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Article

Unveiling the Potential of Sheffer Polynomials: Exploring Approximation Features with Jakimovski–Leviatan Operators

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Abstract: In this article, we explore the construction of Jakimovski–Leviatan operators for Durrmeyer-type approximation using Sheffer polynomials. Constructing positive linear operators for Sheffer polynomials enables us to analyze their approximation properties, including weighted approximations and convergence rates. The application of approximation theory has earned significant attention from scholars globally, particularly in the fields of engineering and mathematics. The investigation of these approximation properties and their characteristics holds considerable importance in these disciplines.

Keywords: Durrmeyer-type Jakimovski–Leviatan operators; Sheffer polynomials; modulus of continuity; order of convergence; weighted space

MSC: 33C45; 33C99; 33E20

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1. Introduction and Preliminaries

The Approximation theory, which serves as a vital connection between pure and practical mathematics, has recently benefitted a variety of computer techniques. It covers the optimal method for approximating functions utilizing more straightforward or accessible functions and processes based on the use of contemporary approximation technology. In many situations requiring the approximation of continuous functions, positive approximation techniques naturally emerge. This is especially true when additional qualitative features, such as monotonicity, convexity, pattern preservation, etc., are required.

To address practical and theoretical inquiries in various fields, such as measurement theory, partial differential equations (PDEs), and probability theory, the positive approximation methods introduced by Korovkin [1] have emerged as crucial techniques. Korovkin's 1953 development of a simple and effective criterion for determining if a given series of positive linear operators on a space, $C[0,1]$, is an approximation process, i.e., if $\mathcal{K}_l(h) \to h$ uniformly on $[0,1]$ for every $h \in C[0,1]$, has had significant implications. This methodology has been extended to abstract spaces, like Banach lattices, Banach algebras, and Banach spaces, making it applicable in diverse contexts. Numerous researchers have explored the properties and convergence rates of Korovkin-type approximations. Notable contributions can be found in works such as [1–11]. These studies have provided valuable insights into approximation theory. Additionally, the literature offers further valuable information on approximation theories, as can be seen in references such as [12–16].

Appell polynomials [17] are a special set of functions used in mathematical analysis and various areas of applied mathematics. They are often used to represent and approximate other functions in a given context. These polynomials have important properties that make them useful in different mathematical and engineering applications. These polynomials are given by the generating relation:

$$J(\rho)\exp(u\rho) = \sum_{k=0}^{\infty} J_k(u)\frac{\rho^k}{k!}, \tag{1}$$

where $J(\rho)$ is expressed as follows:

$$J(\rho) = \sum_{k=0}^{\infty} J_k \frac{\rho^k}{k!}, \qquad \rho_0 \neq 0. \tag{2}$$

Recently, substantial strides and breakthroughs have emerged in the expansion of mathematical physics, with a particular focus on the realm of special functions. This modern evolution has established a robust analytical framework, serving as the bedrock for solving a multitude of intricate problems in the spheres of mathematical physics and engineering. These strides have reverberated across an array of domains, showcasing their versatile applications. Notably, the introduction of special functions endowed with approximation properties stands as a pivotal leap forward within the special functions theory. These functions have garnered acknowledgment for their paramount significance and pertinence in both pragmatic real-world applications and the realm of pure mathematics.

Karaisa [18] introduced Durrmeyer-type Jakimovski–Leviatan operators for a specified special polynomial sequence and, thus, are designed to operate on these special class of mathematical functions called Appell polynomials, given by Expression (1), denoted as $j_k(u)$, where k is a non-negative integer or zero.

The operators introduced by Karaisa are named after Durrmeyer, Jakimovski, and Leviatan, which suggests that these operators may have been inspired by or are related to the works of these mathematicians. The specific details of these operators and their properties are elaborated in [18].

The scope of application for these operators is restricted to real-valued continuous and bounded functions. In other words, the functions they operate on are defined for all non-negative real numbers and do not exhibit extreme behavior or fluctuations. This choice of function class is common in mathematical analysis, as it allows for well-behaved and manageable mathematical operations.

Moreover, the functions on which these operators act are defined on the interval, $[0, \infty)$. This interval spans from 0 to positive infinity and is often used when dealing with functions that have positive domain values or grow without bounds. These operators were specifically designed for real-valued continuous and bounded functions, f, defined on the interval, $[0, \infty)$:

$$\mathbf{L}_n(f;u) = \frac{e^{-nu}}{r(1)} \sum_{k=0}^{\infty} \frac{j_k(nu)}{\mathcal{B}(n+1,k)} \int_0^{\infty} \frac{\rho^{k-1}}{(1+\rho)^{n+k+1}} f(\rho)d\rho, \quad u \geq 0. \tag{3}$$

Here, $\mathcal{B}(k+1,n)$ denotes the Beta function, which is defined as follows:

$$\mathcal{B}(\eta,\vartheta) = \int_0^{\infty} \frac{\rho^{\eta-1}}{(1+\rho)^{\eta+\vartheta}} d\rho = \frac{\Gamma(\eta)\Gamma(\vartheta)}{\Gamma(\eta+\vartheta)}, \qquad (\eta,\vartheta \geq 0).$$

2. Construction of Operators

Building upon Karaisa's research [18], we develop positive linear operators utilizing Sheffer polynomials [19]. These polynomials possess generating expressions of the following form:

$$r(\rho)e^{u\mathbb{H}(\rho)} = \sum_{k=0}^{\infty} \mathcal{S}_k(u)\rho^k, \tag{4}$$

where the analytical functions r and $\mathbb{H}$ are defined as follows:

$$r(\rho) = \sum_{k=0}^{\infty} r_k \rho^k, \ \ r_0 \neq 0, \tag{5}$$

$$\mathbb{H}(\rho) = \sum_{k=0}^{\infty} h_k \rho^k, \ \ \ h_1 \neq 0 (k \geq 0). \tag{6}$$

The Sheffer polynomials fulfilling our restriction are listed as follows:

(i) $r(1) \neq 0, \ \frac{r_{m-k} h_k}{r(1)} \geq 0, \ 0 \leqslant k \leqslant m, m = 0, 1, 2, \cdots$

(ii) The generating function (3) and the power series presented above converge for values of

$$|\rho| < \mathcal{R}; \ \ \mathcal{R} > 1. \tag{7}$$

Additionally, the following method is used to introduce the positive linear operators using $\mathcal{S}_k(u)$ polynomials while taking the aforementioned limitations into account:

$$\frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \frac{\mathcal{S}_k(nu)}{\mathcal{B}(n+1,k)} \int_0^{\infty} \frac{\rho^{k-1}}{(1+\rho)^{n+k+1}} f(\rho) d\rho =: \mathbf{T}_n(f;u), \tag{8}$$

where $u \geq 0$ and $n \in \mathbb{N}$.

Remark 1. *For $\mathbb{H}(\rho) = e^{\rho}, \ \ \mathbb{H}(1) = 1$ and $\mathcal{S}_k(nu) = j_k(nu)$, in this case the operator (8) reduces to the operator given by (3).*

Remark 2. *For $\mathbb{H}(\rho) = e^{\rho}$, $g(\rho) = 1$ and $\mathcal{S}_k(nu) = \frac{(nu)^k}{k!}$ we obtain the Sźasz operators [10].*

3. Approximation Properties of $\mathbf{T}_n$ Operators

Significant advancements in the convergence of sequences, $(\mathcal{K}n(f,u))_{n=1}^{\infty}$, where $\mathcal{K}_n(f,u)$ represents positive linear operators, were made by Korovkin. Notably, if $\mathcal{K}_n(f,u)$ uniformly converges to f in specific scenarios, such as $1, \rho, \rho^2$ or $1, \cos\rho, \sin\rho \equiv f(\rho)$, it also exhibits this convergence behavior for any given function, f.

Our goal is to prove the convergence theorem and determine the order of convergence for the operators $\mathbf{T}_n(f;u)$, as given in Equation (8).

Lemma 1. *We extract the succeeding generating expression from (3) as follows:*

$$\begin{aligned}
r(1) e^{nu\mathbb{H}(1)} &= \sum_{k=0}^{\infty} \mathcal{S}_k(nu), \\
\big(r'(1) + nur(1)\big) e^{nu\mathbb{H}(1)} &= \sum_{k=0}^{\infty} k\mathcal{S}_k(nu), \\
\Big[n^2u^2 r(1) + \Big(2r'(1) + r(1)\mathbb{H}''(1) + r(1)\Big) nu + \Big(r''(1) + r'(1)\Big)\Big] e^{nu\mathbb{H}(1)} &= \sum_{k=0}^{\infty} k^2 \mathcal{S}_k(nu),
\end{aligned}$$

$$\begin{aligned}
&\Big[r(1)n^3u^3 + \Big(r(1)\mathbb{H}'''(1) + 3r'(1)\mathbb{H}''(1) + 3r''(1) + r(1) + 6r'(1) + 3r(1)\mathbb{H}''(1)\Big) nu \\
&+ \Big(3r(1)\mathbb{H}''(1) + 3r'(1) + 3r(1)\Big) n^2u^2 + \Big(r'''(1) + 3r''(1) + r'(1) + 3r''(1)\Big)\Big] e^{nu\mathbb{H}(1)} \\
&= \sum_{k=0}^{\infty} k^3 \, \mathcal{S}_k(nu), \\
&\Big[r(1)n^4u^4 + \Big(4r'(1) + 6r(1)\big(\mathbb{H}''(1) + 1\big)\Big) n^3u^3 + \Big(6r''(1) + r'(1)\big(12\mathbb{H}''(1) + 18\big)
\end{aligned}$$

$$+\big(3(\mathbb{H}''(1))^2+4\mathbb{H}'''(1)+18\mathbb{H}''(1)+7\big)r(1)\Big)n^2u^2+\Big(4r'''(1)+\big(6\mathbb{H}''(1)+18\big)r''(1)$$
$$+\big(4\mathbb{H}'''(1)+18\mathbb{H}''(1)+14\big)r'(1)+\big(\mathbb{H}''''(1)+6\mathbb{H}'''(1)+7\mathbb{H}''(1)+1\big)r(1)\Big)nu$$
$$+\Big(r''''(1)+6r'''(1)+7r''(1)+r'(1)\Big)\Big]e^{nu\mathbb{H}(1)}=\sum_{k=0}^{\infty}k^4\mathcal{S}_k(nu).$$

Lemma 2. *For all $u\in[0,\infty)$ and $e_i=\rho^i$, we have the following:*

$$\mathbf{T}_n(e_0;u) = 1, \quad (9)$$
$$\mathbf{T}_n(e_1;u) = u+\frac{\mathbb{A}_0}{n}, \quad (10)$$
$$\mathbf{T}_n(e_2;u) = \frac{1}{F_1^-}\Big[n^2u^2+nu\,\mathbb{B}_0+\mathbb{C}_0\Big], \quad (11)$$
$$\mathbf{T}_n(e_3;u) = \frac{1}{F_2^-}\Big[n^3u^3+n^2u^2\mathbb{A}_1+nu\,\mathbb{A}_2+\mathbb{A}_3\Big], \quad (12)$$
$$\mathbf{T}_n(e_4;u) = \frac{1}{F_3^-}\Big[n^4u^4+n^3u^3\mathbb{B}_1+n^2u^2\mathbb{B}_2+nu\,\mathbb{B}_3+\mathbb{B}_4\Big], \quad (13)$$

where

$$e_n=\rho^n,\mathbb{A}_0=\frac{r'(1)}{r(1)},\mathbb{B}_0=\frac{2r(1)+2r'(1)\mathbb{H}''(1)r(1)}{r(1)},\mathbb{C}_0=\frac{r''(1)+2r'(1)}{r(1)} \quad (14)$$

and

$$\mathbb{A}_1 = \frac{3r'(1)+r(1)(6+3\mathbb{H}''(1))}{r(1)},$$
$$\mathbb{A}_2 = \frac{3r''(1)+r'(1)(12+3\mathbb{H}''(1))+r(1)(\mathbb{H}'''(1)+6\mathbb{H}''(1)+5)}{r(1)},$$
$$\mathbb{A}_3 = \frac{r'''(1)+6r''(1)+5r'(1)}{r(1)}, \quad (15)$$

$$\mathbb{B}_1 = \frac{4r'(1)+r(1)(12+6\mathbb{H}''(1))}{r(1)},$$
$$\mathbb{B}_2 = \frac{6r''(1)+r'(1)(36+12\mathbb{H}''(1))+r(1)(4\mathbb{H}'''(1)+3(\mathbb{H}''(1))^2+36\mathbb{H}''(1)+32)}{r(1)},$$
$$\mathbb{B}_3 = \frac{4r^3(1)+36r''(1)+64r'(1)+17r(1)}{r(1)},$$
$$\mathbb{B}_4 = \frac{r^4(1)+12r^3(1)+32r''(1)+17r'(1)}{r(1)}$$

and

$$F_n^- = \Pi_{i=0}^{n}(n-i). \quad (16)$$

Proof. Substituting $f(\rho)=1$ into operator Equation (8), we obtain the following expression:

$$\mathbf{T}_n(e_0;u) = \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)}\sum_{k=0}^{\infty}\frac{\mathcal{S}_k(nu)}{\mathcal{B}(n+1,k)}\int_0^{\infty}\frac{\rho^{k-1}}{(1+\rho)^{n+k+1}}d\rho$$

$$
\begin{aligned}
&= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \frac{\mathcal{S}_k(nu)\,\mathcal{B}(k,n+1)}{\mathcal{B}(n+1,k)} \\
&= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} e^{nu\mathbb{H}(1)\; r(1)} \\
&= 1. \qquad (17)
\end{aligned}
$$

By substituting $f(\rho) = \rho$ into operator Equation (8), we obtain the following expression:

$$
\begin{aligned}
\mathbf{T}_n(e_1;u) &= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \frac{\mathcal{S}_k(nu)}{\mathcal{B}(n+1,k)} \int_0^{\infty} \frac{\rho^k}{(1+\rho)^{n+k+1}} d\rho \\
&= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \frac{\mathcal{S}_k(nu)}{\mathcal{B}(n+1,k)} \mathcal{B}(n,k+1) \\
&= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \mathcal{S}_k(nu) \frac{k}{n} \\
&= \frac{e^{-nu\mathbb{H}(1)}}{n\, r(1)} \sum_{k=0}^{\infty} k\, \mathcal{S}_k(nu) \\
&= u + \frac{r'(1)}{nr(1)} \\
&= u + \frac{\mathbb{A}_0}{n}. \qquad (18)
\end{aligned}
$$

By substituting $f(\rho) = \rho^2$ into operator Equation (8), it follows that

$$
\begin{aligned}
\mathbf{T}_n(e_2;u) &= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \frac{\mathcal{S}_k(nu)}{\mathcal{B}(n+1,k)} \int_0^{\infty} \frac{\rho^{k+1}}{(1+\rho)^{n+k+1}} d\rho \\
&= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \frac{\mathcal{S}_k(nu)}{\mathcal{B}(n+1,k)} \mathcal{B}(n-1,k+2) \\
&= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \mathcal{S}_k(nu) \frac{k+k^2}{F_1^-} \\
&= \frac{e^{-nu\mathbb{H}(1)}}{F_1^- r(1)} \sum_{k=0}^{\infty} k\, \mathcal{S}_k(nu) + k^2\, \mathcal{S}_k(nu) \\
&= \frac{e^{-nu\mathbb{H}(1)}}{F_1^-} \Big[n^2u^2 r(1) + nu\big(2r'(1) + r(1)(2+\mathbb{H}''(1))\big) \\
&\quad + r''(1) + (r'(1))^2 + nur(1) \Big] e^{nu\mathbb{H}(1)} \\
&= \frac{1}{F_1^-}\Big[n^2u^2 + nu\mathbb{B}_0 + \mathbb{C}_0\Big]. \qquad (19)
\end{aligned}
$$

Consequently, by substituting $f(\rho) = \rho^3$ into operator Equation (8), we obtain the following expression:

$$
\begin{aligned}
\mathbf{T}_n(e_3;u) &= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \frac{\mathcal{S}_k(nu)}{\mathcal{B}(n+1,k)} \int_0^{\infty} \frac{\rho^{k+2}}{(1+\rho)^{n+k+1}} d\rho \\
&= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \frac{\mathcal{S}_k(nu)}{\mathcal{B}(n+1,k)} \mathcal{B}(n-2,k+3) \\
&= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \mathcal{S}_k(nu) \frac{k+3k^2+k^3}{F_2^-}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^{-nu\mathbb{H}(1)}}{F_2^- r(1)} \sum_{k=0}^{\infty} k\mathcal{S}_k(nu) + 3k^2\mathcal{S}_k(nu) + k^3\mathcal{S}_k(nu) \\
&= \frac{e^{-nu\mathbb{H}(1)}}{F_2^-} \Big[n^3u^3 + n^2u^2\mathbb{A}_1 + nu\mathbb{A}_2 + \mathbb{A}_3 \Big] \qquad (20)
\end{aligned}
$$

Again, put $f(\rho) = \rho^4$ in operator Equation (8), we have the following:

$$
\begin{aligned}
\mathbf{T}_n(e_4;u) &= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \frac{\mathcal{S}_k(nu)}{\mathcal{B}(n+1,k)} \int_0^{\infty} \frac{\rho^{k+3}}{(1+\rho)^{n+k+1}} d\rho \\
&= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \frac{\mathcal{S}_k(nu)}{\mathcal{B}(n+1,k)} \mathcal{B}(n-3,k+4) \\
&= \frac{(r(1))^{-1}}{\exp nu\mathbb{H}(1)} \sum_{k=0}^{\infty} \mathcal{S}_k(nu) \frac{3k + 7k^2 + 6k^3 + k^4}{F_3^-} \\
&= \frac{e^{-nu\mathbb{H}(1)}}{F_3^- r(1)} \sum_{k=0}^{\infty} 3k\mathcal{S}_k(nu) + 7k^2\mathcal{S}_k(nu) + 6k^3\mathcal{S}_k(nu) + k^4\mathcal{S}_k(nu) \\
&= \frac{e^{-nu\mathbb{H}(1)}}{F_3^-} \Big[n^4u^4 + n^3u^3\,\mathbb{B}_1 + n^2u^2\,\mathbb{B}_2 + nu\,\mathbb{B}_3 + \mathbb{B}_4 \Big]. \qquad (21)
\end{aligned}
$$

□

Lemma 3. *The following identities hold for the operators, $\mathbf{T}_n(f;u)$, and for $u \in [0,\infty)$:*

$$
\mathbf{T}_n(s-u;u) = \frac{\mathbb{A}_0}{n}, \qquad (22)
$$

$$
\begin{aligned}
\mathbf{T}_n((s-u)^2;u) &= \left(\frac{u^2 + 2u + 4\mathbb{H}''(1)}{n-1} \right) \\
&\quad + \left(\frac{2ur'(1)}{n(n-1)r(1)} \right) + \left(\frac{r''(1) + 2r'(1)}{n(n-1)r(1)} \right) \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{T}_n((s-u)^4;u) &= \left(\frac{n^4}{F_3^-} - 4\frac{n^3}{F_2^-} + 6\frac{n^2}{F_1^-} - 4 + 1 \right) u^4 \\
&\quad + \left(\frac{n^3}{F_3^-}\mathbb{B}_1 - 4\frac{n^2}{F_2^-}\mathbb{A}_1 + 6\frac{n}{F_1^-}\mathbb{B}_0 - 4\frac{\mathbb{A}_0}{F_0^-} \right) u^3 \\
&\quad + \left(\frac{n^2}{F_3^-}\mathbb{B}_2 - 4\frac{n}{F_2^-}\mathbb{A}_2 + 6\frac{\mathbb{C}_0}{F_1^-} \right) u^2 \\
&\quad + \left(\frac{n}{F_3^-}\mathbb{B}_3 - 4\frac{\mathbb{A}_3}{F_2^-} \right) u + \frac{\mathbb{B}_4}{F_3^-} \qquad (24)
\end{aligned}
$$

where $\mathbb{A}_0$, $\mathbb{B}_0$, $\mathbb{C}_0$, $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{B}_3$, $\mathbb{B}_4$, and F_1^- are given by Equations (14)–(17).

Proof. Due to the linearity property of $\mathbf{T}_n$, it can be inferred that

$$
\begin{aligned}
\mathbf{T}_n(s-u;u) &= \mathbf{T}_n(s;u) - u\mathbf{T}_n(1;u), \\
\mathbf{T}_n((s-u)^2;u) &= \mathbf{T}_n(s^2;u) - 2u\mathbf{T}_n(s;u) + u^2\mathbf{T}_n(1;u), \\
\mathbf{T}_n((s-u)^4;u) &= \mathbf{T}_n(s^4;u) - 4u\mathbf{T}_n(s^3;u) + 6u^2\mathbf{T}_n(s^2;u), \\
&- 4u^3\mathbf{T}_n(s;u) + u^4\mathbf{T}_n(1;u),
\end{aligned}
$$

By utilizing Lemma (2), we can deduce statements (22), (23), and (24) accordingly. □

Theorem 1. *For $f \in C[0,\infty) \cap E$ and $|f(u)| \leqslant ce^{Au}$, then $\lim_{n\to\infty} \mathbf{T}_n(f;u) = f(u)$ and the operators, $\mathbf{T}_n$, converge uniformly in each compact subset of $[0,\infty)$, where $E := f :$ for all $u \in [0,\infty)$, $A \in \mathbb{R}$, and $c \in \mathbb{R}^+$.*

Proof. Considering Lemma (2), it can be deduced that

$$\lim_{n\to\infty} \mathcal{T}_n(e_i, u) = u^i, \quad i = 0,1,2.$$

Uniform convergence is verified on every compact subset of $[0,\infty)$. By applying Korovkin's theorem, we obtain the desired result. □

4. Order of Convergence

By employing the definitions of modulus II of continuity and Peetre's $\mathbb{K}$-functional, along with the lemmas of Gavrea and Raşa [2] and Zhuk [20], the following conclusions can be drawn:

Theorem 2. *Consider a function, fm defined on $C[0,a]$. We can establish the following results:*

$$|\mathbf{T}_n(f;u) - f(u)| \leq 2h^2 a||f|| + \frac{3}{4}(a+2+h^2)w_2(f;h), \tag{25}$$

where

$$\sqrt[4]{\mathbf{T}_n((s-u)^2;u)} = h := h_n(u).$$

Proof. Let us consider the second-order Steklov function, f_h, associated with f. Consequently, with the presence of Expression (9), we can deduce the following:

$$\begin{aligned} |\mathbf{T}_n(f;u) - f(u)| &\leq |\mathbf{T}_n(f - f_h;u)| + |f_h(u) - f(u)| + |\mathbf{T}_n(f_h;u) - f_h(u)|, \\ &\leq |\mathbf{T}_n(f_h;u) - f_h(u)| + 2||f_h - f||. \end{aligned} \tag{26}$$

By utilizing the inequality, $||g_h - g|| \leq \frac{3}{4}\mathbb{W}_2(g;h)$ becomes

$$|\mathbf{T}_n(f;u) - f(u)| \leq \frac{3}{2}w_2(f;h) + |\mathbf{T}_n(f_h;u) - f_h(u)|. \tag{27}$$

Considering that $f_h \in C^2[0,a]$, we can infer the following from Lemma (12):

$$|\mathbf{T}_n(f_h;u) - f_h(u)| \leq ||f_h'||\sqrt{\mathbf{T}_n((s-u)^2;u)} + \frac{1}{2}\,||f_h''||\,\mathbf{T}_n((s-u)^2;u), \tag{28}$$

By utilizing the inequality, $||g_h''|| \leq \frac{3}{2h^2}\mathbb{W}_2(g;h)$, Expression (28) gives

$$|\mathbf{T}_n(f_h;u) - f_h(u)| \leq ||f_h'||\sqrt{\mathbf{T}_n((s-u)^2;u)} + \frac{3}{4h^2}w_2(f;h)\mathbf{T}_n((s-u)^2;u). \tag{29}$$

Furthermore, by utilizing inequality

$$||f_h'|| \leq \frac{2}{\eta}||f_h|| + \frac{a}{2}||f_h''||,$$

in combination with $||g_h''|| \leq \frac{3}{2h^2}\mathbb{W}_2(g;h)$ gives

$$||f_h'|| \leq \frac{2}{a}||f|| + \frac{3a}{4h^2}w_2(f;h). \tag{30}$$

By incorporating inequality (30) into inequality (29) and setting $h = \sqrt[4]{\mathbf{T}_n((s-u)^2;u)}$, we obtain the following:

$$|\mathbf{T}_n(f_h;u) - f_h(u)| \leq \frac{2}{a}||f||\, h^2 + \frac{3}{4}(a+h^2)w_2(f;h). \tag{31}$$

By employing inequality (31) and inequality (27), we can establish assertion (25). □

Theorem 3. *If $f \in C_B^2[0\infty)$, then*

$$|\mathbf{T}_n(f;u) - f(u)| \leq \xi||f||_{C_B^2}, \tag{32}$$

where

$$\begin{aligned}\xi: &= \xi_n(u)\\ &= \frac{1}{n-1}\left[\frac{u^2}{2} + \left(n(1+\frac{\mathbb{H}''(1)}{2}) + \frac{r'(1)}{r(1)}\right)\frac{u}{n} + \frac{\frac{r''(1)}{2}+r'(1)}{nr(1)}\right].\end{aligned}$$

Proof. By leveraging the linearity property of the operator, $\mathbf{T}_n$; Taylor's expansion of f; and statement (9), the following becomes apparent:

$$\mathbf{T}_n(f;u) - f(u) = f'(u)\mathbf{T}_n(s-u;u) + \frac{1}{2}f''(\eta)\mathbf{T}_n((s-u)^2;u), \quad \eta \in (u,s). \tag{33}$$

Based on Lemma (2), the following can be observed:

$$\mathbf{T}_n(s-u;u) = \frac{\mathbb{A}_0}{n} \geq 0 \tag{34}$$

For $s \geq u$, by incorporating Lemma (2) and (11) into (33), we can express this as follows:

$$\begin{aligned}|\mathbf{T}_n(f;u) - f(u)| &\leq \left\{\frac{r'(1)}{n\,r(1)}\right\}||f'||_{\mathbb{C}_B}\\ &+ \frac{1}{2}\left\{\frac{u^2+2u+u\mathbb{H}''(1)}{n-1} + \frac{2ur'(1)}{n(n-1)r(1)} + \frac{r''(1)+2r'(1)}{n(n-1)r(1)}\right\}||f||_{C_B^2}.\end{aligned}$$

This concludes the proof. □

Corollary 1. *Let $f \in \mathbb{C}_B[0,\infty)$; thus, one has*

$$|\mathbf{T}_n(f;u) - f(u)| \leqslant 2M\{w_2(f;\sqrt{\delta}) + min(1,\delta)||f||_{\mathbb{C}_B}\}, \tag{35}$$

where

$$\delta := \frac{1}{2}\xi_n(u) = \delta_n(u)$$

Moreover, the constant, $M \geq 0$, remains fixed and is independent of both f and δ. Additionally, $\xi_n(u)$ exhibits similarity to that of Theorem (3).

Proof. Let us consider $g \in C_B^2[0,\infty)$. According to Theorem (3), we obtain the following:

$$\begin{aligned}|\mathbf{T}_n(f;u) - f(u)| &\leqslant |\mathbf{T}_n(f-r;u)| + |\mathbf{T}_n(r;u) - r(u)| + |r(u) - f(u)|\\ &\leqslant 2||f-r||_{\mathbb{C}_B} + \xi||r||_{C_B^2} = 2\left[||f-r||_{\mathbb{C}_B} + \delta||r||_{C_B^2}\right]\end{aligned} \tag{36}$$

As the left-hand side is not influenced by the choice of $r \in C_B^2[0,\infty)$, we can conclude the following:

$$|\mathbf{T}_n(f;u) - f(u)| \leqslant 2K(f;\delta), \tag{37}$$

Assertion (35) is established by utilizing Peetre's k-functional. □

5. Weighted Approximation

Weighted approximation is a mathematical technique that involves modifying the traditional notion of approximation by assigning varying degrees of importance or significance (weights) to different parts of a given function or data set. This approach has several advantages and significant applications across various fields of mathematics, science, and engineering, as many real-world phenomena exhibit variations and complexities that are not adequately captured by simple uniform approximations. This allows for a more flexible and nuanced representation of such phenomena by assigning different weights to different regions or points, emphasizing their importance or relevance. It further enables the customization of approximation methods to meet specific requirements or constraints. By assigning appropriate weights, it becomes possible to prioritize certain characteristics or properties of the function being approximated, resulting in a more accurate representation within a defined context. Further, it can enhance numerical stability by reducing the influence of oscillations or rapid changes in the function being approximated. This can lead to more stable and robust algorithms, especially in numerical analysis and computational mathematics.

In the subsequent section, we present certain approximation properties for the operator, $\mathbf{T}_n$, within a space of weighted continuous functions. Specifically, we focus on a particular class of functions defined on $[0,\infty)$.

Let us consider the set of functions, h, that satisfy the condition, $|\mathbb{H}(y)| \leq M_{\mathbb{H}}(1+u^2)$, where M_h is a constant dependent on h. Here, $\mathbb{B}u^2[0,\infty)$ denotes the space defined on $[0,\infty)$, and $\mathbb{C}u^2[0,\infty)$ represents the subspace of $\mathbb{B}_u^2[0,\infty)$ consisting of all continuous functions. Moreover, $C_{u^2}^*[0,\infty)$ as the subspace of $h \in \mathbb{C}_{u^2}[0,\infty)$, for which $\lim_{|u|\to\infty} \frac{\mathbb{H}(u)}{1+u^2}$ is finite. It is evident that $C_{u^2}^*[0,\infty) \subset \mathbb{C}_{u^2}[0,\infty) \subset \mathbb{B}_u^2[0,\infty)$. The norm on $C_{u^2}^*[0,\infty)$ is given as follows:

$$||h||_{u^2} = \sup_{u\in[0,\infty)} \frac{|\mathbb{H}(u)|}{1+u^2}. \tag{38}$$

Lemma 4. *Let the weight function $\rho(u) = 1+u^2$. If $h \in \mathbb{C}_{y^2}[0,\infty)$, then*

$$||\mathbf{T}_n(\rho;u)||_{u^2} \leq 1+M.$$

Proof. Using Equations (9) and (11) from Lemma (2), we can deduce the following expressions for $n>1$:

$$\mathbf{T}_n(\rho;u) = 1 + \frac{1}{F_1}\Big[n^2u^2 + nu\,\mathbb{B}_0 + \mathbb{C}_0\Big]. \tag{39}$$

Then, we deduce

$$||\mathbf{T}_n(\rho;u)||_{u^2} = \sup_{u\geq 0}\Bigg[\frac{1}{1+u^2} + \frac{nu^2}{(1+u^2)(n-1)} + 2u\Bigg(\frac{r'(1)+r(1)\big(1+\frac{\mathbb{H}''(1)}{2}\big)}{r(1)(1+u^2)(n-1)}\Bigg) + \frac{r''(1)+2r'(1)}{(1+u^2)r(1)(n-1)}\Bigg]$$

$$\leq \sup_{u\geq 0}\Bigg\{1 + \frac{n}{(n-1)} + \frac{r'(1)+r(1)\big(1+\frac{\mathbb{H}''(1)}{2}\big)}{r(1)(n-1)} + \frac{r''(1)+2r'(1)}{r(1)(n-1)}\Bigg\}.$$

We have
$$\lim_{n\to\infty}\frac{n}{n-1}=1,\ \lim_{n\to\infty}\frac{1}{n-1}=0\ \textit{and}\ \lim_{n\to\infty}\frac{1}{n^2-n}=0.$$
Thus, in view of these assumptions, there exists a positive constant, M, such that
$$||\mathbf{T}_n(\rho;u)||_{u^2}\le 1+M.$$
Thus, the proof is deduced. □

Theorem 4. *Consider the operators, $\mathbf{T}_n$, defined by Equation (8) and the weight function, $\rho(u)=1+u^2$. For any $f\in C^*u^2[0,\infty)$, the following holds:*
$$\lim_{n\to\infty}||\mathbf{T}_n(f;u)-f(u)||_{u^2}=0.$$

Proof. Applying the weighted Korovkin theorem as stated by Gadzhiev [21], it suffices to verify the following conditions:
$$||\mathbf{T}_n(1;u)-1||_{u^2}=0. \tag{40}$$
Expression (10) follows
$$||\mathbf{T}_n(e_1;u)-e_1(u)||_{u^2}=\sup_{u\ge 0}\left|\frac{u}{1+u^2}+\frac{r'(1)}{nr(1)(1+u^2)}-\frac{u}{1+u^2}\right|. \tag{41}$$
Thus,
$$\lim_{n\to\infty}||\mathbf{T}_n(e_1;u)-e_1(u)||_{u^2}=0, \tag{42}$$
Then, in view of (11), it follows that
$$\begin{aligned}||\mathbf{T}_n(e_2;u)-e_2(u)||_{u^2} &= \sup_{u\ge 0}\left|\frac{nu^2}{(n-1)(1+u^2)}\right.\\ &\quad+\left.\frac{nu\mathbb{B}_0}{(n^2-n)(1+u^2)}+\frac{\mathbb{C}_0}{n(n-1)(1+u^2)}-\frac{u^2}{1+u^2}\right|.\end{aligned}$$
Therefore,
$$\lim_{n\to\infty}||\mathbf{T}_n(e_2;u)-e_2(u)||_{u^2}=0.$$
Therefore, the proof is deduced. □

6. Concluding Remarks

This article delved into the construction of Jakimovski–Leviatan operators for Durrmeyer-type approximations utilizing Sheffer polynomials. By developing positive linear operators based on Sheffer polynomials, we conducted a comprehensive analysis of their approximation properties, encompassing weighted approximations and convergence rates. The field of approximation theory has received substantial attention from researchers worldwide, particularly within the realms of engineering and mathematics. The study of these approximation properties and their underlying characteristics holds significant importance in advancing these disciplines.

One can examine the error estimation for the approximation using the Sheffer family of operators. Additionally, it is possible to illustrate the approximate solution, $\check{f}(t)$, of any continuous function using positive linear operators, $\mathbf{T}_n(f;u)$. In a subsequent study, extensions of these operators, such as Sheffer polynomials and other members of the Kantrovich–Stancu type, will be utilized to investigate the Kantrovich–Stancu type more extensively.

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