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Article

Generalized Taylor's Formula and Steffensen's Inequality

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Abstract: New Steffensen-type inequalities are obtained by combining generalized Taylor expansions, Rabier and Pečarić extensions of Steffensen's inequality and Faà di Bruno's formula for higher order derivatives of the composition.

Keywords: Steffensen's inequality; generalized Taylor's formula; Faà di Bruno's formula; Euler polynomial; convex functions

MSC: 26A51; 26D15

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1. Introduction

In several research areas such as mathematical statistics, qualitative theory of integrals [1], information theory [2], differential equations [3], engineering [4] and economics [5], mathematical inequalities play a significant role and have several applications. Numerous mathematical inequalities have attracted the attention of many mathematicians who have worked hard to refine, prove and generalise them. As a result of this rapid expansion, mathematical inequalities are now regarded as a separate branch of analysis. The Hermite–Hadamard inequality, the Jensen's and Jensen–Mercer inequality and Steffensen's inequality are a few notable ones among the many interesting inequalities that have been examined, (see [6–10] and references therein). Mathematical inequality researchers continue to be interested in many versions of these inequalities involving certain families of functions [11–15]. Among other techniques, some significantly used tools to prove integral inequalities are interpolating polynomials. Researchers have used different interpolating polynomials such as Hermite interpolation [16,17], Abel–Gontscharoff interpolation [18] and other interpolations [19,20] to prove integral inequalities.

Steffensen's inequality, proved in [10], has been vastly studied due to its vital role in the branch of mathematical analysis [21–23] along with other research directions; for example, its role in estimating Chebyshev's functional, the difference between the product of integrals and the integral of the product [24] and to asses bounds for expectations of order and record statistics [25,26]. Thus, due to these characteristics, the development of many variants and generalizations of the Steffensen's inequality is still important [14,27–30]. Although, a generalization of Steffensen's inequality [31], which is several years before a generalization given in [21], but interestingly via appropriate substitution, one may obtain the result of [21] from [31]. By keeping in view the importance of [21,31], another generalization of Steffensen's inequality was proved in [32]. In fact, the results presented in [32] provide generalizations of all [10,21,31]. A few other variants of Steffensen's inequality by using interpolating polynomials can be seen in [33–35]. Moreover, to elaborate the importance of Hardy-type inequalities in the theory of function spaces, we recommend [36] to the readers.

In this paper, we prove new Steffensen-type inequalities by combining generalized Taylor expansions, Rabier [21] and Pečarić [31] extensions of Steffensen's inequality and Faà di Bruno's formula. The generalization proved in this paper recaptures the results of [10,21,31,32]. Further, we prove consequences of the main results, which as a special case produce some inequalities from [32] which are related to Hardy-type inequalities, see also [36]. At the end, we prove some inequalities involving Euler polynomials.

2. Materials and Methods

In this section, we include necessary notions and known results which are necessary to describe and achieve the objectives of this paper. We set that for any $k \in \mathbb{N}$ and for any k times differentiable function ψ, the kth order derivative of the function ψ is denoted by $\psi^{(k)}$. We start with the Steffensen's inequality. Steffensen [10] established an inequality, demonstrated as follows: if $\theta, \zeta : [\alpha, \beta] \to \mathbb{R}$, ζ is decreasing and $0 \le \theta \le 1$, then

$$\int_{\alpha}^{\beta} \zeta(\mu)\theta(\mu)\, d\mu \le \int_{\alpha}^{\alpha+\gamma} \zeta(\mu)\, d\mu, \qquad \text{where } \gamma = \int_{\alpha}^{\beta} \theta(\mu)\, d\mu. \tag{1}$$

There are many extensions and generalizations of Steffensen's inequality (1); Rabier [21] has provided a notable contribution in recent times.

Theorem 1. *Let ψ be a real valued, continuous and convex function on $[0, \infty)$ with $\psi(0) = 0$. If $d > 0$ and $\theta \in L^{\infty}(0, d), \theta \ge 0$ and $\|\theta\|_{\infty} \le 1$, then $\theta\psi^{(1)} \in L^1(0, d)$ and*

$$\psi\Big(\int_0^d \theta(\mu)\, d\mu\Big) \le \int_0^d \theta(\mu)\psi^{(1)}(\mu)\, d\mu. \tag{2}$$

Interestingly, Theorem 1 is closely associated with a generalization of (1) proved by Pečarić [31].

Theorem 2. *Let ζ be a real-valued, nondecreasing and differentiable function on $[c, d]$ and $g : J \to \mathbb{R}$ is a nondecreasing function with ($J \subset \mathbb{R}$ be an interval and $c, d, \zeta(c), \zeta(d) \in I$).*

(a) If $\zeta(\mu) \le \mu$, then

$$\int_c^d g(\mu)\zeta^{(1)}(\mu)\, d\mu \ge \int_{\zeta(c)}^{\zeta(d)} g(\mu)\, d\mu. \tag{3}$$

(b) If $\zeta(\mu) \ge \mu$, then the inequality mentioned above holds in reverse.

The inequality denoted as Steffensen's inequality (1) can be derived by employing substitutions $\zeta(\mu) \mapsto \int_c^{\mu} \theta(\nu + \alpha - c)\, d\nu + c$ and $g(\mu) \mapsto -\zeta(\mu + \alpha - c)$ and also by taking $d = \beta - \alpha + c$, based on Theorem 2.

Theorem 1 follows from Theorem 2 (under slightly weaker assumptions, see [32]) by taking $c = 0$, $\psi(\mu) = \int_0^{\mu} g(\nu)d\nu$ and $\zeta(\mu) = \int_0^{\mu} \theta(\nu)d\nu$. Since $0 \le \theta \le 1$, the function ζ satisfies $\zeta(\mu) \le \mu$. On the other hand, a function $\zeta : [0, d] \to \mathbb{R}$ can satisfy $\zeta(\mu) \le \mu$ without satisfying $0 \le \zeta^{(1)}(\mu) \le 1$, so Theorem 2 is broader than Theorem 1.

The following is the concept for the Theorem 2 proof: take $\psi(\mu) = \int_c^{\mu} g(\nu)d\nu$ and make the substitution $z = \zeta(\mu)$ in the integral below

$$\begin{aligned}\psi(\zeta(d)) - \psi(\zeta(c)) &= \int_{\zeta(c)}^{\zeta(d)} g(z)\, dz \\ &= \int_c^d g(\zeta(\mu))\zeta^{(1)}(\mu)\, d\mu \le \int_c^d g(\mu)\zeta^{(1)}(\mu)\, d\mu,\end{aligned} \tag{4}$$

and the last inequality is satisfied when $\zeta(\mu) \le \mu$.

By replacing the equality

$$\psi(\zeta(d)) = \psi(\zeta(c)) + \int_{\zeta(c)}^{\zeta(d)} \psi^{(1)}(\mu)\, d\mu,$$

utilizing the m-th order Taylor expansion of the composition $\psi \circ \zeta$, a generalization of Theorem 2 was obtained in [32].

Theorem 3. *Let $m \in \mathbb{N}$. Let $\zeta : [c,d] \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$ (where $J \in \mathbb{R}$ be a interval in such a way that $c, d, \zeta(c), \zeta(d) \in J$) be two m times differentiable functions such that $\zeta^{(1)}, \zeta^{(2)}, \ldots \zeta^{(m)}$, $\psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(m)}$ are nondecreasing functions. If $\zeta(\mu) \leq \mu$, then*

$$(\psi \circ \zeta)(d) \leq (\psi \circ \zeta)(c) + \sum_{\kappa=1}^{m-1} \psi^{(\kappa)}(\zeta(c)) \sum_{j=\kappa}^{m-1} (-1)^{j+1} B_{j,\kappa}(\zeta^{(1)}(c), \ldots, \zeta^{(j-\kappa+1)}(c)) \frac{(d-c)^j}{j!} + \int_c^d \frac{(d-\mu)^{m-1}}{(m-1)!} \sum_{k=1}^{m} \psi^{(\kappa)}(\mu) B_{m,\kappa}(\zeta^{(1)}(\mu), \ldots, \zeta^{(m-\kappa+1)}(\mu))\, d\mu.$$

Here, $B_{m,\kappa}(g^{(1)}(\mu), \ldots, g^{(m-\kappa+1)}(\mu))$ corresponds to the Bell polynomials.

The Bell polynomial $B_{n,j}(\nu_1, \nu_2, \ldots, \nu_{n-j+1})$, with variables $\nu_1, \nu_2, \ldots \nu_{n-j+1}$ is

$$B_{n,j}(\nu_1, \nu_2, \ldots, \nu_{n-j+1}) := \sum \frac{n!}{i_1! i_2! \cdots i_{n-j+1}!} \left(\frac{\nu_1}{1!}\right)^{i_1} \left(\frac{\nu_2}{2!}\right)^{i_2} \cdots \left(\frac{\nu_{n-j+1}}{(n-j+1)!}\right)^{i_{n-j+1}},$$

where the sum is calculated over all sequences of non-negative integers $i_1, i_2, \ldots, i_{n-\kappa+1}$ such as

$$i_1 + i_2 + \ldots = j \quad \text{and} \quad i_1 + 2i_2 + 3i_3 + \ldots = n.$$

Faa' di Bruno's formula, which provides higher order derivatives of the composition function $\psi \circ \zeta$, contains the Bell polynomials (see, for example, [37])

$$\frac{d^n}{dx^n} \psi(\zeta(\nu)) = \sum_{j=1}^{n} \psi^{(j)}(\zeta(\nu)) B_{n,j}(\zeta^{(1)}(\nu), \ldots, \zeta^{(n-j+1)}(\nu)). \tag{5}$$

The use of these formulae and relevant works is still a topic of interest [38,39].

This paper aims to derive general Steffensen-type inequalities by using the generalized Taylor's formula from [40].

Theorem 4. *Let $\{P_n\}$ be the polynomials form a harmonic sequence, implying*

$$P'_m(\nu) = P_{m-1}(\nu), \quad \text{for } m \in \mathbb{N} \text{ and } P_0(\nu) = 1.$$

Moreover, consider a closed interval $I \subset \mathbb{R}$, and let $c \in I$. If $\Phi : I \to \mathbb{R}$ be any function such that $\Phi^{(m-1)}$ is absolutely continuous for some $m \in \mathbb{N}$, then for any $\nu \in I$.

$$\Phi(\nu) = \Phi(c) + \sum_{j=1}^{m-1} (-1)^{j+1} \left[P_j(\nu)\Phi^{(j)}(\nu) - P_j(c)\Phi^{(j)}(c) \right] + (-1)^{m-1} \int_c^{\nu} P_{m-1}(\mu)\Phi^{(m)}(\mu) d\mu. \tag{6}$$

Some immediate consequence of generalized Taylor are evident in [41,42], where a fractional version of Taylor's formula can be seen [43].

3. Main Results

We will first start with an identity from which we will then derive our inequalities.

Lemma 1. *Let $m \in \mathbb{N}$ and let $\zeta : [c,d] \to J$ and $\psi : J \to \mathbb{R}$ be two m times differentiable functions with the assumption that $\psi^{(m-1)}$ is absolutely continuous. Then, the following identity holds*

$$
\begin{aligned}
(\psi\circ\zeta)(d) &= (\psi\circ\zeta)(c) \\
&\quad + \sum_{\kappa=1}^{m-1}\psi^{(\kappa)}(\zeta(d))\sum_{j=\kappa}^{m-1}(-1)^{j+1}B_{j,\kappa}(\zeta^{(1)}(d),\ldots,\zeta^{(j-\kappa+1)}(d))P_j(d) \\
&\quad - \sum_{\kappa=1}^{m-1}\psi^{(\kappa)}(\zeta(c))\sum_{j=\kappa}^{m-1}(-1)^{j+1}B_{j,\kappa}(\zeta^{(1)}(c),\ldots,\zeta^{(j-\kappa+1)}(c))P_j(c) \\
&\quad + (-1)^{m-1}\int_c^d P_{m-1}(\mu)\sum_{\kappa=1}^{m}\psi^{(\kappa)}(\zeta(\mu))B_{m,\kappa}(\zeta^{(1)}(\mu),\ldots,\zeta^{(m-\kappa+1)}(\mu))\,d\mu. \qquad (7)
\end{aligned}
$$

Proof. The generalized Taylor's Formula (6) for Faá di Bruno's Formula (5) and the composition $\Phi = \psi\circ\zeta$ give

$$
\begin{aligned}
(\psi\circ\zeta)(d) &= (\psi\circ\zeta)(c) \\
&\quad + \sum_{j=1}^{m-1}(-1)^{j+1}P_j(d)\sum_{\kappa=1}^{j}\psi^{(\kappa)}(\zeta(d))B_{j,\kappa}(\zeta^{(1)}(d),\ldots,\zeta^{(j-\kappa+1)}(d)) \\
&\quad - \sum_{j=1}^{m-1}(-1)^{j+1}P_j(c)\sum_{\kappa=1}^{j}\psi^{(\kappa)}(\zeta(c))B_{j,\kappa}(\zeta^{(1)}(c),\ldots,\zeta^{(j-\kappa+1)}(c)) \\
&\quad + (-1)^{m-1}\int_c^d P_{m-1}(\mu)\sum_{\kappa=1}^{m}\psi^{(\kappa)}(\zeta(\mu))B_{m,\kappa}(\zeta^{(1)}(\mu),\ldots,\zeta^{(m-\kappa+1)}(\mu))\,d\mu.
\end{aligned}
$$

Rearranging the terms yields the stated identity (7). □

The next theorem contains our primary result.

Theorem 5. *Let $m\in\mathbb{N}$ and let $\{P_n\}$ be a harmonic sequence of polynomials such that $(-1)^{m-1}P_{m-1}(\nu)\geq 0$ for $\nu\in[c,d]$. Let $\zeta:[c,d]\to\mathbb{R}$ and $\psi:J\to\mathbb{R}$ (where $J\in\mathbb{R}$ be a interval in such a way that $c,d,\zeta(c),\zeta(d)\in J$) be two m times differentiable functions such that $\zeta,\zeta^{(1)},\ldots\zeta^{(m-1)}$, $\psi^{(1)},\psi^{(2)},\ldots,\psi^{(m)}$ are nondecreasing functions and $\psi^{(m-1)}$ is absolutely continuous.*

(a) If $\zeta(\mu)\leq\mu$, then

$$
\begin{aligned}
(\psi\circ\zeta)(d) &\leq (\psi\circ\zeta)(c) \\
&\quad + \sum_{\kappa=1}^{m-1}\psi^{(\kappa)}(\zeta(d))\sum_{j=\kappa}^{m-1}(-1)^{j+1}B_{j,\kappa}(\zeta^{(1)}(d),\ldots,\zeta^{(j-\kappa+1)}(d))P_j(d) \\
&\quad - \sum_{\kappa=1}^{m-1}\psi^{(\kappa)}(\zeta(c))\sum_{j=\kappa}^{m-1}(-1)^{j+1}B_{j,\kappa}(\zeta^{(1)}(c),\ldots,\zeta^{(j-\kappa+1)}(c))P_j(c) \\
&\quad + (-1)^{m-1}\int_c^d P_{m-1}(\mu)\sum_{\kappa=1}^{m}\psi^{(\kappa)}(\mu)B_{m,\kappa}(\zeta^{(1)}(\mu),\ldots,\zeta^{(m-\kappa+1)}(\mu))\,d\mu. \qquad (8)
\end{aligned}
$$

(b) If $\zeta(\mu)\geq\mu$, then the inequality mentioned above holds in reverse.

Furthermore, if $(-1)^{m-1}P_{m-1}(\nu)\leq 0$ for $\nu\in[c,d]$, then the inequalities in (a) and (b) are reversed.

Proof. Identity (7) holds and, according to the theorem's presumptions, the Bell polynomials evaluated at ζ derivatives in (7) are nonnegative because $\zeta^{(j)}\geq 0$ for $j=1,\ldots,m$. Therefore, for $\zeta(\mu)\leq\mu$ the inequality

$$
\begin{aligned}
&(-1)^{m-1}\int_c^d P_{m-1}(\mu)\sum_{\kappa=1}^{m}\psi^{(\kappa)}(\zeta(\mu))B_{m,\kappa}(\zeta^{(1)}(\mu),\ldots,\zeta^{(m-\kappa+1)}(\mu))\,d\mu \\
&\qquad \leq (-1)^{m-1}\int_c^d P_{m-1}(\mu)\sum_{\kappa=1}^{m}\psi^{(\kappa)}(\mu)B_{m,\kappa}(\zeta^{(1)}(\mu),\ldots,\zeta^{(m-\kappa+1)}(\mu))\,d\mu,
\end{aligned}
$$

holds, but the reverse inequality exists when $\zeta(\mu) \geq \mu$. □

The previous theorem has the following corollary as a special case.

Corollary 1. *Let $m \in \mathbb{N}$ and let $\{P_n\}$ be a harmonic sequence of polynomials such that $(-1)^{m-1}P_{m-1}(\nu) \geq 0$ for $\nu \in [c,d]$. Let $\zeta : [0,d] \to [0,+\infty)$ be $m-1$ differentiable function and $\psi : J \to \mathbb{R}$ (where J is an interval in $\mathbb{R}$ such that $0, d, \int_0^d \zeta(\mu)d\mu \in J$) be m times differentiable function such that $\zeta, \zeta^{(1)}, \dots \zeta^{(m-2)}, \psi^{(1)}, \psi^{(2)}, \dots, \psi^{(m)}$ are nondecreasing functions and $\psi^{(m-1)}$ is absolutely continuous.*

(a) If $\int_0^\nu \zeta(\mu)d\mu \leq \nu$ for every $\nu \in [0,d]$, then

$$\begin{aligned}\psi\left(\int_0^d \zeta(\mu)d\mu\right) &\leq \psi(0) \\ &+ \sum_{\kappa=1}^{m-1} \psi^{(\kappa)}\left(\int_0^d \zeta(\mu)d\mu\right) \sum_{j=\kappa}^{m-1} (-1)^{j+1} B_{j,\kappa}(\zeta(d), \zeta^{(1)}(d), \dots, \zeta^{(j-\kappa)}(d))P_j(d) \\ &\quad - \sum_{\kappa=1}^{m-1} \psi^{(\kappa)}(0) \sum_{j=\kappa}^{m-1} (-1)^{j+1} B_{j,\kappa}(\zeta(0), \zeta^{(1)}(0), \dots, \zeta^{(j-\kappa)}(0))P_j(0) \\ &+ (-1)^{m-1} \int_c^d P_{m-1}(\mu) \sum_{\kappa=1}^{m} \psi^{(\kappa)}(\mu) B_{m,\kappa}(\zeta(\mu), \zeta^{(1)}(\mu), \dots, \zeta^{(m-\kappa)}(\mu))\, d\mu. \qquad (9)\end{aligned}$$

(b) If $\nu \leq \int_0^\nu \zeta(\mu)d\mu$ for every $\nu \in [0,d]$, then the inequality mentioned above holds in reverse.

Proof. It can be deduced from Theorem 5 by substituting $c = 0$ and replacing ζ with $\nu \mapsto \int_0^\nu \zeta(\mu)d\mu$. □

Example 1. *The polynomials*

$$P_{m-1}(\nu) = \frac{(\nu - d)^{m-1}}{(m-1)!},$$

form a harmonic sequence. Identity (6) for these polynomials reduces to the classical Taylor's formula. These polynomials satisfy the assumptions of Theorem 5 and inequality (8) becomes

$$\begin{aligned}(\psi \circ \zeta)(d) &\leq (\psi \circ \zeta)(c) \\ &+ \sum_{\kappa=1}^{m-1} \psi^{(\kappa)}(\zeta(c)) \sum_{j=\kappa}^{m-1} B_{j,\kappa}(\zeta^{(1)}(c), \dots, \zeta^{(j-\kappa+1)}(c)) \frac{(d-c)^j}{j!} \\ &+ \int_c^d \frac{(d-\mu)^{m-1}}{(m-1)!} \sum_{\kappa=1}^{m} \psi^{(\kappa)}(\mu) B_{m,\kappa}(\zeta^{(1)}(\mu), \dots, \zeta^{(m-\kappa+1)}(\mu))\, d\mu,\end{aligned}$$

which is a result given in [32].

Example 2. *The polynomials*

$$P_{m-1}(\nu) = \frac{1}{(m-1)!}\left(\nu - \frac{c+d}{2}\right)^{m-1},$$

form a harmonic sequence. These polynomials satisfy the assumptions of Theorem 5 for odd m and inequality (8) becomes

$$
\begin{aligned}
(\psi\circ\zeta)(d) \leq (\psi\circ\zeta)(c) & \\
&+ \sum_{j=1}^{m-1}\frac{(d-c)^j}{j!2^j}\Bigg[\sum_{\kappa=1}^{j}\psi^{(\kappa)}(\zeta(c))B_{j,\kappa}(\zeta^{(1)}(c),\ldots,\zeta^{(j-\kappa+1)}(c)) \\
&+(-1)^{j+1}\sum_{\kappa=1}^{j}\psi^{(\kappa)}(\zeta(d))B_{j,\kappa}(\zeta^{(1)}(d),\ldots,\zeta^{(j-\kappa+1)}(d))\Bigg] \\
&+\int_c^d\frac{(d-\mu)^{m-1}}{(m-1)!}\sum_{\kappa=1}^{m}\psi^{(\kappa)}(\mu)B_{m,\kappa}(\zeta^{(1)}(\mu),\ldots,\zeta^{(m-\kappa+1)}(\mu))\,d\mu.
\end{aligned}
$$

Example 3. *We will use Euler polynomials in this example and we will first recall some of their properties (all the results cited here can be found, for example, in Chapter 23 of [44]). The series expansion can be used to define the Euler polynomials*

$$\frac{2e^{\mu x}}{e^x+1}=\sum_{m=0}^{\infty}\frac{E_m(\mu)}{m!}x^m,\quad |x|<\pi,\mu\in\mathbb{R}.$$

The first few Euler polynomials are

$$E_0(\mu)=1,\ E_1(\mu)=\mu-\frac{1}{2},\ E_2(\mu)=\mu^2-\mu,\ E_3(\mu)=\mu^3-\frac{3}{2}\mu^2+\frac{1}{4},\ \ldots$$

The Euler polynomials are uniquely determined by the following two properties

$$E'_m(\mu)=mE_{m-1}(\mu),\quad \text{for } m\in\mathbb{N};\ E_0(\mu)=1, \tag{10}$$

$$E_m(\mu+1)+E_m(\mu)=2\mu^m,\quad \text{for } m\in\mathbb{N}_0. \tag{11}$$

The property (10) implies that the polynomials given by

$$P_{m-1}(\mu)=\frac{(d-c)^{m-1}}{(m-1)!}E_{m-1}\left(\frac{\mu-c}{d-c}\right), \tag{12}$$

satisfy $P'_m(\mu)=P_{m-1}(\mu)$ and $P_0(\mu)=1$, i.e., they form a harmonic sequence of polynomials.

The properties

$$E_m(1-\mu)=(-1)^mE_m(\mu),\qquad \text{for } m\in\mathbb{N}_0, \tag{13}$$

$$(-1)^mE_{2m}(\mu)>0,\qquad \text{for } 0<\mu<\frac{1}{2},\ m\in\mathbb{N}, \tag{14}$$

$$(-1)^mE_{2m-1}(\mu)>0,\qquad \text{for } 0<\mu<\frac{1}{2},\ m\in\mathbb{N}. \tag{15}$$

yield that

$$E_{4n}(\mu)\geq 0\quad \text{and}\quad E_{4n+2}(\mu)\leq 0,\qquad \text{for } 0\leq\mu\leq 1,\ n\in\mathbb{N}_0. \tag{16}$$

Alternatively, the Euler polynomials for even index have constant sign on $[0,1]$, so for odd m the polynomials P_{m-1} from (12) have constant sign.

Further, the Euler polynomials satisfy

$$E_m(1)=-E_m(0)=\frac{2}{m+1}(2^{m+1}-1)B_{m+1},\qquad \text{for } m\in\mathbb{N}, \tag{17}$$

where B_m are the Bernoulli numbers. Since $B_{2m+1}=0$, we have $P_j(c)=P_j(d)=0$ for even j.

For odd $m=2n+1$ the assumptions of Theorem 5 are satisfied and inequality (8) for even n becomes

$$
\begin{aligned}
(\psi \circ \zeta)(d) \leq\ & (\psi \circ \zeta)(c) \\
& + \sum_{j=1}^{n} (-1)^{2j-1} \frac{(d-c)^{2j-1}}{(2j-1)!} E_{2j-1}(1) \Bigg[\sum_{k=1}^{2j-1} \psi^{\kappa}(\zeta(d)) B_{j,\kappa}(\zeta^{(1)}(d), \ldots, \zeta^{(j-\kappa+1)}(d)) \\
& \qquad + \sum_{\kappa=1}^{2j-1} \psi^{(\kappa)}(\zeta(c)) B_{j,\kappa}(\zeta^{(1)}(c), \ldots, \zeta^{(j-\kappa+1)}(c)) \Bigg] \\
& \qquad + \frac{(d-c)^{m-1}}{(m-1)!} E_{m-1}(1) \int_c^d \sum_{\kappa=1}^{m} \psi^{(\kappa)}(\mu) B_{m,\kappa}(\zeta^{(1)}(\mu), \ldots, \zeta^{(m-\kappa+1)}(\mu))\, d\mu,
\end{aligned}
$$

In the case of odd values for n, the inequality reversed.

4. Conclusions

In this paper, we first proved the generalized Taylor expansion for the composition functions. Then, under suitable assumptions, we proved a new generalization of Steffensen's inequality, which under special cases coincides with the well-known generalizations of Steffensen's inequality [10,21,31,32]. Moreover, we proved consequences of the main results, which as a special case produce some inequalities from [32] which are related to Hardy-type inequalities. At the end, we proved some inequalities involving Euler polynomials.

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