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Czerwik Vector-Valued Metric Space with an Equivalence Relation and Extended Forms of Perov Fixed-Point Theorem

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Abstract: In this article, we shall generalize the idea of vector-valued metric space and Perov fixed-point theorem. We shall introduce the notion of Czerwik vector-valued $\mathcal{R}$-metric space by involving an equivalence relation. A few basic concepts and properties related to Czerwik vector-valued $\mathcal{R}$-metric space shall also be discussed that are required to obtain a few extended types of Perov fixed-point theorem.

Keywords: fixed points; vector-valued metric space; Czerwik vector-valued metric space; Czerwik vector-valued $\mathcal{R}$-metric space

MSC: 47H10; 30L15

1. Introduction

The concept of metric space provides a significant contribution to research activities related to mathematical analysis. This meaningful concept was presented by Maurice Fréchet [1] who was a famous French mathematician. This concept was extended by many mathematicians according to their requirements: for example, b-metric space [2], partial metric space [3], cone metric space [4], vector-valued metric space [5], vector-valued b-metric space [6,7], order (ordered vector) metric space [8], order (ordered vector) pseudo-metric space [9], graphical metric space [10], and graphical b-metric space [11] etc.

The Banach contraction principle is the most basic result of the metric fixed-point theory, and it has been generalized by considering all the above-mentioned extended forms of metric space. The literature also contains several other generalizations of this famous result obtained through involving the concepts of partial order, graph, binary relation, or orthogonality relation associated with contraction mapping, see [12–14]. This technique of generalization raised the question: why not consider the concepts of partial order, graph, or binary relation to generalize the notion of metric space and then drive a generalization of the Banach contraction principle? The work presented in [10,11] is based on the answer to that question.

Perov [5] presented the matrix/vector version of the Banach contraction principle by introducing the notion of vector-valued metric spaces. This vector-valued metric space was extended to vector-valued b-metric space by Boriceanu [6], with a constant scalar multiple in the triangle inequality of vector-valued b-metric space. Ali and Kim [7] modified the triangle inequality of vector-valued b-metric space by replacing a constant scalar multiple with a constant matrix multiple. A couple of interesting results in the context of Perov have been derived by several researchers, for example, Bucur et al. [15] derived fixed-point theorems to generalize Perov's result that discuss the existence of fixed points of set-valued maps. Filip and Petrusel [16] modified the contraction-type inequality to generalize the

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results of Perov and Bucur et al. [15]. Ali et al. [17] used the admissibility concept of single-valued maps to improve the result of Perov, Altun et al. [18] used the technique of θ-contraction to modify the result of Perov for single-valued maps. Guran et al. [19] used the concept of generalized w-distance and Hardy–Rogers-type contraction inequality to generalize the work of Perov. Guran et al. [20] extended the work of Perov for set-valued maps using a set-valued Hardy–Rogers-type contraction inequality. Martínez-Moreno and Gopal [21] defined the concept of Perov fuzzy metric space and studied the existence of common fixed points for compatible single-valued maps. The aim of this article is to introduce a notion of Czerwik vector-valued $\mathcal{R}$-metric space that is a generalized concept of vector-valued b-metric space. A few results confirming the existence of fixed points for certain types of maps are also derived using this notion. The idea of this article follows from the above-mentioned question.

2. Preliminaries

Throughout this article, we consider H as a nonempty set, $\mathbb{R}_+$ as the set of all non-negative real numbers, $M_{m,m}(\mathbb{R}_+)$ as a collection of all $m \times m$ matrices with non-negative real elements, $\bar{0}$ as an $m \times m$ zero matrix, I as $m \times m$ identity matrix, and $\mathbb{R}_m$ as the set of all $m \times 1$ real matrices. If $W, P \in \mathbb{R}_m$, that is $W = (w_1, w_2, \ldots, w_m)^T$ and $P = (p_1, p_2, \ldots, p_m)^T$, then

(i) $W \leq P$ means that $w_i \leq p_i$ for each $i \in \{1, 2, \ldots, m\}$,
(ii) $W < P$ means that $w_i < p_i$ for each $i \in \{1, 2, \ldots, m\}$,
(iii) $W \geq c \in \mathbb{R}_+$ means that $w_i \geq c$ for each $i \in \{1, 2, \ldots, m\}$.

A matrix $C \in M_{m,m}(\mathbb{R}_+)$ is called convergent to zero (or zero matrix) if $C^n \to \bar{0}$ as $n \to \infty$ (see Varga [22]). Also, note that $C^0 = I$. The following matrices are convergent to zero.

$$C := \begin{pmatrix} c & c \\ d & d \end{pmatrix}, \text{ where } c, d \in \mathbb{R}_+ \text{ and } c + d < 1;$$
$$D := \begin{pmatrix} c & d \\ 0 & e \end{pmatrix}, \text{ where } c, d, e \in \mathbb{R}_+ \text{ and } \max\{c, e\} < 1.$$

Czerwik vector-valued metric space was presented by Ali and Kim [7] in the following ways.

Definition 1. *A mapping $d_C \colon H \times H \to \mathbb{R}_m$ is called a Czerwik vector-valued metric on H, if for each $h_1, h_2, h_3 \in H$ the following axioms hold:*

(d_1) $d_C(h_1, h_2) \geq 0$;
$d_C(h_1, h_2) = 0$ *if and only if* $h_1 = h_2$;
(d_2) $d_c(h_1, h_2) = d_C(h_2, h_1)$;
(d_3) $d_C(h_1, h_3) \leq Q[d_C(h_1, h_2) + d_C(h_2, h_3)]$

where $Q = (q_{ij}) \in M_{m,m}(R_+)$ *is a matrix with*

$$q_{ij} = \begin{cases} q, & i = j \\ 0, & i \neq j \end{cases}$$

and $q \geq 1$. *Then, the triple* (H, d_C, Q) *is called Czerwik vector-valued metric space, or Czerwik generalized metric space.*

Note that the Cauchyness and convergence of a sequence in Czerwik vector-valued metric spaces are defined in a similar manner as in b-metric spaces/metric spaces.

If the matrix $Q = (q_{ij}) \in M_{m,m}(R_+)$ is defined by

$$q_{ij} = \begin{cases} 1, \ i = j \\ 0, \ i \neq j \end{cases}$$

then the Czerwik vector-valued metric space becomes a vector-valued metric space. Perov [5] presented the matrix/vector version of the Banach contraction principle on vector-valued metric space in the following way.

Theorem 1 ([5])**.** *Let (H, d_C) be a complete vector-valued metric space and $G\colon H \to H$ be a mapping such that*

$$d_C(Gh, Gk) \leq Ad_C(h,k) \ \forall h, k \in H,$$

where $A \in M_{m,m}(R_+)$ is a matrix convergent to zero. Then, G has a unique fixed point.

The above result was generalized by Ali and Kim [7] in the following way.

Theorem 2. *Let (H, d_C, Q) be a complete Czerwik vector-valued metric space. Let $G\colon H \to H$ be a mapping such that*

$$d_C(Gh, Gk) \leq Ad_C(h,k) + Bd_C(k, Gh) \ \forall h, k \in H$$

where $A, B \in M_{m,m}(\mathbb{R}_+)$. Also assume that the matrix QA converges to zero. Then, G has a fixed point.

3. Main Results

This section begins with the definition of Czerwik vector-valued $\mathcal{R}$-metric space.

Definition 2. *Let H be a nonempty set equipped with an equivalence relation $\mathcal{R}$. A mapping $d_C\colon H \times H \to \mathbb{R}_m$ is called a Czerwik vector-valued $\mathcal{R}$-metric on H if for each $h_1, h_2, h_3 \in H$ the following axioms hold:*

(d_1) $d_C(h_1, h_2) \geq 0$;
$d_C(h_1, h_2) = 0$ if and only if $h_1 = h_2$;
(d_2) $d_C(h_1, h_2) = d_C(h_2, h_1)$;
(d_3) $d_C(h_1, h_3) \leq Q[d_C(h_1, h_2) + d_C(h_2, h_3)]$ provided that $(h_1, h_2), (h_2, h_3) \in \mathcal{R}$

where $Q = (q_{ij}) \in M_{m,m}(R_+)$ is a matrix with

$$q_{ij} = \begin{cases} q, \ i = j \\ 0, \ i \neq j \end{cases}$$

and $q \geq 1$. Then, the $(H, \mathcal{R}, d_C, Q)$ is called a Czerwik vector-valued $\mathcal{R}$-metric space, or Czerwik generalized $\mathcal{R}$-metric space.

Remark 1. *It is important to note that the triangular property (d_3) of Definition 2 should hold for those elements of the set H that are related to each other under an equivalence relation $\mathcal{R}$. From this, an important question arises: why are the reflexive and symmetric conditions added along with the transitive condition on a binary relation involved in Definition 2? The answer is simple:*

(i) The reflexive condition is required for the topology generated by d_C.
(ii) The symmetric condition is essential for the concept of $\mathcal{R}$-convergence of $\mathcal{R}$-sequence.

Remark 2. *It is easy to see that Definition 2 reduces to Definition 1 by defining $\mathcal{R} = H \times H$. Thus, every Czerwik vector-valued metric space generates a Czerwik vector-valued $\mathcal{R}$-metric space. But the converse is not true in general.*

In the following, we present an example of Czerwik vector-valued $\mathcal{R}$-metric space.

Example 1. *Consider $H = \mathbb{N}$ and an equivalence relation $\mathcal{R} = \{(x,y) : x,y \in \{1,2,3\}\} \cup \{(x,x) : x \in \{4,5,\cdots\}\}$ on H. Define $d_C \colon H \times H \to \mathbb{R}_2$ by*

$$\begin{aligned}
&d_C(1,2) = d_C(2,1) = (2,2)^T\\
&d_C(1,3) = d_C(3,1) = (2,2)^T\\
&d_C(2,3) = d_C(3,2) = (5,5)^T\\
&d_C(x,x) = (0,0)^T \forall x \in H\\
&d_C(x,y) = d_C(y,x) = (1/|x-y|, 1/|x-y|)^T, \quad \text{if either } x \geq 4 \text{ or } y \geq 4 \text{ and } x \neq y.
\end{aligned}$$

One can check that $(H, \mathcal{R}, d_C, Q)$ is a Czerwik vector-valued $\mathcal{R}$-metric space with $Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Remark 3. *Note that the above-defined d_C is not a Czerwik vector-valued metric on H, because Axiom (d_3) of Definition 1 does not exist, for instance,*

$$d_C(2,3) > Q[d_C(2,4) + d_C(4,3)].$$

Example 2. *Consider $H = \mathbb{R}$, and an equivalence relation on H is defined by*

$$\mathcal{R} = \{(h_1,h_2) : h_1,h_2 \in [0,\infty)\} \cup \{(h,h) : h \in \mathbb{R}\}.$$

Define $d_C \colon H \times H \to \mathbb{R}_2$ by

$$d(h_1,h_2) = \begin{cases} \begin{pmatrix} |h_1 - h_2| \\ |h_1 - h_2| \end{pmatrix}, & \text{if } h_1,h_2 \geq 0 \\ \begin{pmatrix} \frac{|h_1-h_2|}{1+|h_1-h_2|} \\ 0 \end{pmatrix}, & \text{otherwise.} \end{cases}$$

It is easy to check that $(H, \mathcal{R}, d_C, Q)$ is a Czerwik vector-valued $\mathcal{R}$-metric space with $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Remark 4. *Note that the above-defined d_C does not satisfy Axiom (d_3) of Definition 1, for instance,*

$$d_C(1,5) \npreceq Q[d_C(1,-1) + d_C(-1,5)].$$

For any $\epsilon > 0$ and for any element h of the Czerwik vector-valued $\mathcal{R}$-metric space $(H, \mathcal{R}, d_C, Q)$, the d_C-open ball having center h and radius ϵ is defined by

$$B_{d_C}(h,\epsilon) = \{h_a \in H : (h,h_a) \in \mathcal{R}, d_C(h,h_a) < \epsilon\}.$$

$\mathcal{R}$ is a reflexive relation, thus $B_{d_C}(h,\epsilon) \neq \emptyset$ for each $h \in H$ and $\epsilon > 0$. Thus, the set $\{B_{d_C}(h,\epsilon) : h \in H, \epsilon > 0\}$ provides a neighbourhood system for the topology $\tau_{\mathcal{R}}$ on H induced by the Czerwik vector-valued $\mathcal{R}$-metric space.

Definition 3. *Let $(H, \mathcal{R}, d_C, Q)$ be Czerwik vector-valued $\mathcal{R}$-metric space. Then*

- *A sequence (h_n) in H is said to be an $\mathcal{R}$-sequence if $(h_n, h_{n+1}) \in \mathcal{R}$ for each $n \in \mathbb{N}$.*
- *An $\mathcal{R}$-sequence (h_n) in H is said to be $\mathcal{R}$-convergent to h in H if $\lim_{n\to\infty} d_C(h_n,h) = 0$ and $(h_n,h) \in \mathcal{R}$ $\forall n \geq k$ for some natural number k.*
- *An $\mathcal{R}$-sequence (h_n) in H is said to be $\mathcal{R}$-Cauchy if $\lim_{n,m\to\infty} d_C(h_n,h_m) = 0$.*

- $(H, \mathcal{R}, d_C, Q)$ *is said to be* $\mathcal{R}$*-complete if each* $\mathcal{R}$*-Cauchy sequence in* H *is* $\mathcal{R}$*-convergent in* H*.*

Theorem 3. *Each* $\mathcal{R}$*-convergent sequence in* $(H, \mathcal{R}, d_C, Q)$ *has a unique limit point.*

Proof. Assume that the $\mathcal{R}$-sequence (h_n) is $\mathcal{R}$-convergent to h and l in H. That is,

$$\lim_{n\to\infty} d_C(h_n, h) = 0 \text{ and } (h_n, h) \in \mathcal{R}\ \forall n \geq k_1$$

and

$$\lim_{n\to\infty} d_C(h_n, l) = 0 \text{ and } (h_n, l) \in \mathcal{R}\ \forall n \geq k_2.$$

Then, for each $n \geq k = \max\{k_1, k_2\}$, we have $(h_n, h) \in \mathcal{R}$ and $(h_n, l) \in \mathcal{R}\ \forall n \geq k$. Thus, by (d_3), we obtain

$$d_C(h, l) \leq Q[d_C(h, h_n) + d_C(h_n, l)]\ \forall n \geq k.$$

Hence, by the above inequality, as $n \to \infty$, we conclude that $d_C(h, l) = 0$. That is, the limit point of the $\mathcal{R}$-convergent sequence is unique. □

Theorem 4. *Each* $\mathcal{R}$*-convergent sequence in* $(H, \mathcal{R}, d_C, Q)$ *is* $\mathcal{R}$*-Cauchy.*

Proof. Consider that an $\mathcal{R}$-sequence (h_n) is $\mathcal{R}$-convergent to h in H. That is,

$$\lim_{n\to\infty} d_C(h_n, h) = 0 \text{ and } (h_n, h) \in \mathcal{R}\ \forall n \geq k_1$$

Then, for each $n, k \geq k_1$, we have $(h_n, h) \in \mathcal{R}$, and $(h_k, h) \in \mathcal{R}\ \forall n, k \geq k_1$. Thus, by (d_3), we obtain

$$d_C(h_n, h_k) \leq Q[d_C(h_n, h) + d_C(h, h_k)]\ \forall n, k \geq k_1.$$

Hence, from the above inequality, we obtain $\lim_{n,k\to\infty} d_C(h_n, h_k) = 0$. □

We are now going to state and prove our first result that is a generalized form of the result presented by Perov [5].

Theorem 5. *Let* $(H, \mathcal{R}, d_C, Q)$ *be an* $\mathcal{R}$*-complete Czerwik vector-valued* $\mathcal{R}$*-metric space and let* $G : H \to H$ *be a mapping. Also, assume that*

(i) *There exists* $h \in H$ *with* $(h, Gh) \in \mathcal{R}$*;*
(ii) $\mathcal{R}$ *is G-closed, that is, for each* $h_1, h_2 \in H$ *with* $(h_1, h_2) \in \mathcal{R}$*, we have* $(Gh_1, Gh_2) \in \mathcal{R}$*;*
(iii) *Either*
(a) If $\{h_n\}$ *is* $\mathcal{R}$*-convergent to* $h \in H$*, then* $\{Gh_n\}$ *is* $\mathcal{R}$*-convergent to* Gh*;*
or
(b) For each $\mathcal{R}$*-convergent sequence* $\{h_n\}$ *in* H *with* $h_n \to h$*, we have* $d_C(h_n, \cdot) \to d_C(h, \cdot)$ *as* $n \to \infty$*;*
(iv) *For each* $(h, k) \in \mathcal{R}$*, we have*

$$d_C(Gh, Gk) \leq A_1 d_C(h, k) + A_2 d_C(h, Gh) + A_3 d_C(k, Gk) + A_4 d_C(h, Gk) + B d_C(k, Gh) \quad (1)$$

where $A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ *such that* $(I - (A_3 + A_4 Q))^{-1}$ *exists and the matrix* $Q[(I - (A_3 + A_4 Q))^{-1}(A_1 + A_2 + A_4 Q)]$ *is convergent to zero.*

Then, G has a fixed point.

Proof. Using hypothesis (i), we have $h_0 \in H$ with $(h_0, Gh_0) \in \mathcal{R}$. Starting from h_0, we can obtain an iterative sequence $\{h_n\}$, that is, $h_n = Gh_{n-1} = G^n h_0$ for each $n \in \mathbb{N}$. Since $\mathcal{R}$ is G-closed, thus, we conclude $(h_{n-1}, h_n) \in \mathcal{R}$ for all $n \in \mathbb{N}$. From (1), we obtain

$$\begin{aligned} d_C(Gh_{n-1}, Gh_n) &\leq A_1 d_C(h_{n-1}, h_n) + A_2 d_C(h_{n-1}, Gh_{n-1}) + A_3 d_C(h_n, Gh_n) \\ &\quad + A_4 d_C(h_{n-1}, Gh_n) + B d_C(h_n, Gh_{n-1}) \ \forall n \in \mathbb{N}. \end{aligned} \tag{2}$$

That is,

$$\begin{aligned} d_C(h_n, h_{n+1}) &\leq A_1 d_C(h_{n-1}, h_n) + A_2 d_C(h_{n-1}, h_n) + A_3 d_C(h_n, h_{n+1}) \\ &\quad + A_4 d_C(h_{n-1}, h_{n+1}) + B d_C(h_n, h_n) \ \forall n \in \mathbb{N}. \end{aligned}$$

This implies that

$$(I - (A_3 + A_4 Q)) d_C(h_n, h_{n+1}) \leq (A_1 + A_2 + A_4 Q) d_C(h_{n-1}, h_n) \ \forall n \in \mathbb{N}.$$

The above inequality yields that

$$d_C(h_n, h_{n+1}) \leq (I - (A_3 + A_4 Q))^{-1} (A_1 + A_2 + A_4 Q) d_C(h_{n-1}, h_n) \ \forall n \in \mathbb{N}. \tag{3}$$

Putting $M = (I - (A_3 + A_4 Q))^{-1} (A_1 + A_2 + A_4 Q)$ in (3), we obtain

$$d_C(h_n, h_{n+1}) \leq M d_C(h_{n-1}, h_n) \ \forall n \in \mathbb{N}. \tag{4}$$

From (4), we conclude that

$$d_C(h_n, h_{n+1}) \leq M^n d_C(h_0, h_1) \forall n \in \mathbb{N}. \tag{5}$$

As $(h_{n-1}, h_n) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $\mathcal{R}$ is an equivalence relation, then by repeated application of the triangle inequality, i.e., Axiom (d_3), of Definition 2, we obtain

$$d_C(h_n, h_m) \leq \sum_{i=n}^{m-1} Q^i d_C(h_i, h_{i+1}) \ \forall m > n \in \mathbb{N}.$$

Thus, the above inequality and (5) yield the following inequality.

$$\begin{aligned} d_C(h_n, h_m) &\leq \sum_{i=n}^{m-1} Q^i d_C(h_i, h_{i+1}) \\ &\leq \sum_{i=n}^{m-1} Q^i M^i d_C(h_0, h_1) \\ &= \sum_{i=n}^{m-1} [QM]^i d_C(h_0, h_1) \ \forall m > n \ \text{(By Remark 5)} \\ &\leq [QM]^n (I - QM)^{-1} d_C(h_0, h_1). \end{aligned}$$

This proves that $\{h_n\}$ is an $\mathcal{R}$-Cauchy sequence in H. Considering the $\mathcal{R}$-completeness of H, we say that $\{h_n\}$ is an $\mathcal{R}$-convergent to $h_a \in H$, that is, $\lim_{n\to\infty} d_C(h_n, h_a) = 0$, and $(h_n, h_a) \in \mathcal{R}$ for all $n \geq k_0$, for some natural k_0. Now, consider Axiom (iii-a) exists, then we obtain $\lim_{n\to\infty} d_C(Gh_n, Gh_a) = 0$, and $(Gh_n, Gh_a) \in \mathcal{R}$, for all $n \geq k_0$.

Thus, through the triangle inequality, for each $n \geq k_0$, we obtain

$$d_C(h_a, Gh_a) \leq Q[d_C(h_a, h_{n+1}) + d_C(h_{n+1}, Gh_a)].$$

This yields $d_C(h_a, Gh_a) = 0$ as $n \to \infty$. That is, $h_a = Gh_a$. We now proceed with Axiom (iii-b). As $\{h_n\}$ is $\mathcal{R}$-convergent to $h_a \in H$, that is, $\lim_{n\to\infty} d_C(h_n, h_a) = 0$, and $(h_n, h_a) \in \mathcal{R}$ for all $n \geq k_0$ for some natural k_0. By (1), for each $n \geq k_0$, we obtain

$$\begin{aligned} d_C(Gh_n, Gh_a) &\leq A_1 d_C(h_n, h_a) + A_2 d_C(h_n, Gh_n) + A_3 d_C(h_a, Gh_a) \\ &\quad + A_4 d_C(h_n, Gh_a) + B d_C(h_a, Gh_n) \\ &\leq A_1 d_C(h_n, h_a) + A_2 d_C(h_n, Gh_n) + A_3 d_C(h_a, Gh_a) \\ &\quad + A_4 Q[d_C(h_n, h_{n+1}) + d_C(h_{n+1}, Gh_a)] + B d_C(h_a, Gh_n). \end{aligned}$$

That is,

$$\begin{aligned} d_C(h_{n+1}, Gh_a) &\leq A_1 d_C(h_n, h_a) + A_2 d_C(h_n, h_{n+1}) + A_3 d_C(h_a, Gh_a) \\ &\quad + A_4 Q[d_C(h_n, h_{n+1}) + d_C(h_{n+1}, Gh_a)] + B d_C(h_a, h_{n+1}). \end{aligned} \tag{6}$$

Applying the limit $n \to \infty$ in (6) we obtain

$$d_C(h_a, Gh_a) \leq A_3 d_C(h_a, Gh_a) + A_4 Q d_C(h_a, Gh_a).$$

This gives $d_C(h_a, Gh_a) = 0$ because $(I - A_3 - A_4 Q)^{-1}$ exists. Hence, $h_a = Gh_a$. $\square$

Remark 5. *If Q is a diagonal matrix and its nonzero elements are the same, then $Q^i M^i = (QM)^i$ $\forall i \in \mathbb{N}$.*

Example 3. *Consider $H = \mathbb{R}^2$, and the equivalence relation on H is defined by*

$$\mathcal{R} = \{((h_1, h_2), (k_1, k_2)) : h_1, h_2, k_1, k_2 \in [0, 3]\} \cup \{((h, k), (h, k)) : h, k \in \mathbb{R}\}.$$

Define a Czerwik vector-valued $\mathcal{R}$-metric on H by

$$d_C((h_1, h_2), (k_1, k_2)) = \begin{cases} \begin{pmatrix} (h_1 - k_1)^2 \\ (h_2 - k_2)^2 \end{pmatrix}, & \text{if } h_1, h_2, k_1, k_2 \in [0, 3] \\ \begin{pmatrix} \frac{|h_1 - k_1|}{1 + |h_1 - k_1|} \\ \frac{|h_2 - k_2|}{1 + |h_2 - k_2|} \end{pmatrix}, & \text{otherwise} \end{cases}$$

with $Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Define a mapping $G : H \to H$ by

$$G(h_1, h_2) = \begin{cases} \left(\frac{h_1}{6} - \frac{h_2}{3} + 2, \frac{h_2}{3} + 2 \right), & \text{if } h_1, h_2 \geq 0 \\ ((h_1 + h_2)^2, (h_2)^2), & \text{otherwise.} \end{cases}$$

Readers can easily verify the following points:

- *For $h = (0, 0)$, we have $Gh = (2, 2)$, thus, we say that $(h, Gh) \in \mathcal{R}$.*
- *For each $h_1, h_2 \in [0, 3]$, we have $\frac{h_1}{6} - \frac{h_2}{3} + 2, \frac{h_2}{3} + 2 \in [0, 3]$.*
  Thus, we say that $(G(h_1, h_2), G(k_1, k_2)) \in \mathcal{R}$, provided $((h_1, h_2), (k_1, k_2)) \in \mathcal{R}$.
- *For each sequence $(h_n^1), (h_n^2)$ with $h_n^1, h_n^2 \in [0, 3]$ and $h_n^1 \to h^1$, $h_n^2 \to h^2$, it is obvious that $h^1, h^2 \in [0, 3]$, we say that $\frac{h_n^1}{6} - \frac{h_n^2}{3} + 2 \to \frac{h^1}{6} - \frac{h^2}{3} + 2$, and $\frac{h_n^2}{3} + 2 \to \frac{h^2}{3} + 2$. Thus, we conclude that if $\{h_n\}$ is $\mathcal{R}$-convergent to $h \in H$, then $\{Gh_n\}$ is $\mathcal{R}$-convergent to Gh.*

- *For each* $((h_1,h_2),(k_1,k_2)) \in \mathcal{R}$ *with* $(h_1,h_2) \neq (k_1,k_2)$, *we have*

$$\begin{aligned} d_C(G(h_1,h_2),G(k_1,k_2)) &= d_C\left(\left(\frac{h_1}{6}-\frac{h_2}{3}+2,\frac{h_2}{3}+2\right),\left(\frac{k_1}{6}-\frac{k_2}{3}+2,\frac{k_2}{3}+2\right)\right) \\ &= \begin{pmatrix} \left((\frac{h_1}{6}-\frac{h_2}{3}+2-\frac{k_1}{6}+\frac{k_2}{3}-2\right)^2 \\ \left(\frac{h_2}{3}+2-\frac{k_2}{3}-2\right)^2 \end{pmatrix} \\ &\leq \begin{pmatrix} 2/36 & 2/9 \\ 0 & 2/9 \end{pmatrix}\begin{pmatrix} (h_1-k_1)^2 \\ (h_2-k_2)^2 \end{pmatrix} \\ &= \begin{pmatrix} 2/36 & 2/9 \\ 0 & 2/9 \end{pmatrix} d_C((h_1,h_2),(k_1,k_2)). \end{aligned}$$

- *For each* $((h_1,h_2),(k_1,k_2)) \in \mathcal{R}$ *with* $(h_1,h_2) = (k_1,k_2)$, *we have*

$$\begin{aligned} d_C(G(h_1,h_2),G(k_1,k_2)) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2/36 & 2/9 \\ 0 & 2/9 \end{pmatrix} d_C((h_1,h_2),(k_1,k_2)). \end{aligned}$$

Thus, it can be concluded that the axioms of Theorem 5 exist. Therefore, G has a fixed point.

Remark 6. *Note that the above-defined* d_C *is a Czerwik vector-valued* $\mathcal{R}$*-metric on* H*, but not a Czerwik vector-valued metric on* H*. Thus, the related fixed-point results on Czerwik vector-valued metric space from the existing literature are not applicable to this example.*

The following corollary is an extended form of Theorem 2 given in the introduction of this article.

Corollary 1. *Let* (H, d_C, Q) *be a complete Czerwik vector-valued metric space and let* $G : H \to H$ *be a mapping. Also, assume that*

(i) For each $h, k \in H$, *we have*

$$d_C(Gh,Gk) \leq A_1 d_C(h,k) + A_2 d_C(h,Gh) + A_3 d_C(k,Gk) + A_4 d_C(h,Gk) + B d_C(k,Gh) \tag{7}$$

where $A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ *such that* $(I-(A_3+A_4Q))^{-1}$ *exists and the matrix* $Q[(I-(A_3+A_4Q))^{-1}(A_1+A_2+A_4Q)]$ *is convergent to zero;*

(ii) Either

(a) If $\{h_n\}$ *is convergent to* $h \in H$*, then* $\{Gh_n\}$ *is convergent to* Gh*;*
or
(b) For each convergent sequence $\{h_n\}$ *in* H *with* $h_n \to h$*, we have* $d_C(h_n,\cdot) \to d_C(h,\cdot)$ *as* $n \to \infty$.

Then, G has a fixed point.

The conclusion of this result follows from Theorem 5 by considering an equivalence relation on H by $\mathcal{R} = H \times H$.

If we define an equivalence relation on H by $\mathcal{R} = H \times H$, then the complete Czerwik vector-valued metric space will also be an $\mathcal{R}$-complete Czerwik vector-valued $\mathcal{R}$-metric space. As we consider $\mathcal{R} = H \times H$, then Axioms (i) and (ii) of Theorem 5 trivially hold. Also, Axioms (i) and (ii) of the above theorem imply the existence of Axioms (iv) and (iii) of Theorem 5, respectively. Hence, the conclusion of the above results follows from Theorem 5.

The following corollary is an extended form of Theorem 1 given in the Section 1.

Corollary 2. *Let (H, d_C, Q) be a complete Czerwik vector-valued metric space and let $G : H \to H$ be a mapping such that for each $h, k \in H$, we have*

$$d_C(Gh, Gk) \leq A_1 d_C(h, k) \tag{8}$$

where $A_1 \in M_{m,m}(\mathbb{R}_+)$ such that the matrix QA_1 is convergent to zero. Then, G has a fixed point.

The conclusion of this result follows from Corollary 1, since (8) implies the existence of (7) and implies the existence of (iii-a).

In the following results, we will study the existence of fixed points for multi-valued mappings. We denote by $N(H)$ the collection of all nonempty subsets of H.

Theorem 6. *Let $(H, \mathcal{R}, d_C, Q)$ be an $\mathcal{R}$-complete Czerwik vector-valued $\mathcal{R}$-metric space and let $G : H \to N(H)$ be a mapping. Also, assume that*

(i) There exist $h \in H$ and $h_* \in Gh$ with $(h, h_*) \in \mathcal{R}$;

(ii) $\mathcal{R}$ is G-closed, that is, for each $h, k \in H$ with $(h, k) \in \mathcal{R}$, we have $(q, w) \in \mathcal{R}$ $\forall q \in Gh$ and $w \in Gk$;

(iii) $Graph(G) = \{(h, k) : k \in Gh\}$ is $\mathcal{R}$-closed, that is, for all $\mathcal{R}$-convergent sequences $\{h_n\}$ and $\{k_n\}$ in H with $h_n \to h_* \in H$ and $k_n \to k_* \in H$, we have $(h_*, k_*) \in Graph(G)$, whenever $(h_n, k_n) \in Graph(G)$ $\forall n \geq k_0$ for some k_0;

(iv) For each $(h, k) \in \mathcal{R}$ and $q \in Gh$, there exists $w \in Gk$ with

$$d_C(q, w) \leq A_1 d_C(h, k) + A_2 d_C(h, q) + A_3 d_C(k, w) + A_4 d_C(h, w) + B d_C(k, q) \tag{9}$$

where $A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ such that $(I - (A_3 + A_4 Q))^{-1}$ exists and the matrix $Q[(I - (A_3 + A_4 Q))^{-1}(A_1 + A_2 + A_4 Q)]$ is convergent to zero.

Then, G has a fixed point.

Proof. Assumption (i) of the theorem implies that there is some $h_0 \in H$ with $h_1 \in Gh_0$ and $(h_0, h_1) \in \mathcal{R}$. By using (9), for $(h_0, h_1) \in \mathcal{R}$ and $h_1 \in Gh_0$, there exists $h_2 \in Gh_1$ satisfying

$$d_C(h_1, h_2) \leq A_1 d_C(h_0, h_1) + A_2 d_C(h_0, h_1) + A_3 d_C(h_1, h_2) + A_4 d_C(h_0, h_2) + B d_C(h_1, h_1). \tag{10}$$

As $(h_0, h_1) \in \mathcal{R}$, then by assumption (ii), we obtain $(h_1, h_2) \in \mathcal{R}$. Now, (10) yields the following inequality:

$$d_C(h_1, h_2) \leq A_1 d_C(h_0, h_1) + A_2 d_C(h_0, h_1) + A_3 d_C(h_1, h_2) + A_4 Q[d_C(h_0, h_1) + d_C(h_1, h_2)].$$

That is,

$$d_C(h_1, h_2) \leq (I - A_3 - A_4 Q)^{-1}(A_1 + A_2 + A_4 Q) d_C(h_0, h_1).$$

By defining $M = (I - A_3 - A_4 Q)^{-1}(A_1 + A_2 + A_4 Q)$ in the above inequality, we obtain

$$d_C(h_1, h_2) \leq M d_C(h_0, h_1). \tag{11}$$

Again, by using (9), for $(h_1, h_2) \in \mathcal{R}$ and $h_2 \in Gh_1$, there exists $h_3 \in Gh_2$ with

$$d_C(h_2, h_3) \leq A_1 d_C(h_1, h_2) + A_2 d_C(h_1, h_2) + A_3 d_C(h_2, h_3) + A_4 d_C(h_1, h_3) + B d_C(h_2, h_2). \tag{12}$$

Since $(h_1, h_2) \in \mathcal{R}$, by assumption (ii), we obtain $(h_2, h_3) \in \mathcal{R}$. Now, (12) implies that

$$d_C(h_2, h_3) \leq A_1 d_C(h_1, h_2) + A_2 d_C(h_1, h_2) + A_3 d_C(h_2, h_3) + A_4 Q[d_C(h_1, h_2) + d_C(h_2, h_3)].$$

That is,

$$d_C(h_2, h_3) \leq M d_C(h_1, h_2) \tag{13}$$

where $M = (I - (A_3 + A_4 Q))^{-1}(A_1 + A_2 + A_4 Q)$. By (11) and (13), we obtain

$$d_C(h_2, h_3) \leq M^2 d_C(h_0, h_1). \tag{14}$$

Proceeding with the same methodology, we obtain a sequence $\{h_n\}$ such that $(h_{n-1}, h_n) \in \mathcal{R}$, $h_n \in Gh_{n-1}$ for all $n \in \mathbb{N}$ and

$$d_C(h_n, h_{n+1}) \leq M^n d_C(h_0, h_1) \ \forall n \in \mathbb{N}. \tag{15}$$

By using the triangular inequality and (15) we obtain

$$\begin{aligned} d_C(h_n, h_m) &\leq \sum_{i=n}^{m-1} Q^i d_C(h_i, h_{i+1}) \\ &\leq \sum_{i=n}^{m-1} [QM]^i d_C(h_0, h_1) \forall m > n. \end{aligned}$$

This yields that $\{h_n\}$ is an $\mathcal{R}$-Cauchy sequence in H. Thus, $\{h_n\}$ is $\mathcal{R}$-convergent to $h_a \in H$, that is $\lim_{n\to\infty} d_C(h_n, h_a) = 0$ and $(h_n, h_a) \in \mathcal{R}$, for all $n \geq k_0$ for some natural k_0. The construction of $\{h_n\}$ implies that $(h_n, h_{n+1}) \in Graph(G) \ \forall n \in \mathbb{N}$, since $Graph(G)$ is $\mathcal{R}$-closed, thus we obtain $(h_a, h_a) \in Graph(G)$, that is $h_a \in Gh_a$. This completes the proof of the result. □

In the following result, we assume that $B(H)$ is the collection of all those subsets of H that are d_C-bounded with respect to $(H, \mathcal{R}, d_C, Q)$, that is, for $A \in B(H)$, $\delta_C(A) = \overline{\sup}\{d_C(a,b) : a, b \in A\}$ exists. Also, we define $\delta_C(A, B) = \overline{\sup}\{d_C(a,b) : a \in A, b \in B\}$ and $\delta_C(a, B) = \overline{\sup}\{d_C(a,b) : b \in B\}$. Note that for the set $W = \{(a_{11}^j, a_{21}^j, \cdots, a_{n1}^j)^T : j \in I\}$, for some index set I,

$$\overline{\sup} W = \left(\sup_{j\in I} a_{11}^j, \sup_{j\in I} a_{21}^j, \cdots, \sup_{j\in I} a_{n1}^j \right)^T.$$

Theorem 7. *Let $(H, \mathcal{R}, d_C, Q)$ be an $\mathcal{R}$-complete Czerwik vector-valued $\mathcal{R}$-metric space and let $G : H \to B(H)$ be a mapping. Also, assume that*

(i) There exist $h \in H$ and $h_* \in Gh$ with $(h, h_*) \in \mathcal{R}$;

(ii) $\mathcal{R}$ is G-closed, that is, for each $h, k \in H$ with $(h, k) \in \mathcal{R}$, we have $(q, w) \in \mathcal{R} \ \forall q \in Gh$ and $w \in Gk$;

(iii) $Graph(G) = \{(h, k) : k \in Gh\}$ is $\mathcal{R}$-closed, that is, for all $\mathcal{R}$-convergent sequences $\{h_n\}$ and $\{k_n\}$ in H with $h_n \to h_* \in H$ and $k_n \to k_* \in H$, we have $(h_*, k_*) \in Graph(G)$, whenever $(h_n, k_n) \in Graph(G) \ \forall n \geq k_0$ for some k_0;

(iv) For each $(h, k) \in \mathcal{R}$, we have

$$\delta_C(Gh, Gk) \leq A_1 d_C(h, k) + A_2 \delta_C(h, Gh) + A_3 \delta_C(k, Gk) \tag{16}$$

where $A_1, A_2, A_3 \in \mathbb{M}_m(\mathbb{R}_+)$ such that $(I - A_3)^{-1}$ exists and the matrix $Q[(I - A_3)^{-1}(A_1 + A_2)]$ is convergent to zero.

Then, G has a fixed point.

Proof. Using hypothesis (i), we have $h_0 \in H$ and $h_1 \in Gh_0$ such that $(h_0, h_1) \in \mathcal{R}$. Hypothesis (ii) now gives $(q, w) \in \mathcal{R} \ \forall q \in Gh_0$ and $w \in Gh_1$. Thus, we write $(h_1, h_2) \in \mathcal{R}$ with $h_1 \in Gh_0$ and $h_2 \in Gh_1$. Further, we can construct a sequence $\{h_n\}$ with $h_n \in Gh_{n-1}$, and $(h_{n-1}, h_n) \in \mathcal{R}$ for all $n \in \mathbb{N}$. By (16), we obtain

$$\delta_C(Gh_{n-1}, Gh_n) \leq A_1 d_C(h_{n-1}, h_n) + A_2 \delta_C(h_{n-1}, Gh_{n-1}) + A_3 \delta_C(h_n, Gh_n) \ \forall n \in \mathbb{N}. \tag{17}$$

That is,

$$\delta_C(h_n, Gh_n) \le A_1\delta_C(h_{n-1}, Gh_{n-1}) + A_2\delta_C(h_{n-1}, Gh_{n-1}) + A_3\delta_C(h_n, Gh_n)\ \forall n \in \mathbb{N}.$$

This implies that

$$(I - A_3)\delta_C(h_n, Gh_n) \quad \le \quad (A_1 + A_2)\delta_C(h_{n-1}, Gh_{n-1})\ \forall n \in \mathbb{N}.$$

This inequality yields that

$$\delta_C(h_n, Gh_n) \quad \le \quad (I - A_3)^{-1}(A_1 + A_2)\delta_C(h_{n-1}, Gh_{n-1})\ \forall n \in \mathbb{N}. \tag{18}$$

Letting $M = (I - A_3)^{-1}(A_1 + A_2)$ in (18), we obtain

$$\delta_C(h_n, Gh_n) \quad \le \quad M\delta_C(h_{n-1}, Gh_{n-1})\ \forall n \in \mathbb{N}. \tag{19}$$

From the above inequality, we conclude the following inequalities:

$$\delta_C(h_n, Gh_n) \quad \le \quad M^n\delta_C(h_0, Gh_0)\ \forall n \in \mathbb{N} \tag{20}$$

and

$$d_C(h_n, h_{n+1}) \quad \le \quad M^n\delta_C(h_0, Gh_0)\ \forall n \in \mathbb{N}. \tag{21}$$

By considering the triangular inequality and (21), we obtain the following inequality:

$$\begin{aligned} d_C(h_n, h_m) \quad &\le \quad \sum_{i=n}^{m-1} Q^i d_C(h_i, h_{i+1}) \\ &\le \quad \sum_{i=n}^{m-1} [QM]^i \delta_C(h_0, Gh_0) \forall m > n. \end{aligned}$$

This proves that $\{h_n\}$ is an $\mathcal{R}$-Cauchy sequence in H. The $\mathcal{R}$-completeness of H ensures that $\{h_n\}$ is $\mathcal{R}$-convergent to $h_a \in H$, that is, $\lim_{n\to\infty} d_C(h_n, h_a) = 0$ and $(h_n, h_a) \in \mathcal{R}$, for all $n \ge k_0$ for some natural k_0. The construction of $\{h_n\}$ implies that $(h_n, h_{n+1}) \in Graph(G)\ \forall n \in \mathbb{N}$, since $Graph(G)$ is $\mathcal{R}$-closed, thus we obtain $(h_a, h_a) \in Graph(G)$, that is, $h_a \in Gh_a$. This completes the proof of the result. □

Example 4. *Consider $H = \mathbb{R}$ and an equivalence relation on H is defined by*

$$\mathcal{R} = \{(h_1, h_2) : h_1, h_2 \in [0, \infty)\} \cup \{(h, h) : h \in \mathbb{R}\}.$$

Define the Czerwik vector-valued $\mathcal{R}$-metric $d_C \colon H \times H \to \mathbb{R}_2$ by

$$d_C(h_1, h_2) = \begin{cases} \begin{pmatrix} (h_1 - h_2)^2 \\ (h_1 - h_2)^2 \end{pmatrix}, \text{ if } h_1, h_2 \ge 0 \\ \begin{pmatrix} \frac{|h_1 - h_2|}{1 + |h_1 - h_2|} \\ 0 \end{pmatrix}, \text{ otherwise.} \end{cases}$$

with $Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Define $G : H \to B(H)$ by

$$Gh = \begin{cases} \{(h+1)/2\},\ h \ge 0 \\ \{0\},\ \ h < 0. \end{cases}$$

The reader can easily verify that all the conditions of Theorem 7 are satisfied for this example. Hence, G has a fixed point.

4. Application

In this section, by the graph G we mean that $G = (V, E)$ is an undirected graph on a nonempty set H, such that $V = H$ and $E \subset H \times H$ contains all loops, that is, $(h, h) \in E$ for all $h \in H$, without any parallel edges. Define a path relation P_G on H equipped with the graph G: $(h, k) \in P_G$ if and only if there is a path from h to k in G. The relation P_G on H equipped with the graph G is reflexive, symmetric, and transitive, that is, $(h, h) \in P_G$ $\forall h \in H$, $(h, k) \in P_G \implies (k, h) \in P_G$ and $(h, k), (k, l) \in P_G$ implies $(h, l) \in P_G$.

Definition 4. *Let H be a nonempty set and let G be the graph on H. A mapping $d_C \colon H \times H \to \mathbb{R}_m$ is called a Czerwik vector-valued graphical metric on H, if for each $h_1, h_2, h_3 \in H$ the following axioms hold:*

(d_1) $d_C(h_1, h_2) \geq 0$;
$d_C(h_1, h_2) = 0$ *if and only if* $h_1 = h_2$;
(d_2) $d_C(h_1, h_2) = d_C(h_2, h_1)$;
(d_3) $d_C(h_1, h_3) \leq Q[d_C(h_1, h_2) + d_C(h_2, h_3)]$ *provided that* $(h_1, h_2), (h_2, h_3) \in P_G$,

where $Q = (q_{ij}) \in M_{m,m}(R_+)$ is a matrix with

$$q_{ij} = \begin{cases} q, \ i = j \\ 0, \ i \neq j \end{cases}$$

and $q \geq 1$. Then, (H, P_G, d_C, Q) is called a Czerwik vector-valued graphical metric space, or Czerwik generalized graphical metric space.

Remark 7. *Readers can note the following facts:*

(i) Czerwik vector-valued graphical metric space is a particular case of Czerwik vector-valued $\mathcal{R}$-metric space.
(ii) Czerwik vector-valued graphical metric space provides an extended concept of graphical b-metric space [11] as well as graphical metric space [10] over an undirected graph.

Definition 5. *Let (H, P_G, d_C, Q) be a Czerwik vector-valued graphical metric space. Then*

- *A sequence (h_n) in H is said to be P_G-sequence if $(h_n, h_{n+1}) \in E$ for each $n \in \mathbb{N}$.*
- *A P_G-sequence (h_n) in H is said to be P_G-convergent to h in H if $\lim_{n\to\infty} d_C(h_n, h) = 0$ and $(h_n, h) \in E$ $\forall n \geq k$ for some natural number k.*
- *A P_G-sequence (h_n) in H is said to be P_G-Cauchy if $\lim_{n,m\to\infty} d_C(h_n, h_m) = 0$.*
- *(H, P_G, d_C, Q) is said to be P_G-complete if each P_G-Cauchy sequence in H is P_G-convergent in H.*

Theorem 8. *Let (H, P_G, d_C, Q) be a P_G-complete Czerwik vector-valued graphical metric space and let $T : H \to H$ be a mapping. Also, assume that*

(i) There exists $h \in H$ with $(h, Th) \in E$;
(ii) For each $h_1, h_2 \in H$ with $(h_1, h_2) \in E$, we have $(Th_1, Th_2) \in E$;
(iii) Either
(a) If $\{h_n\}$ is P_G-convergent to $h \in H$, then $\{Th_n\}$ is P_G-convergent to Th;
or
(b) For each P_G-convergent sequence $\{h_n\}$ in H with $h_n \to h$, we have $d_C(h_n, \cdot) \to d_C(h, \cdot)$ as $n \to \infty$;
(iv) For each $(h, k) \in E$, we have

$$d_C(Gh, Gk) \leq A_1 d_C(h, k) + A_2 d_C(h, Gh) + A_3 d_C(k, Gk) + A_4 d_C(h, Gk) + B d_C(k, Gh) \quad (22)$$

where $A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ *such that* $(I - (A_3 + A_4Q))^{-1}$ *exists and the matrix* $Q[(I - (A_3 + A_4Q))^{-1}(A_1 + A_2 + A_4Q)]$ *is convergent to zero.*

Then, T *has a fixed point.*

The proof of this result is similar to the proof of Theorem 5.

5. Conclusions

This article presents the notion of Czerwik vector-valued $\mathcal{R}$-metric space, which is a generalized form of Czerwik vector-valued metric space. In Czerwik vector-valued $\mathcal{R}$-metric space, the triangle inequality is discussed only for comparable elements under an equivalence relation. The limit point of an $\mathcal{R}$-convergent sequence is unique in the Czerwik vector-valued $\mathcal{R}$-metric space. The set $\{B_{d_C}(h, \epsilon) : h \in H, \epsilon > 0\}$ provides a neighbourhood system for the topology on H induced by the Czerwik vector-valued $\mathcal{R}$-metric space. The existence of fixed points for single-valued and set-valued maps is also discussed using the Czerwik vector-valued $\mathcal{R}$-metric space.

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References

1. Fréchet, M. Sur quelques points du calcul fonctionnel. *Rend. Circ. Mat. Palermo* **1906**, *22*, 1–74. [CrossRef]
2. Czerwik, S. The contraction mapping principle in quasimetric spaces. *Acta Math. Univ. Ostrav.* **1993**, *1*, 5–11.
3. Matthews, S.G. Partial metric topology, In: Proc. 8th Summer Conference on General Topology and Applications. *Ann. N. Y. Acad. Sci.* **1994**, *728*, 183–197. [CrossRef]
4. Huang, L.G.; Zhang, X. Cone metric spaces and fixed point theorems of contractive mappings. *J. Math. Anal. Appl.* **2007**, *332*, 1468–1476. [CrossRef]
5. Perov, A.I. On the Cauchy problem for a system of ordinary differential equations. *Pviblizhen. Met. Reshen. Differ. Uvavn.* **1964**, *2*, 115–134.
6. Boriceanu, M. Fixed point theory on spaces with vector-valued b-metrics. *Demonstratio Math.* **2009**, *42*, 831–841.
7. Ali, M.U.; Kim, J.K. An extension of vector-valued metric spaces and Perov's fixed point theorem. *Nonlinear Anal. Convex Anal. Rims* **2019**, *2114*, 12–20.
8. Li, J.L.; Zhang, C.J.; Chen, Q.Q. Fixed point theorems on ordered vector spaces. *Fixed Point Theory Appl.* **2014**, *2014*, 109. [CrossRef]
9. Ali, M.U.; Postolache, M. On vector valued pseudo metrics and applications. *Trans. Razmadze Math. Inst.* **2018**, *172*, 309–317. [CrossRef]
10. Shukla, S.; Radenovic, S.; Vetro, C. Graphical metric space: A generalized setting in fixed point theory. *RACSAM* **2017**, *111*, 641–655. [CrossRef]
11. Chuensupantharat, N.; Kumam, P.; Chauhan, V.; Singh, D.; Menon, R. Graphic contraction mappings via graphical b-metric spaces with applications. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 3149–3165. [CrossRef]
12. Nieto, J.J.; Rodriguez-Lopez, R. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* **2005**, *22*, 223–239. [CrossRef]
13. Jachymski, J. The contraction principle for mappings on a metric space with a graph. *Proc. Am. Math. Soc.* **2008**, *136*, 1359–1373. [CrossRef]
14. Alam, A.; Imdad, M. Relation-Theoretic contraction principle. *J. Fixed Point Theory Appl.* **2015**, *17*, 693–702. [CrossRef]
15. Bucur, A.; Guran, L.; Petrusel, A. Fixed points for multivalued operators on a set endowed with vector-valued metrics and applications. *Fixed Point Theory* **2009**, *10*, 19–34.
16. Filip, A.D.; Petrusel, A. Fixed point theorems on spaces endowed with vector-valued metrics. *Fixed Point Theory Appl.* **2010**, *2010*, 281381. [CrossRef]
17. Ali, M.U.; Tchier, F.; Vetro, C. On the existence of bounded solutions to a class of nonlinear initial value problems with delay. *Filomat* **2017**, *31*, 3125–3135. [CrossRef]

18. Altun, I.; Hussain, N.; Qasim, M.; Al-Sulami, H. H. A new fixed point result of Perov type and its application to a semilinear operator system. *Mathematics* **2019**, *7*, 1019. [CrossRef]
19. Guran, L.; Bota, M.; Naseem, A. Fixed point problems on generalized metric spaces in Perov's sense. *Symmetry* **2020**, *12*, 856. [CrossRef]
20. Guran, L.; Bota, M.; Naseem, A.; Mitrović, Z.D.; Sen, M.d.l.; Radenović, S. On some new multivalued results in the metric spaces of Perov's type. *Mathematics* **2020**, *8*, 438. [CrossRef]
21. Martínez-Moreno, J.; Gopal, D. A Perov version of fuzzy metric spaces and common fixed points for compatible mappings. *Mathematics* **2021**, *9*, 1290. [CrossRef]
22. Varga, R.S. *Matrix Iterative Analysis*; Springer Series in Computational Mathematics; Springer: Berlin/Heidelberg, Germany, 2000; Volume 27.