambda)}\int_{a_1}^{\tau}\frac{F(v,s)g(v)h(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}} \\
&+ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g(\tau)\frac{1}{k\Gamma_k(\lambda)}\int_{a_1}^{\tau}\frac{F(v,s)f(v)h(v)dv}{[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fgh)(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(1) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(1)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(fgh)(\tau) \\
&\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fh)(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}g(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(gh)(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}f(\tau) \\
&- {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(fg)(\tau) - {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau) \\
&{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}h(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(gh)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(fh)(\tau).
\end{aligned}
$$

□

Remark 4. *If in the previous theorem we take the kernel $F(u,s) = u^s$, this is reduced to Theorem 3.2 of [17]. Analogously with $F(u,s) = \frac{1}{u}$ and $k = 1$ in this result, we obtain Theorem 3.2 of [21].*

If in Theorem 6 we consider $\alpha = \lambda$, then we have the following results.

Corollary 1. *Let f and g be two synchronous functions on $[0,\infty)$, $h \geq 0$. Then, for all $\tau > a_1 \geq 0$, $\alpha > 0$, we obtain the following inequality:*

$$
\begin{aligned}
&{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fgh)(\tau) \geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fh)(\tau)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(gh)(\tau)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau) \\
&- {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau).
\end{aligned}
$$

Remark 5. *If in the previous result $F(u,s) = u^s$, we have Corollary 3.3 of [17].*

More refined outcomes can be achieved by introducing supplementary constraints on the function h in the preceding theorem.

Theorem 7. *Consider three monotonically increasing functions, denoted as f, g, and h, defined over the interval $[0,\infty)$. These functions fulfill the subsequent inequality for all $u, v \in [a_1,\tau]$:*

$$(f(v) - f(u))(g(v) - g(u))(h(v) - h(u)) \geq 0.$$

Thus, for all $\tau > a_1 \geq 0$, $\alpha > 0$, $\lambda > 0$, we have that

$$
\begin{aligned}
&{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fgh)(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(1) - {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(1)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(fgh)(\tau) \\
&\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fh)(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}g(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(gh)(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}f(\tau) \\
&- {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(fg)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}h(\tau) \\
&- {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(gh)(\tau) - {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(fh)(\tau).
\end{aligned}
$$

Proof. We use the same arguments as in the proof of Theorem 6. □

An inequality concerning the squares of functions f and g can be expressed in the following manner.

Theorem 8. *Consider functions f and g defined over the interval $[0,\infty)$. For any values of $\tau > a_1 \geq 0$, $\alpha > 0$, and $\lambda > 0$, the ensuing set of inequalities in terms of integrals holds:*

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f^2(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(1) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(1)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}g^2(\tau) \geq 2\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}g(\tau). \tag{28}$$

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f^2(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}g^2(\tau) + {}^{s}J^{\frac{\lambda}{k}}_{F,a_1}f^2(\tau)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g^2(\tau) \geq 2\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau)\,{}^{s}J^{\frac{\lambda}{k}}_{F,a_1}(fg)(\tau) \tag{29}$$

Proof. Since $(f(u)-g(v))^2 \geq 0$, we have

$$f^2(u)+g^2(v) \geq 2f(u)g(v). \tag{30}$$

Now, if we multiply both sides of inequality (30) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$ and $\frac{F(v,s)}{k\Gamma_k(\lambda)[\mathbb{F}(\tau,v)]^{1-\frac{\lambda}{k}}}$, subsequently integrating the result obtained in terms of u and v over the interval (a_1,τ), respectively, we arrive at expression (28).

Furthermore, since $(f(u)g(v)-f(v)g(u))^2 \geq 0$, consequently, employing the identical reasoning as previously, we obtain (29). □

Remark 6. *Theorem 3.5 of [17] is obtained from the previous result, making $F(u,s)=u^s$.*

If we consider $\alpha=\lambda$, we obtain the following consequence.

Corollary 2. *Let f and g be on $[0,\infty)$; then, for all $\tau > a_1 \geq 0$, $\alpha > 0$, we obtain*

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(1)[\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f^2(\tau)+\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g^2(\tau)] \geq 2\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g(\tau), \tag{31}$$

and

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f^2(\tau)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}g^2(\tau) \geq [\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(fg)(\tau)]^2. \tag{32}$$

A result in a different direction is that which shows the following result.

Theorem 9. *Let $f:\mathbb{R}\to\mathbb{R}$ with $\bar{f}(u)=\int_{a_1}^{u}F(\tau,s)f(\tau)d\tau$, $u > a_1 \geq 0$, $s \neq -1$. Then, for $\alpha \geq k > 0$ we have*

$$ {}^{s}J^{\frac{\alpha+k}{k}}_{F,a_1}f(u)=\frac{1}{k}\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\bar{f}(u).$$

Proof. Here,

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\bar{f}(u)=\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{u}F(\tau,s)\mathbb{F}(u,\tau)^{\frac{\alpha}{k}-1}\int_{a_1}^{\tau}F(z,s)f(z)dzd\tau.$$

Then, by the Dirichlet formula, we see that the last expression becomes

$$\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{u}F(z,s)f(z)\int_{z}^{u}F(\tau,s)\mathbb{F}(u,\tau)^{\frac{\alpha}{k}-1}d\tau dz \tag{33}$$

$$=\frac{1}{\alpha\Gamma_k(\alpha)}\int_{a_1}^{u}F(z,s)f(z)\mathbb{F}(u,z)^{\frac{\alpha}{k}}dz \tag{34}$$

$$=\frac{1}{\Gamma_k(\alpha+k)}\int_{a_1}^{u}F(z,s)f(z)\mathbb{F}(u,z)^{\frac{\alpha}{k}}dz \tag{35}$$

$$=k\,{}^{s}J^{\frac{\alpha+k}{k}}_{F,a_1}f(u). \tag{36}$$

This complete the proof. □

Remark 7. *Theorem 3.7 of [17] is obtained from this result if we consider the kernel $F(u,s)=u^s$.*

2. Applications

Following the idea presented in Example 1, we can obtain various inequalities in terms of other integral operators, fractional or not. In [22], the following functional was considered:

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{37}$$

From Theorem 1, with $F\equiv 1$ and $\alpha=k=1$, we have a direct estimation of (37). On the other hand, if we take $F\equiv 1$ and $k=1$, it is clear that Theorem 3.1 of [23] is a particular case of our result. Under the last assumption, we can easily formulate a more general result than Theorem 4 of [24] and all the results of [25].

The generality of our results can also be checked if we apply our integral operator to the results of [26], which can be easily generalized, as readers can check if we consider $F(\tau, s) = (\tau - a_1)^{1-s}$ and $F(\tau, s) = (a_2 - \tau)^{1-s}$ for the left- and right-sided integrals.

3. Conclusions and Recommendations

Within this research, we introduce a comprehensive formulation of the Riemann–Liouville fractional integral formulation, encompassing numerous integral operators documented in the existing literature. Within this framework, we unveil a variety of integral inequalities that extend the scope of various well-known inequalities.

We aim to emphasize the robustness of Definition 7, elucidating the following points. If we consider the kernel $F(\tau, s) = \tau^{1-s}$, we obtain a variant of the (k,s)-Riemann–Liouville fractional integral [17]:

$$ {}^{s}_{a_1}I^{\alpha}_{u} f(u) = \frac{(2-s)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_{a_1}^{u} (u^{2-s} - \tau^{2-s})^{\frac{\alpha}{k}-1} \tau^{1-s} f(\tau)\, d\tau. $$

This opens up a wide range of possibilities in obtaining new integral inequalities.

On the other hand, taking into account some recent results (see, for example, [27,28]), this integral operator in our work, and its corresponding differential operator, can be used in investigations related to very general fractional differential equations.

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