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Review

Graphs Defined on Rings: A Review

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Abstract: The study on graphs emerging from different algebraic structures such as groups, rings, fields, vector spaces, etc. is a prominent area of research in mathematics, as algebra and graph theory are two mathematical fields that focus on creating and analysing structures. There are numerous studies linking algebraic structures and graphs, which began with the introduction of Cayley graphs of groups. Several algebraic graphs have been defined on rings, a fast-growing area in the literature. In this article, we systematically review the literature on some variants of Cayley graphs that are defined on rings and highlight the properties and characteristics of such graphs, to showcase the research in this area.

Keywords: unitary Cayley graphs; Euler totient Cayley graphs; unitary addition Cayley graphs; unit graphs; absorption Cayley; nilpotent Cayley graphs; zero-divisor Cayley graphs; mixed unitary Cayley graphs; divisor Cayley graphs; involutory Cayley graphs; quadratic residue Cayley graphs

MSC: 05C25; 05C75; 05C10; 05C22; 05C69; 05C09; 05C15

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1. Introduction

Graph theory and algebra are two disciplines of mathematics which concentrate on building and investigating structures. Algebra is a fundamental branch of mathematics, whose roots can be traced back to the early 16th century, whereas graph theory is a flourishing mathematical research area which began unfolding in the early 18th century, as the Swiss mathematician solved the famous eponymous *Königsberg bridge problem* by representing the structure of the bridge and the landmass surrounding it as a graph. Hence, this subject emerged as a consequence of modeling real-life problems in terms of graphs, as it gives a comprehensive visual representation of the problem, aiding in obtaining optimal and feasible solutions to the problem. It is interesting to note that, along with the increase in applications of the developed subtheories, the theory by itself has evolved independently over time and has established itself as a flourishing mathematical discipline.

An algebraic structure is a non-empty set along with one or more operations (usually binary) defined on it, and, by the very definition of a graph, it can be noticed that a graph can be realised as a structural representation of a relation defined on a (vertex) set. Relating these two structural aspects, a synergy between algebraic and graphical structures is studied in the field of *algebraic graph theory*. It has become a stimulating research field, yielding numerous intriguing results, as these two disciplines—algebra and graph theory—interact in many ways to mutually extend the tools of one subject for the benefit of the other. In fact, powerful combinatorial methods found in graph theory have been used to prove specific, significant, and well-known results in group theory. For example, all finite groups can be represented as the automorphism group of a connected graph (c.f. [1]).

Any algebraic structure can be interpreted as a graph, and there are multiple ways to associate an algebraic structure with a graph. In the past few decades, several graphs have been constructed from algebraic structures based on different properties possessed by the algebraic structures, and these algebraic graphs have been studied extensively, in order to understand the algebraic structure more clearly, thereby making this an enthralling area of research (c.f. [2–7]).

This association of an algebraic structure with a graph began at the end of the 19th century, when Arthur Cayley connected graph theory and group theory by introducing the *Cayley graph* of a group (c.f. [8]), which encodes the algebraic information of a group as a graphical structure. The *Cayley graph* for a group $\mathscr{G}$ is a graph with the vertex set as the elements of the group $\mathscr{G}$, which is invariant under the right translation by elements of $\mathscr{G}$. Cayley graphs are, by far, the most well-known graphs associated with algebraic structure. There is massive yet still growing segment of the literature dedicated to convincing the community that algebraic graph theory is only the study of Cayley graphs of finite groups (see [9–14]).

The studies on Cayley graphs paved the path to the construction and investigation of certain graph classes such as circulant and transitive graphs (refer to [15,16]). It was proven that all circulant graphs are Cayley graphs and every Cayley graph is vertex-transitive. This aided researchers to construct various families of circulant and transitive graphs, according to the requirements concerned, as circulant graphs are highly stable and reliable networks that are used in modeling real-life situations.

Another important class of algebraic graph construction is the construction of graphs from rings, as the study of graphs constructed from rings contributes to an interplay between the ring structure and the corresponding graph structure. One can sometimes translate the algebraic properties of the rings in terms of graph-theoretic properties and vice versa, which can help in exploring some interesting results related to the graphs as well as to the rings. Graphs defined on rings either have vertices as the set of elements of the ring or they are intersection graphs, such that each vertex represents some subset of the ring or some well-known sub-structure of the ring such as ideals, subring, etc.; additionally, the edges are defined with respect to an algebraic condition on the elements of the vertex set.

The study of graphs defined from rings began with the introduction of the zero-divisor graph, which is one of the most well-studied graphs defined on commutative rings, having a massive (and still augmenting) portion of the literature (see [17–19]). Apart from the zero-divisor graphs, there are several other graphs, such as total graphs, annihilating graphs, comaximal graphs, unit graphs, Jacobson graphs, generalised total graphs, etc. They all have a substantial and growing segment within the literature (c.f. [18–25]). A few decades back, algebraic graph theory was just a theory that did not apply to ordinary human activities, whereas it has now been successfully used in transmitting encrypted information with high security and privacy through public communication networks (c.f. [15]).

Though Cayley graphs were initially constructed on groups, their graph construction has been extended to rings as well. As rings possess several symmetric subsets such as the set of all zero-divisors, units, idempotent and nilpotent elements, etc., many variants of Cayley graphs (using the symmetric subsets of rings) were constructed and studied. This literature review intends to present an overview of these variants of Cayley graphs that are defined on rings, that is, the graphs defined such that their vertex sets are the ring elements, and their adjacency relation is similar to the adjacency condition given in the Cayley graph with respect to some symmetric subset of the ring.

It can be seen that there are many survey papers, review papers, and books on graphs defined on rings (see [17,22,23]), but many of them cover only a few well-studied graphs. Furthermore, review papers that focus on a particular property of graphs defined on rings can also be found in the literature (c.f. [26–28]), whereas there was no comprehensive review found related to the variants of Cayley graphs defined on rings. This motivated us to create a literature hub for these graphs defined on common grounds and to systematically analyse the studies that have been carried out on these graphs to understand the pattern and dynamics of research in this area.

In this manuscript, we list the significant results on the structural properties of the concerned graphs, such as girth, diameter, traversal properties, connectivity, connectedness, and perfection, along with certain characterisations that have led to further research and open problems in that direction. In addition to this, the spectral properties of these graphs are also highlighted in this article.

This systematic review also helps to identify unsolved open problems that have been proposed in the literature, revealing the future scopes of study on this topic. Furthermore, this article aims to resolve the ambiguity over different graphs with similar names and the same graphs with different names that have been defined and studied independently by different authors, which fall under these criteria.

The outline of the article is as follows. The graph theoretic and algebraic preliminaries that are required to proceed further are given in Section 1.1. Comprehensive reviews on the unitary Cayley graph of $\mathbb{Z}_n$ and unitary Cayley graph of a ring, where the former is a particular case of the latter, are given in Sections 2 and 3, respectively. These are followed by reviews on the unitary addition Cayley graph in Section 4 and the unit graph of a ring in Section 5, where, again, the first class of graphs forms a subset of the second one. Finally, a review on other variants of Cayley graphs, for which detailed investigations have not yet been performed, is given in Section 6. We conclude the article by proposing the research gaps that we have found over the course of this review, along with several possible avenues for further research in Section 7.

1.1. Preliminaries

This subsection aims to familiarise the reader with the terminology and notation that are used in the article. It also includes definitions and results which are required to understand the study. Unless otherwise noted, all definitions relating to algebra are from [29] and all definitions relating to graph theory are from [30].

We let $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, and complex numbers. A non-empty set together with a binary operation (termed as addition) is called a *group* if the properties of closure, associativity, existence of a unique identity (additive identity) of the set and a unique inverse for each element in the set are satisfied. In addition to this, if the group elements commute with each other under the defined binary operation, then the group is said to be an *Abelian group*.

The structure of a group endowed with another binary operation called multiplication gave rise to the abstract concept of rings in the mid 19th century. A non-empty set R with two binary operations of addition and multiplication, denoted by $+$ and $\cdot$, respectively, is said to be a *ring* or an *associative ring* if R is a commutative group under addition and the properties of associativity and distributivity hold for the multiplication.

In general, the binary operation of multiplication need not be commutative and the ring need not have an identity element under multiplication. If the ring is commutative under multiplication, then the ring called a *commutative ring*, and when a ring has an identity element under multiplication, called the *multiplicative identity*, the ring is termed as a *ring with identity*, where this multiplicative identity is denoted by 1. Similarly, the existence of a multiplicative inverse for a non-zero element in a ring with identity is not guaranteed. If a non-zero element in a ring has a multiplicative inverse, then it is called a *unit element* of the ring and the set of all unit elements of the ring R form a group under multiplication, which is called the *multiplicative group of units*. For a ring R, we denote this group of units of R by R^*. In other words, if R is a ring with identity and $x \in R$, x is a unit of R when there exists a $y \in R$ such that $xy = yx = 1$ and $R^* = \{x \in R : xy = yx = 1, y \in R\}$ is the group of units of R.

An element $x \in R$ is a left (right) *zero-divisor* if there exists a $y \in R$ such that $xy = 0$ ($yx = 0$) and $y \neq 0$. Note that the additive identity 0 of a ring R is a trivial zero-divisor and, for a commutative ring, the notions of left and right zero-divisors are the same, so we can simply call them zero-divisors. An *integral domain* is a commutative ring with identity such that there are no non-zero zero-divisors, and a *field* is a commutative ring with identity such that every non-zero element is a unit. Therefore, it can be concluded that every integral domain is a field. In addition, a field can be interpreted as a ring that forms an Abelian group with respect to both addition and multiplication. The *characteristic* of a ring R, denoted by $char(R)$, is the smallest integer k such that $\underbrace{1+1+\ldots+1}_{k-times} = 0$ in R and, if there exists no such k, then R is said to have characteristic 0.

A *subring* of a ring R is a subset of R, which is a ring by itself with the operations defined on R. A subset I of a ring R is called a left (right) *ideal* of R if $(I,+)$ is a subgroup of R and $yx \in I$ ($xy \in I$), for all $x \in I$ and $y \in R$. For an element $x \in R$, the set $\langle x \rangle = Rx = \{yx : y \in R\}$ ($\langle x \rangle = xR = \{xy : y \in R\}$) is an ideal of R, called the *principal left (right) ideal* generated by x. A left (right) ideal I of a ring R is said to be a *maximal left (right) ideal* of R if, whenever I_1 is a left (right) ideal of R and $I \subseteq I_1 \subseteq R$, then $I_1 = I$ or $I_1 = R$, that is, the only ideal that properly contains a maximal ideal is the ring itself. Note that the notions of left and right are the same for a commutative ring.

A commutative ring with identity is called a *local or quasilocal ring* if it has a unique maximal ideal. A *division ring* is a non-trivial ring in which division by non-zero elements is defined. In other words, a field is a commutative division ring, and all division rings that are not fields are non-commutative rings in which the non-zero elements have a multiplicative inverse, either with respect to left or right multiplication. The *Jacobson radical* of a ring R, denoted by J_R, is the intersection of all the maximal ideals of R. For a ring R and an ideal I of R, $\frac{R}{I} = \{x + I : x \in R\}$ is called a *quotient ring* of R by I. For a commutative ring R, $R[x] = \{\sum\limits_{i=0}^{n} a_i x^i : a_i \in R, n \in \mathbb{Z}\}$ is called the *ring of polynomials* over R in the indeterminate x.

A ring R is said to be left (right) *Artinian* if every strictly descending chain of left (right) ideals in R is finite. The *structure theorem* for Artinian rings says that an Artinian ring R is uniquely (up to isomorphism) a finite direct product of Artinian local rings, where the direct product $R_1 \times R_2 \times \ldots \times R_k$ of rings $R_1, R_2, \ldots, R_k$ is the set of all ordered pairs $\{(r_1, r_2, \ldots, r_k) : r_i \in R_i, 1 \leq i \leq k\}$, such that the binary operations of addition and multiplication are defined element-wise. A *simple ring* is a non-zero ring that has no non-zero proper ideals. By $\mathbb{Z}_n$, we denote the ring of integers modulo n, with the usual operations of addition modulo n and multiplication modulo n, that is, $\mathbb{Z}_n = (\mathbb{Z}_n, +_n, \cdot_n)$. The units of the ring $\mathbb{Z}_n$, denoted by $\mathbb{Z}_n^*$, are the set of all integers that are relatively prime to n and are less that n, that is, $\mathbb{Z}_n^* = \{k \in \mathbb{Z}_n : \gcd(k,n) = 1\}$, and the cardinality of this set is given by the arithmetic function called the *Euler totient function*, denoted by $\phi(n)$.

A *ring-homomorphism* $f : R_1 \to R_2$ between two rings R_1 and R_2 is a mapping that preserves the two ring operations, that is, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in R_1$, where we assume that $f(1) = 1$. A one-to-one and onto ring-homomorphism is a *ring-isomorphism* and if two rings R_1 and R_2 are isomorphic; it is denoted by $R_1 \cong R_2$. Note that other related definitions are given within the article on the basis of requirement.

For a graph G with the vertex set $V(G)$ and edge set $E(G)$, the order and the size of the graph are $|V(G)| = n$ and $|E(G)| = m$, respectively. A graph in which there exists an edge joining a vertex to itself, called a *loop*, is known as a *pseudograph*, and a graph in which the edges are ordered pairs of vertices is called a *directed graph*. A subgraph H of a graph G is said to be a *spanning subgraph* if, with $V(H) = V(G)$ and for any subset $S \subseteq V(G)$, the subgraph induced by S, denoted by $\langle S \rangle$, is the maximal subgraph of G, with vertex set S. The *complement* $\overline{G}$ of a graph G is the graph such that $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : uv \notin E(G)\}$.

The set $N(v) = \{u \in V(G) : uv \in E(G)\}$ is called the *open neighbourhood* of a vertex $v \in V(G)$ and, for each vertex $v \in V(G)$, the set $N[v] = N(v) \cup \{v\}$ is the *closed neighbourhood* of v. The *degree* of a vertex $v \in V(G)$, denoted by $deg_G(v)$ or $d(v)$, is the number of vertices adjacent to v in G, that is, $deg(v) = |N(v)|$ and $\Delta(G) = \sup\{|N(v)| : v \in V(G)\}$ is the maximum degree of a graph G.

A graph G is called *connected* if there is a path between any two distinct vertices in G; otherwise, G is said to be *disconnected*. A graph is called *Eulerian* if it contains a closed trail containing every edge, and a graph is *Hamiltonian* if it contains a spanning cycle. Let G be a connected graph and, for two vertices $u, v \in V(G)$, the length of a shortest path from u to v is denoted by $d(u,v)$ and the diameter of the graph G, $diam(G) = \sup\{d(u,v) : u, v \in$

$V(G)\}$. The *girth* of a graph G is the length of the smallest induced cycle in G and, if the graph is acyclic, the girth of the graph is taken as ∞.

An *isomorphism* between two graphs G and H is a bijective function $f : V(G) \to V(H)$, such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H; an isomorphism from a graph G to itself is called an *automorphism*. The set of all automorphisms of a graph G forms a group called the *automorphism group* of G, denoted by $\text{Aut}(G)$. Since each graph has a unique automorphism group, it is called the *algebraic invariant* of the graph.

The *adjacency matrix* $A(G)$ of a graph G is a binary matrix of order n, such that the ij-th entry is 1 if $v_iv_j \in E(G)$, or 0 otherwise. The set of all eigenvalues in this real symmetric adjacency matrix of a graph G, along with their multiplicities, is called the *spectra* of the graph G. A graph G is said to be *perfect* if the clique number and the chromatic number are equal for all the induced subgraphs of G. A graph is said to be *planar* if it can be drawn on a surface such that no two edges cross each other. The other graph parameters and concepts that are investigated for different graphs are defined herein on the basis of requirement.

For more definitions and concepts related to algebra, see [31,32], and [29], specifically for ring theory. For fundamental concepts in graph theory, refer to [30], and for algebraic and spectral aspects in graphs, see [15,33]. For the theory of domination in graphs, refer to [34]. For more details on concepts related to the planarity of graphs, see [35], and for all basic definitions and results required to understand the study of graphs defined on rings in both graph theory as well as ring theory, we refer the reader to Chapter 1 within [17].

As the ring of integers modulo n is a standard ring that has an easily understandable structure, almost all graphs defined on rings are examined on $\mathbb{Z}_n$, whose elements are the integers modulo n. Therefore, to examine the graphs defined on $\mathbb{Z}_n$ and its related rings, proficiency in ring theory, graph theory, and elementary number theory are essential. Therefore, for fundamental concepts in number theory, we refer the reader to [36,37].

2. Unitary Cayley Graph of $\mathbb{Z}_n$

One of the most-studied graphs defined on rings, especially on $\mathbb{Z}_n$, is the *unitary Cayley graph*. As the name suggests, the unitary Cayley graph can be seen as a restriction of or a variation on the broadly defined Cayley graph. As this graph is specifically defined on $\mathbb{Z}_n$, it can be seen that the number-theoretic definition of the graph leads to several interesting results that are obtained using number-theoretic properties, and, often, the innate structure of the graph gives rise to pleasing combinatorial results.

A graph of order n is said to be *representable modulo k* if its vertices can be labeled using distinct integers between 0 and k, such that the difference between the labels of two vertices are relatively prime to k if and only if the vertices are adjacent; the smallest k for which the graph is representable modulo k is called the *representation number* of the graph (see [38]). The problem of determining the representation number of a given graph and analysing the properties of graphs that have a given representation number, along with its relation between the order of the graph, was one of prominence, that was put forth as the graph representation problem in the last decade of the 20th century (refer to [39]), as it was proven that every graph is representable modulo for some positive integer (c.f. [38]). The main motivation to study the unitary Cayley graph of $\mathbb{Z}_n$ was to investigate the representation problem of graphs, as put forth in [38], which is closely related to the definition of the unitary Cayley graph on $\mathbb{Z}_n$ given below. Following the definition, an example of a unitary Cayley graph is given in Figure 1.

Definition 1 ([40])**.** *The* unitary Cayley graph *of the ring $\mathbb{Z}_n$, denoted by $X_n = Cay(\mathbb{Z}_n, \mathbb{Z}_n^*)$, is a graph with vertices set as the elements of the ring; $0, 1, \ldots, n-1$ and two distinct vertices are adjacent if their difference is a unit of the ring; that is, for all $x, y \in V(X_n)$, $xy \in E(X_n)$ when $|x - y| \in \mathbb{Z}_n^*$, where $\mathbb{Z}_n^*$ is the set of all relatively prime integers to n, which are the units of $\mathbb{Z}_n$.*

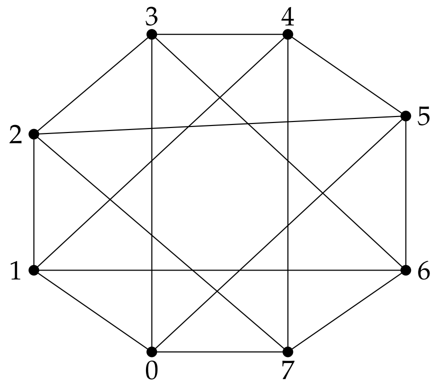


Figure 1. The unitary Cayley graph X_8.

Note that the definition of the unitary Cayley graph on $\mathbb{Z}_n$ is closely associated with the definition of a graph to be representable modulo n; this motivated researchers to gain insight into the graph representation problem, and, therefore, unitary Cayley graphs were investigated. It can be observed that, post-introduction of the unitary Cayley graph X_n, the definition of a graph to be representable modulo n was given in terms of X_n. In other words, a graph is said to be representable modulo k if it is isomorphic to an induced subgraph of X_n (refer to [41]).

Though the representation problem is stated in terms of the unitary Cayley graph X_n and the results obtained on the investigation of the representation problem may be related to the graph X_n; note that we do not consider them in this review, as the results may only address certain induced subgraph structures of the graph X_n, which may or may not have all the properties of X_n.

The unitary Cayley graph of $\mathbb{Z}_n$ was introduced in [40] as a specific case of the Cayley graph, defined using the generating sets of $\mathbb{Z}_n$, as the set $\mathbb{Z}_n^*$ generates $\mathbb{Z}_n$. The other variants of Cayley graphs, defined based on such generating sets in [42], were complete graphs, and, based on colouring the edges of these complete graphs in a symmetric fashion, the realisation of the induced subgraphs of these complete graphs as totally multicoloured (TMC) subgraphs, that is, a subgraph of a graph in which no two edges have the same colour, was studied in [42].

Motivated to investigate the possibilities of obtaining totally multi-coloured Cayley graphs, the unitary Cayley graph was defined on $\mathbb{Z}_n$, and its basic properties were investigated in [40]. By Definition 1, it can be seen that the graph X_n is $\phi(n)$-regular, where $\phi(n)$ is the Euler totient function that gives the number of integers less than n that are relatively prime to n. The symmetric nature of the graph can be observed from the adjacency pattern as well as the regularity, as it is closely related to the number-theoretic concepts of modular arithmetic (c.f. [43]). This symmetry of the unitary Cayley graphs gives rise to several applications in modelling networks and encourages investigation of the graph in several directions.

The primary focus of the study in [40] was to examine the existence of triangles and enumerate them, in the newly defined unitary Cayley graph, as the intended study was to explore the possibilities of obtaining totally multi-coloured graphs. This study on the triangles present in the graph helped to identify TMC graphs, but it can be seen that the study would not be significant when the graph turned out to be a complete graph. Therefore, the first result obtained on X_n classifies the values of n for which X_n is a complete graph. Since bipartite graphs are characterised based on the existence of odd cycles, the values of n for which X_n is bipartite and complete bipartite were also obtained, as follows.

Theorem 1 ([40])**.**

(i) *A unitary Cayley graph X_n is isomorphic to a complete graph K_n and a complete bipartite graph $K_{2^{t-1},2^{t-1}}$ when n is prime and $n = 2^t$, $t \geq 1$, respectively.*

(ii) *A unitary Cayley graph X_n is a bipartite graph if n is even.*

It can be observed that the graphs X_{2^t}, $t \in \mathbb{N}$ are regular, with each vertex having degree equal to half the number of vertices, making the size of the graph the square of the sum of degrees of all vertices in the graph. Since the chromatic uniqueness of complete bipartite graphs was proven in [44], the graphs X_{2^t}, $t \in \mathbb{N}$ are called *chromatically unique unitary Cayley graphs*. Note that, for a graph G, the polynomial that gives the number of graph colourings as a function of the number of colours is a *chromatic polynomial* (see [30]), and two graphs G_1 and G_2 are *chromatically equivalent* if they have the same chromatic polynomial, that is, $P_\alpha(G_1) = P_\alpha(G_2)$. A graph G_1 is said to be *chromatically unique* if $P_\alpha(G_1) = P_\alpha(G_2)$ implies that $G_1 \cong G_2$ (see [45]).

As the graph X_n is triangle-free for even n, the enumeration of triangles was restricted to X_n for odd n. As a first step, the number of triangles in X_n with two common vertices was enumerated, following which the total number of triangles in the graph was determined. The number of triangles with two common vertices was obtained as the cardinality of the set $\{u \in \mathbb{Z}_n^* : (u-1) \in \mathbb{Z}_n^*\}$. This is because the vertex set of any triangle in X_n with two common vertices can be taken as $\{0, 1, u : u \in \mathbb{Z}_n^*\}$, owing to the fact that the difference between the vertices of any edge in the graph is a unit. Therefore, the third vertex that differs, for the triangles with two common vertices, will always be a unit, and, hence, the number of triangles with two common vertices is obtained as

$$n \prod_{p|n} \left(1 - \frac{2}{n}\right),$$

where the product is run over all the prime factors of n.

To enumerate the number of triangles in the graph X_n, the group action of the group $\mathbb{Z}_n^* \times \mathbb{Z}_n$ on the set of all triangles of the graph, that is, if $(u', x) \in \mathbb{Z}_n^* \times \mathbb{Z}_n$, then the action $(u', x)\{0, 1, u\} = \{u'x, u'(1+x), u'(u+x)\}$ that gives the orbits of the triangles corresponding to different pairs $(u', x) \in \mathbb{Z}_n^* \times \mathbb{Z}_n$ was considered. As orbits partition a set, the sum of the cardinalities of these orbits obtained through the given group action aided in determining the total number of triangles in the graph X_n. Using the orbits obtained through the group action, the edges of the triangles were also coloured to obtain the edge colouring of the graph, and this led to the enumeration of triangles having different possible combination of colours, that is, the triangles that had all three edges coloured with different colours, all three edges coloured with the same colour, and two edges coloured with same colour were termed as scalene-coloured triangles, equilateral-coloured triangles and isosceles-coloured triangles, after which they were enumerated.

The enumeration of triangles in unitary Cayley graphs gave rise to the problem of counting the number of induced cycles of any given length k. Additionally, it was seen that, to prove the chromatic uniqueness of a graph, it is important to count the number of induced k cycles in the graph, as some of the coefficients in the chromatic polynomials are related with the number of such induced cycles (see [46]). Therefore, this problem of counting the induced k cycles was proposed in [47], and the induced cycles of length four were enumerated using the concept of the *multiplicative arithmetic property* (map) of the graphs X_n.

A sequence of Cayley graphs $Cay(\Gamma_t, S_t)$, where Γ_t is an Abelian group and S_t is a symmetric subset of Γ_t, is said to have the *multiplicative arithmetic property* if, for each pair of positive relatively prime integers (n_1, n_2), there is a group isomorphism ϕ_{n_1,n_2} from $\Gamma_{n_1 n_2}$ to $\Gamma_{n_1} \times \Gamma_{n_2}$, such that ϕ_{n_1,n_2} maps $S_{n_1 n_2}$ onto $S_{n_1} \times S_{n_2}$ (see [47]). In [47], the multiplicative arithmetic properties on all the Cayley graphs defined on Abelian groups were discussed and, since $\mathbb{Z}_n$ is also an Abelian group and $\mathbb{Z}_n^*$ is a symmetric subset of $\mathbb{Z}_n$, the unitary Cayley graphs were also examined in [47].

In [47], all Cayley graphs defined on Abelian groups were proven to have the multiplicative arithmetic property by obtaining the corresponding multiplicative arithmetic functions. A construction of sequences of Cayley graphs with the multiplicative arithmetic property, based on the number-theoretic concepts such as the Chinese reminder theorem, was also given in the article. As an application of proving the multiplicative arithmetic property of the unitary Cayley graphs, the number of induced cycles of length three (triangles) and four were enumerated. Though the formula for the number of triangles had been obtained previously in [40] by using the group actions, the same result was deduced in this article using the multiplicative arithmetic property of the graph.

Along with the results obtained, the authors also posted many open problems, among which were the possibility to obtain a generalised expression to find the number of induced k cycles in the graph X_n for any given n and to characterise the chromatic uniqueness in X_n as it pertained to unitary Cayley graphs. These open problems were partially addressed by the same authors in [48], by establishing a connection between the existence of an induced k cycle in X_n and the number of prime divisors of n, as follows.

Theorem 2 ([48])**.** *Given $r \in \mathbb{N}$, there is a natural number $M(r) \in \mathbb{N}$, depending only on r, such that the number of induced k cycles in X_n is zero for all $k \geq M(r)$ and for all n, with at most r different prime divisors.*

This result was proven based on the results obtained in [47], that established the multiplicative arithmetic property of unitary Cayley graphs. By Theorem 2, it was deduced that X_n is a complete p-partite graph on n vertices with the maximum number of edges and is chromatically unique when $n = p^t$, where p is prime and $t \in \mathbb{N}$, with the partitions $P_i = \{x : x \cong i \mod p, 0 \leq i \leq p-1\}$. In [40], it was obtained that X_n is chromatically unique when $n = 2^t$, for some $t \in \mathbb{N}$ based on the structure of the graph, and this result extends the class of chromatically unique unitary Cayley graphs from n being only 2^t to any prime power p^t, and this result was also proven based on the multiplicative arithmetic property. Along with this, the bounds for the value $M(r)$ were also obtained as follows.

Theorem 3 ([48])**.** *For $r \in \mathbb{N}$, there is a natural number $M(r) \in \mathbb{N}$ that depends only on r, such that $(r-1)\ln(r-1) \leq M(r) \leq 9r!$.*

The bounds given in Theorem 3 show the existence of induced k cycles in X_n for arbitrarily large r, which adds credibility to Theorem 2. Additionally, a large gap between the bounds of $M(r)$ opened an avenue to find better estimates, which were computed in [49]. The main problem addressed in [49] was to determine the length of the longest induced cycle in X_n for a given n and, to address this problem, a representation of the vertices in X_n based on their residues modulo the prime factors of n, called the *residue representation*, was introduced as follows.

Definition 2 ([49])**.** *For $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r}$, where p_i, $1 \leq i \leq r$ are distinct primes and $\alpha_i \in \mathbb{N}$, if $x \in V(X_n)$, such that $x \equiv \alpha_i \mod p_i$, for $1 \leq i \leq r$ and $0 \leq \alpha_i < p_i$, the* residue representation *of x is the unique string $\alpha_1 \alpha_2 \ldots \alpha_r$.*

This representation simplifies the problem of finding the induced cycles in the graph to that of checking the similarity conditions between consecutive vertices, that is, to check if any pair of non-consecutive vertices has at least one identical index in the representation, as it can be observed that, for any $x, y \in V(X_n)$, $xy \in E(X_n)$ if and only if $x \equiv y \mod p_i$ for all $1 \leq i \leq r$. In this article, the number $M(r)$ defined in [48] is given in terms of $m(n)$, which denotes the longest induced cycle in X_n as $M(r) = \max_n\{m(n)\}$, where the maximum is taken over all n values with r distinct prime divisors. Since $M(r)$ was proven to depend only on r in [48], $m(n)$ was also proven to depend only on r in [49], so that there arose no ambiguity in the given definition of $M(r)$ in terms of $m(n)$. Significant questions as to the relation between the values $m(n)$ and $M(r)$ were also answered in [49]: the conditions under which these values of $m(n)$ and $M(r)$ were equal were obtained, as given below.

Theorem 4 ([49]). *For $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$, where p_i, $1 \leq i \leq r$ are distinct primes and are large, $m(n) = M(r)$.*

Theorem 5 ([49]). *For $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$ and $n' = p_1 p_2 p_3 \dots p_r$, where p_i, $1 \leq i \leq r$ are distinct primes and $r \neq 1$, $m(n) = M(n')$.*

Theorem 5 reduces the complexity of calculating $m(n)$ for large values of n, as it considers only the values of n whose prime powers are square-free. These results aided in improving the tightness of the bounds of $M(r)$ in [49], which is given below.

Theorem 6 ([49]). *For all positive integers n with $r > 1$ distinct prime divisors, $2^r + 2 \leq M(r) \leq 6r!$.*

To prove Theorem 6, an induced subgraph of X_n with $2^r + 2$ vertices was constructed for all n, and it was proven that the construction depended only on the number of prime divisors, r of n, and not on the value of the prime divisors, thus providing a lower bound for $m(n)$. It was natural to examine the properties of X_n that contributed to the results that were obtained and to explore the possibilities of constructing similar graphs. On analysing these properties, it was noted that the above results on the length of the longest cycles could be extended to the direct product of any number of complete k-partite graphs, and this extension can be seen as an immediate consequence of the fact that, for any $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$, $X_n \cong X_{p_1^{\alpha_1}} \times X_{p_2^{\alpha_2}} \times \dots \times X_{p_r^{\alpha_r}}$, as X_n is a complete p-partite graph for $n = p^t$ when p is prime. Note that the unitary Cayley graphs are referred to as *unitary circulant graphs* in [49].

A *random walk* on a finite, connected graph is a Markov chain (a *Markov chain* is a sequence of random variables, such that the next move depends only the current position and not on any of the previous ones—refer to [50] for more details.) that jumps from a current vertex v to one of its k neighbours with a uniform probability (refer to [51]). The *hitting time* T_v of a vertex v is the minimum number of steps that a random walk takes to reach back to the same vertex, and the expected value of T_v for a vertex is known as the *expected hitting time*. The expected hitting times for the random walks in the unitary Cayley graph X_n and the direct product of two unitary Cayley graphs X_{n_1} and X_{n_2}, where $n_1 = p^{t_1}$ and $n_2 = p^{t_2}$,$t_1, t_2 \in \mathbb{N}$, were studied in [52] and [53], respectively, as an extension of the study on the expected hitting time of the edge transitive graphs by the same authors in [51]. Though the high symmetry of the graph X_n can be realised from the graph's construction, the unitary Cayley graphs were formally proven to be arc-transitive in [52] by obtaining an automorphism of the graph that satisfied the condition of arc transitivity as follows.

Theorem 7 ([52]). *The function $\psi(x) = wx + z$, where $w \in \mathbb{Z}_n^*$, $z \in \mathbb{Z}_n$ and $x \in V(X_n)$ are fixed, is an automorphism of the graph X_n.*

A graph G is said to be a *vertex-transitive* (*edge-transitive*) *graph* if its automorphism group acts transitively on $V(G)$ $(E(G))$. In other words, a graph G is vertex-transitive (edge-transitive) if there exists an automorphism between any two distinct vertices (edges) of G. Similarly, a graph G is *arc-transitive* if there exists an automorphism between any two distinct edges of G such that the direction of the edges are preserved.

As it can be observed that an arc-transitive graph is both vertex-transitive and edge-transitive, this automatically proves that unitary Cayley graphs are both vertex- and edge-transitive. The main focus of the article [52] was to determine the expected hitting time of the edge-transitive graphs, when the diameter of the graphs were two and three, and to tighten the results when the graphs followed certain adjacency patterns. Since Theorem 7 proves the edge-transitivity of the unitary Cayley graphs, the expected hitting times of these graphs were explicitly computed in [52] by classifying the graphs that had diameter two and three as follows.

Theorem 8 ([52]). *The diameter of X_n,*

$$diam(X_n) = \begin{cases} 2, & \text{if } n = 2 \text{ or } n \text{ is odd and composite;} \\ 3, & \text{if } n = 2^l k, \text{ where } l \geq 1 \text{ and } m > 1 \text{ is odd.} \end{cases}$$

By the definition of a random walk, it can be noted that the study of a random walk in a regular graph tends to give a uniform distribution, as the number of neighbours to which the vertex can jump is equal for all the vertices in the graph. Additionally, the unitary Cayley graphs considered in the study were the graphs X_n, $n = p^k$, where p is a prime, which were already proven to be complete k-partite graphs in [48]. To determine the hitting times of these graphs, the degree and distance between each pair of vertices in the graph must be known and, therefore, the degree and distance between each pair of vertices in the graph X_n, when n is a prime power, was determined in [52], along with the diameter of the graph $X_{n_1} \times X_{n_2}$, where $n_1 = p^{r_1}$ and $n_2 = p^{r_2}$; for $r_1, r_2 \in \mathbb{N}$, it was also determined as 2 in [53]. As the graphs $X_{n_1} \times X_{n_2}$ were of diameter two, the hitting time of the vertices of these graphs were also computed and are given as follows.

Theorem 9 ([53]).

(i) *The expected hitting time between the vertices at distance 1 is* $|V(X_{n_1} \times X_{n_2})| - 1 = p^{n_1+n_2} - 1$.

(ii) *The expected hitting time between the vertices at distance 2 is*
- *(a)* $|V(X_{n_1} \times X_{n_2})| = p^{n_1+n_2}$, *when no pair of vertices are at distance 1 in the graphs* X_{n_1} *or* X_{n_2};
- *(b)* $|V(X_{n_1} \times X_{n_2})| + \frac{1}{p-2} = p^{n_1+n_2} + \frac{1}{p-2}$, *otherwise.*

Though unitary Cayley graphs were officially introduced in [40] in the year 1995, not many studies emerged on unitary Cayley graphs until 2007, when [54] was published. It was the first study that laid a strong foundation for the study of unitary Cayley graphs, as it had an in-depth investigation on the properties of unitary Cayley graphs, after which the fast-growing literature on this topic commenced. The study in [54] begins with a brief review on the previous investigations of unitary Cayley graphs, following which the chromatic number, clique number, and the vertex connectivity of X_n were computed as follows.

Theorem 10 ([54]). *If p is the smallest prime divisor of n, then $\chi(X_n) = \omega(X_n) = p$, where χ and ω denote the chromatic and clique number, respectively.*

Theorem 11 ([54]). *The vertex connectivity $\kappa(X_n)$ of the unitary Cayley graph X_n is $\phi(n)$, where $\phi(n)$ is the Euler totient function.*

An arc-transitive graph in which the vertex connectivity is its degree makes the unitary Cayley graph highly reliable and stable for network models. Additionally, the regularity of the graph X_n implies that its complement is also regular and highly symmetric, an,d therefore, by Theorem 10, the chromatic and clique number $\chi(\overline{X}_n)$ and $\omega(\overline{X}_n)$ of the complement of X_n are computed as $\frac{n}{p}$, where p is the smallest prime divisor of n. Based on these results on the complement of the unitary Cayley graphs, the following realisation was obtained.

Theorem 12 ([54]). *A unitary Cayley graph X_n is self-complementary if and only if $n = 1$ or $n = 2$. That is, $X_n \cong \overline{X}_n$ if and only if $n = 1$ or $n = 2$.*

Based on the investigation of the complement of the graph X_n and its regularity, the number of common neighbours between the vertices was enumerated in [54] by partitioning the vertex based on different conditions for different values of n. On obtaining the chromatic and the clique number of the graphs, perfection in the unitary Cayley graphs was studied by investigating the existence of odd cycles of length five or more in the graph X_n, and the unitary Cayley graphs that were perfect were characterised as follows.

Theorem 13 ([54])**.** *A unitary Cayley graph X_n is perfect if and only if n is even or n is odd and has at most two distinct prime divisors.*

The investigation of the spectral properties of the unitary Cayley graphs began in [54], where the adjacency matrix of the graph X_n was obtained. It is known that there are multiple adjacency matrices for any graph, which are given based on different ordering of the vertices. With the natural order of vertices $0, 1, 2, \ldots, n-1$, the adjacency matrices of the unitary Cayley graphs were obtained as circulant matrices—that is, matrices in which the entries of their first row generate the entries of the other rows by a cyclic shift—which established that unitary Cayley graphs are circulant graphs: graphs with circulant adjacency matrices (c.f. [15]).

Using the explicit formula to obtain the eigenvalues of a circulant matrix given in [55], the eigenvalues of the adjacency matrix of X_n were obtained in terms of an arithmetic function $c(r,n)$ called the Ramanujan sum (For $k_1, k_2 \in \mathbb{N}$, the *Ramanujan Sum*, $c(k_1,k_2) = \sum\limits_{1\leq q\leq k_1} e^{2\pi i \frac{q}{k_1}} n$, where the summation is taken over all integers q, such that $\gcd(k_1,q) = 1$—for more details, refer to [37,56]), which takes only integral values for the given integers $r, n, n > 0$.

Therefore, it was concluded that all eigenvalues of unitary Cayley graphs are integers and, hence, unitary Cayley graphs fall under the class of graphs called the *integral circulant graphs*: circulant graphs whose eigenvalues are integers (see [57]). Further investigation on the eigenvalues of the graph X_n, based on their symmetry and the number-theoretical properties led to the following interesting results related to the eigenvalues of the graphs.

Theorem 14 ([54])**.** *Let $\phi(n)$ denote the Euler totient function and $\mu(n)$ denote the Mobiüs function on a natural number $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, defined as*

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ -1, & \text{if } \alpha_1 = \alpha_2 = \ldots = \alpha_r = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then, the following hold:

(i) Every non-zero eigenvalue of X_n, $n > 1$ is a divisor of $\phi(n)$;

(ii) Let p be the maximal square-free divisor of n. Then, $\lambda_{min} = \mu(p)\frac{\phi(n)}{\phi(p)}$ is a non-zero eigenvalue of X_n, $n > 1$ of minimal absolute value and multiplicity $\phi(p)$;

(iii) Every eigenvalue of X_n, $n > 1$ is a multiple of λ_{min};

(iv) If $n > 1$ is odd, then λ_{min} is the only non-zero eigenvalue of X_n with minimal absolute value;

(v) If $n > 1$ is even, then $-\lambda_{min}$ is also an eigenvalue of X_n, with multiplicity $\phi(n)$.

Theorem 15 ([54])**.**

(i) There is an eigenvalue -1 or 1 of X_n if and only if n is square-free.;

(ii) If n is square-free, then X_n has an eigenvalue $\mu(n)$ with multiplicity $\phi(n)$;

(iii) The unitary Cayley graph X_n has both eigenvalues 1 and -1 with multiplicity $\phi(n)$ if and only if n is square-free and even.

Fascinated by the spectral properties of unitary Cayley graphs and their close relation with number theory, a generalisation of the unitary Cayley graphs, called the *GCD*-graphs, was defined, in which the set of all positive, proper divisors of an integer $n > 1$ are considered to form the symmetric subset, to define the adjacency condition. The formal definition of the graph is given below.

Definition 3 ([54])**.** *The GCD-graph, denoted by $X_n(D_n^*)$, is a graph with its vertices set as the elements of the ring $\mathbb{Z}_n$; $0, 1, \ldots n-1$ and two distinct vertices are adjacent if the* gcd *of their difference and n is a positive proper divisor of n, that is, for all $x, y \in V(X_n(D_n^*))$, $xy \in E(X_n(D_n^*))$ when $\gcd(x-y, n) \in D_n^*$, where D_n^* is the set of all positive, proper divisors of the integer $n > 1$. An example of a GCD-graph is given in Figure 2.*

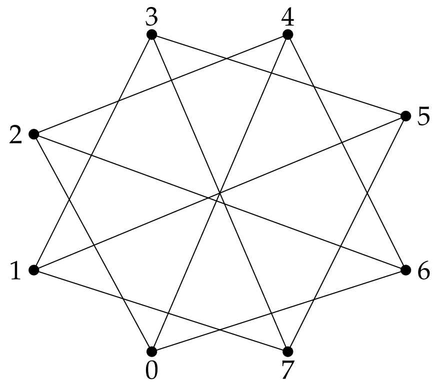


Figure 2. The GCD-graph $X_8(D_8^*)$.

Observe that the set D_n^* consists of only all the proper positive divisors because, when one is included as a divisor, the graph obtained is the complement of X_n for certain values of n. The analysis on the spectra of GCD-graphs in [58] proved that the GCD-graphs also have integral eigenvalues. On further exploration of the properties of these graphs with integral spectra, the authors came up with a slightly modified definition of the graphs, based on this basic definition of GCD-graphs that was put forth by them in [54], to obtain multiple smaller graphs which fell under this broad category with similar properties as follows.

Definition 4 ([59])**.** *For a positive integer n, let D_n be the set of all its divisors. Define the graph $G_n(d)$, where $d \in D_n$, with the vertices set as the elements of the ring $\mathbb{Z}_n$, and two distinct vertices x, y in the graph are adjacent when the $\gcd(x-y, n) = d$. The graph $G_n(d)$ is extended by increasing the number of divisors and modifying the adjacency condition of any two distinct vertices x, y to be $\gcd(x-y, n) \in D$, where $D \subseteq D_n$; this graph is represented as $G_n(D)$. These graphs are known as* gcd-graphs*.*

Note that, if $|D_n| = k$, then 2^{k-1} gcd-graphs $G_n(D)$ are possible for any integer n, where the graphs X_n and $X_n(D_n)$ are also among them. An illustration of some gcd-graphs emerging from $\mathbb{Z}_{12}$, for the subsets $D \subset D_n$, apart from $D = \{1\}$ and $D = D_n$, is given in Figure 3.

This new generalised definition was simultaneously given in [58] in the process of characterising integral circulant graphs, where it was proven that a graph is an integral circulant graph if and only if it can be realised as the graph $G_n(D)$ for some $D \subseteq D_n$. It can be observed that, when the set of all proper divisors are considered, the gcd-graphs G_n are the GCD-graph defined in [54], and, when, $D = \{1\}$, $G_n(1) = X_n$.

Therefore, it can be seen that unitary Cayley graphs can be realised as a special case of GCD-graphs, as well as the gcd-graphs from their definitions, and any study on gcd-graphs can be considered to obtain results on unitary Cayley graphs. Additionally, based on the characterisation of the integral circulant graphs as gcd-graphs and the fact that $G_n(1) = X_n$, the results established for the integral circulant graphs also hold for unitary Cayley graphs. The integral circulant graphs, or the graphs $G_n(D)$, have a huge, growing body within the literature, owing to their spectral properties that have applications in fields such as chemistry, quantum physics, radiology, etc. (c.f. [57]).

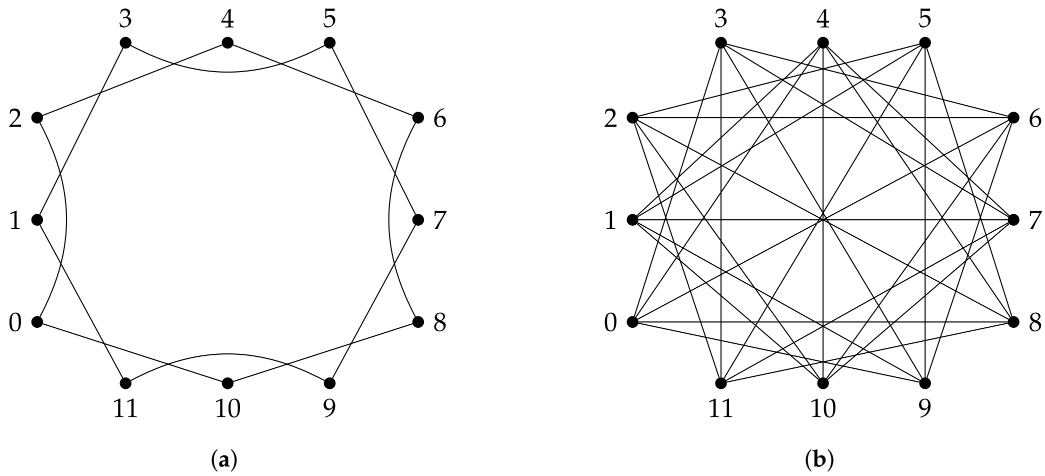


Figure 3. gcd-graphs of $\mathbb{Z}_{12}$. (**a**) The gcd-graph $G_{12}(2)$; (**b**) The gcd-graph $G_{12}(\{3,4,6\})$.

As already seen, the unitary Cayley graph X_n is a special case of integral circulant graphs, or gcd-graphs, and, hence, all the properties that are investigated for the latter hold for X_n, but the bounds and results obtained for unitary Cayley graphs are more specific and tight than results obtained for these broader classes of graphs. Therefore, in this article, we present a review of the studies which are specifically made on the unitary Cayley graphs and the results that were explicitly stated for the graph X_n as an application or a corollary in the articles that studies the integral circulant graphs or gcd-graphs.

In [54], an open problem to determine the automorphism group of the unitary Cayley graphs X_n, for $n > 6$, had been posited by the authors, which led to the investigation on the automorphisms of X_n. Though the problem was not fully addressed, a necessary and sufficient condition for a bijective mapping to possess the structure of an automorphism of the graph X_n was given in [60] as follows.

Theorem 16 ([60])**.** *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$, where p_i, $1 \leq i \leq r$ are distinct primes, $\alpha_i \in \mathbb{N}$, and r is the number of distinct prime divisors of n. Then, a bijective mapping induces an automorphism of the graph X_n if and only if it preserves congruence modulo p_i for all i.*

Apart from the above-mentioned result, that was obtained on the automorphism of X_n, a characterisation of planar unitary Cayley graphs was obtained along with the crossing number (the lowest number of edges that cross in a planar graph drawing) of X_n, for a few values of n for which the graph structure was a well-known graph class, by using the existing results on the crossing number of these graph classes. The traversal properties of X_n were also discussed in the article along with which the edge chromatic number, and the edge connectivity of the graph was also determined as given below, where $\phi(n)$ denotes the Euler totient function.

Theorem 17 ([60])**.** *The graph X_n is planar if and only if $n \in \{1, 2, 3, 4, 6\}$.*

Theorem 18 ([60])**.** *The graph X_n, $n \geq 3$ is Eulerian as well as Hamiltonian, and each such X_n can be decomposed into $\frac{\phi(n)}{2}$ edge-disjoint Hamiltonian cycles.*

Theorem 19 ([60])**.** *The edge connectivity of the graph X_n is $\phi(n)$.*

Theorem 20 ([60])**.** *For the graph X_n, the edge chromatic number is $\phi(n)$ and $\phi(n) + 1$, when n is even or odd, respectively.*

The property of the graph X_n having both its edge and vertex connectivity equal to its degree of regularity, as well as the graph being integral-circulant, increases the application of the graphs in the field of networks, especially in areas that require a stable and strong network. This increased the significance of studying the graph for various application purposes, and this also gave researchers curiosity to investigate other properties of the graphs and construct similar graphs. Extending the study further, the authors studied the basic graph properties of the unitary Cayley graph of a ring, which was obtained as a finite direct product of the rings $\mathbb{Z}_n$ for different values of n. This extension gave rise to the idea of generalising the unitary Cayley graphs of $\mathbb{Z}_n$ to any ring R, a detailed review of which is given in Section 3.

The open problem to determine the automorphism group of X_n, put forth in [54], was solved in [61] by obtaining the automorphism groups of X_n and their cardinality for different values of n, as a tool to generalise the automorphism groups of the integral circulant graphs. The results obtained are given below, showing that the structure of the automorphism groups become highly sophisticated as the value of n increases.

Theorem 21 ([61])**.** *For $n = p^k$, where p is a prime number and $k \geq 1$, the size of the automorphism group of X_n, $|Aut(X_n)| = p!((p^{k-1})!)^p$.*

Theorem 22 ([61])**.** *For $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$, where p_i, $1 \leq i \leq r$, are distinct primes and $\alpha_i \in \mathbb{N}$, the size of the automorphism group of X_n, $|Aut(X_n)| = \prod_{i=1}^{r} p_i!((\frac{n}{\prod_{i=1}^{r} p_i})!)^{\prod_{i=1}^{r} p_i!}$.*

The structure of the automorphism group of X_n was proven by partitioning the vertices of X_n based on the residue modulo primes, which is similar than the residue representation introduced in [49], and the permutations on these residue classes were considered to obtain automorphisms of the graph, using the notion of modular arithmetic and the Chinese remainder theorem. According to the construction of automorphisms of X_n in the proof of Theorem 22, it was concluded that the automorphism group is isomorphic to the wreath product of the permutation group (refer to [62]) of the graphs of residue classes modulo r and the permutation groups of vertices in each class, as given below.

Theorem 23 ([61])**.** *For $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$, where p_i, $1 \leq i \leq r$ are distinct primes and $\alpha_i \in \mathbb{N}$, the automorphism group of X_n, $Aut(X_n) \cong (S_{p_1} \times S_{p_2} \times \dots \times S_{p_r}) \wr S_{\frac{n}{r}}$, where S_k represents the group of permutations on k elements and $\wr$ denotes the wreath product of groups.*

The same problem of determining the automorphism group of the unitary Cayley graph was solved in [21,63] using different approaches. The study in [21] began with a motive to investigate the automorphism group of X_n, but the authors, upon observing the symmetric pattern of X_n in several aspects, extended the concept of unitary Cayley graphs to any ring R, and the automorphism groups of the unitary Cayley graphs defined on a ring R were investigated, which, in the special case $R = \mathbb{Z}_n$, gave the automorphism group of X_n. The main idea of their algebraic proof, where the dependence of the automorphisms on the underlying algebraic structure of the rings concerned was emphasized, was different from the proof given in [61], which used a number-theoretical approach. The authors of [63] investigated the automorphism group of the rational circulant graphs—circulant graphs with rational spectra—in which the integral circulant graphs became a subclass by developing a framework based on Schur rings (for more details, refer to [64,65]). The approach is highly complex, as it is built for all rational circulant graphs, but it is claimed, in [63], that the automorphism group of X_n could have been traced a few decades ago if the framework of the approach presented in [63] was followed.

The results on the spectra of the unitary Cayley graphs obtained in [54] fascinated the researchers, leading them to explore other parameters and properties of the unitary Cayley graph X_n that were closely associated with its adjacency matrix and its eigenvalues. The first of such properties to be investigated was the perfect state transfer in unitary Cayley graphs. For a graph G with adjacency matrix A, $H(t)$ is defined as the operator $e^{(itA)}$, called the transition operator. A *perfect state transfer* between the vertices u and v is said to happen at time τ if the uv-entry of $|H(\tau)_{u,v}| = 1$. This perfect state transfer is used in several areas that deal with allocation and assignment factors; in particular, it has been efficiently applied to key distribution in commercial cryptosystems and in assignment of objects in quantum spin networks (see [57]). This notion was introduced to circulant graphs in [66], and the perfect state transfer in the integral circulant graphs was studied in [57]. Based on these studies, the class of unitary Cayley graphs that allowed perfect state transfer was characterised in [57] as follows.

Theorem 24 ([57])**.** *The only unitary Cayley graphs that allow perfect state transfer are X_2 and X_4.*

Following the study on perfect state transfer in the unitary Cayley graph X_n, the properties related to the energy of the graph, which is the sum of the absolute values of the eigenvalues of the adjacency matrix of the graph, was determined in [67,68] as follows.

Theorem 25 ([67,68])**.** *For $n = p^t$, where p is a prime and $t \in \mathbb{N}$, the energy of X_n, $\mathscr{E}(X_n) = 2\phi(n)$, where $\phi(n)$ represents the Euler totient function.*

Theorem 26 ([67,68])**.** *For $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$ and $n' = p_1 p_2 p_3 \dots p_r$, where p_i, $1 \leq i \leq r$ are distinct primes and $r \neq 1$, the energy of X_n, $\mathscr{E}(X_n) = 2^r \phi(n)$, where $\phi(n)$ represents the Euler totient function.*

Theorem 26 arises as a consequence of Theorem 25 along with the fact that, for $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$ and $n' = p_1 p_2 p_3 \dots p_r$, where p_i, $1 \leq i \leq r$ are distinct primes and $r \neq 1$, $X_n \cong X_{p_1^{\alpha_1}} \times X_{p_2^{\alpha_2}} \times \dots \times X_{p_r^{\alpha_r}}$. Based on the energy of the graph X_n obtained, the hyperenergetic unitary Cayley graphs, along with their complements, were characterised in [67,68] as follows. Note that a graph G of order n is called *hyperenergetic* if its energy, $\mathscr{E}(G)$, is greater than the energy of the complete graph of order n, that is, $\mathscr{E}(G) > \mathscr{E}(K_n) = 2(n-1)$ (see [67]).

Theorem 27 ([67,68])**.** *The graph X_n is hyperenergetic if and only if n has at least two prime factors greater than 2 or at least three distinct prime factors.*

Theorem 28 ([67,68])**.** *The graph $\overline{X}_n$ is hyperenergetic if and only if n has at least two distinct prime factors and $n \neq 2p$, where p is a prime number.*

Both [67,68] discuss the energy and hyperenergercity of the graphs X_n and $\overline{X}_n$, and the same results, using similar proof techniques, were obtained independently. In addition to these results, the ratio $\frac{\mathscr{E}(X_n)}{2(n-1)}$, that measures the degree of hyperenergecity of X_n, which can be seen to grow exponentially as the number of distinct prime divisors of n increases, was given in [68]. In the process of proving the above results, the nullity of the graph was discussed, which was also independently proven in [69]. After the publication of [68], a comment on the article was released, wherein a one line proof to determine the energy of unitary Cayley graphs, as was determined in Theorems 25 and 26, using the notion of Ramanujan sums, was given.

This was followed by a discussion on the eigenspace of the Unitary Cayley graphs in [70], where, in a specific case, the class of graphs called Hamming graphs were proven to be isomorphic to the unitary Cayley graphs and, using the results obtained on the spectra of these unitary Cayley graphs, the eigenspaces of Hamming graphs were determined. Note that, for non-negative integers k, r, s, the *Hamming graph* $HG(l_1, l_2, \dots l_r; s)$ is a graph which is constructed based on the number of words formed by considering r out of a given k letters, which have a Hamming distance s. In other words, given k letters, the k^r possible words with $r \leq k$ letters are the vertices of a Hamming graph, and two vertices are joined by an edge if their associated words differ in exactly s positions (see [70]).

Theorem 29 ([70])**.** *For $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$ and $n' = p_1 p_2 p_3 \dots p_r$, where p_i, $1 \leq i \leq r$ are distinct primes and $r \neq 1$, $X_n \cong HG(p_1, \dots, p_r; r)$.*

A k-regular graph G is said to be a *Ramanujan graph* if and only if the second largest absolute value of the eigenvalues of the adjacency matrix of G, $\lambda_2(G) \geq 2\sqrt{k-1}$ (c.f. [71]). This idea of realising a graph as a Ramanujan graph was explored in unitary Cayley graphs and their complement, using the spectra of the graphs that were obtained in the previous literature, and a complete characterisation of the cases in which the unitary Cayley graph and its complement were Ramanujan graphs were obtained in [71,72], respectively, as follows.

Theorem 30 ([71])**.** *The graph X_n is a Ramanujan graph if and only if n satisfies one of the following conditions for some distinct odd primes $p_1 < p_2$ and for $s \in \mathbb{N}$:*

(i) $n = 2s$, *for some* $s > 2$;

(ii) $n = p_1$;

(iii) $n = 2^s p$, *where* $p > 2^{s-3} + 1$;

(iv) $n = p_1^2\,, 2p_1^2, 4p_1^2$;

(v) $n = p_1 p_2\,, 2p_1 p_2$, *where* $p - 1 < p_2 \leq 4p_1 - 5$;

(vi) $n = 4p_1 p_2$, *where* $p - 1 < p_2 \leq 2p_1 - 3$.

Theorem 31 ([72])**.** *For $n \geq 2$, the graph $\overline{X}_n$ is a Ramanujan graph if and only if n has one of the following forms:*

(i) *n is a prime power;*

(ii) $n = 2^{t_1} 3^{t_2}$, *where* $1 \leq t_1 \leq 3$ *when* $t_2 = 1$, *or* $t_1 = 1$ *when* $t_2 = 1, 2$;

(iii) $n = 10$ *or* 30;

(iv) $n = p_1 p_2$, *where* $p_1 = 3, 5$ *and* $p_2 = 5, 7$.

Further investigations on some variants of energy, namely, the distance energy, colour energy, minimum covering Gutman energy, the minimum edge dominating energy, and the Seidal Laplacian energy of the unitary Cayley graphs, were conducted in [9,73–77], respectively. As already known, *energy* of a graph is the sum of the absolute values of the eigenvalues of a matrix. Based on the defined matrix, the corresponding spectra and the energies can be computed. Therefore, the distance energy is obtained from the distance matrix of the graph, which is a square matrix in which the ij-th entry gives the shortest distance between the vertices v_i and v_j in the graph (see [74]). The *colour energy* of a graph G corresponds to the energy of the A_L-matrix of G (c.f. [9,76]), whose entries are based on a proper vertex colouring of the graph G, say c, such that

$$a_{L_{ij}} = \begin{cases} 1, & \text{if } v_i v_j \in E(G) \text{ and } c(v_i) \neq c(v_j); \\ -1, & \text{if } v_i v_j \notin E(G) \text{ with } c(v_i) = c(v_j); \\ 0, & \text{if } v_i = v_j \text{ or } v_i v_j \notin E(G) \text{ with } c(v_i) \neq c(v_j). \end{cases}$$

A *minimum covering set* $C \subseteq V(G)$ of a graph G is a subset of vertices, such that each edge of the graph is incident to at least one vertex in the subset, and the minimum number of vertices in such a set is called the *minimum covering number* of the graph (c.f. [75]). A *minimum covering matrix* $MC_C(G)$ of a graph G of order n is an $n \times n$ matrix, defined based on the adjacency of the vertices in a minimum covering set C, such that the diagonal entries of the adjacency matrix of the graph G are 1 if the corresponding vertex belongs to the minimum covering set considered (see [78]). The *Gutman matrix* $GM(G)$ of a graph G of order n is a square matrix of order n, whose entries are 0 and $d_i d_j d_{ij}$, where d_i and d_j are the degrees of the vertices v_i and v_j, and d_{ij} is the shortest distance between v_i and v_j, corresponding to the conditions if the vertices $v_i = v_j$ and $v_i \neq v_j$ (c.f. [79]).

The *minimum covering Gutman energy* of a graph G is computed based on the minimum covering Gutman matrix $MCG(G)$ defined in [75], which, as observed, is defined as a combination of the minimum covering matrix and Gutman matrix as follows:

$$mcg_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E(G) \text{ and } c(v_i) \neq c(v_j); \\ 0, & \text{if } i = j \text{ and } v_i \notin C, \text{ where } C \text{ is a minimum covering set;} \\ d_i d_j d_{ij}, & \text{otherwise, where } d_i \text{ and } d_j \text{ are the degrees of the vertices } v_i \text{ and } v_j \\ & \text{and } d_{ij} \text{ is the shortest distance between } v_i \text{ and } v_j. \end{cases}$$

Similarly, the *minimum edge dominating energy* of a graph G is the sum of the absolute values of eigenvalues of the minimum edge dominating matrix of G, which is a binary matrix of order $m \times m$, where m is the size of G in which the entries are based on the adjacency of the edges and the minimum edge-dominating set of the graph. A subset $F \subseteq E(G)$ is an *edge-dominating set* of a graph G if every edge not in F is adjacent to at least one edge in F, an edge-dominating set with the least cardinality is called a minimum edge-dominating set of the graph, and cardinality is the *edge domination number* of the graph (c.f. [34]).

The study on minimum covering Gutman energy of X_n involved the discussion of this energy for unitary Cayley graph X_n, for the values of n for which X_n was a common graph class such as complete graph, complete multipartite graph, etc. A similar situation was encountered on the discussion of the minimum edge-dominating energy of the unitary Cayley graphs in [73], except for a few bounds that were deduced instead of calculating the exact values.

The distance spectra, along with the corresponding energy of the unitary Cayley graphs, were computed in [74], as a part of the study on the same integral circulant graphs, and it was proven that the integral circulant graphs, including X_n, had integral distance spectra. On investigating the distance energies of both these graphs, a construction of infinite families of distance equi-energetic graphs (graphs, possibly isomorphic, that have the same energy) emerged, which were the first ones to be derived without using construction methods, that is, without taking graph products nor iterated line graphs (defined in the later part of this section). The results on the distance energy of X_n and the construction obtained in [74] are given below.

Theorem 32 ([74])**.** *The distance energy of X_n,*

$$DE(X_n) = \begin{cases} 2(n-1), & \text{if } n \text{ is prime;} \\ 4(n-2), & \text{if } n = 2^t, \text{for some } t \in \mathbb{N}. \end{cases}$$

Theorem 33 ([74])**.** *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$, where p_i, $1 \leq i \leq r$ are distinct primes and $\alpha_i \in \mathbb{N}$, be an odd composite number and $m = p_1 p_2 \dots p_r$ be the maximal square-free divisor of n. The distance energy of X_n,*

$$DE(X_n) = 2\left[2n + \phi(n)(2^{r-1} - 1) - m - 2 + \prod_{i=1}^{k}(2 - p_i)\right],$$

where $\phi(n)$ is the Euler totient function.

Theorem 34 ([74]). *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$, where p_i, $1 \leq i \leq r$ are distinct primes and $\alpha_i \in \mathbb{N}$, be an even number with odd prime divisor and $m = p_1 p_2 \dots p_r$ be the maximal square-free divisor of n. The distance energy of X_n,*

$$DE(X_n) = \frac{9n}{2} - 2m + 1 + \phi(n)(2^{k+1} - 6) + |2\phi(n) - 2 - \frac{n}{2}|,$$

where $\phi(n)$ is the Euler totient function.

In Theorem 34, the value of $|2\phi(n) - 2 - \frac{n}{2}|$ cannot be resolved, since it takes all positive, zero, and negative values, and, for specific n values, the solution of the problem relates to the still-open conjecture on the Euler totient function (refer to [80]), for which obvious solutions involve prime Fermat numbers, where a *Fermat number* is a positive integer of the form $2^{2^n} + 1$, $n \in \mathbb{N}$ (see [36]).

Theorem 35 ([74]). *Let $n = p_1 p_2$, where p_1 and p_2 are odd primes. The unitary Cayley graph X_n is equi-energetic with the* gcd*-graph $G_n(1, p_1)$, that is, $DE(X_n) = DE(G_n(1, p_1))$.*

The colour energy of the unitary Cayley graph and its complement were studied in [9,76]. The eigenvalues of the A_L matrix, defined with respect to the proper colourings of the graphs, were examined, and the corresponding energy was obtained in terms of the chromatic number of the graph and the Euler totient function, using the notion of Ramanujan Sums. A study on a few other matrices of the unitary Cayley graphs, along with their eigenvalues and energy, was conducted in [81], where a small-world network depending on the unitary Cayley graph was proposed, with an intent to decrease the delay and increase the reliability in data transfer used to create and analyse network communication.

The Seidal Laplacian energy of the unitary Cayley graph X_n was computed in [77] by obtaining the eigenvalues of the Seidal Laplacian matrix $SL(X_n) = S(X_n) - DS(X_n)$ of X_n, where $SL(X_n)$ is the Seidal Laplacian matrix of X_n, $S(X_n)$ is the Seidal matrix of X_n, and $DS(X_n)$ is an $n \times n$ diagonal matrix of X_n, which has its diagonal entries $n - 1 - 2deg(v_i)$, $1 \leq i \leq n$. The *Seidal matrix* of a graph G is an $n \times n$ matrix with entries $1, -1$ corresponding to whether the vertices $v_i v_j \in E(G)$ or $v_i v_j \notin E(G)$, or 0 otherwise (refer to [77]).

An *algebra* over a field is an algebraic structure consisting of a set, together with the operations of addition, multiplication, and scalar multiplication by elements of a field, that satisfies the axioms of a vector space with a bilinear operator; where, a *bilinear operator* is a function of two variables which are linear with respect to each of their variables. In other words, an algebra over a field is a vector space equipped with a bilinear operator (c.f. [82]).

For a positive integer n, the set of all $n \times n$ matrices over the field of complex numbers $\mathbb{C}$ forms an algebra $\mathbb{M}_n(\mathbb{C})$, with the usual matrix multiplication. As the adjacency matrix of a graph $A(G)$ is a well-known square matrix, the adjacency algebra of a graph is defined as the subalgebra of $\mathbb{M}_n(\mathbb{C})$ which consists of all polynomials of $A(G)$ with coefficients from $\mathbb{C}$, where a *subalgebra* is a subset of the algebra, which is an algebra by itself under the same bilinear operator (refer to [16]).

The adjacency algebra of the unitary Cayley graph X_n was investigated in [82]. Since every element of the adjacency algebra of a graph is a linear combination of the powers of its adjacency matrix, the results on the adjacency algebra of a graph was obtained using the powers of the adjacency matrix. Therefore, using the existing results on the adjacency matrix of the graph X_n, the adjacency algebra of X_n was discussed in [82], and it was proven that the adjacency algebra of unitary Cayley graphs is a coherent algebra, that is, it is a subalgebra of $\mathbb{M}_n(\mathbb{C})$ containing I, J, where I is the identity matrix and J is the matrix with all its entries being 1, which is closed under Hadamard product and conjugate transposition. For any two square matrices M_1 and M_2 of order n, their *Hardamard product* $M_1 \circ M_2$ is also an $n \times n$ matrix, such that $(m_1 \circ m_2)_{ij} = m_{1ij} m_{2ij}, 1 \leq i, j \leq n$, where m_{1ij} and m_{2ij} are the entries of M_1 and M_2, respectively(c.f. [82]).

For a graph G with an adjacency matrix $A(G)$, its *coherent closure*, denoted by $\mathscr{CC}(G)$, is the smallest coherent algebra containing $A(G)$, and a graph G is said to be a *pattern polynomial graph* if its adjacency algebra is its coherent closure. On proving that the unitary Cayley graphs had a coherent adjacency algebra, the authors proved that every unitary Cayley graph was a pattern polynomial graph, and, using this, certain properties of the unitary Cayley graphs were derived based on the properties of pattern polynomial graphs, in [83]. To prove that all unitary Cayley graphs are pattern polynomial graphs, the following characterisations on the structure of the graphs were obtained.

Theorem 36 ([82])**.** *The graph X_n is a strongly regular graph if and only if n is a prime power.*

Recall that a k-regular graph G of order n is *strongly regular* with parameters (n, k, r, s) if any two adjacent vertices have exactly r common neighbours and any two non-adjacent vertices have exactly s common neighbours, and also that a *crown graph*, $C_{r,r}$ is a bipartite graph with vertex set such that $V(C_{r,r}) = V_1 \cup V_2$ and $|V_1| = |V_2| = r$, with $V_1 = \{v_1, v_2, \ldots, v_r\}$ and $V_2 = \{u_1, u_2, \ldots, u_r\}$, such that $v_i u_j \in E(C_{r,r})$ if and only if $i \neq j$.

Theorem 37 ([82])**.** *The graph X_n is crown graph if and only if $n = 2p$, where p is an odd prime.*

Appropriate representation of the circulant graphs on a Euclidean plane unveils the rotational symmetry of the graph. As previously known, unitary Cayley graphs are integral circulant graphs and, therefore, such a suitable representation or drawing, called the unit circle drawing of a unitary Cayley graph, was examined in [84]. The *unit circle drawing* of the graph X_n is simply a drawing of the graph X_n such that the vertices are placed equidistantly on a unit circle on the complex plane $\mathbb{C}$ and the edges are drawn as line segments. This representation gives a hole-like structure in the middle of the graph, which is called the *central hole* or the *geometric kernel* of the graph. Just as the spectrum of a graph provides vital information on the graph, the size of the geometric kernel in the unit circle drawing of an integral circulant graph, which is measured through the *kernel radius*, also provides the arithmetic properties of the graph.

It was proven in [85] that the central hole in the unit circle drawing of any circulant graph on $n > 3$ vertices is a regular n-gon. Therefore, only the size of the geometric kernel for X_n, which is already known to be an n-gon, had to be determined, in [84], by computing the kernel radius, given by the formula $\max\{k : 1 \leq k < \frac{n}{2}, \gcd(k, n) = 1\}$. Only integers less than $\frac{n}{2}$ were considered because there is no central hole when the edge $(k, \frac{k}{2})$ exists in the unit circle drawing of a graph. It was observed that the kernel radius of X_n is a strictly decreasing function on the range $\left(0, \frac{n}{2}\right]$.

Apart from this, computation of certain graph parameters of the unitary Cayley graph were carried out in [86–93]; certain topological indices of the unitary Cayley graphs were computed in [90–92], and a few graph polynomials for the unitary Cayley graphs were determined in [86], using the results that were given in [54], as graph polynomials are also graph invariants that code numerical information about the underlying graph (c.f. [94]).

It was already seen that unitary Cayley graphs are highly reliable networks that can be used in modeling situations which require stable networks. To assert this, and to study the degree of reliability of these networks, a few vulnerability parameters which measure the vulnerability of a graph were computed for unitary Cayley graphs in [88]. This study on computing vulnerability parameters paved the way to examine the parameters related to graph covering, as performed in [93,95].

The graph covering problem is one of the most classical topics in graph theory, where the minimum number of the entities of a graph, such as vertices, edges, etc., with a particular property having a given graph as their union is determined. One such covering parameter is the *tree covering number*, which is defined as the minimum cardinality among all tree covers of the graph, where a family of mutually edge-disjoint trees in a graph is called a *tree cover* of the graph if each edge is an edge of a tree in the family. This tree covering number was determined for the unitary Cayley graph X_n and its complement $\overline{X}_n$ in [93], from which the Nordhaus–Gaddum-type inequalities, that is, bounds on the sum and the product of the invariant for a graph and its complement, for the tree covering number were obtained. The exact value of the tree covering number of X_n was computed as given in Theorem 38, whereas, for the complement $\overline{X}_n$, the bounds according to different values of n were obtained. Based on these bounds, the Nordhaus–Gaddum-type inequalities were also obtained for different cases of n, depending on its prime factorisation.

Theorem 38 ([93])**.** *The tree covering number of a unitary Cayley graph X_n is $\frac{\phi(n)}{2} + 1$, where $\phi(n)$ is the Euler totient function.*

The other aspect related to covering that was discussed for the unitary Cayley graphs in [95] was the property of the well-coveredness of a graph. A graph G is said to be *well-covered* if all its maximal independent sets are of the same size. In [95], the well-coveredness of the graphs X_n and $\overline{X}_n$, along with its vertex decomposability, were examined, and the conditions under which the graphs were well-covered and vertex decomposable (refer to [96] for more details on vertex decomposable graphs) were given. The number of walks between any pair of two vertices in the unitary Cayley graphs was enumerated in [87], and, as an application of this result, it was shown that there exists a bijection between walks in X_n and the ordered sums of units in $\mathbb{Z}_n$, from which the number of representations of a fixed residue class mod n as the sum of k units in $\mathbb{Z}_n$ was determined.

A function which is defined on the set of positive integers to a subset of the set of complex numbers is an *arithmetic function*. An arithmetic function h is *multiplicative* if it is not identically zero and, for any $r, s \in \mathbb{N}, h(rs) = h(r)h(s)$, whenever $\gcd(r, s) = 1$. For each non-negative integer r and prime p, the r-th *Schemmel totient function* ST_r is a multiplicative arithmetic function that satisfies

$$ST_r(p^\alpha) = \begin{cases} p^{\alpha-1}(p-r), & \text{if } p \geq r; \\ 0, & \text{otherwise}, \end{cases}$$

where α is a positive integer. From the name Schemmel totient function, it can be seen that this function, introduced by Schemmel, is a generalisation of the Euler totient function $\phi(n)$ (c.f. [97]). It can be seen that $ST_0(n) = n$ and $ST_1(n) = \phi(n)$ for all integers n. Since most of the graph invariants of the unitary Cayley graph X_n are computed and expressed in terms of $\phi(n)$ and $ST_r(n)$ is its generalisation, this opened an avenue to check the possibility of expressing the parameters in terms of $ST_r(n)$ and, in [98], a simple formula for the number of cliques of any order in the unitary Cayley graph X_n was obtained as follows.

Theorem 39 ([98])**.** *For a given integer k, the number of cliques of order k in the unitary Cayley graph X_n is given by the expression $\prod_{i=1}^{k} \frac{ST_{i-1}(n)}{k}$, where $ST_{i-1}(n)$ is the Schemmel totient function.*

This formula naturally gives the number of triangles in the graph X_n in terms of the Schemmel totient function as $\frac{ST_0(n)}{1}\frac{ST_1(n)}{2}\frac{ST_2(n)}{3}$, which is more generalised and simple than the same expression which was computed independently in [40,47,48,54].

The *k-th power $G^{(k)}$ of a graph G* is a graph whose vertex set is the same as the vertex set of G, and there is an edge between two vertices in the graph $G^{(k)}$ if and only if there is a path of length at most k between them in G. The k-th powers of the unitary Cayley graphs were examined in [89], where the energies of these graphs were determined and all the powers of unitary Cayley graphs that were Ramanujan graphs were classified. Note that, in [89], the k-th powers of a unitary Cayley graph are addressed as the the distance powers of the graph. Using the results obtained on the energies of distance powers of unitary Cayley graphs, infinitely many pairs of non-cospectral equi-energetic graphs were constructed, and all the hyperenergetic distance powers of unitary Cayley graph X_n were characterised. It can be noticed that the k-th power of any graph G can be defined for the values $1 \leq k \leq diam(G)$ and $diam(X_n) \leq 3$. Therefore, the investigation is limited to the unitary Cayley graphs that have diameter three, in which case there exists only the value $k = 2$, for which the discussion of the k-th power of the graph X_n is non-trivial.

Apart from Cayley graphs, the literature on power graphs of groups is growing, giving rise to several survey papers (c.f. [2,99–101]). Note that the *power graph* of a finite group is a graph with the vertex set as the elements of the group, and two vertices are adjacent if one is a power of the other, which are not to be confused with the k-th power of a graph, as both the graphs are referred to as power graphs in the literature. Owing to the huge literature on power graphs of finite groups, an open problem to explore the relation between power graphs and Cayley graphs was put forth in [99]. This problem was addressed in [102], and it was shown that, for certain values of n, the vertex-deleted subgraphs of power graphs of $\mathbb{Z}_n$ span subgraphs or the complement of the vertex-deleted subgraphs of certain unitary Cayley graphs. Using these relations, the relationships between the energy of power graphs and Cayley graphs were also obtained in [102]. The following theorem gives a relation between the power graph $\mathscr{P}(\mathbb{Z}_n)$ and unitary Cayley graph X_n of $\mathbb{Z}_n$ for some values of n.

Theorem 40 ([102])**.**

(i) *For any prime p, $\mathscr{P}(\mathbb{Z}_p) \cong X_p \cong K_p$;*

(ii) *If $n = p_1^{\alpha_1}$, for a prime p_1 and $\alpha_1 > 1$, X_n is a regular spanning subgraph of $\mathscr{P}(\mathbb{Z}_n)$;*

(iii) *When $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where p_1, p_2 are distinct primes, and α_1, α_2 are positive integers, $\mathscr{P}^*(\mathbb{Z}_n)$ is a spanning subgraph of $\overline{X^*}_n$, where $\mathscr{P}^*(\mathbb{Z}_n)$ is the vertex-deleted subgraph, $\mathscr{P}(\mathbb{Z}_n) - \{\mathbb{Z}_n^* \cup 0\}$ and X_n^* is the vertex-deleted subgraph, $X(\mathbb{Z}_n) - \{\mathbb{Z}_n^* \cup 0\}$. The graphs $\overline{X^*}_n \cong \mathscr{P}^*(\mathbb{Z}_n)$ if and only if $\alpha_1 = \alpha_2 = 1$.*

Recall that the study on unitary Cayley graphs began with the investigation of the edge colouring of the graph, in order to obtain a total multi-coloured graph. This motivated researchers to study different colourings of the graph and to investigate the related parameters and properties. The total colouring and the strong edge colouring of unitary Cayley graphs were studied in [103–105]. A *total colouring* of a graph G is a proper colouring on both the edges and vertices, such that no two adjacent entities (both vertices and edges) are assigned the same colour, and the *total chromatic number* is the minimum number of colours required in the total colouring of the graph (see [105]). The total colouring conjecture given in [106] states that the total chromatic number of a graph G is at most $\Delta(G) + 2$, where $\Delta(G)$ is the maximum degree of G; this was proven for the unitary Cayley graphs in [105], as part of the investigation on the total colouring of some regular graphs. Additionally, the total chromatic number of unitary Cayley graphs was determined; along with this, a pattern to assign colours to obtain an optimal total colouring of unitary Cayley graphs for some values of n was given in [103].

A *strong edge colouring* of a graph G is a proper edge colouring of G such that every colour class induces a matching, and the minimum number of colours required is the *strong chromatic index*. In [104], the strong chromatic index of all unitary Cayley graphs was determined, and the colouring technique revealed the underlying product structure from which the unitary Cayley graphs emerge.

Following the notion of colouring, domination in unitary Cayley graphs was investigated in [107–110]. In [98], the domination number, upper domination number, and total domination number (refer to [34]) of the unitary Cayley graphs were investigated based on the structural property of the unitary Cayley graph X_n to be realised, as a direct product of its factor graphs that were complete. The bounds for these domination parameters were obtained in terms of an arithmetic function called the *Jacobsthal function* $g(n)$, that denotes the smallest positive integer r, such that every set of r consecutive integers contains an element that is relatively prime to n (see [111]). By the definition of $g(n)$ and X_n, it can be deduced that the set $\{0, 1, \ldots, g(n) - 1\}$ is a dominating set as well as a total dominating set of X_n, the cardinality of which gives a tight bound on the total domination number and the domination number of X_n. It was proven that the domination number of X_n need not necessarily be equal to $g(n)$ by identifying the cases when the equality $\gamma(X_n) = g(n)$ does not hold. Additionally, the rate at which the tightness of the bound decreases as the n value increases was also discussed in [108], as given below.

Theorem 41 ([108])**.** *For each positive integer j, there is an integer n with more than j distinct prime factors, such that $\gamma(X_n) \leq \gamma_t(X_n) \leq g(n)$, where $\gamma(X_n)$, $\gamma_t(X_n)$, and $g(n)$ denote the domination number of X_n, total domination number of X_n, and the Jacobsthal function.*

Theorem 42 ([108])**.** *If $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$ is an integer with a square-free canonical representation ($\alpha_i < 2$, for all $1 \leq i \leq r$), having fewer than 3 distinct primes, then the domination number of X_n is at most 4.*

Theorem 43 ([108])**.** *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_r^{\alpha_r}$, where p_i, $1 \leq i \leq r$ are distinct primes and $\alpha_i \in \mathbb{N}$. If $r \leq 3$ and $\alpha_j \geq 2$ for some $1 \leq j \leq r$, then the domination number of X_n is at least $\frac{p_1}{p_1 - 1}$.*

Theorem 44 ([108])**.** *If the number of distinct prime factors of n is at most 3, such that n is not square-free, then the domination number of X_n is $g(n)$, where $g(n)$ denotes the Jacobsthal function.*

The proof of Theorems 38 and 43 establishes that, for infinitely many n, the domination number of X_n is strictly less than the Jacobsthal function evaluated at n, and this gives rise to a tighter bound on the total domination number (for definition, refer to Section 4) of X_n, $\gamma_t(X_n)$; $\gamma_t(X_n) \leq g(n)$, whenever n has at most three distinct prime factors. These results also affirm the fact that as the number of prime factors of n increases, the domination number as well as the total domination number of X_n are never equal to the Jacobsthal function $g(n)$, by showing that there exists an integer n with arbitrarily many distinct prime factors, such that the bound $\gamma(X_n) \leq \gamma_t(X_n) < g(n)$ holds.

Additionally, the possibility of the value $g(n) - \gamma(X_n)$ being arbitrarily large was not explored in the article, meaning that the open problems to determine the existence of integers n with arbitrarily large number of distinct prime factors such that $\gamma(X_n) \leq g(n) - 2$ and to find a single integer n such that $\gamma_t(X_n) \leq g(n) - 2$ were posited. Apart from this, it was also conjectured that the upper domination number of X_n is $\frac{n}{p_1}$, where p_1 is the smallest prime factor of n, and the conjecture was proven for certain values of n, based on their number-theoretical properties. The approach in [107] to determine the domination parameters of the unitary Cayley graphs were built in order to investigate the solutions of the two open problems posed in [108]. These open problems were solved in [107] by constructing integers n with arbitrarily many distinct prime factors, such that the unitary Cayley graph X_n contains a dominating cycle of size $g(n) - 2$, thus answering both questions, because a dominating cycle is a total dominating set.

Recall that a dominating set which is independent is called an *independent dominating set* and the minimum cardinality of such a set is called the *independent domination number*. Additionally, a set $S \subseteq V(G)$ is called *irredundant* if, for each $v \in S$, either v is isolated in S or v has a neighbour $u \notin S$ such that u is not adjacent to any vertex of $S - \{v\}$. The minimum size of a maximal irredundant set is called the *irredundance number* of graph G (c.f. [34]). The bounds on other domination parameters such as the irredundance number ($ir(X_n)$), independent domination number ($i(X_n)$), etc. of unitary Cayley graphs were determined in [107], as a special case of these bounds, obtained for the direct products of complete graphs. This result gave rise to the construction of some infinite families of integers n, where $ir(X_n) = \gamma(X_n) = i(X_n)$, as given below.

Theorem 45 ([107])**.** *For a unitary Cayley graph X_n, $ir(X_n) = i(X_n)$, when $n = p$, $n = 2p$, or $n = 3p$ for some prime p, or when n is square-free with exactly three prime divisors.*

The problem of finding other square-free integers n for which the equality is achieved in the lower portion of the domination chain (see [34]) was posed, along with two other open problems similar to the ones posed in [108], to check the existence of infinitely many integers n such that $\gamma_c(X_n) > g(n)$, and, if so, to check if such integers can have arbitrarily many distinct prime factors and if there exists a single integer n such that $\gamma_t(X_n) \geq g(n) - 3$, where $\gamma_c(X_n)$ and $\gamma_t(X_n)$ are the connected and total domination number of X_n, respectively. Note that the *connected domination number* of a graph is the cardinality of a minimum dominating set whose induced subgraph is connected (refer to [34]).

The study on the domination parameters of the unitary Cayley graph X_n was extended in [110], where the open problem to find an integer n such that $\gamma_t(X_n) \geq g(n) - 3$ was solved, using the updated results on the nature of Jacobsthal function in the literature. The problem was solved not just for $\gamma_t(X_n) \geq g(n) - 3$, but also the existence of n with arbitrarily many prime factors that satisfied $\gamma_t(X_n) \geq g(n) - 16$ was also proven in [110]. In addition to this, new lower bounds on the domination numbers of direct products of complete graphs were presented in [110], from which new asymptotic lower bounds on the domination number of X_n, when n was a product of distinct primes, were obtained by adopting the proof techniques used in [108].

Two variants of domination, namely, the closed domination and the inverse closed domination, of unitary Cayley graphs were discussed in [109] by determining the corresponding domination parameters. Given a graph G, we choose $v_1 \in V(G)$ and put $S_1 = \{v_1\}$. If $N_G[S_1] \neq V(G)$, we choose $v_1 \in V(G) - S_1$ and put $S_2 = \{v_1, v_2\}$. Where possible, for ≥ 3, we choose $v_k \in V(G) - N_G[S_{k-1}]$ and put $S_k = \{v_1, v_2, \ldots, v_k\}$. At some point, we obtain a positive integer k, such that $N_G[S_k] = V(G)$. A dominating set obtained in the above method is called a *closed dominating set*, and the smallest cardinality of a closed dominating set is called the *closed domination number* of G (c.f. [112]). The dominating set $S \subseteq V(G) - D$ is called an *inverse dominating set* with respect to D. A closed dominating set $S \subseteq V(G) - C$ is called an *inverse closed dominating set* with respect to C, and the minimum cardinality of an inverse closed dominating set is the *inverse closed domination number* of G (c.f. [113]). In the study, the closed and inverse closed domination numbers of unitary Cayley graphs whose structures are standard graph classes, such as complete graphs, complete r-partite graphs, etc., were computed based on the existing results for those graph classes, and hence, they did not contribute to any dynamic results.

On reviewing the literature on the domination of unitary Cayley graphs, it was seen that unitary Cayley graphs were independently investigated under the name *Euler totient Cayley graphs*, and a review of the studies conducted on the graphs X_n under the name Euler totient Cayley graphs is given in the following subsection.

2.1. Euler Totient Cayley Graphs

Definition 5 ([114])**.** *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r}$, where p_i, $1 \leq i \leq r$ are distinct primes, $\alpha_i \in \mathbb{N}$, and r is the number of prime divisors of n. The* arithmetic graph *$\mathscr{V}_n$ is defined as the graph whose vertex set consists of the divisors of n, and two distinct vertices are adjacent in the graph if and only if their* gcd *is a prime divisor of n. In other words, two distinct vertices $u, v \in E(\mathscr{V}_n)$, when* $\gcd(u, v) = p_i$, $1 \leq i \leq r$. *An illustration of an arithmetic graph is given in Figure 4.*

Euler totient Cayley graphs were introduced in [114] as a combination of arithmetic graphs and Cayley graphs. As this was a parallel independent study on the same graph with a different name, various results were repeated in the literature; however, studies on Euler totient Cayley graphs were mainly concentrated on the computation of different domination parameters of the graph. Euler totient Cayley graphs were introduced in [114], in which the basic properties of the graphs were studied, and the values of n for which the graph was a standard graph class were classified and characterised. Using this study, various types of domination were discussed and the corresponding domination parameters were determined in [115–122].

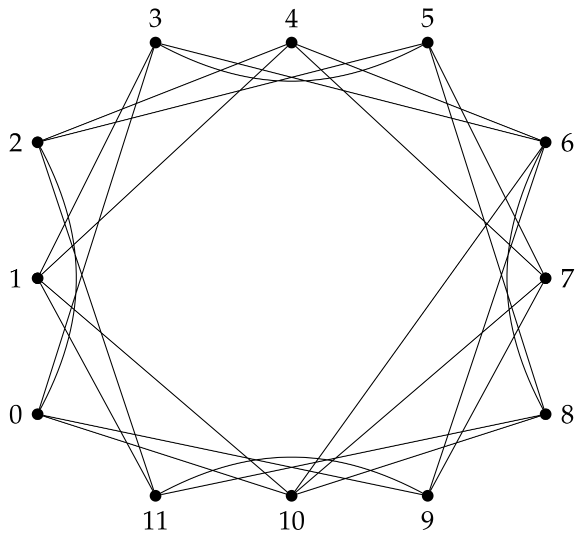


Figure 4. The arithmetic graph $\mathscr{V}_{12}$.

The results on the domination number of the Euler totient Cayley graph proven in [120] were the motivation to investigate the tightness of the bounds of the domination number in terms of the Jacobsthal function, as given in [107,108]. Additionally, on computing the domination parameters of X_n in [107], an error in the bounds obtained in [117] for the independent domination number of the graph was stated and rectified. The independent domination number and the isolate domination number of the Euler totient Cayley graphs were discussed again in [123], in which the bounds obtained in [117] were improved for a few cases, and a few counterexamples to disprove the results in [123] were also obtained. Note that a set-dominating set of a graph G whose induced subgraph has an isolated vertex is called an *isolate dominating set of* G, and the minimum cardinality of such a set is the *isolate domination number* of the graph (c.f. [124]).

Apart from this, the energy of Euler totient Cayley graphs was studied in [123,125], which were a prefatory studies when compared to the study on the energy of unitary Cayley graphs in [67,68]. Additionally, certain functions defined on the vertex set of a graph-like independent and basic minimal-dominating functions (for more details, see Section 6.5.2) were discussed for Euler totient Cayley graphs in [126,127], and the structure and enumeration of cycles in Euler totient Cayley graphs was discussed in [122,128]. Note that a function $f : V \to [0,1]$ is an *independent function* if, for every vertex v with $f(v) > 0$, $\sum_{u \in N(v)} f(u) = 1$, where $N(v)$ is the set of all vertices adjacent to v (see [127]).

As Euler totient Cayley graphs were introduced relating the arithmetic graphs, the different domination numbers that were determined for the Euler totient Cayley graphs were also computed for the different graph products of Euler totient Cayley graphs with the arithmetic graphs in [129–133]. These included the lexicographic product, Cartesian product, direct product, and strong product of the graphs concerned, where the definitions of different graph products studied were as follows.

Definition 6 ([134])**.** *Let G_1 and G_2 be two simple graphs with vertex sets $V(G_1)$ and $V(G_2)$, respectively. The* lexicographic product *$G_1[G_2]$ of G_1 and G_2 is a graph with $V(G_1[G_2]) = V(G_1) \times V(G_2)$, and two vertices (v_1, u_1) and (v_2, u_2) are adjacent in $G_1[G_2]$ if either v_1 is adjacent to v_2 in G_1 or u_1 is adjacent to u_2 in G_2.*

Definition 7 ([134])**.** *For two graphs G_1 and G_2 with vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, the* direct product *of G_1 and G_2, denoted by $G_1 \times G_2$, is a graph with $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, and two vertices (v_1, u_1) and (v_2, u_2) are adjacent in $G_1 \times G_2$ if both $v_1 v_2 \in E(G_1)$ and $u_1 u_2 \in G_2$.*

Definition 8 ([134])**.** *Let G_1 and G_2 be two graphs with vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$. The* Cartesian product *of G_1 and G_2, denoted by $G_1 \square G_2$, is a graph with the vertex set $V(G_1 \square G_2) = V(G_1) \times V(G_2)$, and two vertices (v_1, u_1) and (v_2, u_2) are adjacent in $G_1 \square G_2$ if either $u_1 = u_2$ and $u_1 u_2 \in E(G_1)$ or $v_1 = v_2$ and $u_1 u_2 \in G_2$.*

Definition 9 ([134])**.** *Let G_1 and G_2 be two simple graphs with vertex sets $V(G_1)$ and $V(G_2)$, respectively. The* strong product *$G_1 \boxtimes G_2$ of G_1 and G_2 is a graph with $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$, and two vertices (v_1, u_1) and (v_2, u_2) are adjacent in $G_1 \boxtimes G_2$ if*

- *$u_1 = u_2$ and v_1 is adjacent to v_2 in G_1; or*
- *$v_1 = v_2$ and v_1 is adjacent to v_2 in G_2; or*
- *$v_1 v_2 \in E(G_1)$ and $u_1 u_2 \in E(G_2)$.*

The studies in [131,135] focused on the computation of the domination parameters of the Cartesian product of $X_n \square \mathscr{V}_n$, and, in [129,130,133,136], the domination parameters in the direct product of X_n and $\mathscr{V}_n$ were studied. The domination parameters in the lexicographic product of X_n and $\mathscr{V}_n$ were discussed in [132,137–139], and the matching domination number—the minimum cardinality of a dominating set that induces a matching in a graph—of the strong product of the graphs X_n and $\mathscr{V}_n$ was determined in [140].

The different products of the arithmetic graphs with Euler totient Cayley graphs give rise to various graphs with different structural properties, as per the number-theoretic properties of the values of n. Based on this, the parameters were computed in multiple cases, where it can be observed the results were mainly obtained for the structure of graph products that were standard graph classes, making these studies secondary ones. Additionally, it can be seen that the product structures became complex as the value of n increased and the number of prime factors increased. Therefore, this presents a challenge in studying many other structural parameters, despite the pattern and symmetry of the factor graph.

2.2. Signed Graphs Based on the Unitary Cayley Graphs

A *signed graph* (or a *sigraph*), $S = (G, \sigma)$, is a graph obtained from G, in which every edge is assigned either a positive or a negative sign by a function $\sigma : E(G) \to \{+, -\}$. If the signs assigned to the edges depend on some property, the graph is called an *induced sign graph*. It is very natural to extend the theory of signed graphs into the algebraic graphs by assigning signs to the edges of algebraic graphs, and studies on such signed algebraic graphs (algebraic signed graphs) have been found to be of much interest (see [141,142]).

One such signed algebraic graph is the *signed unitary Cayley graph*. As the assignment of signs can be arbitrary or can depend on any property, there are possibilities for generating several variations of signed graphs from a single algebraic graph. Depending on how the signs are assigned to the edges of the graph X_n, there are four variations of the signed graphs that have emerged from the unitary Cayley graphs to date, and the definitions of these graphs are given below, following which the illustrations of each of them are given in Figure 5. Note that the dashed edges in the figures represent the negative edges, and the other edges are positively signed.

Definition 10 ([143])**.** *The* unitary Cayley join signed graph, *denoted by $S_n^{\vee} = (X_n, \sigma^{\vee})$, is a signed graph whose underlying graph is the unitary Cayley graph X_n, $n \in \mathbb{N}$, and the sign of an edge $v_i v_j \in E(S_n^{\vee})$ is assigned by the function $\sigma^{\vee} : E(X_n) \to \{+, -\}$ as follows. For an edge $v_i v_j$ in X_n,*

$$\sigma^{\vee}(v_i v_j) \begin{cases} +, & \text{if } v_i \in \mathbb{Z}_n^* \text{ or } v_j \in \mathbb{Z}_n^*; \\ -, & \text{otherwise.} \end{cases}$$

Definition 11 ([143])**.** *The* negation of the unitary Cayley join signed graph, *denoted by $S_n^{\overline{\vee}} = (X_n, \sigma^{\overline{\vee}})$, is a signed graph whose underlying graph is the unitary Cayley graph X_n, $n \in \mathbb{N}$,* *and the sign of an edge $v_i v_j \in E(S_n^{\overline{\vee}})$ is assigned by the function $\sigma^{\overline{\vee}} : E(X_n) \to \{+, -\}$ as follows. For an edge $v_i v_j$ in X_n,*

$$\sigma^{\overline{\vee}} \begin{cases} +, & \text{if both } v_i \notin \mathbb{Z}_n^* \text{ and } v_j \notin \mathbb{Z}_n^*; \\ -, & \text{otherwise.} \end{cases}$$

Definition 12 ([143])**.** *The* unitary Cayley meet signed graph*, denoted by $S_n^{\wedge} = (X_n, \sigma^{\wedge})$, is a signed graph whose underlying graph is the unitary Cayley graph X_n, $n \in \mathbb{N}$, and the sign of an edge $v_i v_j \in E(S_n^{\wedge})$ is assigned by the function $\sigma^{\wedge} : E(X_n) \to \{+, -\}$ as follows. For an edge $v_i v_j$ in X_n,*

$$\sigma^{\wedge}(v_i v_j) \begin{cases} +, & \text{if both } v_i \in \mathbb{Z}_n^* \text{ and } v_j \in \mathbb{Z}_n^*; \\ -, & \text{otherwise.} \end{cases}$$

Definition 13 ([143])**.** *The* unitary Cayley ring signed graph*, denoted by $S_n^{\oplus} = (X_n, \sigma^{\oplus})$, is a signed graph whose underlying graph is the unitary Cayley graph X_n, $n \in \mathbb{N}$, and the sign of an edge $v_i v_j \in E(S_n^{\oplus})$ is assigned by the function $\sigma^{\oplus} : E(X_n) \to \{+, -\}$ as follows. For an edge $v_i v_j$ in X_n,*

$$\sigma^{\oplus}(v_i v_j) \begin{cases} +, & \text{if either } v_i \in \mathbb{Z}_n^* \text{ or } v_j \in \mathbb{Z}_n^*; \\ -, & \text{otherwise.} \end{cases}$$

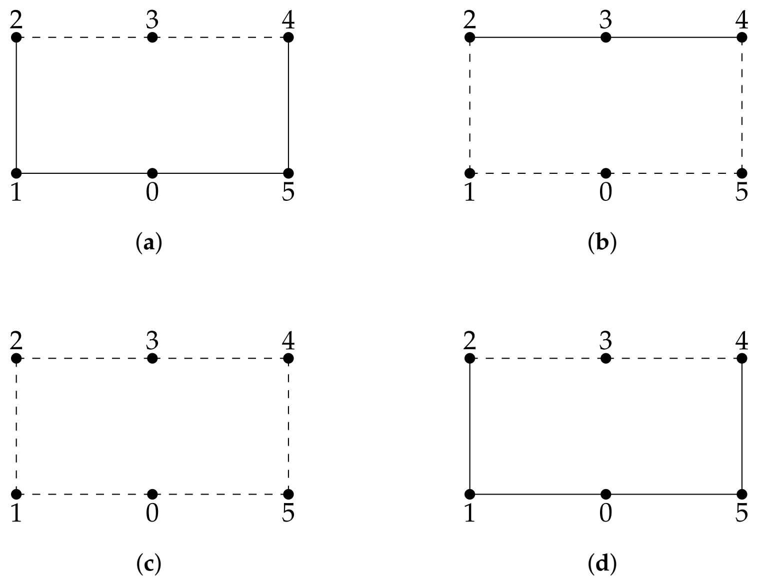


Figure 5. The signed unitary Cayley graphs of X_6. (**a**) The unitary Cayley join signed graph $S_6^{\vee}$. (**b**) The negation of unitary Cayley join signed graph $S_6^{\overline{\vee}}$. (**c**) The unitary Cayley meet signed graph $S_6^{\wedge}$. (**d**) The unitary Cayley ring signed graph $S_6^{\oplus}$.

One of the main properties of a signed graph is its balance and consistency. A signed graph is said to be *balanced* if every cycle in the graph has an even number of negative edges. A *marked sign graph* of a graph G is an ordered pair $S_\mu = (S, \mu)$, where $S = (G, \sigma)$ is a signed graph and the function $\mu : V(S) \to \{+, -\}$ is called a *marking* of the signed graph S. A cycle in S_μ is said to be *consistent* if it contains an even number of negative vertices, and a sign graph S is said to be consistent if every cycle within it is consistent (see [144]). The unique marking μ_σ induced by the sign function $\sigma : E(G) \to \{+, -\}$, such that, for every vertex $v \in V(S)$, $\mu_\sigma(v) = \prod_{e \in E_v} \sigma(e)$, where E_v is the set of all edges incident with v in S, is called the *canonical marking*; a cycle in S is said to be *canonically consistent* if it contains an even number of negative vertices, and the given sigraph is said be *canonically consistent* if every cycle within it is canonically consistent. A sigraph S is *sign-compatible* if there exists a marking of its vertices such that the end vertices of every negative edge receive a negative marking and no positive edge in S has both of its ends assigned a negative sign by the marking; otherwise, the graph is sign-incompatible (see [144]).

The above-mentioned four variations of the signed unitary Cayley graphs were examined in [143,145–147], where the properties of the unitary Cayley join signed graph and its negation were investigated in [146], the unitary Cayley ring signed graph was investigated in [145], and the unitary Cayley meet signed graph was explored in [143,147]. In [146], a characterisation of the balanced unitary Cayley join signed graphs and canonically consistent unitary Cayley join signed graphs $S_n^\vee$, where n had at most two distinct odd prime factors, were obtained, as follows.

Theorem 46 ([146])**.** *The unitary join Cayley signed graph $S_n^\vee$ is balanced if and only if either n is even or if n is odd and it does not have more than one distinct prime factor.*

Theorem 47 ([146])**.** *The negation of a unitary join Cayley sigraph $S_n^\vee$ is balanced if and only if n is even.*

Theorem 48 ([146])**.** *The unitary join Cayley sigraph $S_n^\vee$, where n has at most two distinct odd prime factors, is canonically consistent if and only if n is odd, 2, 6, or a multiple of 4.*

Unitary Cayley ring signed graphs, which are closely associated with unitary Cayley join signed graphs, were examined in [145]. It can be seen that an edge in a unitary Cayley join signed graph is positively signed when at least one of its end vertices is a unit of the ring, that is, either one or both of the end vertices can be units for an edge to be positive. Conversely, an edge in the unitary Cayley ring signed graph is positively signed only when exactly one of its end vertices is a unit of the ring. Therefore, the difference and the relation between the unitary join Cayley signed graph, the unitary ring Cayley signed graph, and the unitary Cayley meet signed graph were given in [145], and the conditions under which they are isomorphic were obtained, as given in Theorems 49 and 50.

Theorem 49 ([145])**.** *For a unitary Cayley graph X_n, the unitary Cayley join sigraph and unitary Cayley ring sigraph are isomorphic if and only if n is even.*

Theorem 50 ([145])**.** *For a unitary Cayley graph X_n, the unitary Cayley join sigraph can never be isomorphic to the unitary Cayley meet sigraph.*

Along with the above-mentioned characterisations of balanced and canonically consistent unitary Cayley ring signed graphs, the characterisations of clusterable and sign-compatible unitary Cayley ring signed graphs were also obtained in [145], as given in Theorems 51 and 52, based on the results on the property of balance. A signed graph is said to be *clusterable* if its vertex set can be partitioned into pairwise disjoint subsets, called clusters, such that every negative edge joins vertices in different clusters and every positive edge joins vertices in the same cluster.

Theorem 51 ([145])**.** *For unitary Cayley graph X_n, the unitary Cayley ring sigraph is balanced if and only if n is even, and is clusterable if and only if the graph is balanced.*

Theorem 52 ([146])**.** *The unitary Cayley ring signed graph $S_n^\vee$ is sign-compatible if and only if either n is even or $n = p^t$, where p is an odd prime and $t \in \mathbb{N}$.*

The unitary Cayley meet signed graphs in which an edge was positively signed only when both of its end vertices were units were investigated in [143,147], where the graphs were characterised based on the similar properties of balance, canonical consistency, sign-compatibility, and clusterability, as given below.

Theorem 53 ([143,147])**.** *For unitary Cayley graph X_n, the unitary Cayley meet sigraph is balanced if and only if n is even or n is a power of an odd prime.*

Theorem 54 ([143,147])**.** *The unitary meet Cayley sigraph $S_n^{\wedge}$, where n has two distinct odd prime factors, is canonically consistent if and only if n is even.*

Theorem 55 ([143,147])**.** *For unitary Cayley graph X_n, the unitary Cayley meet sigraph is always clusterable.*

Theorem 56 ([143,147])**.** *For unitary Cayley graph X_n, the unitary Cayley meet sigraph is sign-compatible if and only if n is even.*

Along with the significant characterisations on the properties of balance, clusterability, etc. of the four different signed graphs defined from the unitary Cayley graphs, a few cursory studies on certain derived signed graphs from the signed graphs corresponding to each of the definitions of the signed graphs were also carried out in [143,145–147], which included the discussions on different variations of the line signed graphs, as the canonical marking served as the signs of the edges in the line signed graphs. Moreover, the property of canonical consistency of the signed graph can be used to investigate the properties, such as balance, clusterability, etc., of the line signed graphs.

3. Unitary Cayley Graph of a Ring

The definition of the unitary Cayley graph X_n on the ring $\mathbb{Z}_n$, naturally fostered an extension of the definition to any associative ring R, in order to explore the properties of the ring and to obtain similar graphs to that of X_n with the same properties. It can be seen that all investigations on the unitary Cayley graphs of rings were inspired from the investigations of the same concepts on X_n, and a particular case of the study or the results obtained on the unitary Cayley graph of a ring R produces the existing results on the graph X_n, which can be seen as a factor of verification of the obtained results on the unitary Cayley graph of any ring, as well as a validation of the existing results on the graphs X_n. This definition of the unitary Cayley graph for a ring R, which is mentioned below, was first put forth in [148]. Following the definition, an illustration of a unitary Cayley graph of a ring is given in Figure 6.

Definition 14 ([148])**.** *Let R be a ring and R^* be the group of units in R. The* unitary Cayley graph*, denoted by $G(R) = Cay(R, R^*)$, is a graph with the vertices set as the elements of the ring, and any two distinct vertices u and v are adjacent in the graph if their difference is a unit, that is, for $u, v \in V(G(R))$, $uv \in E(G(R))$, when $u - v \in R^*$.*

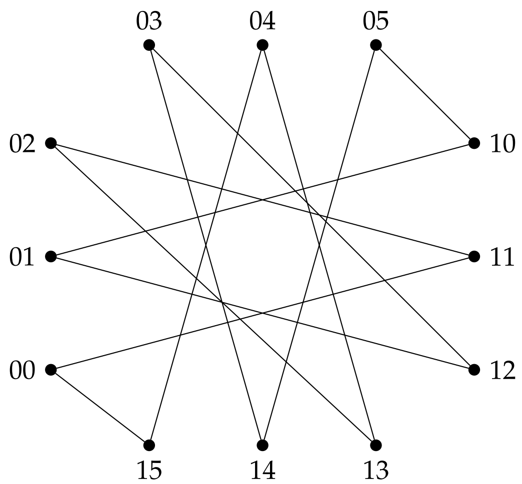


Figure 6. The unitary Cayley graph on $\mathbb{Z}_2 \times \mathbb{Z}_6$.

Before the introduction of the graph as the unitary Cayley graphs in [21], a graph that was constructed using the property of the elements of an Artinian ring, to be expressed as the sum of two units under certain conditions, had the same definition in [148], where a short introductory study on the graph was performed to understand the nature of the graph. The two main results obtained in the study were that, for an Artinian ring R, the number of connected components of the constructed graph $G(R)$ is always a power of 2 and is Hamiltonian. Additionally, to answer the question of the existence of algebraic graphs possessing certain properties that have their clique and chromatic numbers equal, a graph construction on the Artinian rings was proposed in [149] using the same notion, that is, the nature of the elements to be expressed as the sum of units, which later emerged as the formal definition of unitary Cayley graphs of rings in [21].

As we restricted our study to finite graphs, the rings considered are taken as finite rings, unless otherwise mentioned. In [49], the unitary Cayley graph of a ring was defined with the motive of extending a few results of X_n to the unitary Cayley graph of any ring R, where the result on the number of induced cycles in the graph X_n that was enumerated was extended to the graph $G(R)$ for some specific rings. To obtain this extension, the rings which were isomorphic to the direct product of local rings were considered first, and it was proven that, if $R \cong R_1 \times R_2 \times \ldots \times R_t$, where each R_i, $1 \leq i \leq t$ is a local ring with M_i as the maximal ideal, called the local factors of R, then $G(R)$ is a direct product of complete k_i-partite graphs for some k_i. As it was also proven in [49] that the result obtained on the length of the longest induced cycle in X_n holds for the direct product of complete k_i-partite graphs for some k_i (which need not be necessarily finite), the longest induced cycles in $G(R)$, for a ring R which is isomorphic to the direct product of the local rings, were investigated in [49].

To prove the structure of the graph $G(R)$ as the direct product of complete k_i-partite graphs when R is the direct product of local rings, the graph $G(R_i)$ for each local ring R_i was first obtained as a complete k_i-partite graph, where $k_i = |\frac{R_i}{M_i}|$, by partitioning the vertex set of the graph into k_i residue classes modulo. In this partition, two vertices, say, $u, v \in V(G(R_i))$, $1 \leq i \leq t$, belong to the same residue class modulo k_i only when $u - v \in M_i$ and, hence, $u - v \notin R^*$. This implies that two vertices $u, v \in V(G(R_i))$ belong to different partite sets only when they are adjacent and, hence, their difference is a unit, according to the definition of the graph. This partition gives a complete k_i partition for the unitary Cayley graph of each of the local rings, such that the partite sets are the cosets of M_i in the additive group R. Following this, the graph $G(R)$ was proven to be isomorphic to the direct product $G(R_1) \times G(R_2) \times \ldots \times G(R_t)$, based on a similar argument. As a corollary of this result, the same direct product structure of the unitary Cayley graphs of a Dedekind ring, that is, the quotient ring of a Dedekind domain, was also discussed, as the Dedekind rings are local rings.

In algebraic graph theory, realisation of an algebraic structure through the structure of the graph defined on the corresponding algebraic structure is a fundamental problem considered for any new algebraic graph defined. That is, to investigate the relation between the isomorphism of the algebraic structure and the corresponding graphs defined, in order to understand the properties of the algebraic structure that induces the properties of the graph. This problem of realising rings through the graph $G(R)$ was addressed in [150], by proving that the unitary Cayley graphs of rings are isomorphic when the corresponding rings on which they are defined are isomorphic, with respect to certain conditions on the structure of the ring.

A ring R_1 is said to be a determined by the unitary Cayley graph $G(R_1)$ if R_2 is also a ring, such that $G(R_1) \cong G(R_2)$ implies $R_1 \cong R_2$. The *Jacobson radical* of a ring R, denoted by J_R, is defined as the intersection of all the maximal ideals of R, and a ring R is said to be *reduced* if it has no non-zero nilpotent elements.

Successively, the unitary Cayley graph of finite rings was investigated in [150], where the study, as a whole, aimed to discuss the unitary Cayley graphs of all finite rings; however, the results obtained were mainly focused on the unitary Cayley graphs of some specific finite rings and finite commutative rings. For these rings, the graph invariants of $G(R)$, such as the clique and the chromatic number, were also obtained when $R = \mathbb{M}_n(\mathbb{F})$, where $\mathbb{F}$ is a field. Additionally, for a ring R, it was proven that the clique and the chromatic number of $G(R)$ are equal to the clique and the chromatic number of the graph $G(\frac{R}{J_R})$, the unitary Cayley graph of the ring $\frac{R}{J_R}$. A stronger result was proven on the isomorphism of these graphs in [150], as given below.

Theorem 57 ([150])**.** *Let R_1 and R_2 be finite rings such that $G(R_1) \cong G(R_2)$. Then, $G(\frac{R_1}{J_{R_1}}) \cong G(\frac{R_2}{J_{R_2}})$. Additionally, $|J_{R_1}| = |J_{R_2}|$.*

As an application of Theorem 57, a similar result was proven in the case of commutative rings, which aided in proving that a commutative reduced ring can be determined by the unitary Cayley graph. Along with the proof of this theorem, an example of the ring $R = \mathbb{Z}_4$ was also given to show that not all commutative rings can be determined by the unitary Cayley graphs. Finally, a conjecture on the isomorphism between the reduced rings $\frac{R_1}{J_{R_1}}$ and $\frac{R_2}{J_{R_2}}$, when their unitary Cayley graphs are isomorphic, was given in [150].

Following this, the diameter of unitary Cayley graphs of rings was investigated in [151], and it was proven that, for each integer $n \geq 1$, there exists a ring R such that $diam(G(R)) = n$. The proof of this result revealed that the connectedness of the graph $G(R)$ is closely related to the property of the ring R to be generated additively by its units. The diameter of the unitary Cayley graphs of a few extensions of rings such as the power series ring over a ring, polynomial ring over a ring, and self-injective rings were also investigated based on the main results that were obtained. Note that a ring R is called right (left) *self-injective* if every homomorphism from a right (left) ideal of R into R can be extended to a homomorphism of R to itself (refer to [152]).

An element of a ring R is said to be *k-good* if it can be expressed as a sum of k units of the ring R, and a ring is said to be k-good if every element is k-good. The *unit sum number*, $usn(R)$, of a ring R is the smallest number l such that every element can be written as the sum of at most l units. If some element of R is not k-good for any $k \geq 1$, then $usn(R)$ is ∞ (c.f. [153]). A few characterisations of rings with their unitary Cayley graphs having different values of diameter were obtained based on the definitions of the unit sum number of a ring, as follows.

Theorem 58 ([150])**.** *Let R be any ring with the unitary Cayley graph $G(R)$ and unit sum number $usn(R)$. Then, the following hold:*

(i) $diam(G(R)) = 1$ *if and only if R is a division ring;*

(ii) $diam(G(R)) = 2$ *if and only if $usn(R) = 2$ and R is not a division ring;*

(iii) $diam(G(R)) = k$ *if and only if $usn(R) = k$, for $k \geq 3$.*

In [21], the unitary Cayley graph of finite commutative rings with a non-zero unit element was considered for the study, where the properties of the graph $G(R)$ were investigated in a similar pattern, such as how the properties of X_n were discussed in [54], but using an algebraic approach. That is, the proof techniques of the results on the unitary Cayley graph of finite commutative rings emphasize the algebraic structure of the rings, which, in some cases, were comparatively simpler and more efficient that the proofs given in [54] for the graphs X_n. The structure of the graph $G(R)$ was first discussed by obtaining results on its regularity, the number of common neighbours between the vertices of the graph, and the basic graph parameters such as diameter, girth, the number of triangles, chromatic number, clique number, edge and vertex connectivity, etc., as follows.

Theorem 59 ([21])**.** *For any ring R with the group of units R^*, $G(R)$ is a r-regular graph, where $r = |R^*|$.*

Theorem 60 ([21]). *Let R be a local ring with maximal ideal M. Then, $G(R)$ is a complete graph if and only if R is a field.*

Theorem 61 ([21]). *Let $G(R)$ be the unitary Cayley graph of an Artinian ring R. The neighbourhoods of two vertices $u, v \in V(G(R))$ are equal if and only if $u - v$ belongs to the ideal of all nilpotent elements of R.*

Recall that a finite ring R is Artinian, and the structure theorem of Artinian rings (refer to [31]) states $R \cong R_1 \times R_2 \times \ldots \times R_t$, where each R_i, $1 \leq i \leq t$ is a finite local ring with the corresponding maximal ideal M_i, $1 \leq i \leq t$, such that the decomposition is unique up to the permutation of factors. Here, the finite residue field is $\frac{R_i}{M_i}$, and the mapping $\pi_i : R_i \to \frac{R_i}{M_i}$ is the quotient map. With appropriate permutation of the factors, $f_1 \leq f_2 \leq \ldots \leq f_t$, where $f_i = |\frac{R_i}{M_i}|$, for $1 \leq i \leq t$ can be obtained. Note that these notations are used in the following theorems and the notation is maintained throughout the paper whenever R is mentioned as a finite or an Artinian ring.

Theorem 62 ([21]). *Let $G(R)$ be the unitary Cayley graph of an Artinian ring $R \cong R_1 \times R_2 \times \ldots \times R_t$. Then, the diameter of $G(R)$,*

$$diam(G(R)) = \begin{cases} 1, & \text{if } t = 1 \text{ and } R \text{ is a field;} \\ 2, & \text{if } t = 1 \text{ and } R \text{ is not a field;} \\ 3, & \text{if } t \geq 2, f_1 \geq 3 \text{ or } t \geq 2, f_1 = 2, f_2 \geq 3; \\ \infty, & \text{if } t \geq 2, f_1 = f_2 = 2. \end{cases}$$

Theorem 63 ([21]). *Let $G(R)$ be the unitary Cayley graph of an Artinian ring $R \cong R_1 \times R_2 \times \ldots \times R_t$. Then, the girth of $G(R)$,*

$$gir(G(R)) = \begin{cases} 3, & \text{if } f_1 \geq 3; \\ 6, & \text{if } R \cong \mathbb{Z}_2^r \times \mathbb{Z}_3, \text{ for some } r \geq 1; \\ \infty, & \text{if } R \cong \mathbb{Z}_2^r, \text{ for some } r \geq 1; \\ 4, & \text{otherwise.} \end{cases}$$

Theorem 64 ([21]). *Let $G(R)$ be the unitary Cayley graph of an Artinian ring $R \cong R_1 \times R_2 \times \ldots \times R_t$. Then,*

(i) The clique number, $\omega(G(R)) = \chi(G(R)) = f_1$, where $\chi(G(R))$ denotes the chromatic number of $G(R)$;

(ii) The independence number, $\alpha(G(R)) = \frac{|R|}{f_1}$;

(iii) The edge chromatic number,

$$\chi'(G(R)) = \begin{cases} |R^*| + 1, & \text{if } |R| \text{ is odd;} \\ |R^*|, & \text{otherwise;} \end{cases}$$

(iv) The vertex and the edge connectivity of $G(R)$, $\kappa(G(R)) = \kappa'(G(R)) = |R^*|$.

Along with the computation of these parameters, the planarity and perfection of the graph $G(R)$ was also discussed in [21], and a characterisation of planar and perfect unitary Cayley graphs of finite commutative rings was obtained, as mentioned in Theorems 66 and 67. To investigate the perfection of the graph, the clique and the chromatic numbers of the complement $(\overline{G(R)})$ of the graph $G(R)$ were also determined in [21], as given below.

Theorem 65 ([21])**.** *The clique number of the graph $\overline{G(R)}$, $\omega(\overline{G(R)}) = \chi(\overline{G(R)}) = \alpha(G(R)) = \frac{|R|}{f_1}$, where χ and α represent the chromatic and the independence number.*

Theorem 66 ([21])**.** *Let R be an Artinian ring. Then, $G(R)$ is perfect if and only if $f_1 = 2$, R is local, or R is a product of two local rings.*

Theorem 67 ([21])**.** *Let R be a finite ring and s be a non-negative integer. Then, the graph $G(R)$ is planar if and only if R is one of the following rings:*
(i) $(\frac{\mathbb{Z}}{2\mathbb{Z}})^s$;
(ii) $\frac{\mathbb{Z}}{3\mathbb{Z}} \times (\frac{\mathbb{Z}}{2\mathbb{Z}})^s$;
(iii) $\frac{\mathbb{Z}}{4\mathbb{Z}} \times (\frac{\mathbb{Z}}{2\mathbb{Z}})^s$;
(iv) $\mathbb{F}_4 \times (\frac{\mathbb{Z}}{2\mathbb{Z}})^s$*, where $\mathbb{F}_4$ is a field with 4 elements.*

Following this, the algebraic properties such as the automorphism group and the spectra of the graph $G(R)$ were obtained using the concept of reduction of a graph, given in [38], as follows.

Two vertices of a graph G are said to be *equivalent* if their open neighbourhoods are equal, and this defines an equivalence relation on the vertices of the graph, as two vertices are adjacent only if they are in different equivalence classes; the induced subgraph of the vertices of two equivalence classes is either a complete bipartite graph or an edgeless graph. The *reduction* of a graph G is said to be the graph in which vertices are the equivalence classes of G; two classes are adjacent if and only if their union induces a complete bipartite graph, and a graph is said to be *reduced* if it is isomorphic to its reduction. Recall that a ring is said to be *reduced* if it has no non-zero nilpotent element, and hence, a finite commutative reduced ring is a finite product of finite fields.

An interesting relation between the reduction of the unitary Cayley graph $G(R)$ of a ring R and the structure of the reduced ring R was obtained in [21], which decreases the complexity of answering general questions about unitary Cayley graphs of finite rings to answering the questions for the corresponding finite reduced rings, as follows.

Theorem 68 ([21])**.** *Let R be an Artinian ring. Then, the reduction $(G(R))_{red} \cong G(R_{red})$, where $(G(R))_{red}$ is the reduced graph of $G(R)$ and $R_{red} \cong \frac{R}{\mathcal{N}_R}$, where $\mathcal{N}_R$ is the maximal ideal of R containing the nilpotent elements in the reduced ring R and $G(R_{red})$ is the unitary Cayley graph of the ring $R_{red} \cong \frac{R}{\mathcal{N}_R}$.*

The above established relation aids in determining the automorphism group of the graph $G(R)$, by reducing the problem of determining the automorphism group of the reduced graph of $G(R)$. In that case, an isomorphism $f : \mathrm{Aut}(G(R)) \to \mathrm{Aut}(G(R_{red})) \times (S_n)^{\frac{R}{\mathcal{N}_R}}$ is established between the structures of the automorphism group of the graph $G(R)$ and its reduced graph because any $\sigma \in \mathrm{Aut}(G(R))$ permutes the cosets of $\mathcal{N}_R$ and induces an automorphism $\overline{\sigma} \in \mathrm{Aut}(G(R_{red}))$, as a consequence of Theorem 61. As the automorphism group of the reduced graph is known through this process, the automorphism group of the graph was determined using this in [21], as follows.

Theorem 69 ([21])**.** *Let $t \in \mathbb{N}$ and $r_1, r_2, \ldots, r_t$ be prime power integers, such that $2 \le r_1 < r_2 < \ldots < r_t$ and $R \cong \prod_{i=1}^{t} (F_i)^{n_i}$, where F_i denotes a field with r_i elements and $n_i \in \mathbb{Z}$, for each $1 \le i \le t$. Then, $\mathrm{Aut}(G(R)) \cong \prod_{i=1}^{t} S_{r_i} \times \prod_{i=1}^{t} S_{n_i}$.*

As mentioned previously, the spectra of the unitary Cayley graph $G(R)$ of a ring R was also determined based on the properties of the ring, by grouping the rings into three cases. Firstly, the spectra of $G(R)$ when R is a field were computed in the case that the graph $G(R)$ is a complete graph. Followed by that, the spectra of $G(R)$ when R is not a field were computed as follows.

Theorem 70 ([21])**.** *Let R be a finite local ring which is not a field, having a non-zero maximal ideal of size s and $f = \frac{|R|}{s}$. Then,*

$$Spec(G(R)) = \begin{pmatrix} -s & 0 \\ f & f(s-1) \end{pmatrix}.$$

Theorem 71 ([21])**.** *Let $R \cong R_1 \times R_2 \times \ldots \times R_t$ be a finite ring having t local factors of which none are fields. Then,*

$$Spec(G(R)) = \begin{pmatrix} -1^t(|\mathscr{N}_R|) & 0 \\ |R_{red}| & |R| - |R_{red}| \end{pmatrix},$$

where $\mathscr{N}_R$ is the maximal ideal of R containing the nilpotent elements and R_{red} is the reduced ring of R.

On computing the eigenvalues of the graph $G(R)$, the properties related to the spectra such as energy, perfect state transfer, etc. of the graph were studied. It could be seen that all these properties that were examined on the unitary Cayley graph of a finite commutative ring were inspired from the study of the same properties on the unitary Cayley graph on $\mathbb{Z}_n$. The energy of the unitary Cayley graph of finite commutative rings, as well as their complements, was determined in [154], and the rings that had hyperenergetic unitary Cayley graphs were characterised as follows.

Theorem 72 ([154])**.** *Let R be a finite commutative ring such that $R \cong R_1 \times R_2 \times \ldots \times R_t$, where each R_i, $1 \leq i \leq t$ is a local ring with the corresponding maximal ideal M_i. Then, the energy, $\mathscr{E}(G(R)) = 2^t|R^*|$, where R^* is the group of units in R.*

Theorem 73 ([154])**.** *Let R be a finite commutative ring such that $R \cong R_1 \times R_2 \times \ldots \times R_t$, where each R_i, $1 \leq i \leq t$ is a local ring with the corresponding maximal ideal M_i and assume that $f_1 \leq f_2 \leq \ldots \leq f_t$, where $f_i = |\frac{R_i}{M_i}|$, for $1 \leq i \leq t$. Then,*

(i) For $s = 1$, $G(R)$ is not hyperenergetic;
(ii) For $s = 2$, $G(R)$ is hyperenergetic if and only if $f_1 \geq 3$ and $f_2 \geq 4$;
(iii) For $s \geq 3$, $G(R)$ is hyperenergetic if and only if $f_{s-2} \geq 3$ or $f_{s-1} \geq 3$ and $f_s \geq 4$.

The study on the energy of the unitary Cayley graph $G(R)$ was followed by the characterisation of finite commutative rings R, for which $G(R)$ and its complement $\overline{G(R)}$ were Ramanujan graphs, in [155], as given in Theorems 74 and 75. In addition to this, the energy of the line graph $\mathscr{L}(G(R))$ of the unitary Cayley graph $G(R)$ of a ring R, its hyperenergecity, and its spectral moments were also determined in [155]. Note that, for an integer $k \geq 0$, the k-th *spectral moment* of a graph G of order n with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ is given by the the value $sm_k(G) = \sum_{i=1}^{n} \lambda_i^k$, which was found to be related to many combinatorial properties of the graph (see [156]).

Theorem 74 ([155])**.** *Let R be a finite local ring with maximal ideal M of order s. Then, $G(R)$ is a Ramanujan graph if and only if either $|R| = 2s$ or $|R| = \left(\frac{m}{2} + 1\right)^2$ and $m \neq 2$.*

Theorem 75 ([155])**.** *The complement $\overline{G(R)}$ of the unitary Cayley graph, $G(R)$, of a finite local ring R is always a Ramanujan graph.*

All the characterisations obtained in [155] were given separately for the cases of R being a local ring and R being a finite product of local rings, where the characterisation on the latter involved the number-theoretic properties of the cardinalities of the quotient ring $|\frac{R_i}{M_i}|$. This is mainly because of the variation in the spectra of the unitary Cayley graph of these two types of rings, which reveals the innate algebraic structure of the rings. This could be observed explicitly because, on proving these characterisations, several other results on the structure of the graph which completely relied on the structure of the rings were obtained in the process. For example, it was proven that the graph $G(R)$ is connected if and only if there is at most one factor R_i such that $\frac{R_i}{M_i} \cong \mathbb{F}_2$, that is, a field with 2 elements. This result on the connectedness of the graph can also be seen as a consequence of the well-known fact that, for an r-regular graph G, the multiplicity of r as an eigenvalue gives the number of connected components of G, and, in view of the same, it was also concluded that the unitary Cayley graph of a finite local ring R is always connected.

In the sequence of studying the graph properties based on the spectra, the perfect state transfer in the unitary Cayley graphs of rings, that is, the problem of finding if the network admits data transfer without a loss of information, so that the probability of transfer is 1, were investigated in [157,158]. The rings were characterised based on the existence of the perfect state transfer in their unitary Cayley graphs, along with which the time of transfer, which was also obtained for the unitary Cayley graph of a finite local ring as follows.

Theorem 76 ([157])**.** *Let R be a finite local ring with maximal ideal M of size s. Then, $G(R)$ has a perfect state transfer if and only if $R = \mathbb{F}_2$ or $s = 2$, where $\mathbb{F}_2$ is a field with 2 elements. In particular, a perfect state transfer occurs at time $t = \frac{\pi}{2}$.*

One of the interesting aspects of research in spectral graph theory is to find non-cospectral (non-isospectral) equi-energetic graphs. One such problem is to find families of regular graphs which are equi-energetic with their own complements. With unitary Cayley graphs being regular, an attempt to obtain such non-cospectral equi-energetic regular graphs was performed in [159,160], and it was proven that, if $R \cong R_1 \times R_2 \times \ldots \times R_t$ has an even number of local factors, then $G(R)$ and $\overline{G(R)}$ are complementary equi-energetic if and only if R is the product of two finite fields and, in this case, the graphs are strongly regular. It was also given that the classification of such complementary equi-energetic unitary Cayley graphs for R, when it has an odd number of local factors greater than three, remains open.

A similar problem of finding integral equi-energetic non-isospectral graphs was addressed with the properties of unitary Cayley graphs $G(R)$, their complements $\overline{G(R)}$, and the unit graphs $G^+(R)$ (refer to Section 5 for details on the unit graphs of rings) in [159]. The conditions under which the unit and unitary Cayley graph of a finite commutative ring are equi-energetic were obtained in [159] and, in addition to that, using the results on the equi-energetic complements of the unitary Cayley graphs given in [160], all integral equi-energetic non-isospectral triple $\{G(R), \overline{G(R)}, G^+(R)\}$, such that all three graphs are also Ramanujan graphs, was characterised in [159].

It was first proven that, for a ring R, $G(R)$ and $G^+(R)$ are equi-energetic, as the group of units considered for the adjacency criteria is a symmetric subset of R. Following this, the conditions on the structure of the ring R, the spectrum of $G(R)$ and $G^+(R)$, and the corresponding graphs were obtained, in order to prove that the unitary Cayley and the unit graphs of the ring concerned are non-isospectral. Using this, it was shown that $G(R)$ and $G^+(R)$ are integral equi-energetic non-isospectral connected non-bipartite graphs under certain conditions, and, as an application, the graphs $G(R)$ and $G^+(R)$, which are strongly regular, were characterised. This characterisation of all finite commutative rings for which their unitary Cayley graphs are strongly regular was also obtained independently in [161] as follows.

Theorem 77 ([159,161])**.** *The unitary Cayley graph $G(R)$ of a finite commutative ring R is strongly regular if and only if R is a local ring or $R \in \{\mathbb{Z}_2^k, \mathbb{F} \times \mathbb{F}\}$, where $\mathbb{F}$ is a finite field with $|\mathbb{F}| \geq 3$.*

Other important spectra of the graph that arise from the adjacency and the degree matrices of the graph are the Laplacian and the signless Laplacian spectra. These Laplacian and signless Laplacian eigenvalues for the unitary Cayley graph of a commutative ring, along with their corresponding energies for the graph $G(R)$ and its line graph $\mathscr{L}(G(R))$, were determined in [162].

It can be noted that the properties of the Laplacian and the signless Laplacian spectra are in parallel with the properties of the adjacency spectra, as the Laplacian matrix and the signless Laplacian matrix of a graph G are given by the relation $L(G) = A(G) - Deg(G)$ and $L(G) = A(G) + Deg(G)$, respectively, where $A(G)$ is the adjacency matrix and $Deg(G)$ is the degree matrix of the graph G. The *degree matrix* $Deg(G)$ of a graph G of order n is an $n \times n$ matrix whose only non-zero entries are the diagonal entries that give the degree of the vertices.

The study of groups admitting planar Cayley graphs can be traced back over almost 120 years, and there is a long history of studying infinite planar Cayley graphs which satisfy additional special conditions (For example, see [35,163]). Regarding the unitary Cayley graphs of rings, a list of finite commutative rings whose unitary Cayley graphs are planar was given in [21,164]. This result only dealt only with finite graphs, and the main algebraic tool used in its proof was the Wedderburn–Artin Theorem. The Wedderburn—Artin theorem states that an Artinian semisimple ring R is isomorphic to a product of finitely many $n_i \times n_i$ matrix rings $M_{N_i}(D_i)$ over the division rings D_i, for some integers n_i, both of which are uniquely determined up to permutation of the index i (c.f. [31])). In [164,165], the unitary Cayley graph of arbitrary rings was considered for investigation, for which the unitary Cayley graphs were mostly infinite.

Though the list of finite planar unitary Cayley graphs was given in [21], the difference in the technique of investigating the planarity of a finite graph and an infinite graph was visible in observing the proof techniques used to prove the results in [164,165]. One distinguishing example is, for a finite planar graph, the minimal degree of the graph is at most five, whereas it was proven, in [166], that there exists a k-regular planar infinite graph for any positive integer k.

A thorough analysis of the group of units of the associated ring structures was conducted in [165], and it was shown that a ring with a planar unitary Cayley graph has either at most four units or exactly six units. This result served as a key to obtain a complete characterisation of the rings whose unitary Cayley graphs were planar in [165], as given in Theorem 78. Using Theorem 78, the semilocal rings with planar unitary Cayley graphs were completely determined. Note that a *semilocal ring* is a commutative Noetherian ring with finitely many maximal ideals, where a ring is called *Noetherian* if every strictly ascending chain of ideals in the ring is finite.

Theorem 78 ([155,165])**.** *Let R be a ring with the group of units R^*. Then, $G(R)$ is planar if and only if one of the following holds:*

(i) $|R^*| \leq 3$ *and* $|R| \leq |\mathbb{R}|$;

(ii) $|R^*| = 4$, $Char(R) = 0$ *and* $|R| \leq |\mathbb{R}|$;

(iii) $|R^*| = 6$ *and R contains a subring isomorphic to* $\frac{\mathbb{Z}[t]}{(t^2-t+1)}$ *with* $|R| \leq |\mathbb{R}|$, *where $\mathbb{Z}[t]$ is the polynomial ring over a ring $\mathbb{Z}$ in the indeterminate t.*

An orientable surface is said to be of genus g if it is topologically homeomorphic to a sphere with g handles. The *genus* of a graph is the minimum number of handles that must be added to a plane to embed the graph without any crossings. A *planar graph* is a graph with genus zero, and a *toroidal graph* is a graph with genus one (c.f. [35]). It should be noted that this investigation on the planarity of unitary Cayley graphs of rings was restricted to finite commutative rings, owing to the complexity of the structure of the unitary Cayley graphs emerging from finite as well as infinite arbitrary rings, due to the diversity in their properties.

As an extension of the characterisation of planar unitary Cayley graphs, minimal non-planar unitary Cayley graphs were investigated in [165,167]. In [167], the structure of the finite commutative rings whose unitary Cayley graphs had genus at most three was examined, and it was proven that, for any given positive integer g, there are at most finitely many finite commutative rings whose unitary Cayley graphs have genus g.

A graph G is a *ring graph* if each block of G which is not a bridge or a vertex can be constructed from a cycle by successively adding H-paths of length at least two, meeting the graph H in two adjacent vertices. Here, given a graph H, we call a path P an H-path if P is non-trivial and meets H exactly at its ends (For more details, refer to [168]). By definition, it is clear that the ring graphs are planar. An *outerplanar graph* is a graph that has a planar drawing for which all vertices are in the outer face of the drawing.

Based on the characterisations of planar unitary Cayley graphs on rings, the rings for which the unitary Cayley graphs are outerplanar and the ring graphs were characterised in [169], as follows.

Theorem 79 ([169])**.** *Let R be a finite ring. Then, $G(R)$ is a ring graph if and only if it is a planar graph.*

This gives the same list of rings for which $G(R)$ is planar as given in Theorem 67. It was proven in [168] that every outerplanar graph is a ring graph. The following theorem on the characterisation of outerplanar unitary Cayley graphs serves as a counterexample for the converse of the theorem, as the existence of a ring R for which $G(R)$ is a ring graph but not outerplanar could be seen.

Theorem 80 ([169])**.** *Let R be a finite ring and s be a non-negative integer. Then, $G(R)$ is outerplanar if and only if R is one of the following rings:*

(i) $(\frac{\mathbb{Z}}{2\mathbb{Z}})^s$;

(ii) $\frac{\mathbb{Z}}{3\mathbb{Z}} \times (\frac{\mathbb{Z}}{2\mathbb{Z}})^s$;

(iii) $\frac{\mathbb{Z}}{4\mathbb{Z}} \times (\frac{\mathbb{Z}}{2\mathbb{Z}})^s$.

The same study of examining the rings for which the line graph of the unitary Cayley graphs are planar, outerplanar, and ring graphs was performed in [170], and it was proven that $\mathscr{L}(G(R))$ is planar if and only if $G(R)$ is planar, and $\mathscr{L}(G(R))$ is outerplanar if and only if it is a ring graph. Both of these conditions are similar to the outerplanarity conditions of the unitary Cayley graph itself.

Following the investigation on the planarity of line graphs of the unitary Cayley graphs, the planarity parameters on the iterated line graphs were investigated in [171]. The k-th *iterated line graph* of a graph G, denoted by $\mathscr{L}^k(G)$, is defined inductively as $\mathscr{L}^0(G) = G$, $\mathscr{L}^1(G) = \mathscr{L}(G)$, and $\mathscr{L}^k(G) = \mathscr{L}^{k-1}(\mathscr{L}(G))$. The *planarity (outerplanarity) index* of a graph G, denoted by $\zeta(G)$ $(\eta(G))$, is the smallest integer k such that $\mathscr{L}^k(G)$ is non-planar (non-outerplanar). The results obtained on these parameters of the unitary Cayley graph on R are given as follows.

Theorem 81 ([171])**.** *For a finite commutative ring R,*

(i) $\zeta(G(R)) = \infty$ *if and only if $G(R)$ is outerplanar;*

(ii) $\zeta(G(R)) = 2$ *if and only if $G(R)$ is a non-outerplanar ring graph;*

(iii) $\zeta(G(R)) = 0$, *otherwise.*

Theorem 82 ([171])**.** *For a finite commutative ring R,*

(i) $\eta(G(R)) = \infty$ *if and only if* $G(R)$ *is outerplanar;*

(ii) $\eta(G(R)) = 0$*, otherwise.*

Equivalently, it can also be told as $\eta(G(R)) = \infty$ if and only if $\zeta(G(R)) = \infty$ and, if not, $\eta(G(R)) = 0$, to establish the significance of the relation between the planarity and outerplanarity indices of the graph. Note that we have rephrased the above results from[171] in terms of the planarity and outerplanarity of the unitary Cayley graphs to emphasise the relation and similarity between the concepts. Along with this, the studies in [169–171] also determined the same properties and parameters related to planarity and outerplanarity of graphs, as well as line graphs, for the unit graphs of the rings, and similar results were obtained, as their structures are similar to each other, according to the graph construction.

By identifying the vertices in a simple graph G as the variables of the polynomial ring $R = \mathbb{F}[x_1, x_2, \ldots, x_n]$ over a field $\mathbb{F}$, the edge set of the graph becomes an ideal I for the ring R, and the quotient ring $\frac{R}{I}$ is called the *edge ring* of the graph G. A *simplicial complex* Ω on a vertex set $V = \{x_1, x_2, \ldots, x_n\}$ is a set of subsets of V that satisfies the following conditions, where the elements of Ω are called its faces:

(i) If $F \in \Omega$ and $F_1 \subseteq F$, then $F_1 \in \Omega$;

(ii) For each $i = 1, 2, \ldots, n$, $\{x_i\} \in \Omega$.

Using the above given definitions, the properties of a graph being Cohen–Macaulay and Gorenstien are defined based on the Cohen–Macaulay and Gorenstien ring structures (refer to [172]). It was already seen that the property of well-coveredness of the graphs X_n was examined in [95]. The same was extended to the unitary Cayley graphs of finite commutative rings in [173], in which a characterisation of the rings that had well-covered unitary Cayley graphs was obtained in terms of the unitary Cayley graph of its reduced ring, as given in Theorem 83, along with an equivalence relation of the properties of Cohen–Macauleyness, shellability, and Gorenstien, which state that all the Cohen–Macaulay unitary Cayley graphs are shellable and Gorenstein.

Theorem 83 ([173])**.** *Let R be a finite ring. Then,* $G(R)$ *is a well-covered graph if and only if* $G(\frac{R}{J(R)})$ *is well covered.*

It was seen that several variants of domination numbers and other domination related parameters were computed for the graph X_n, as the computation of domination parameters for algebraic graphs is a very common study. Interestingly, for the unitary Cayley graphs of rings, the literature has discussions only on the Roman domination number $\gamma_{rom}(G(R))$ (refer to [174]) of these graphs in [175], where the following characterisation of the unitary Cayley graphs with Roman domination number of at most four was obtained.

Theorem 84 ([161])**.** *Let R be a finite commutative ring with non-zero identity. Then, the following properties are satisfied:*

(i) *For the graph* $G(R)$, $\gamma_{rom}(G(R)) = 2$ *if and only if R is a field;*

(ii) *For the graph* $G(R)$, $\gamma_{rom}(G(R)) = 3$ *if and only if R is a local ring with the maximal ideal M, such that* $|M| = 2$*;*

(iii) *For the graph* $G(R)$, $\gamma_{rom}(G(R)) = 4$ *if and only if either R is a local ring with the maximal ideal M such that* $|M| \geq 3$ *or* $R \cong \mathbb{Z}_2 \times \mathbb{F}$*, where* $\mathbb{F}$ *is a field.*

Over the course of the study on the unitary Cayley graph of a ring, the extension of the graph's definition to an algebraic signed graph was given in [176]. The unitary Cayley signed graph was defined based on the definition of unitary Cayley graphs on finite commutative rings, as given in Definition 15, and the graphs were characterised based on the properties of balance and canonical consistence of the graph.

Definition 15 ([176])**.** *Let R be a finite commutative ring with the group of units R^*. The* unitary Cayley signed graph, *denoted by $S_R = (G(R), \sigma)$, is a signed graph whose underlying graph is the unitary Cayley graph $G(R)$, and the sign of an edge $v_i v_j \in E((G(R))$ is assigned by the function $\sigma : E(G(R)) \to \{+, -\}$, as follows. For an edge $v_i v_j$ in $(G(R)$,*

$$\sigma(v_i v_j) : \begin{cases} +, & \text{if } v_i \in R^* \text{ or } v_j \in R^*; \\ -, & \text{otherwise.} \end{cases}$$

The spectra and energy of the signed graphs and also their corresponding line signed graph were computed, and the characterisation of all finite commutative rings for which the graph S_R was hyperenergetically balanced was given. Additionally, it was obtained, in [176], that, for a finite local ring, the adjacency matrix of the unitary Cayley graph and the adjacency matrix of the unitary Cayley signed graph coincide. Using this, the perfect state transfer in this signed graph S_R was examined in [158].

It was seen in [21] that the structures of unitary Cayley graphs were determined by the appropriate reduction structures of the graph as well as the rings. The properties of the graph as well as the ring reduction give further scope to examine the rings and the unitary Cayley graphs of the rings by studying the properties of the subgraph induced by the unit elements in the unitary Cayley graph; that is, for a finite commutative ring R with the unitary Cayley graph $G(R)$, the induced subgraph $\Gamma(G(R))$ is the graph with $V(\Gamma(G(R))) = R^*$, and two vertices are adjacent if their difference is a unit, where R^* is the group of all units of the ring R. This graph was introduced in [177], and the basic properties of the graph $\Gamma(G(R))$ were investigated. Some characterisation results based on the graph invariants such as girth, chromatic number, chromatic index (edge chromatic number) and genus were also given in [177].

The main motivation of the study in [177] was to examine the possibility of determining the structure of the reduced ring of a ring R using $\Gamma(G(R))$, for which the outcome was positive. This was proven by showing that, for two finite commutative rings R_1 and R_2, $\Gamma(G(R_1)) \cong \Gamma(G(R_2))$ if and only if $\frac{R_1}{J_{R_1}} \cong \frac{R_2}{J_{R_2}}$, where J_{R_1} and J_{R_2} are the Jacobson radical of R_1 and R_2, respectively, using the algebraic properties of the spectrum of the graph.

In distinction from the extensive studies on the unitary Cayley graphs over commutative rings, it can be seen that not much work was performed on unitary Cayley graphs over non-commutative rings, for which a possible reason is the complicated structures of non-commutative rings, compared to commutative rings. The first class of non-commutative ring that was specifically considered to construct the graph $G(R)$ and study its properties is the matrix ring.

The unitary Cayley graphs of matrix algebras, that is, the set of all square matrices of order n over a finite field $\mathbb{F}$, denoted by $\mathbb{M}_n(\mathbb{F})$, was studied specially in [178–180]. Though, in [21,150], certain properties of the graph $G(\mathbb{M}_n(\mathbb{F}))$ were discussed for these rings as a special case, [178–180] re-iterated them and gave a broader proof. As known, the unit group of $\mathbb{M}_n(\mathbb{F})$ is the set of all invertible matrices of order n, which is also called the *general linear group*, denoted by $GL_n(\mathbb{F})$. The graph invariants of $G(\mathbb{M}_n(\mathbb{F}))$ were already discussed in [150,161], as given below, and they can also be deduced as a special case from the existing results of the graphs $G(R)$.

Theorem 85 ([150])**.**

(i) *The clique number of the unitary Cayley graph on $\mathbb{M}_n(\mathbb{F})$ is $|\mathbb{F}|^n$;*

(ii) *The independence number of the unitary Cayley graph on $\mathbb{M}_n(\mathbb{F})$ is $|\mathbb{F}|^{n^2-n}$;*

(iii) *The diameter of $G(\mathbb{M}_n(\mathbb{F}))$ is 1 when $n = 1$, or 2 otherwise.*

In [180], an analogous notion to the representation problem of graphs put forth in [38] was given, as the representation of graphs by matrices was defined to investigate whether every graph in any family was an induced subgraph of $G(\mathbb{M}_n(\mathbb{F}))$, and it was conjectured that there is a graph G such that, for each finite field $\mathbb{F}$, the graph G is not an induced subgraph of $G(\mathbb{M}_n(\mathbb{F}))$. Additionally, the characterisation of the $G(\mathbb{M}_n(\mathbb{F}))$ to be strongly regular was obtained in [180] as follows.

Theorem 86 ([180])**.** *The graph $G(\mathbb{M}_n(\mathbb{F}))$ is strongly regular if and only if $n = 2$ and $\mathbb{M}_2(\mathbb{F})$ is strongly regular with the parameters $(q^4, q^4 - q^3 - q^2 + q, q^4 - 2q^3 - q^2 + 3q, q^4 - 2q^3 + q)$, where $q = |\mathbb{F}|$.*

In [180], Theorem 86 has been proven only by considering two special cases of n, when $n = 2, 3$, and has failed to cover the other general cases. This was quoted and rectified in [179], and the same result was re-established by proving that the graph $G(\mathbb{M}_n(\mathbb{F}))$ cannot be strongly regular for any $n > 2$. Following this, the spectral properties of the graph $G(\mathbb{M}_n(\mathbb{F}))$ were studied in [178], where the three eigenvalues of the graph were determined using the additive property of the ring $\mathbb{M}_n(\mathbb{F})$, along with its energy and the conditions for hyperenergecity of the graphs, which were determined without explicitly computing the spectrum of the graph. The characterisation of rings $\mathbb{M}_n(\mathbb{F})$ by determining the value of n for which $G(\mathbb{M}_n(\mathbb{F}))$ were Ramanujan graphs were also obtained in [178], as given below.

Theorem 87 ([178])**.** *The graph $G(\mathbb{M}_n(\mathbb{F}))$ is a Ramanujan graph if and only if $n = 2$ or $n = 3$ and $\mathbb{F} = \mathbb{Z}_2$.*

The study on the unitary Cayley graphs of matrix rings was extended in [181], where explicit formulas for all the eigenvalues of the graphs $G(\mathbb{M}_n(\mathbb{F}))$ and $G(\mathbb{M}_n(R))$, where R is a finite commutative local ring that is not a field, were obtained using an alternate approach to the one that was followed in [178]. Using this, the energy, the Kirchhoff index, and the number of spanning trees of the graphs $G(\mathbb{M}_n(\mathbb{F}))$ and $G(\mathbb{M}_n(R))$ were also derived. Note that the *Kirchoff index* of a graph G of order n is the value $n \sum_{i=2}^{n} \frac{1}{\lambda_i}$, where λ_i, $2 \leq i \leq n$, denotes the eigenvalues of the Laplacian matrix of the graph (see [182,183]).

For a vertex v in a graph G, the *first and the second subconstituent* of G at v are the subgraph of G induced by the neighbours and the non-neighbours of v (except v), respectively. The subconstituents of strongly regular graphs have been studied for several graphs, as they have many interesting properties associated with the structure of the graph (see [184,185]). Moreover, the problem of finding graphs which have strongly regular subconstituents is a problem of interest to researchers, as several properties, including the eigenvalues of these subconstituents, were used to prove the uniqueness of the parameters of some strongly regular graphs (c.f. [184,185]). This notion of subconstituents of the unitary Cayley graphs of the ring $G(\mathbb{M}_n(R))$ was investigated in [186].

On examining the subconstituents of the unitary Cayley graphs of a finite ring R with identity $1 \neq 0$, it can be seen that both the first and the second subconstituent of the additive identity 0 are the graph isomorphisms that map v to $u - v$, where $u, v \in V(G(\mathbb{M}_n(R)))$. Hence, a complete study on the subconstituents of 0 in $G(\mathbb{M}_n(\mathbb{R}))$ was performed, particularly when R was a finite field $\mathbb{F}$, that is, the subconstituents of the 0 element in the graph $G(\mathbb{M}_n(\mathbb{F}))$ were investigated. It can be observed that the first constituent of the 0 element in the graph $G(\mathbb{M}_n(\mathbb{F}))$ was nothing but the graph with the vertex set as the group $G(GL_n(\mathbb{F}))$ (can be correlated as the graph $\Gamma(G(GL_n(\mathbb{F}))))$, and the second constituent was defined on the set of non-zero non-invertible matrices over $\mathbb{F}$. The structures of these subconstituents were determined, from which the spectra, energy, and other related spectral properties such as hyperenergeticity and Ramanujan property for both graphs were studied. In addition to these, the clique number, chromatic number, and independence number of these subconstituents were also computed in [186].

The next ring for which unitary Cayley graphs were investigated in [187] was the quotient ring $\frac{R}{I}$, where R was a Dedekind domain and I was an ideal of R, giving a finite and non-trivial $\frac{R}{I}$. The unitary Cayley graph defined on this Dedekind ring is a very close generalisation to that of the graph X_n, and, hence, the unitary Cayley graphs of such

Dedekind rings $\frac{R}{I}$ are called *generalised totient graphs*. Recall that the Schemmel totient function ST_r is a generalisation of the Euler totient function, defined for each non-negative integer r and prime p, as a multiplicative arithmetic function that satisfies

$$ST_r(p^\alpha) = \begin{cases} p^{\alpha-1}(p-r), & \text{if } p \geq r; \\ 0, & \text{otherwise,} \end{cases}$$

where α is a positive integer (c.f. [36]).

To study the properties of the generalised totient graphs, the Schemmel totient function was used, and, in particular, one of the two extensions of the Schemmel totient function was used to obtain a formula for the number of cliques of any order k in a given generalised totient graph. This formula had not been used in the literature even for Euler totient Cayley graphs before this article, and, after a couple of years, the formula to obtain the number of cliques of any order k was given using the Schemmel totient functions in [98].

Using this formula of the number of cliques, the clique domination number of the generalised totient graphs was determined, which aided in the correction of an erroneous claim that had been made regarding this topic in [119], and also to provide a counterexample for the result on the strong domination (refer to Section 4 for definition) of the graph X_n that was given in [114]. The study in [187] can be seen to have built on the basis of [54], as similar results and proof techniques were adopted. The paper concluded by suggesting further scopes of research pertaining to the topic, of which some have been investigated for all finite commutative rings.

A *dual number* is a number $x + \epsilon y$, where $x, y \in \mathbb{R}$ and ϵ is a matrix with the property that $\epsilon^2 = 0$ (refer to [188]). As the set of all dual numbers is an Artinian local ring, the unitary Cayley graph associated with ring of dual numbers was investigated in [188], where the exact values of the diameter, chromatic number, and chromatic index were determined, along with which a classification of all perfect unitary Cayley graphs of this ring was given.

Definition 16 ([189])**.** *The set of all complex numbers $a + ib$, where $a, b \in \mathbb{Z}$, is the ring of* Gaussian integers*, denoted by $\mathbb{Z}[i]$. For any $k \in \mathbb{N}$, if $[k]$ is the principal ideal generated by k in $\mathbb{Z}[i]$, then the factor ring $\frac{\mathbb{Z}[i]}{[k]}$ is isomorphic to $\mathbb{Z}_k[i]$, where $\mathbb{Z}_k[i]$ is the set of all complex numbers $a + ib$, in which $a, b \in \mathbb{Z}_k$, and the ring $\mathbb{Z}_k[i]$ is called the* ring of Gaussian integers modulo *k.*

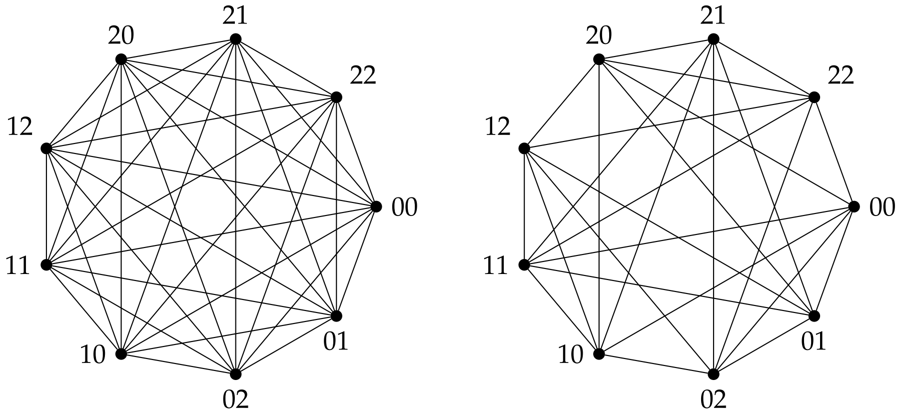
Definition 17 ([190])**.** *The set of all complex numbers $a + b\omega$, where $a, b \in \mathbb{Z}$ and $\omega = \frac{1}{2}(-1 + i\sqrt(3))$ is a primitive third root, forms an integral domain called the* ring of Eisenstein integers*, denoted by $\mathbb{Z}^e[i]$. For any $k \in \mathbb{N}$, if $[k]$ is the principal ideal generated by k in $\mathbb{Z}^e[i]$, then the factor ring $\frac{\mathbb{Z}^e[i]}{[k]}$ is isomorphic to $\mathbb{Z}_k^e[i]$, where $\mathbb{Z}_k^e[i]$ is the set of all complex numbers $a + b\omega$, in which $a, b \in \mathbb{Z}_k$, and the ring $\mathbb{Z}_k^e[i]$ is called the* ring of Einstein integers modulo *k.*

To understand the unitary Cayley graphs of these rings, the nature of the units of these rings must be known. Both the rings have n^2 elements, and they form a ring with respect to the operations of usual addition modulo n and multiplication modulo n. The structure of the units of the ring depends on the norm defined and is given below in the following theorems. An illustration of the unitary Cayley graph on both the rings, $\mathbb{Z}_k[i]$ and $\mathbb{Z}_k^e[i]$, is given in Figure 7.

In [191,192], the unitary Cayley graphs of the rings $\mathbb{Z}_k[i]$ and $\mathbb{Z}_k^e[i]$ were studied individually, in which the basic graph invariants were obtained for the unitary Cayley graphs of these rings. In addition, the traversal properties of these graphs were explored, and it was proven that the unitary Cayley graphs of both these rings were Hamiltonian; furthermore, certain necessary and sufficient conditions for the graph $G(\mathbb{Z}_k[i])$ to be Eulerian were obtained in [191].

Theorem 88 ([189])**.** *An element $a + ib \in \mathbb{Z}_n$ is a unit in the ring $\mathbb{Z}_n$ if and only if $a^2 + b^2$ is a unit in $\mathbb{Z}_n$.*

Theorem 89 ([190])**.** *An element $a + b\omega \in \mathbb{Z}_n^e$ is a unit in the ring $\mathbb{Z}_n^e$ if and only if $a^2 + b^2 - ab$ is a unit in $\mathbb{Z}_n$.*



(**a**) (**b**)

Figure 7. Unitary Cayley graphs of the rings Gaussian and Einstein integers modulo n. (**a**) The unitary Cayley graph of $\mathbb{Z}_3[i]$. (**b**) The unitary Cayley graph of $\mathbb{Z}_3^e[i]$.

It can be seen that the properties of the unitary Cayley graph of rings highly depend on the properties of the rings, which is the reason that not many properties of the graphs were discussed, unlike for the graphs X_n. This is because the feasibility of condensing all the rings under the same roof and investigating many properties is lower; however, several avenues are still open for further research.

4. Unitary Addition Cayley Graph

The conventional definition of a Cayley graph on any algebraic structure, with respect to any of its symmetric subsets, is a graph with the vertices set as the elements of the algebraic structure, and there exists an edge between two vertices in the graph if their difference is an element of the symmetric subset considered. A slight modification on this adjacency condition in the usual Cayley graph, for the sum of two elements to belong to the symmetric subset instead of their difference, made its way to the concept of *addition Cayley graphs*, also known as *Cayley-sum graphs* in [193], which almost have the same properties and symmetric nature as usual Cayley graphs.

Though these addition Cayley graphs were termed as a twin to Cayley graphs, it can be seen that they have received far less attention in the literature when compared to Cayley graphs. To some extent, this situation can be explained based on the fact that the addition Cayley graphs are comparatively more difficult to study than Cayley graphs. For example, the connectivity of a Cayley graph on a finite Abelian group was obtained as an immediate consequence of its adjacency pattern, whereas determining the connectivity of an addition Cayley graph was a non-trivial problem that was exclusively solved in [194].

In the literature, though the addition Cayley graph was first defined for groups in [193], it was extended to many algebraic structures. The addition Cayley graph of an algebraic structure $\mathscr{A}$, with a symmetric subset S, is given in Definition 18, after which its illustration is given in Figure 8.

Definition 18 ([193])**.** *An* addition Cayley graph *of an algebraic structure $\mathscr{A}$ is a graph with the vertices set as the elements of $\mathscr{A}$, and any two vertices u and v in the graph are adjacent when $u + v \in S$, where S is a symmetric subset of $\mathscr{A}$. This addition Cayley graph on $\mathscr{A}$ with respect to its symmetric subset S is usually denoted by $Cay^+(\mathscr{A}, S)$.*

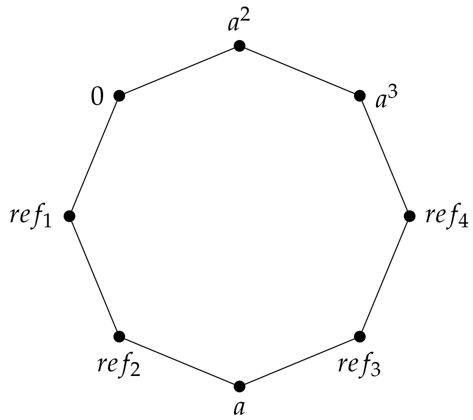


Figure 8. The addition Cayley graph of the dihedral group D_4, $Cay^+(D_4, \{a^2, b_1^2\})$.

Combining the notions of the addition Cayley graph with the definition of the graph X_n, that is, the unitary Cayley graphs of $\mathbb{Z}_n$, the concept of *unitary addition Cayley graphs* was introduced in [195], as given below, with an example of a unitary addition Cayley graph given in Figure 9.

Definition 19 ([195])**.** *The* unitary addition Cayley graph*, denoted by* $X_n^+ = Cay^+(\mathbb{Z}_n, \mathbb{Z}_n^*)$*, is a graph with the vertices set as the elements of the ring* $\mathbb{Z}_n$*;* $0, 1, \ldots, n-1$*, and two vertices are adjacent if their sum is a unit of the ring, that is, for all* $u, v \in V(X_n^+)$*,* $uv \in E(X_n^+)$ *when* $|u + v| \in \mathbb{Z}_n^*$*, where* $\mathbb{Z}_n^*$ *is the set of all relative prime integers to n, which are the units of* $\mathbb{Z}_n$*.*

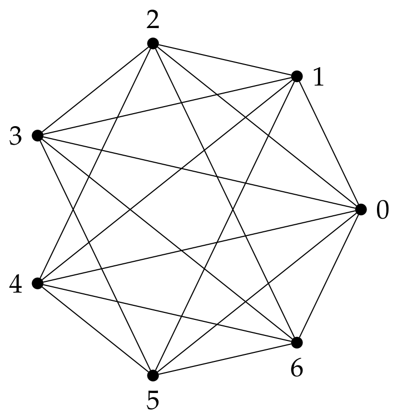


Figure 9. Unitary addition Cayley graph X_7^+.

Though the graph was defined and officially introduced with the name unitary addition Cayley graph in [195], this graph was already defined by Grimaldi in [24], from which the unit graphs of rings (refer to Section 5) were defined and studied. Since a unitary addition Cayley graph is a unit graph of $\mathbb{Z}_n$, researchers focused on studying the unit graphs of all rings, rather than a particular one. Over a period of time, as the unitary Cayley graph on $\mathbb{Z}_n$ marked its high significance in this area of research, its claimed twin, the unitary addition Cayley graph, was defined independently and is still currently being studied.

In [24], the basic results on the regularity of the graph X_n^+ and the decomposition of the graph into Hamiltonian cycles were given, along with which the challenging nature of investigating different graph properties for the unitary Cayley graphs with odd order, despite a clear understanding of the structure of the graph, was discussed.

On re-introducing the unit graph of $\mathbb{Z}_n$ as the unitary addition Cayley graph, the basic properties such as the regularity, girth, size, etc. of the graph were investigated in [195], along with their traversal properties, as mentioned in Theorem 90. The structural characterisations of the graph on their k-partiteness or planarity were also obtained, which are given below.

Theorem 90 ([195])**.** *Let X_n^+ be the unitary addition Cayley graph of the ring $\mathbb{Z}_n$ and $\phi(n)$ be the Euler totient function. Then, the following properties hold:*

(i) *The graph X_n^+ is $(\phi(n), \phi(n)-1)$- semiregular, when n is odd;*

(ii) $|E(X_n^+)| = \frac{(n-1)\phi(n)}{2}$*, when n is odd;*

(iii) $gir(X_n^+) = 3$ *for odd* $n > 3$, 4 *for even* $n > 2$, *and* $n \not\equiv 0 \mod 3$.

Theorem 91 ([195])**.** *The unitary addition Cayley graph is planar if and only if the value of n is 1, 2, 3, 4, or 6, and it is outerplanar if and only if it is planar.*

As the graph is obtained from a unitary Cayley graph, a natural and an important question of the relation between the unitary addition Cayley graph X_n^+ and its so-called twin, the unitary Cayley graph X_n, had to be answered. This was solved by obtaining the characterisation that $X_n \cong X_n^+$ if and only if n is even, and this characterisation reduces the problem of investigating the properties and the structure of X_n^+ to only being necessary for the odd values of n. Owing to this, the results on the unitary addition Cayley graphs explicitly mentioned in this section are only for odd values of n.

This characterisation naturally motivates the researchers to extend the investigation on all similar problems and properties that were addressed for the unitary Cayley graphs to the unitary addition Cayley graphs for two different reasons: one is to understand how the structure and properties of the unitary addition Cayley graphs differ for odd values of n, and the other reason is to obtain parallel results with the help of a similar methodology that was already existent in the literature, especially in a similar context which could also be verified without much challenge.

The study in [195] was extended in [196] by more clearly establishing the structure of the unitary addition Cayley graph as a k-partite graph for odd n, as given in Theorem 92, which aided in computing several numerical parameters of the graph in [196]. Note that the parameters of the graph X_n^+ that were computed in [196] are given below only for odd n.

Theorem 92 ([196])**.** *The unitary addition Cayley graph X_n^+, for an odd n is a $\frac{\phi(n)}{2} + r$-partite graph, where r is the number of distinct prime factors of n.*

Theorem 93 ([196])**.** *Let X_n^+ be the unitary addition Cayley graph on $\mathbb{Z}_n$, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, such that $p_i < p_j$, for $i < j$ and $\alpha_i \in \mathbb{N}$, for all $1 \le i \le r$. Then, the following are true:*

(i) *The independence number, $\alpha(X_n^+) = 2$, when n is prime and $\alpha(X_n^+) = \frac{n}{p_1}$, when n is an odd composite number;*

(ii) *The vertex covering number, $\alpha_0(X_n^+) = n-2$, when n is prime and $\alpha_0(X_n^+) = n - \frac{n}{p_1}$, when n is an odd composite number;*

(iii) *The edge covering number, $\alpha_1(X_n^+) = \frac{n+1}{2}$, when n is odd;*

(iv) *The matching number, $\beta_1(X_n^+) = \frac{n-1}{2}$, when n is odd;*

(v) *The edge connectivity, $\kappa_1(X_n^+) = \phi(n) - 1$, when n is odd;*

(vi) *The edge chromatic number, $\chi'(X_n^+) = \phi(n)$, for all n.*

Based on Theorem 92, the bounds for the chromatic number and clique number of the unitary addition Cayley graph were obtained in [196], from which it was obtained that a unitary addition Cayley graph X_n^+ is perfect if and only if n is even or a prime power. This characterisation was obtained by proving that, for all the other values of n, the unitary addition Cayley graph contained an induced cycle of length five, according to its chromatic partition.

A more detailed study on the chromatic number of the unitary addition Cayley graph was performed in [197], where tighter bounds for the clique and the chromatic number of the unitary addition Cayley graph X_n^+ for different values of n, based on their number-theoretic properties, were obtained. A colouring pattern that satisfied the bound was also given, along with some examples of the unitary addition Cayley graphs, to show that the bounds were sharp as well as strict.

This was followed by a study on the achromatic number of the unitary addition Cayley graph in [198], whose relation with the chromatic number of the graph is visible from the definition given hereafter. The *achromatic number* of G, denoted by $\chi_{ach}(G)$, is the maximum number of colours that can be assigned to the vertices of the graph such that the adjacent vertices are assigned different colours and any two different colours are assigned to some pair of adjacent vertices. It therefore follows that, for any graph G, $\chi_{ach}(G) \geq \chi(G)$ (c.f. [199]).

Though the lower bounds of the chromatic number obtained in [197] can serve as the lower bounds for the achromatic number, better bounds were computed as per the maximization condition in [198], and, in a similar way, colouring patterns were given to establish the bounds as well as their tightness. In certain cases, the exact value of the achromatic number was also determined, as given below.

Theorem 94 ([198])**.** *The achromatic number of a unitary addition Cayley graph,*

$$\chi_{ach}(X_n^+) = \begin{cases} 2, & \text{if } n = 2^k, \text{for some } k \in \mathbb{N}; \\ 1 + \frac{\phi(n)}{2} & \text{if } n = p^k, \text{for and odd prime } p \text{ and } k \in \mathbb{N}. \end{cases}$$

Ensuing this, the domination parameters of the unitary addition Cayley graph were determined in [200,201]. In [201], the exact values of the domination number of the unitary addition Cayley graph were determined for a few values of n, as given in Theorem 95, and, in [201], the strong domination and the total strong domination of the graph X_n^+ were studied, where the parameters were computed for similar cases of n, which are also given in Theorem 95.

For a graph G without isolated vertices, a *total dominating set* of the graph is a dominating set in which every vertex of the graph is adjacent to at least one vertex in the dominating set (c.f. [34]). A vertex $v \in V(G)$ *strongly dominates* a vertex $u \in V(G)$ in a graph G if $uv \in E(G)$ and $deg(u) \geq deg(v)$. A dominating set $S \subseteq V(G)$ in which every vertex $u \in V - S$ is strongly dominated by some vertex $v \in S$ is said to be a *strong dominating set* of the graph G, and the minimum cardinality of a strong dominating set is the *strong domination number* $\gamma_s(G)$ of the graph G (see [202]). A total dominating set $S \subseteq V(G)$ in which every vertex $u \in V - S$ is strongly dominated by some vertex, $v \in S$ is said to be a *total strong dominating set* of a graph G, and the minimum cardinality of total strong dominating set of G is called the *total strong dominating number* of the graph, denoted by γ_{ts} (refer to [202]).

Theorem 95 ([200,201])**.** *Let X_n^+ be the unitary addition Cayley graph and $\phi(n)$ represent the Euler totient function. Then, the following hold:*

(i) $\gamma(X_n^+) = 2$, *when* $n = 2^r$, *for some integer* $r \geq 2$;
(ii) $\gamma(X_n^+) = \gamma_s(X_n^+) = 1$ *and* $\gamma_{ts}(X_n^+) = 2$, *when n is prime;*
(iii) $\gamma(X_n^+) = \gamma_s(X_n^+) = 2$, *when* $n = 2k$, *where k is an odd prime;*
(iv) $\gamma(X_n^+) = \gamma_s(X_n^+) = \lceil \frac{n}{3} \rceil$, *when n is even such that* $\phi(n) = 2$;
(v) $\gamma_{ts}(X_n^+) = \gamma_s(X_n^+) = 2$, *when n is a prime power.*

Proceeding with the study on other computational parameters of the unitary addition Cayley graphs, a few topological indices for the graph were computed in [203,204]. The *Wiener index* of a graph, which is the sum of shortest paths between all pairs of vertices in the graph, and the *hyper-Wiener index* of a graph, which is the sum of the shortest distance, as well as their square between every pair of vertices in the graph, were computed in [204]. The *reverse Wiener index* of the graph G, given by the value $\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} diam(G) - d(u,v)$, where $d(u,v)$ is the shortest distance between two distinct vertices u and v in the graph G, was computed for the unitary addition Cayley graph in [203].

By the above-mentioned definition of the topological indices, it can be seen that the reverse Wiener index of a graph is closely related with the previously computed Wiener and hyper-Wiener indices. As the computation of all these topological indices requires the distance between the vertices, the number of common neighbours between any two vertices in the unitary addition Cayley graph was computed in [204]. The values of all three topological indices for the graph X_n^+ that were obtained in [203,204], based on the values of n, are given in Table 1, where $\phi(n)$ denotes the Euler totient function.

Table 1. Topological indices of the unitary addition Cayley graph X_n^+.

n Values	Wiener Index	Hyper-Wiener Index	Reverse-Wiener Index
n is a prime integer	$\frac{n^2-1}{2}$	$(n-1)(n+2)$	$\frac{(n-1)^2}{2}$
$n = 2^t$, for some integer $t > 1$	$\frac{3n^2}{4} - 4$	$2(n^2 - \frac{3n}{2})$	$(\frac{n}{2})^2$
n is a composite odd number	$(n-1)(n - \frac{\phi(n)}{2})$	$(n-1)(3n - 2\phi(n))$	$\frac{(n-1)\phi(n)}{2}$
$n = 2t$, for some integer $t > 1$ having odd prime divisors	$\frac{5n^2}{4} - n(\phi(n) - 1)$	$\frac{n(9n-10\phi(n)-6)}{2}$	$\frac{n(n-2+4\phi(n))}{4}$

The Wiener index of the graph X_n^+ was independently computed in [205], using an algorithm and a program. Programs to draw the unitary addition Cayley graphs as well as the unitary Cayley graph of the given order, and also to find the adjacency matrix and the energy of a unitary addition Cayley graph, were given in [205]. Additionally, a few other topological indices for the unitary addition Cayley graphs were computed in [206,207], whose values could be derived from the entries of different matrices associated with the graph.

Apart form the study of these computational parameters, the spectra associated with different matrices defined on the graph, along with their corresponding spectral properties, were investigated in [208–211]. In [210], spectral studies related to the adjacency and the Laplacian matrix were conducted, where the eigenvalues and the Laplacian eigenvalues of the unitary addition Cayley graph X_n^+ and its complement $\overline{X^+}_n$ were determined. Additionally, the bounds for the energy and Laplacian energy for both these graphs were computed, and it was proven that the unitary addition Cayley graph is hyperenergetic if and only if n is an odd composite number that is not a power of three or if n is even and has at least three distinct prime factors. The characterisation for the complement of the unitary addition Cayley graph to be hyperenergetic was also given, explained as follows.

Theorem 96 ([210])**.** *The graph $\overline{X^+}_n$ is hyperenergetic if and only if n is odd and has at least 2 distinct prime factors.*

On comparing the degree of hyperenergeticity of the unitary Cayley graph X_n with the unitary addition Cayley graph X_n^+, it was seen that X_n^+ is more hyperenergetic than X_n. A high number-theoretical approach can be seen in the proof of the results in which both the adjacency and the Laplacian spectra and their corresponding energies were obtained in [210]. This was followed by a discussion on the signless Laplacian spectrum for the graph in [211], where the results obtained can be seen to be closely related to the results in [210].

The signless Laplacian energy of the unitary addition Cayley graph was also independently examined in [209], which again had the same results as similar proof techniques.

In [209], along with the signless Laplacian energy, other derived forms of Laplacian energies such as the distance Laplacian and the signless distance Laplacian energy for the unitary addition Cayley graphs were investigated. The *distance Laplacian energy* and the *signless distance Laplacian energy* of a graph are the sum of the absolute values of the eigenvalues of the distance Laplacian and the signless distance Laplacian matrix, respectively. The *distance Laplacian matrix* and the *signless distance Laplacian matrix* are correspondingly given as $D(v) - Dis(G)$ and $D(v) + Dis(G)$, where $Dis(G)$ denotes the distance matrix of the graph G, and $D(v)$ denotes the diagonal matrix in which each diagonal element corresponding to a vertex v is the sum of the shortest distances from the vertex v to all the vertices of the graph (refer to [209]).

These derived Laplacian spectra were computed for the unitary addition Cayley graph X_n^+ and its complement $\overline{X^+}_n$, and the bounds for these energy values for different n were also determined. This was followed by the investigation of the A_α *matrix* of the unitary addition Cayley graph in [208]. The A_α-matrix of a graph G is defined as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, $\alpha \in [0,1]$, where $D(G)$ and $A(G)$ are the degree and the adjacency matrices of G (see [208]).

In [208], the eigenvalues of the A_α matrix for the unitary addition Cayley graph X_n^+ and its complement were computed, along with some bounds for these eigenvalues, when n was odd. Consequently, the A_α-energy of both X_n^+ and its complement, when n was a prime power and n was even, were determined, along with some bounds for the A_α-energy of X_n^+ and $\overline{X^+}_n$, when n was odd. From these, the A_α-borderenergetic and A_α-hyperenergetic graphs were defined as the graphs having their A_α-energy equal to the A_α-energy of a complete graph and the graphs having their A_α energy greater than the A_α-energy of a complete graph, respectively; then, a few unitary addition Cayley graphs were classified as A_α-borderenergetic and A_α-hyperenergetic.

An incidence structure $\mathscr{D} = (P, B, J)$, with a point set P, block set B, and an incidence relation J, is a $t-(r,k,s)$-design, where $|P| = r$, every block in B is incident with precisely k points, and every t distinct points are incident with precisely s blocks. The *code* $C_{\mathbb{F}}(\mathscr{D})$ of the structure $\mathscr{D}$ over the finite field $\mathbb{F}$ is the space spanned by the incidence vectors of the blocks over F (c.f. [212]). The notion of codes is given in higher design theory to study the relation between the elements in a design, but, in the case of this restriction to the discrete structure of graphs, it reduces to the notions related to the incidence and adjacency in a graph, such as the adjacency design, incidence design, neighbourhood design, etc. (refer to [213]).

If G is a k-regular graph, then the $1-(|E|,k,2)$ design with the incidence matrix of G is called the *incidence design* of G, where the incidence matrix, $B(G)$, of the graph G is a $|V(G)| \times |E(G)|$ binary matrix, such that the entry $b_{ij} = 1$, if v_i is incident with e_j and 0 otherwise. A *code* $C_{|\mathbb{F}|}(G)$ of a graph G over a finite field $\mathbb{F}$ is the row span of the incidence matrix of the graph over $\mathbb{F}$, and the dimension of the code is the rank of the matrix over $\mathbb{F}$.

As the unitary addition Cayley graphs are regular, linear codes from the incidence matrix of the unitary addition Cayley graph X_n^+ over the field $\mathbb{Z}_2$ were determined in [214], by computing the main parameters of the code for the values $n = p, 2p$, where p is prime. Since the incidence matrix is a binary matrix, the field considered to determine the linear code is $\mathbb{Z}_2$. To determine these binary linear codes, the edge connectivity, regularity, and size of the graphs were taken from the existing results, as stated in [196], that the incidence code of a graph G over a field with two elements is a $[|E|, |V|-1, (\kappa_1(G))]_2$ code, where the subscript two denotes that the binary conversions of these integers are to be considered.

In [215–217], the properties of the unitary addition Cayley graph of the ring of Gaussian integers modulo n, $\mathbb{Z}_n[i]$ (refer to Definition 16) were investigated, where the exact values and bounds of certain parameters of the graph $\mathbb{Z}_n[i]$ were obtained. Note that the number of elements in the ring $\mathbb{Z}_n[i]$ is n^2, as there are n ways to fill both the real and the complex part of the number $a + ib$. Correspondingly, the number of units of the ring differs, based on the value of n.

The degrees of the vertices, size, diameter, and girth of the unitary addition Cayley graph on $\mathbb{Z}_n[i]$ was given in [215], based on the value of n, as mentioned in Theorem 97, from which it was determined that the unitary addition Cayley graph on $\mathbb{Z}_n[i]$ is a complete bipartite graph if and only if $n = 2^t$, $t \in \mathbb{N}$. The traversal properties of the graph were also investigated in [215]; it was proven that the unitary addition Cayley graph on $\mathbb{Z}_n[i]$ is always Hamiltonian and, when n is even, the graph is Eulerian. It was also found that the unitary addition Cayley graph on $\mathbb{Z}_n[i]$ is planar only for $n = 1, 2$.

Theorem 97 ([215])**.**

(i) *The diameter of the unitary addition Cayley graph on $\mathbb{Z}_n[i]$ is 3 if $n = kp$, where k is even and p is an odd prime, or 2 otherwise;*

(ii) *The girth of the unitary addition Cayley graph on $\mathbb{Z}_n[i]$ is 3 if n is odd, and 2 when n is even.*

Adding to the study, the basic graph invariants for the unitary addition Cayley graph on $\mathbb{Z}_n[i]$ were computed in [216,217]. Some bounds for the chromatic and the clique number of the graph were given in [217] as well as [216], which coincided with each other. In [216], the clique covering number of the unitary addition Cayley graph on $\mathbb{Z}_n[i]$ was determined by determining the independence number of its complement, and, in [217], the domination number of the graph was obtained as either 1, 2, or 3, based on the value of n. A similar study was conducted on the unitary addition Cayley graphs of the ring Einstein integers modulo n, $\mathbb{Z}_n^e[i]$ (refer to Definition 17), in [218], where, along with the basic properties and parameters of the unitary addition Cayley graphs of $\mathbb{Z}_n^e[i]$, a comparison between the unitary addition Cayley graphs of the rings $\mathbb{Z}_n[i]$ and $\mathbb{Z}_n^e[i]$ was also given, to enable a better comprehension of the structure of the rings, graphs, and their properties. For understanding the structure of the unitary Cayley graphs on the rings $\mathbb{Z}_n[i]$ and $\mathbb{Z}_n^e[i]$, an illustration is given in Figure 10.

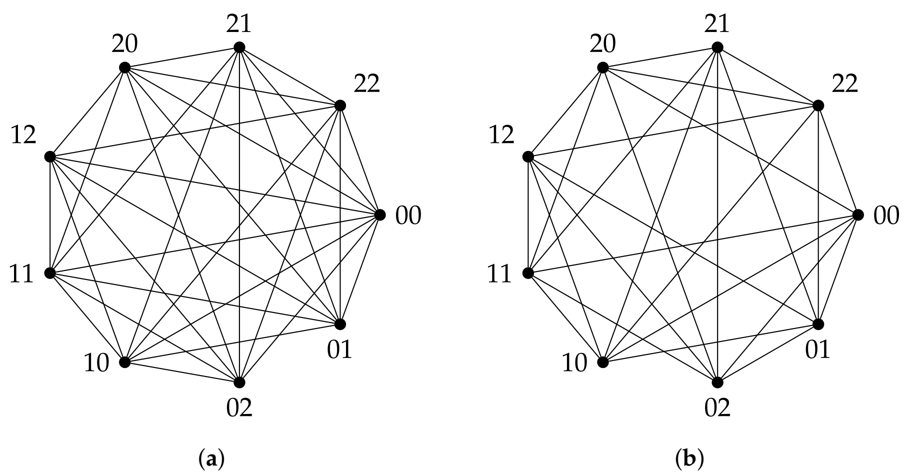


Figure 10. Unitary addition Cayley graphs (unit graphs) of the rings with Gaussian and Einstein integers modulo n. (**a**) The unitary addition Cayley graph of $\mathbb{Z}_3[i]$. (**b**) The unitary addition Cayley graph of $\mathbb{Z}_3^e[i]$.

From the literature, it can be seen that these unitary addition Cayley graphs of the rings $\mathbb{Z}_n[i]$ and $\mathbb{Z}_n^e[i]$ were independently examined in [191,192], respectively, as the unit graphs of the corresponding rings, where almost the same invariants and the properties were examined in more detail. In the next section (Section 5), it can be seen that the unit graphs are nothing but an extension of the same definition of a unitary addition Cayley graph to a ring R, similar to how the unitary Cayley graph X_n of $\mathbb{Z}_n$ was extended to all the rings R in the graph $G(R)$.

For a graph G, $S \subseteq V(G)$ is a *perfect code* (different from the notions of a code of a graph) of the graph if S is an independent set, such that every vertex in $V(G) - S$ is adjacent to exactly one vertex in S (see [219]). The perfect codes in an induced subgraph of the unitary addition Cayley graph containing the vertices that represent the idempotent elements of the ring $\mathbb{Z}_n$ were examined in [220], where the question of when a subset of the idempotent elements in the ring $\mathbb{Z}_n$ are a perfect code in this induced subgraph of a unitary addition Cayley graph was answered.

It was shown, in [220], that the subgraph of X_n^+ induced by the idempotent elements of the ring $\mathbb{Z}_n$ admits a perfect code of size two if n is a product of two prime powers, where one of the primes is even; a perfect code of size one if n is the product of k factors of odd prime powers; and a perfect code of size 2^{t-1} for the unitary addition Cayley graph on a ring which is the direct product of the factors of $\mathbb{Z}_{p^k}$.

Analogous to the previously discussed unitary Cayley graphs, the notion of signed algebraic graphs was also investigated for the unitary addition Cayley graphs. Similar to the case of the unitary Cayley graphs on $\mathbb{Z}_n$, multiple signed graphs were defined on the unitary addition Cayley graph in [221–223]. These definitions are given below, followed by an example of these graphs, given in Figure 11.

Definition 20 ([222])**.** *The* unitary addition Cayley signed graph*, denoted by $S_n^{\vee +} = (X_n^+, \sigma^{\vee +})$, is a signed graph whose underlying graph is the unitary addition Cayley graph X_n^+, $n \in \mathbb{N}$, and the sign of an edge $v_i v_j \in E(X_n^+)$ is assigned by the function $\sigma^{\vee +} : E(X_n^+) \to \{+, -\}$ as follows. For an edge $v_i v_j$ in X_n^+,*

$$\sigma^{\vee +}(v_i v_j) \begin{cases} +, & \text{if } v_i \in \mathbb{Z}_n^* \text{ or } v_j \in \mathbb{Z}_n^*; \\ -, & \text{otherwise.} \end{cases}$$

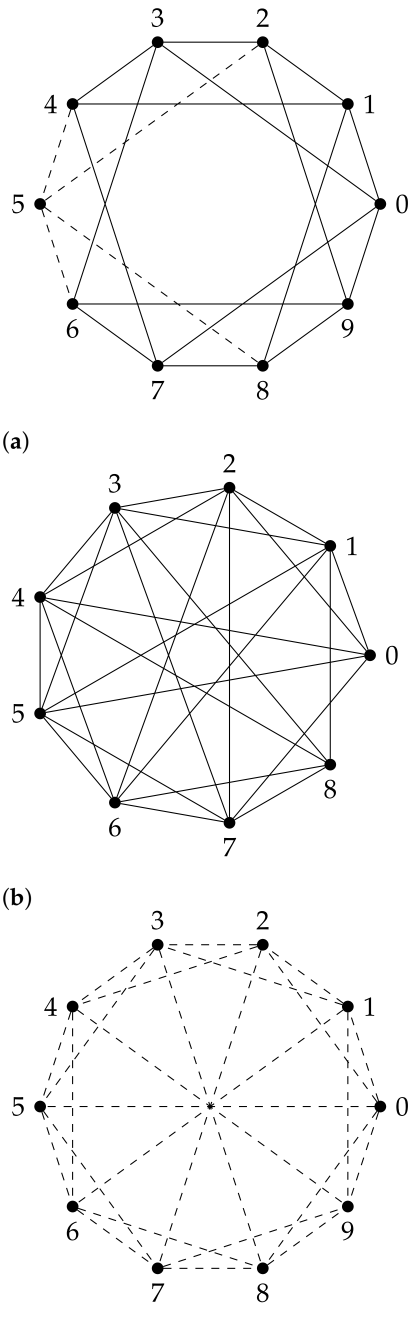
Definition 21 ([221])**.** *The* unitary addition Cayley ring signed graph*, denoted by $S_n^{\oplus +} = (X_n^+, \sigma^{\oplus +})$, is a signed graph whose underlying graph is the unitary addition Cayley graph X_n^+, $n \in \mathbb{N}$, and the sign of an edge $v_i v_j \in E(X_n^+)$ is assigned by the function $\sigma^{\oplus +} : E(X_n) \to \{+, -\}$ as follows. For an edge $v_i v_j$ in X_n^+,*

$$\sigma^{\oplus +}(v_i v_j) \begin{cases} +, & \text{if either } v_i \in \mathbb{Z}_n^* \text{ or } v_j \in \mathbb{Z}_n^*; \\ -, & \text{otherwise.} \end{cases}$$

Definition 22 ([223])**.** *The* addition signed Cayley graph*, denoted by $S_n^{\wedge +} = (X_n, \sigma^{\wedge +})$, is a signed graph whose underlying graph is the unitary addition Cayley graph X_n^+, $n \in \mathbb{N}$, and the sign of an edge $v_i v_j \in E(X_n^+)$ is assigned by the function $\sigma^{\wedge +} : E(X_n^+) \to \{+, -\}$ as follows. For an edge $v_i v_j$ in X_n^+,*

$$\sigma^{\wedge +}(v_i v_j) \begin{cases} +, & \text{if both } v_i \in \mathbb{Z}_n^* \text{ and } v_j \in \mathbb{Z}_n^*; \\ -, & \text{otherwise.} \end{cases}$$

For all the above defined signed graphs, the properties of balance, clusterability, sign-compatibility, and canonical consistency were studied in the corresponding articles. As the the graphs X_n and X_n^+ coincide when n is even, the corresponding sign graphs also coincide, as do their properties and characterisations. In [222], the unitary addition Cayley sigraph was introduced, and the above-mentioned properties were studied, from which the following characterisations were obtained.



(**a**)

(**b**)

(**c**)

Figure 11. Examples of signed unitary addition Cayley graphs. (**a**) The unitary addition Cayley signed graph $S_{10}^{\vee+}$. (**b**) The unitary addition Cayley ring signed graph $S_9^{\overline{\oplus}+}$. (**c**) The addition signed Cayley graph $S_{10}^{\wedge+}$.

Theorem 98 ([222])**.**

(i) *The unitary addition Cayley sigraph $S_n^{\vee+}$ is balanced if and only if either n is even or it does not have more than one distinct prime factor;*

(ii) *The unitary addition Cayley sigraph $S_n^{\vee+}$ is clusterable if and only if it is balanced;*

(iii) *The unitary addition Cayley sigraph $S_n^{\vee+}$, where n has at most two distinct odd prime factors, is canonically consistent if and only if n is either odd or n is 2, 6, or a multiple of 4;*

(iv) *Every unitary addition Cayley sigraph $S_n^{\vee+}$ is sign-compatible.*

It has been shown, in [224], that all line signed graphs are sign-compatible. Hence, in view of (iv) in Theorem 98, the question of realising a unitary addition Cayley sigraph as a line sigraph had come up, and this was answered by characterising all the unitary addition Cayley sigraphs that could be realised as a line graph and also a line signed graph, as given in Theorem 99.

Theorem 99 ([222])**.** *A unitary addition Cayley graph is a line graph if and only if* $n \in \{2, 3, 4, 6\}$ *and is a line signed graph if and only if it is a line graph.*

Similarly, the unitary addition Cayley ring signed graph and the addition signed Cayley graph were introduced, and similar properties were studied in [221,223], respectively. Through the results obtained on all these signed graphs defined on the unitary addition Cayley graphs, it can be seen that, even though the definitions of the signed graphs differ, the properties are almost similar to each other, with the exception of a few. It can also be noticed that, in some cases, the properties of the signed graphs defined on the unitary Cayley graphs coincide with the properties of the corresponding signed graph defined on the unitary addition Cayley graphs. Along with the characterisation of the signed graphs based on the above-mentioned four properties, the characterisations of these properties of balance, clusterability, etc. in certain derived signed graphs from the signed graphs, such as the negation of the signed graph, and some variations of line signed graphs were also investigated in [218,221,222].

5. Unit Graph of a Ring

As mentioned earlier, Grimaldi introduced the unitary addition Cayley graph as the unit graph of $\mathbb{Z}_n$ in [24], which remained latent for some years. This definition of the unitary addition Cayley graph on $\mathbb{Z}_n$ was generalised to all rings as the *unit graph* of a ring in [225] as follows. Note that these graphs may be referred to as Grimaldi graphs in the literature by some authors, owing to the fact that the unit graph of rings is generalised based on the graph formerly introduced by Grimaldi in [24]. Following the definition of the unit graph and the closed unit graph of a ring, examples of these graphs are given in Figure 12.

Definition 23 ([24])**.** *The* unit graph *of a ring R, denoted by* $G^+(R) = Cay^+(R, R^*)$*, is a graph with the vertices set as the elements of the ring, and two distinct vertices are adjacent if their sum is a unit of the ring, that is, for all* $u, v \in V(G^+(R))$*,* $uv \in E(G^+(R))$ *when* $u + v \in R^*$*, where* R^* *is the group of units of the ring R. If the word"distinct" is omitted from this definition, it gives the definition of the* closed unit graph *of a ring R, that is, a* closed unit graph *of a ring R is the unit graph of R, where there may be a loop from the vertex to itself in the graph if the sum of an element with itself is a unit.*

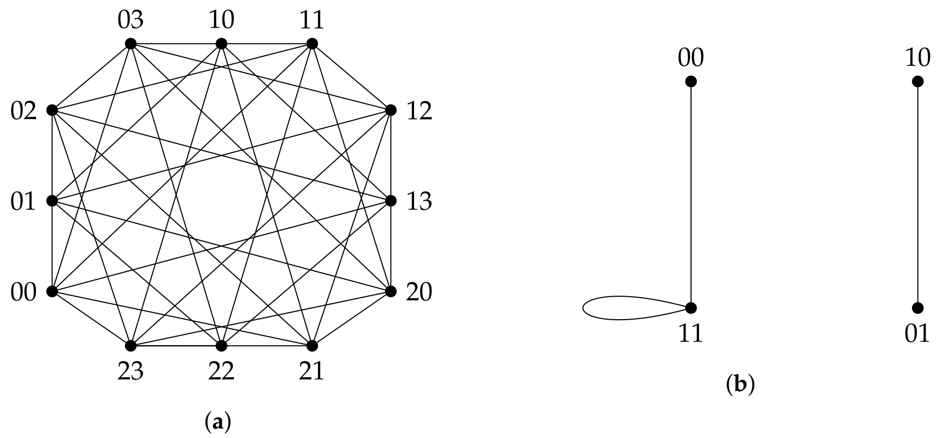


Figure 12. Examples of unit and closed unit graphs of rings. (**a**) The unit graph of $\mathbb{Z}_3 \times \mathbb{Z}_4$. (**b**) The closed unit graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Though this definition of the unit graphs is given for any associative ring with unity, it can be seen that, for most of the studies, only a finite commutative ring with unity is considered, owing to the symmetric structure of these rings. Furthermore, very limited study on the unit graphs of associative rings can be seen, as the structure of an arbitrary ring is very sophisticated to comprehend. This sophisticated structure of the ring gives rise to highly complex and diverse graphs, whose structures cannot be generalised. Therefore, it can be seen in the literature reviewed in this section that, in several instances, different authors have considered rings with specific properties to obtain the results pertaining to the unit graphs of rings in their study.

Note that the unit graphs of rings are the complement of the *total graphs* defined on rings, which have the vertices of the graph set as the elements of the ring, and two vertices are adjacent if their sum is a zero divisor. This relation between the unit and the total graph of a rings is because of the fact that every element in a ring is either a unit or the zero-divisor of the ring. The literature pertaining to total graphs is huge (c.f. [17,22,226]), where certain properties of the complement of total graphs have also been investigated. Though the complement of total graphs of rings represents the unit graphs, in this article, we review the literature that has discussed the properties of unit graphs of rings under this name only.

On observing the definition of the unit graph of a ring, it can be noticed that it is a subgraph of the *comaximal graph* defined on a ring R, in which the vertices are the elements of the ring, and any two vertices u and v are adjacent in the graph if $Ru + Rv = R$ (refer to [25]). Though certain properties of comaximal graphs (when restricted to their subgraphs) also hold for the unit graphs, this article only focuses on the results that are specifically obtained for the unit graphs of rings.

In [225], discussions on the unit graph of rings were initiated, where the properties such as the regularity and connectedness were investigated for the unit graphs of all associative rings, and some properties such as diameter, girth, and planarity were investigated for the unit graphs of finite commutative rings. The unit graph of a ring was found to be either $|R^*|$-regular or $(|R^*|, |R^*| - 1)$-biregular, based on the unit elements of the ring.

Recall that an element of a ring R is said to be *k-good* if it can be expressed as a sum of k units of the ring R, and a ring is said to be k-good if every element is k-good. The connectedness of the graph was characterised based on the unit sum number and the k-goodness property of the ring as given below, and this discloses the fact that the unit graphs are generally not connected, not all rings have finite unite set numbers. Additionally, an interesting relation between the dominating set and the connectedness of the unit graph of rings was also obtained in [225], as stated below.

Theorem 100 ([225])**.** *The unit graph $G^+(R)$ of a ring R is connected if and only if the ring is k-good for some integer $k \geq 1$ or the ring R is not k-good but every element of R is k-good for some $k \geq 1$, that is, the units additively generate R.*

Theorem 101 ([225])**.** *If the set of all vertices that corresponds to the units of the ring form a dominating set of the unit graph of the ring, then the unit graph is connected.*

The connectedness of the unit graphs of some particular rings was investigated based on the abovementioned characterisation that was obtained on the connectedness of the unit graphs. The chromatic index of the unit graph of an associative ring was also computed as $\Delta + 1$, where Δ is the maximum degree of the vertices in the unit graph, and certain structural characterisations of the unit graph, related to when the unit graph of a ring can be a cycle, path, bipartite, and complete bipartite graph, were obtained in [225], which are given below.

Theorem 102 ([225])**.** *The unit graph $G^+(R)$ of a ring R is a cycle if and only if R is $\mathbb{Z}_4$, $\mathbb{Z}_6$, or the set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, where $a, b \in \mathbb{Z}_2$.*

Theorem 103 ([225]).

(i) *The unit graph of a ring R is a complete graph if and only if R is a division ring with characteristic 2;*

(ii) *The unit graph of a ring R is a complete bipartite graph if and only if R is a local ring with the maximal ideal M, such that* $|\frac{R}{M}| = 2$.

Following this, the structure of the cliques and co-cliques (independent sets) in the unit graph of a finite commutative ring R was studied in relation with its Jacobson radical J_R and the corresponding quotient ring $\frac{R}{J_R}$. In addition to this, characterisations of finite commutative rings based on their diameter, girth, and planarity were also obtained in [225]. Using this structure of cliques and co-cliques, as well as the structural realisations obtained in in [225], the unit graph of a finite commutative ring was proven to be weakly perfect in [227], that is, for a finite commutative ring R, $\chi(G^+(R)) = \omega(G^+(R))$, where χ and ω denote the chromatic and the clique numbers of the graph.

This was proven by using a series of lemmas, where finite commutative rings having different algebraic properties were considered, and the corresponding unit graphs were proven to be weakly perfect by computing their clique and chromatic numbers. Owing to the fact that every finite commutative ring R is isomorphic to the direct product of local rings and their quotient ring $\frac{R}{J_R}$ is isomorphic to the direct product of fields, the proof of the main theorem was given in two cases, based on the structure of the fields that were present in the direct product of the quotient ring $\frac{R}{J_R}$. The first case was considered as no field in the local factors of $\frac{R}{J_R}$ had its characteristic equal to two, and the second one was the existence of at least one field in the local factors of $\frac{R}{J_R}$ with characteristic two in the direct product.

The structure of the unit graphs of the quotient rings $\frac{R}{J_R}$ in these cases followed the values of the clique and the chromatic number of the unit graph of obtained in [24], which correlated the structure of a ring R and its quotient ring $\frac{R}{J_R}$. Using this result, the parameters were computed and the final result was proven. This discussion of the weak perfect property led to the discussion of the property of perfection in the unit graphs of rings in [228], where the perfection of the unit graphs of finite commutative Artinian rings were examined and the results on classification of rings whose unit graphs were perfect and not-perfect were obtained.

The girth of the unit graph of any finite commutative ring R was proven to be either 3, 4, 6, or ∞ in [225]. This result was extended in [229] to the unit graph of any arbitrary ring and the same values were obtained as the girth of the corresponding unit graphs. On obtaining these restricted values for the girth of unit graphs, the exact girth values of the unit graph of specific rings were computed, and relations between the girth of the unit graph of a ring R and $\frac{R}{J_R}$ were also established. The rings R with semipotent quotient rings $\frac{R}{J_R}$, such that the girth of the unit graph of the ring R was either six or ∞, were determined, and some necessary conditions on the group of unit elements of a ring were obtained to realise the unit graph of the corresponding ring based to its girth. Note that a *semipotent ring* is a ring such that every left ideal that is not contained in the Jacobson radical of the ring contains a non-zero idempotent element.

In an analogous manner, it was proven that the diameter of the unit graphs of finite commutative rings take the values 1, 2, 3, or ∞ in [225], and this result was extended to the unit graphs of rings that had a self-injective quotient ring $\frac{R}{J_R}$ in [230]. Recall that a ring is called *self-injective* if every homomorphism from the principal ideal to the ring extends to a homomorphism of the ring to itself.

As the diameter of a graph is associated with its connectedness, certain discussions on the connectedness of the unit graphs of some rings, based on their unit sum numbers, were given, following which all rings that had a self-injective quotient ring $\frac{R}{J_R}$ were classified based on the values of the diameter of their unit graphs. Furthermore, characterisation of rings based on the diameter values of their unit graphs were also obtained, as given in

Theorem 104, and it was proven that, for any integer $n \geq 1$, there exists a ring R such that $n \leq diam(G^+(R)) \leq 2n$.

Theorem 104 ([230]). *For a ring R with its unit graph $G^+(R)$, $diam(G^+(R)) = 2$ if and only if $usn(R) = 2$ and R is not a division ring with $char(R) = 2$.*

As an extension to the discussions on the diameter of the unit graphs of rings, the radii of the unit graphs were investigated in [231]. It can be seen that the studies on the radii of algebraic graphs are rare when compared to the studies on the diameter, though they are closely related. This is because several graphs tend to have a minimum eccentricity equal to one. In [231], the relation between the unit graph of a ring R and its corresponding quotient ring $\frac{R}{J_R}$ was obtained, and some characterisations of rings having the radius of their unit graphs 1, 2, 3, or ∞ were given. It was also proven that, for every positive integer n, there exists a ring R such that the radius of its unit graph is n. It can be seen that the investigations in [231] on the radii of the unit graphs of rings were made in a similar pattern of discussion as followed in [229,230].

This was followed by a cursory investigation on the connectedness of the complement of unit graphs of finite commutative rings in [232], where the complement of the unit graph was proven to always be connected, and the following equivalent statements were obtained by relating connectedness to the dominating set and the number of the maximal ideals of the ring, based on the results obtained in [24], relating the same notions.

Theorem 105 ([232]). *Given a finite commutative ring R with the set of all maximal ideals of the ring $\mathscr{M}$, then the following statements are equivalent:*

(i) *The complement of the unit graph $\overline{G^+(R)}$ is connected;*

(ii) $|\mathscr{M}| \geq 2$;

(iii) *$R - \{R^*\}$ is a dominating set of the graph $\overline{G^+(R)}$.*

Note that Theorem 101 states the necessary condition of the set of all units to be just a dominating set, not a minimal or a minimum dominating set, of the unit graph of a ring. This conveys the possibilities of the graph having other minimal dominating sets, which may possibly be a subset of the set of all vertices that also represent the units of the ring; this led to an investigation of the domination numbers in the unit graphs of rings. In [233], the finite commutative rings that had domination number less than four were characterised, as given in Theorem 106, by studying the domination number of the unit graphs of fields, product of fields, rings, local rings, etc. The unit graphs of the product of local rings were also investigated by considering the cases of certain special rings as local factors, where these special rings had unit graphs with structural properties that influenced the structure of the overall unit graph of the ring.

Theorem 106 ([233]). *Let R be a finite commutative ring with the unit graph $G^+(R)$. Then,*

(i) $\gamma(G^+(R)) = 1$ *if and only if R is a field;*

(ii) $\gamma(G^+(R)) = 2$ *if and only if either R is a local ring that is not a field or R is isomorphic to the product of two fields such that only one of them have characteristic 2 or $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a finite field;*

(iii) $\gamma(G^+(R)) = 3$ *if and only if R is not isomorphic to the product of two fields, such that only one of them has characteristic 2 and $R \cong R_1 \times R_2$, where R_1 and R_2 are local rings with maximal ideals M_1 and M_2, respectively, such that their quotient rings are not isomorphic to $\mathbb{Z}_2$.*

The concept of domination in unit graphs was also studied in [234], where the motive of the study was to characterise commutative rings that had the domination number of their unit graphs as half their order, that is, to characterise rings where $\gamma(G^+(R)) = \frac{|R|}{2}$ or

$\gamma(G^+(R)) = \frac{|R|-1}{2}$. A characterisation of the former one was obtained completely as given in Theorem 107, whereas the latter problem was solved partially, considering only the rings of integer modulo n.

Theorem 107 ([233])**.** *Let R be a finite commutative ring with the unit graph $G^+(R)$. Then, $\gamma(G^+(R)) = \frac{|R|}{2}$ if and only if $R \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{t-times} \times S$, $t \geq 0$, where S is is either $\mathbb{Z}_2$, $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x \rangle}$.*

An open problem to determine the existence of a ring R such that, given an integer n, the unit graph has domination number n was put forth in [234]. Though the question is yet to be fully answered, in the same article, it was concluded that, for integers of the form 2^k, $k \geq 0$, there exists a ring R such that $\gamma(G^+(R)) = 2^k$, using the results obtained in that article. Continuing the investigation on the domination number of the unit graphs of rings, the study in [235] examined the domination number of the unit graph $G^+(R)$ of a ring $R \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \mathbb{Z}_{p_3^{\alpha_3}}$, where p_i; $1 \leq p \leq 3$ are primes was computed, and the following characterisations were obtained in [235].

Theorem 108 ([235])**.** *Let $R \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \mathbb{Z}_{p_3^{\alpha_3}}$, where p_i; $1 \leq p \leq 3$ and $p_1 < p_2 < p_3$ are primes and $G^+(R)$ be its unit graph having domination number $\gamma(G^+(R))$. Then,*
(i) $4 \leq \gamma(G^+(R)) \leq 6$;
(ii) $\gamma(G^+(R)) = 4$ if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 1$ or $p_1 > 3$;
(iii) $\gamma(G^+(R)) = 5$ if and only if $\alpha_1\alpha_2\alpha_3 \geq 2$ or $p_1 = 3$;
(iv) $\gamma(G^+(R)) = 6$ if and only if $\alpha_1\alpha_2\alpha_3 \geq 2$ or $p_1 = 2$.

In [236], a relation between the domination number as well as the total domination number of the unit graph of a ring R and its Ore extension $R[x; \alpha_1, \alpha_2]$—the ring of polynomials over R with usual addition and multiplication, defined as the relation $xy = \alpha_1(y)x + \alpha_2(y)$—were studied, and it was discovered that, for all associative rings, $\gamma_t(G^+(R)) = \gamma_t(G^+(R[x; \alpha_1, \alpha_2]))$, where γ_t denotes the total domination number of the graph.

Based on this, an open problem was presented, to investigate if the same equality holds for the domination number of the unit graphs of all associative rings and their Ore extension, that is, to check if $\gamma(G^+(R)) = \gamma(G^+(R[x; \alpha_1, \alpha_2]))$ for all associative rings, was posed in [236]. Note that, in the former study, the rings considered were general associative rings and were not restricted to the finite commutative rings, whereas several bounds for the domination number of the unit graphs of only the finite commutative rings were obtained in [237] by using the existing results on the domination number of the unit graphs of rings.

Examining planarity in algebraic graphs has caught the attention of several researchers, due to which, for any new algebraic graph defined, these algebraic structures are characterised based on the planarity of the algebraic graphs introduced. Such characterisations of finite commutative rings for which the unit graph is planar were obtained in [225] see (Theorem 109). This was followed by characterising any associative ring whose unit graph was planar in [238], which was determined based on mainly the order of the ring and its unit group, along with the structure of the ring, as given in Theorem 110; as an application of the obtained result, all semipotent rings whose unit graphs were planar were characterised in [239], and, based on this, a list of all semilocal rings with planar unit graphs were obtained. Recall that a *semipotent ring* is a ring such that every left ideal that is not contained in the Jacobson radical of the ring contains a non-zero idempotent element, and a *semilocal ring* is a commutative Noetherian ring with finitely many maximal ideals, where a ring is called *Noetherian* if every ideal of the ring is finitely generated.

Theorem 109 ([225]). *Let R be a finite commutative ring with the unit graph $G^+(R)$. Then, $G^+(R)$ is planar if and only if R is $\mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, or S is isomorphic to one of the following rings:*

(i) $\mathbb{Z}_2$;

(ii) $\mathbb{Z}_3$;

(iii) $\mathbb{Z}_4$;

(iv) $\mathbb{F}_4$;

(v) *The set of all* 2×2 *matrices of the form* $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, *where* $a, b \in \mathbb{Z}_2$.

Theorem 110 ([238]). *Let R be an associative ring with the unit graph $G^+(R)$ and the group of units R^*. Then, $G^+(R)$ is planar if and only if one of the following holds:*

(i) $|R^*| < 4$ *and* $|R| \leq |\mathbb{R}|$;

(ii) $|R^*| = 4$ *and* $char(R) = 0$ *with* $|R| \leq |\mathbb{R}|$;

(iii) $R \cong \mathbb{Z}_5$;

(iv) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

The planarity of the unit graphs of some local and quasilocal rings were examined in [240–242], where a commutative ring R, which has only a finite number of maximal ideals, is referred to as a *quasilocal* ring, and a ring with a unique maximal ideal is a *local ring*. In [239], a characterisation of finite quasilocal rings that had planar unit graphs was obtained, and it was proven that, if the unit graph of a quasilocal ring is planar, then the ring is finite. This was proven by considering rings of two cases: when the ring has exactly two maximal ideals, and when the quasilocal ring has more than two but finitely many ideals. These cases were investigated in [241,242], respectively.

In succession to the planar unit graphs, the non-planar unit graphs of finite commutative rings that had genus one were investigated in [243], where all finite commutative rings with non-zero identity whose unit graphs were toroidal were determined up to isomorphism, and it was proven that, for any positive integer k, there are finitely many finite commutative rings with non-zero identity, such that the genus of their unit graph is k. As a continuation of the study on the unit graphs of finite commutative rings with unit genus, the rings having unit graphs with higher order genera were investigated in [244], and all finite rings with unit graphs having genera one, two, and three were characterised.

As the spectra of algebraic graphs are another area of keen interest for researchers, the adjacency spectrum of the closed unit graph was computed in [245], based on the properties of the closed unit graphs obtained in [225]. The cases when the unit and closed unit graphs coincided with each other, as well as a few structural properties of the closed unit graphs when they did not coincide with the unit graph of the corresponding ring, were determined in [225]. Utilising these results and properties from [225], especially the result that established that the closed unit graph of product of two rings was the direct product of the closed unit graphs of the corresponding rings, which arose as a consequence of the structure theorem (refer to [31]), the spectra of the closed unit graphs of arbitrary finite rings and their quotient rings $\frac{R}{J_R}$ were determined. Using the spectral values, it was shown that the unit graphs $G^+(R_1)$ and $G^+(R_2)$ of two arbitrary finite rings R_1 and R_2 are isomorphic if and only if the unit graphs of their corresponding quotient rings $G^+(\frac{R_1}{J_{R_1}})$ and $G^+(\frac{R_2}{J_{R_2}})$ are isomorphic.

As the closed unit graph and unit graph of rings coincide in a good number of cases, these spectra can also be taken as the spectra of the unit graphs and, based on that, the rings whose unit graphs are Ramanujan graphs were determined, from which, a necessary and sufficient condition for the unit graph of a ring to be strongly regular was established in [245], explained as follows.

Theorem 111 ([245]). *For a ring R with the unit graph $G^+(R)$, the following statements are equivalent:*

(i) $G^+(R)$ *is a strongly regular graph;*

(ii) *R is a local ring with the maximal ideal M such that $Char(\frac{R}{M}) = 2$ or $R \in \{\mathbb{Z}_2^t, \mathbb{F} \times \mathbb{F}\}$, where $\mathbb{F}$ is a field with $|\mathbb{F}| = 2^k$, where $t, k \geq 2$.*

A *biclique* is a complete bipartite subgraph of a graph G, and a collection of subgraphs of G is called a *biclique partition covering* of a graph G if every subgraph in the collection is a biclique and, for every edge in the graph, there exists exactly one biclique in the collection to which the edge belongs. The *biclique partition number* of a graph G, denoted by $bp(G)$, is the minimum cardinality among the biclique covers of the graph (refer to [246]). There are several applications of this parameter in networks, but one of the main motivations to study this parameter in graphs is to minimise the storage space, as listing the subgraphs in a minimum complete bipartite decomposition of G consumes less space than the adjacency list representation.

If $a_+(G)$ and $a_-(G)$ denote the number of positive and negative eigenvalues in the adjacency spectrum of the graph G, then the graph is said to be *eigensharp* (almost eigensharp) when $bp(G) = \max\{a_+(G), a_-(G)\}$ ($bp(G) = \max\{a_+(G), a_-(G)\} + 1$) (for more details on the eigensharp properties of graphs, c.f. [247]). In [248], rings that had eigensharp unit graphs were investigated, and, by computing the adjacency spectrum and the corresponding biclique numbers, using the structural properties of the rings determined in [225], it was found that, for prime p, the rings $\mathbb{Z}_p$, $\mathbb{Z}_{2p}$, and $\frac{\mathbb{Z}_p[x]}{\langle x^2 \rangle}$ are eigensharp graphs. The authors had also posited the problem to check if the unit graphs of rings $\mathbb{Z}_{p^n}$, $\mathbb{Z}_{qp}$, and $\frac{\mathbb{Z}_p[x]}{\langle x^n \rangle}$, for prime p and q, are eigensharp, which still remains unsolved.

The other computational parameters that were determined for the unit graph of finite commutative rings are the topological indices, namely, the Wiener index and the hyper-Wiener index. These topological indices were computed for unitary addition Cayley graphs in [204,249]; these results were extended to the unit graphs of all finite commutative rings, and, from these results, the values of these indices for the graph X_n^+ were computed by considering the finite commutative ring R as $\mathbb{Z}_n$.

The other graph properties such as the well-coveredness, Hamiltonicity and chordality of the unit graphs of rings were examined in [250–252], respectively. In [251], a necessary and sufficient condition for the unit graph of a finite commutative ring to be Hamiltonian was derived, by constructing a graph based on the structural properties of the rings, whose unit graph was connected as obtained in [225]. As connectedness of the unit graph of a ring was given based on the unit sum number of the ring, a set of equivalent statements involving all these aspects of the ring was given in [251], as follows.

Theorem 112 ([251]). *Let R be a finite commutative ring R that is not isomorphic to $\mathbb{Z}_2$ and $\mathbb{Z}_3$, with unit graph $G^+(R)$. Then, the following statements are equivalent:*

(i) $G^+(R)$ *is Hamiltonian.*

(ii) *The ring R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient ring;*

(iii) *R is generated by its units;*

(iv) $G^+(R)$ *is connected.*

Followeing the study on Hamiltonicity, the chordality in the unit graphs of finite commutative rings was studied in [252], where the rings having quotient ring $\frac{R}{J_R}$ as a product of fields were characterised based on the chordality of the unit graphs, and, in [250], a necessary and sufficient condition under which the unit graphs of finite commutative rings were well-covered was deduced, from which the unit graphs whose edge rings were Cohen–Macaulay and Gorenstein were characterised, as given in Theorem 113. This characterisation led to the identification of a large class of non-Cohen–Macaulay graphs.

Theorem 113 ([251])**.** *Let R be a finite commutative ring R with unit graph $G^+(R)$. Then,*

(i) $G^+(R)$ *is Cohen–Macaulay if and only if R is a field with characteristic 2 or* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$*;*

(ii) $G^+(R)$ *is Gorenstein if and only if* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$*.*

A graph G is called *realisable* as an algebraic graph (unit graph) if it is isomorphic to the algebraic graph defined (unit graph $G^+(R)$, for some ring R). As already mentioned, two prominent problems that exist for any algebraic graph introduced are to analyse the graph parameters of the newly introduced graph and to check if any given graph G can be realised as the defined algebraic graph. A partial solution to the second problem of realising the given graph structure as a unit graph of a ring was given in [253], where the classes of graphs which can be realised as a unit graph were determined, as given below.

Theorem 114 ([253])**.**

(i) P_n *is realisable as a unit graph if and only if* $n = 2, 3$*;*

(ii) C_n *is realisable as a unit graph if and only if* $n = 4, 6$*;*

(iii) K_n *is realisable as a unit graph if and only if* $n = 2^k$*, for some a positive integer k;*

(iv) K_{s_1,s_2} *is realisable as a unit graph if and only if* $s_1 = s_2 = 2^k$, $k \in \mathbb{N}$ *or* $s_1 = 1$ *and* $s_2 = 3$*.*

It can be seen that the graph realisations in Theorem 114 are given based on the results obtained in [225], where the rings were characterised based on the unit graph's structure, as given in Theorems 102 and 103. While using Theorems 102 and 103, for obtaining further realisations of the unit graphs, the authors of [253] observed that the characterisation of rings whose unit graph was complete bipartite was incomplete, as there emerged an ambiguity if the authors of [225] assumed that the ring R was a local ring with or without the condition $|\frac{R}{M}| = 2$, where M was the maximal ideal of the local ring. In both cases of this assumption, counterexamples of rings with the corresponding properties were obtained in [253], which led to a modification of the existing result.

In the case of such a ring for which $|\frac{R}{M}| \neq 2$ was considered, in [24], to prove the result that was given in [24], a counterexample of a field with four elements, say, $\mathbb{F}_4$, whose unit graph was K_4, which is not complete bipartite, was given in [253], and, on the other hand, if R was considered as a local ring with condition $|\frac{R}{M}| = 2$, the result was proven to be incorrect because, if $R \cong \mathbb{Z}_3$, then $G^+(R) \cong K_{1,2}$, which is a complete bipartite. Based on these observations, the result was modified in [253] by including the condition $|\frac{R}{M}| \neq 2$ or $R \cong \mathbb{Z}_3$, along with the existing statement that was given in [225].

Recollect that, for a graph G, $S \subseteq V(G)$ is a *perfect code* of the graph if S is an independent set, such that every vertex in $V(G) - S$ is adjacent to exactly one vertex in S. A perfect code can also be called an *efficient independent dominating set* (c.f. [219]). By the definition of a perfect code, the investigation of perfect codes can be seen as computing a variant of the domination number of a graph, and, in [254], perfect codes in the unit graphs were examined, where the rings were characterised first based on the existence of a perfect code in their unit graphs or their complements, as finding whether a graph admits perfect code is also a question that remains to be addressed. Following this characterisation of rings, the commutative rings with identities in which their associated unit graphs accepted perfect codes of order one and two were characterised, and a few results relating the structure of the perfect code and the structure of the rings were obtained.

This study was extended to investigate the perfect codes in the induced subgraph of the unit graph of finite commutative rings in [255], where the subgraph of the unit graph of a ring induced by the set of all vertices that represented the elements of the ring that were not units of the ring was considered. Here, the commutative rings in which their associated induced subgraphs of unit graphs admitted the trivial and non-trivial perfect codes were classified, and a characterisation of rings that did not admit perfect codes in this induced subgraph of their unit graph was also deduced. Furthermore, it was proven that the complement of this induced subgraph of the unit graph of finite commutative rings admits only the trivial subring perfect code, where a *subring perfect code* refers to the perfect code on a subgraph induced by a subring of the ring. A similar investigation on some other induced subgraphs of the unit graph of commutative rings was conducted in [256], whose results were analogous to the ones obtained in [255], even though the vertex set of the induced subgraphs differed. This gives an underlying property of the unit graph of the ring itself rather than the subgraphs.

A *Boolean ring* is a ring with identity in which every element is idempotent. Perfect codes in the unit graph of Boolean rings were investigated in [257,258], where the existence of a subring perfect code in the unit graphs associated with the finite Boolean rings was proven in [257], along with a necessary and sufficient condition for a subring of an infinite Boolean ring to admit a perfect code of size infinity in the unit graph. In [258], the perfect codes spanning subgraphs of a unit graph associated with a Boolean ring R of order 2^k, for some positive integer $k \geq 1$, were determined, and, as a consequence of this, sharp lower and upper bounds for the cardinality of a subset of the vertex set to be a perfect code spanning subgraphs of a unit graph were established.

The line graph of a graph is a well-studied derived graph of a graph, and, as already known, several properties of the line graph of a graph are interrelated with the properties of the graph. In this regard, the line graphs of the unit graphs associated with the finite commutative rings were exclusively studied in [259–261]. The basic properties of the line graph of the unit graph of finite commutative rings such as the diameter, girth, clique, and chromatic number, along with some classifications of rings whose unit graphs are planar, as well as the Hamiltonian, were given in [260]. Observe that almost all the results in this article [260] were deduced based on the properties of the unit graphs that were discussed in [225].

An extended investigation on the line graph of the unit graph associated with finite commutative rings was performed in [259], where characterisations of the line graphs of the unit graphs of rings on the basis of their structural properties such as the completeness, bipartiteness, traversability, diameter, girth, and chromatic number were obtained. Additionally, the domination number of this line graph of the unit graph of rings was computed in [234], along with the domination number of the unit graphs of rings. Significant and curious problems of identifying the structure of the unit graph of a given finite commutative ring as a line graph of some graph, as well as identifying the finite commutative rings for which the complement of the unit graph can be realised as a line graph of a graph, were addressed in [261], and the list of rings of order two, three, and four with these realisation conditions were given.

For better understanding of the structure of the graph based on the structure of the ring, the unit graphs of certain specific rings whose structures are well known were investigated in detail. In [262], the unit graph of the ring $\mathbb{Z}_r \times \mathbb{Z}_s$, for any $r, s \in \mathbb{N}$, was discussed exclusively, where the basic structural and traversal properties of the graph $G^+(\mathbb{Z}_r \times \mathbb{Z}_s)$ and its graph invariants were determined. Similarly, in [263], the rings of polynomials and power series over a ring were examined, and all standard properties and invariants of the unit graph of these rings were obtained, along with some results on the planarity of the graph.

In [264], the unit graphs of group rings were discussed, where, if $\mathscr{G}$ is a group and R is a ring, the *group ring* of $\mathscr{G}$ over R, denoted by $R[\mathscr{G}]$, is a generalisation of a given multiplicative group, produced by attaching to each element of the group a "weighting factor" from a given ring. It is a set of mappings with certain properties involving module operations. The basic graph invariants and certain structural properties of the unit graph of these rings were deduced in [264]. As a detailed conceptual understanding of the group rings was obtained with the knowledge of the structure of modules, we refer the reader to [265,266] for more details on group rings.

For most of the study on the unit graphs of rings that had been conducted, it can be seen that the unit graphs of finite commutative rings were considered and, in a few instances, the unit graph of an associative ring was considered. As already mentioned, this is because of the symmetric nature of the commutative rings. In [267], the unit graph of a left Artinian ring was exclusively examined and the connectedness, girth, and diameter of the unit graph of this ring were determined. Additionally, the conditions under which the unit graph of any finite ring was Hamiltonian were obtained in [267] by providing an algorithm that found a spanning cycle of the unit graph, which took the required end points as the inputs and provided the corresponding Hamiltonian cycle. In [268], a short discussion on the unit graphs of non-commutative rings was given, wherein a very few results of the unit graphs of commutative rings were extended by proving them without using the commutative property of the ring. With this study, the challenge to investigate the unit graphs associated with non-commutative rings was clearly visible.

The signed graph of the unit graph of rings was defined in [269], as given in Definition 24, and an example of this graph is given in Figure 13. The rings for which this signed unit graph is balanced were characterised in [269], and the line signed graphs of these signed unit graphs were investigated in [270], where the commutative rings with unity for which a line signed graph of a signed unit graph is balanced and consistent, were characterised by establishing some sufficient conditions for balance and consistency of the line signed graph of signed unit graphs.

Definition 24 ([269])**.** *The* signed unit graph*, denoted by* $S(G^+(R)) = (G^+(R), \sigma^+)$*, is a signed graph whose underlying graph is the unit graph* $G^+(R)$ *of the ring* R*, and the sign of an edge* $v_iv_j \in E(G^+(R))$ *is assigned by the function* $\sigma^+ : E(G^+(R)) \to \{+, -\}$ *as follows. For an edge* v_iv_j *in* $G^+(R)$*,*

$$\sigma^+(v_iv_j)\begin{cases} +, & \text{if } v_i \in R^* \text{ or } v_j \in R^*; \\ -, & \text{otherwise,} \end{cases}$$

where R^* *denotes the group of units of the ring.*

An independent investigation on the signed unit graphs of the rings of the form $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}}$, where p_i, $1 \le i \le r$ are prime numbers and $r \in \mathbb{N}$, was performed in [271]. In this article, the sign compatibility, balance, and clusterability of the unit graphs of these rings were discussed, and the rings were characterised according to the above-mentioned properties.

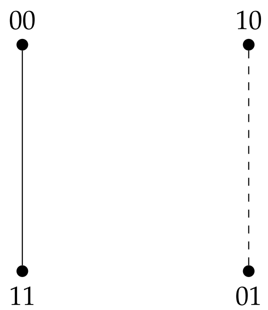


Figure 13. The signed unit graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

It can be seen in the literature that several surveys and brief literature reviews of the investigations on the unit graphs of rings have been performed periodically, since the introduction of these graphs (c.f. [165,272,273]), to understand the dynamics of research problems proposed and addressed related to the unit graphs of rings. Further, since unitary addition Cayley graphs also possess the same definition, the unit graphs of some rings are sometimes addressed as the unitary addition Cayley graphs of the respective ring and are investigated along with the unitary Cayley graphs. Such articles, where more than one graph among the graphs given in this review are discussed, are included in the section of the first graph that is discussed, with appropriate explanation and cross-referencing.

Additionally, it can be noticed that not many investigations have been performed on the closed unit graphs of rings, unlike for unit graphs. This provides an area for further exploration into this pseudo-graph structure.

6. Other Cayley Graphs Defined on Rings

Towards the end of the 18th century, the Cayley graph was defined on groups such that the vertex set of the graph was the elements of the group, and the adjacency condition was defined with respect to a symmetric subset of the group. This was considered as an underlying principle to define a Cayley graph on any algebraic structure, and multiple variations of Cayley graphs were defined on algebraic structures, based on several of its well-known symmetric subsets. In this article, as we deal exclusively with rings, we collect the literature on different Cayley graphs defined on rings, based on various symmetric subsets of the ring, and provide a brief review in this section.

As we can observe, $\mathbb{Z}_n$ is one of the most comprehensible ring structures, and the properties of any symmetric subset of this ring are related to the number-theoretic properties of n. Owing to this, it can be seen that several Cayley graph variations are defined on $\mathbb{Z}_n$ and investigated as the first step, following which the definitions are extended to a general ring, based on the feasibility of investigation. Though almost all the graph definitions on $\mathbb{Z}_n$ can be extended to any ring R, the process of investigating these graphs for any general ring is highly challenging, as the graph properties depend on the algebraic structure of the ring. Additionally, even in the articles where the definitions are extended to a general ring R, it can be observed that the commutative ring with unity, local rings, and rings that can be factorised into products of local rings are mainly considered for determining the properties of these graphs.

In this section, we denote the different Cayley graphs by the notation ξ with an appropriate suffix, corresponding to the property by which the graph is defined, for brevity and uniformity. Additionally, the symmetric subsets considered are denoted by S is all the subsections, where, in each subsection, the set S corresponds to the symmetric subset considered to define the corresponding graph in that subsection.

6.1. Absorption Cayley Graphs

The absorption Cayley graph of the ring $\mathbb{Z}_n$ was introduced and studied in [274,275]. As the name conveys, this variant of Cayley graph was defined based on the absorption property of the elements in the ring, as given below, following which an example of an absorption Cayley graph is given in Figure 14.

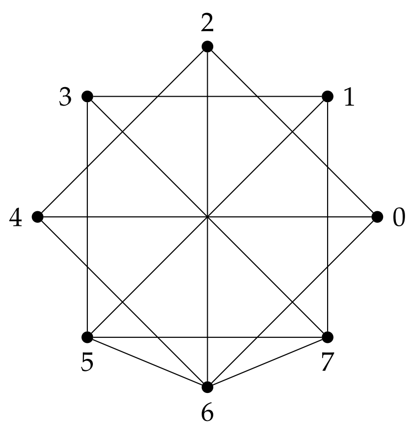


Figure 14. The absorption Cayley graph ξ_8^{acg}.

Definition 25 ([275])**.** *The* Absorption Cayley graph*, denoted by* $\xi_n^{acg} = Cay(\mathbb{Z}_n, S)$*, is a graph with the vertices set as the elements of the ring* $\mathbb{Z}_n$*,* $0, 1, \ldots, n-1$*, and two vertices are adjacent if their sum is an element of the set* S*, where* $S = \{x \in \mathbb{Z}_n : xy = yx = x, \text{ and } x \neq y, y \in \mathbb{Z}_n\}$*,* *that is, for all $u, v \in V(\xi_n^{acg})$, $uv \in E(\xi_n^{acg})$ when $u + v \in S$, where S is the set of all elements in the ring, such that it absorbs some element in the ring (except for itself).*

As the graph is defined on the subset formed by all the elements of the ring that absorbs some other element of the ring, the properties of this set were first discussed in [275]. The cardinality of this set and the properties of the elements in the set were discussed, and it was found that, for $n = 2k$, where k is odd, this subset $S \subseteq \mathbb{Z}_n$ coincides with the set of zero-divisors of the ring. Following this, the subset was proven to be a subgroup of the group $\mathbb{Z}_n$, which verified that the graph was defined with respect to a symmetric subset of the ring $\mathbb{Z}_n$.

We know that both the adjacency matrix of a graph as well as the Cayley table of a group are symmetric, such that each entry in a particular row and the corresponding column is unique. An interesting relation was seen between the adjacency matrix of the absorption Cayley graph on $\mathbb{Z}_n$ and the Cayley table of $\mathbb{Z}_n$: if each element $a \in S$ is replaced with 1 in the Cayley table and all the other elements, including the diagonals, are given 0, the adjacency matrix for the absorption Cayley graph on $\mathbb{Z}_n$ can be obtained. As the absorption Cayley graph is defined based on the sum of two elements belonging to the symmetric subset, an interesting relation between the unitary addition Cayley graphs and the absorption Cayley graphs was given in [275], as follows.

Theorem 115 ([274,275])**.** *Let k be an odd integer. For $n \neq 2k$, the complement of the unitary addition Cayley graphs $\overline{X}_n^+$ is isomorphic to the absorption Cayley graphs ξ_n^{acg}.*

Several graph parameters of the graph ξ_n^{acg} were computed in [275], as given in Theorem 116, along with the investigation on the connectedness, traversal properties, perfection, and planarity of the graph, as given below. Owing to the relation between the unitary addition Cayley graphs and the absorption Cayley graphs, only the results on absorption Cayley graphs, which are not derived exactly from the properties of the unitary addition Cayley graphs, are stated in this subsection.

Theorem 116 ([274,275])**.** *Let $\xi_{acg}^n = Cay(\mathbb{Z}_n, S)$ be the absorption Cayley graph of the ring $\mathbb{Z}_n$. Then,*

(i) The graph ξ_n^{acg} is either $|S| - 1$-regular or $(|S|, |S| - 1)$-semi regular;

(ii) $|E(\xi_n^{acg})| = k\lceil \frac{n-1}{2} \rceil + (|S| - k)\left(\lceil \frac{n-1}{2} \rceil - 1\right)$, where k is the number of odd elements in S;

(iii) $diam(\xi_n^{acg}) = 2$;

(iv) The edge connectivity of ξ_n^{acg}, when connected, is $|S| - 1$;

(v) The girth of ξ_n^{acg} (when connected) is 4, when $n = 6$ or 3, otherwise.

Theorem 117 ([274,275])**.**

(i) An absorption graph ξ_{acg}^n is connected if and only if n has at least two distinct prime factors;

(ii) An absorption graph ξ_{acg}^n is disconnected if and only if $n = p^k$, where p is prime and $k \geq 1$ is an integer;

(iii) The number of components in a disconnected absorption Cayley graph ξ_n^{acg} is $\frac{n-1}{2}$ when n is prime, and 2 otherwise.

Theorem 118 ([274,275])**.**

(i) An absorption Cayley graph is never Eulerian;

(ii) An absorption Cayley graph ξ_n^{acg} is Hamiltonian if $|S| > \frac{n}{2}$, where $n \neq 2k$, for some odd integer k.

It can be observed that, due to the strong perfect graph theorem, which states that a graph is perfect if and only if the graph as well as its complement do not contain any induced cycle of odd length at least 5, and Theorem 115, the conditions for the perfection of the graph ξ_n^{acg} coincide with those of the unitary addition Cayley graphs.

Theorem 119 ([274,275])**.** *The absorption Cayley graph of the ring $\mathbb{Z}_n$ is planar if and only if $n \in \{2, 4, 6, 8, p\}$, where p is a prime number.*

An important question that arises for defining a new algebraic graph is the realisation of a given graph as the defined algebraic graph, that is, in this context, the question is, when can a graph of order n be realised as an absorption Cayley graph of order n? This was answered in [274,275] as follows.

Theorem 120 ([274,275])**.** *A given graph G of order n is isomorphic to an absorption Cayley graph ξ_n^{acg} if and only if there are $|S|$ edge-disjoint subgraphs of the graph G, say, $G_1, G_2, \ldots, G_{|S|}$, whose union is the graph G, such that the following conditions hold:*

(i) $ab \in E(G_i)$ *if and only if* $a + b \equiv i \mod n$;

(ii) $|E(G_i)| = \lceil \frac{n-1}{2} \rceil - 1$, *when i is even and* $\lceil \frac{n-1}{2} \rceil$, *when n is odd.*

Owing to Theorem 120 and the fact that the absorption Cayley graph is disconnected, the structure of the components of a disconnected absorption Cayley graph was also examined in [275], and it was observed that these disconnected components are the union of subgraphs that are generated by the prime factors of n, which are nothing but disjoint cliques. This gave rise to the characterisation that an absorption Cayley graph ξ_n^{acg} is bipartite if and only if n is prime, as $S = \{0\}$ when n is prime.

As the graph coincides with the unitary addition Cayley graph, in some cases, and the zero-divisor Cayley graphs (see Section 6.6), for some values of n, the existing literature on these graphs determines most of their properties, which curtails the scope of unique study on these graphs. Additionally, in the remaining cases, it was seen that the graph was a union of disjoint cliques, which also does not extend the scope for much further exploration.

6.2. Nilpotent Cayley Graphs

The nilpotent Cayley graph of the ring $\mathbb{Z}_n$ was introduced in [276] and was studied in [276,277]. As the name suggests, this variant of Cayley graph is defined based on the subset of all nilpotent elements of the ring, as given below. Recall that an element x of a ring is said to be *nilpotent* if there exists a positive integer k, called the index, such that $x^k = 0$, where 0 is the additive identity of the ring.

Note that there are different graphs defined as the nilpotent and non-nilpotent graph of a ring having different vertex sets, such as the set of all nilpotent elements, non-nilpotent elements, etc., or they have been defined based on the product operation of the ring. We do not include them in this review because we restricted ourselves to the graphs defined on rings that are analogous to Cayley graphs—in other words, when the vertex set of the graph is the elements of the rings, where the adjacency condition is defined based on either the sum or the difference of two elements that must belong to a symmetric subset.

Definition 26 ([276])**.** *The* nilpotent Cayley graph *of the ring $\mathbb{Z}_n$, denoted by $\xi_n^{nil} = Cay(\mathbb{Z}_n, S)$, is a graph with the vertices set as the elements of the ring $\mathbb{Z}_n$, $0, 1, \ldots, n-1$, and two vertices are adjacent if their difference is an element of the set S, where $S = \{x \neq 0 \in \mathbb{Z}_n : x^k = 0$, for some $k \in \mathbb{N}\}$, that is, for all $u, v \in V(\xi_n^{nil})$, $uv \in E(\xi_n^{nil})$, when $u - v \in S$, where S is the set of all non-zero nilpotent elements of the ring. An example of a nilpotent Cayley graph is given in Figure 15.*

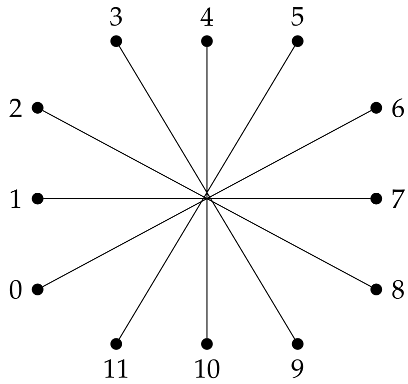


Figure 15. The nilpotent Cayley graph ζ_{12}^{nil} .

The properties of the set of all nilpotent elements and the basic graph properties for the nilpotent Cayley graphs of $\mathbb{Z}_n$ were studied in [276], where the number of nilpotent elements in the ring $\mathbb{Z}_n$ was given, from which the regularity and size of the nilpotent Cayley graph were determined. It was also proven that, for any integer which is a product of distinct prime numbers, the nilpotent Cayley graph is a null graph, which gave rise to the problem of investigating the connectedness of the graph. On solving this problem, it was found that the nilpotent Cayley graph is disconnected in some cases, for which the number of components in the graph was determined in [276], and each component was proven to be a clique. This led to the result that the nilpotent Cayley graph on $\mathbb{Z}_n$ is a union of k disjoint cliques, where k is the product of all distinct prime factors of n. The number of triangles in this graph was also enumerated in [276] based on the number of nilpotent elements in the ring.

The study on the nilpotent Cayley graph on $\mathbb{Z}_n$ was extended in [277] by investigating the neighbourhood set and the neighbourhood graph of the nilpotent Cayley graph. A subset $S \subseteq V(G)$ is called a *neighbourhood set* of the graph G if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the subgraph induced by the closed neighbourhood $N[v]$ of the vertex v, and the cardinality of a minimum neighbourhood set is called the *neighbourhood number* of the graph. The *neighbourhood graph* $N[G]$ of a graph G is a graph with the same vertex as G, and two vertices u and v are adjacent in $N[G]$ if their closed neighbourhood does not intersect (see [277]).

The neighbourhood number of the graph ζ_n^{nil} was determined as the number of distinct prime factors of n in [277], and the structure of the neighbourhood graph of the graph ξ_n^{nil}, along with the properties such as regularity and Hamiltonicity of the graph $N[\zeta_n^{nil}]$, were also discussed in [277]. It is known that all nilpotent elements are zero-divisors of the ring, and the set of all non-zero nilpotent elements form a symmetric subset of a ring. Thus, in several cases, it can be seen that the nilpotent Cayley graphs coincide with the zero-divisor Cayley graphs defined for a ring (see Section 6.6).

Recall that an element x is *idempotent* when $x^2 = x$. Usig this idempotent property of the elements of a ring, the concept of the *idempotent graph* of a ring R was introduced in [278]; the definition is given below, following which an example of an idempotent graph of a ring is provided in Figure 16.

Definition 27 ([278])**.** *The* idempotent graph *of a ring R is defined for all rings R with unity such that the vertex set of the graph is the set of all elements of the ring R, and two vertices u and v are adjacent if and only if $u + v$ is an idempotent element of the ring.*

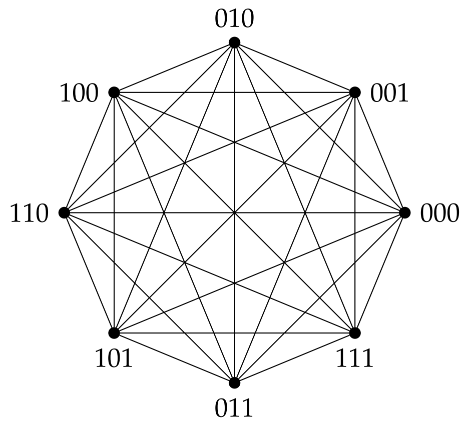


Figure 16. The idempotent graph of the ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

It can be seen that a slight modification to the ring considered and the binary operation of addition in the definition make the graph distinct from a subgraph of the other Cayley graphs defined on a ring. In [278], the structural properties of the idempotent graph of a finite non-local commutative ring R with unity were investigated, and a necessary and sufficient condition on the ring R for its idempotent graph to be planar was obtained. Using this result, it was proven that the idempotent graph of a ring can never be outerplanar. Moreover, when analysing the structure of the idempotent graphs of rings, all the finite non-local commutative rings having their idempotent graph as cograph, split graph, and threshold graph were classified.

Note that a graph is said to be a *cograph* if it has no induced subgraph isomorphic to P_4, and it is a *threshold graph* if it does not contain an induced subgraph isomorphic to P_4, C_4, or $2K_2$. Graphs whose vertex set can be partitioned into a clique and an independent set, where each vertex of the independent set is adjacent to some vertices in the clique, is a *split graph*. As the idempotent graphs have been defined very recently, several avenues, such as investigating their relation with the other related graphs (such as nilpotent Cayley graphs, zero-divisor graphs, etc.) or studying the traversal, structural properties, graph invariants, etc., are open to further exploration.

6.3. Mixed Unitary Cayley Graphs

A *mixed graph* is a graph that contains directed as well as undirected edges. In [279], the *mixed adjacency matrix* $M(G)$ of a graph G of order n is defined as an $n \times n$ matrix on the vertex set of the graph, such that

$$m_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \text{ is an edge or arc;} \\ -1, & \text{if } (v_j, v_i) \text{ is an arc;} \\ 0, & \text{otherwise.} \end{cases}$$

From this, the *mixed energy* of the graph is defined as the sum of the absolute values of eigenvalues of this mixed adjacent matrix. As it was seen that the unitary Cayley graphs have significant spectral properties, investigating the mixed spectra of the unitary Cayley graphs was a curious area to explore. Hence, the mixed Cayley graphs were defined in [279], and their spectra was investigated. The definition of the mixed unitary Cayley, followed by an example, are given in Definition 28 and Figure 17.

Definition 28 ([279])**.** *The* mixed unitary Cayley graph*, denoted by $\zeta_n^{mix} = Cay(\mathbb{Z}_n, \mathbb{Z}_n^+)$, is a graph whose underlying graph is the unitary Cayley graph X_n, and the conditions for an edge uv to be an arc or an edge are defined based on the properties of the end vertices u and v of the edge, as given below:*

(i) *uv is an edge if $\frac{v-u}{n} = 1$;*

(ii) (u, v) *is an arc if* $\frac{v-u}{n} = -1$ *and* $(j - i) < \lceil \frac{n}{2} \rceil$;
(iii) (v, u) *is an arc if* $\frac{v-u}{n} = -1$ *and* $(j - i) > \lceil \frac{n}{2} \rceil$.

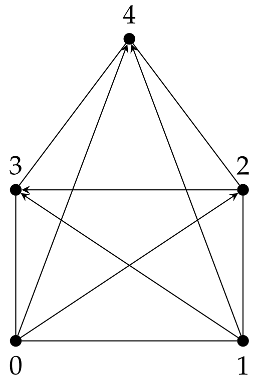


Figure 17. The mixed unitary Cayley graph on $\mathbb{Z}_5$.

Using these definitions of the mixed unitary Cayley graph and the mixed adjacency matrix, the spectra of the graph and the corresponding energy were determined in [280]. This investigation on the mixed spectra was performed for a few values of n, based on their number-theoretic properties, because a general structure of this mixed graph is yet to be studied in detail. As the structures become determined more clearly, other studies can be taken up in the future.

6.4. Divisor Cayley Graphs

The Cayley graph variation defined on the ring $\mathbb{Z}_n$ with respect to the subset of all divisors of n is called the *divisor Cayley graphs*, which were first introduced in [281]. An example of a divisor Cayley graph, following its definition, is given in Figure 18.

Definition 29 ([281])**.** *The* divisor Cayley graph*, denoted by* $\zeta_n^{div} = Cay(\mathbb{Z}_n, S)$*, is a graph with the vertices set as the elements of the ring* $\mathbb{Z}_n$*;* $0, 1, \ldots, n-1$*, and two vertices are adjacent if their difference is an element of the set S, where* $S = \{x, n-x : x \in \mathbb{Z}_n\}$*. That is, for all* $u, v \in V(\zeta_n^{div+})$*,* $uv \in E(\zeta_n^{div+})$*, when* $u - v \in S$*, where S is the set of all divisors of n and their inverse in* $\mathbb{Z}_n$*.*

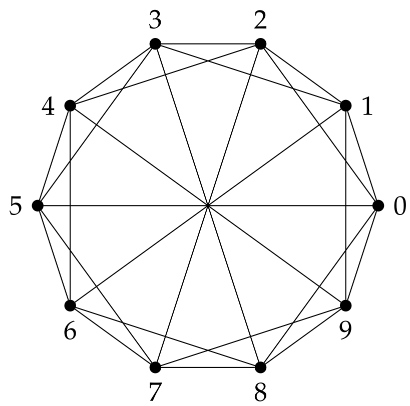


Figure 18. The divisor Cayley graph ζ_{10}^{div}.

Note that the definition of divisor Cayley graphs may seem similar to the gcd-graphs defined in Section 2, but the key difference between these graphs is that, in the definition of a gcd-graph, the subset considered was not a symmetric subset, whereas the divisor Cayley graphs are defined with respect to the symmetric subset of divisors and their inverses.

The graph properties of the divisor Cayley graphs such as regularity, Eulerianness, and Hamiltonicity were examined in [281], and the number of triangles in the divisor

Cayley graph was also enumerated. The number of triangles in the divisor Cayley graph was enumerated by partially following the technique that was used for the enumeration of triangles in the unitary Cayley graphs in [40]. Here, the triangles with vertices $\{0, a, b\}$ were given the term *fundamental triangles*, and the number of fundamental triangles was calculated as an intermediate step to compute the total number of triangles in the graph. This result was substantiated by several examples, which led to an interesting question as to the relationship between the number of divisors of n and the number of triangles in the divisor Cayley graph of the corresponding $\mathbb{Z}_n$, which still remains open.

Following this, the problem of enumerating the disjoint Hamiltonian cycles in the divisor Cayley graph was addressed in [282]. Using the previously determined properties of the divisor Cayley graphs in [281], it was proven that a divisor Cayley graph ζ_n^{div} can be decomposed into disjoint Hamiltonian cycles if and only if n is odd, and, for this case, it was determined that the graph ζ_n^{div} can be decomposed into $k+1$ disjoint Hamiltonian cycles, where k is the number of proper divisors of n.

In [282], an algorithm to find disjoint Hamiltonian cycles in the graph according to the values of n and to enumerate them was also given. This was followed by computing the domination number of the divisor Cayley graphs in [283], where an algorithm to construct a minimal dominating set of the graph was given, from which the domination number of the graph was determined. Certain topological indices of the divisor Cayley graph were computed in [284]. Note that the divisor Cayley graphs are also known as unitary divisor Cayley graphs and are different from difference divisor graphs, which appear to be almost similar to divisor Cayley graphs (see [285]).

Based on the unitary divisor Cayley graph, the unitary divisor addition Cayley graph, denoted by ζ_n^{div+}, was introduced in [286] by modifying the adjacency relation in the unitary divisor graphs to the sum of the elements to be a divisor. An example of a unitary divisor addition Cayley graph is given in Figure 19, which succeeds the definition of the graph, given as follows.

Definition 30 ([286])**.** *The* divisor addition Cayley graph, *denoted by* $\zeta_n^{div+} = Cay^+(\mathbb{Z}_n, S)$*, is a graph with the vertices set as the elements of the ring* $\mathbb{Z}_n$*;* $0, 1, \ldots, n-1$*, and two vertices are adjacent if their difference is an element of the set S, where* $S = \{x, n-x : x \in \mathbb{Z}_n\}$*. That is, for all* $u, v \in V(\zeta_n^{div})$*,* $uv \in E(\zeta_n^{div})$*, when* $u + v \in S$*, where S is the set of all divisors of n and their inverse in* $\mathbb{Z}_n$*.*

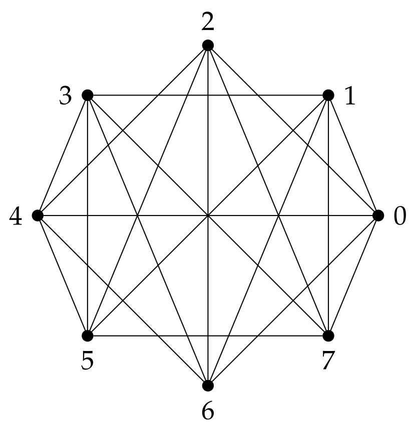


Figure 19. The divisor addition Cayley graph ζ_8^{div+}.

The article [286] is the only study available on the unitary divisor addition Cayley graph in which the graph is defined and the basic invariants of the graph such as the size, diameter, matching number, and the degree of the vertices were computed. In addition to this, unitary divisor addition Cayley graphs were characterised based on their traversal properties, such that the graph ζ_n^{div+} is Eulerian if and only if $n = 2^t$, for some integer $t > 1$, and ζ_n^{div+} is Hamiltonian if and only if n is even. Several properties of the graph and its association with other addition Cayley graphs defined on $\mathbb{Z}_n$, such as gcd-graphs, etc., can be explored further.

6.5. Involutory Cayley Graphs

In mathematics, the term *involution* means an entity which is its own inverse, and the elements of any algebraic structure which is its own inverse are called the *involutory elements* of the structure. This set of all involutory elements of a ring is called the *involution set* of the ring, which is a symmetric subset. With respect to this involution set, the *involutory Cayley graph* of the ring $\mathbb{Z}_n$, denoted by ξ_n^{inv}, was defined in [287], as follows.

Definition 31 ([287])**.** *The* involutory Cayley graph*, denoted by $\xi_n^{inv} = Cay(\mathbb{Z}_n, S)$, is a graph with the vertices set as the elements of the ring $\mathbb{Z}_n$; $0, 1, \ldots, n-1$, and two vertices are adjacent if their difference is an element of the set S, where $S = \{x \neq 0 \in \mathbb{Z}_n : x^2 \equiv 1 \mod n\}$, that is, for all $u, v \in V(\xi_n^{inv})$, $uv \in E(\xi_n^{inv})$, when $u - v \in S$, where S is the set of all involutory elements in the ring.*

Similarly, the addition variant of this Cayley graph, called the *involutory addition Cayley graph* of the ring $\mathbb{Z}_n$, denoted by ξ_n^{inv+}, was defined in [287], as given below. Illustrations of an involutory Cayley graph and an involutory addition Cayley graph are given in Figure 20.

Definition 32 ([288])**.** *The* involutory addition Cayley graph*, denoted by $\xi_n^{inv+} = Cay^+(\mathbb{Z}_n, S)$, is a graph with the vertices set as the elements of the ring $\mathbb{Z}_n$; $0, 1, \ldots, n-1$, and two vertices are adjacent if their difference is an element of the set S, where $S = \{x \neq 0 \in \mathbb{Z}_n : x^2 \equiv 1 \mod n\}$, that is, for all $u, v \in V(\xi_n^{inv+})$, $uv \in E(\xi_n^{inv+})$, when $u + v \in S$, where S is the set of all involutory elements in the ring.*

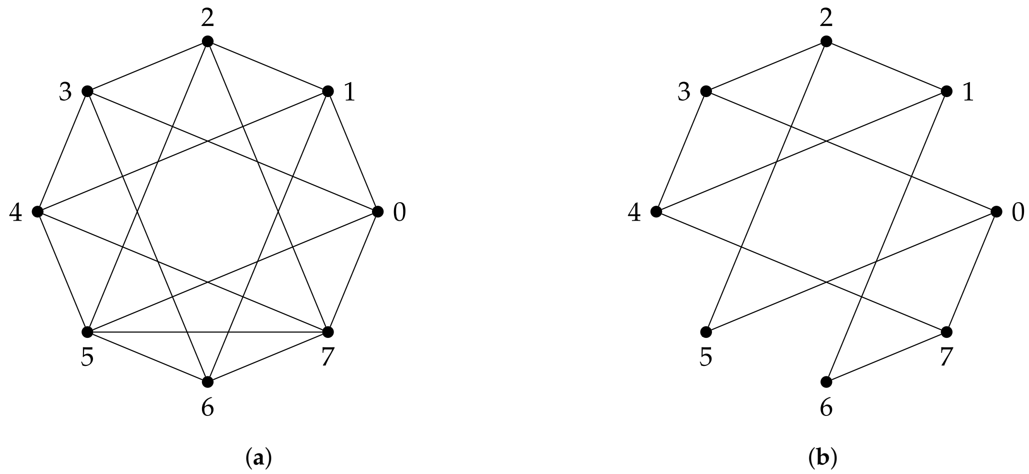


Figure 20. Examples of involutory and involutory addition Cayley graphs. (**a**) Involutory Cayley graph ξ_8^{inv}. (**b**) Involutory addition Cayley graph ξ_8^{inv+}.

The basic properties of the graphs ξ_n^{inv} and ξ_n^{inv+} were discussed in [287,288], respectively. In comparing the properties that were obtained for both graphs, the differences as well as the similarities between the graphs and the values of n for which they coincide can be obtained. The involutory Cayley graph is S-regular, whereas the involutory addition Cayley graphs can be $|S|$-regular or $(|S|, |S|-1)$-semi regular, depending on the value of n. As the degree of each vertex in the involutory addition Cayley graph and the diameter of the graph depend on the value of n, the degree and the diameter of the graph were only explored in the article [288]; conversely, in [287], apart from computing the degree of the vertices in the graph, it was proven that the involutory Cayley graphs are connected, Eulerian, and Hamiltonian. The domination number and related parameters for the involutory Cayley graph were computed in [289], where the parameters were computed for the involutory Cayley graphs that fall under the standard graph classes, using the exact values that had been obtained for these graph classes.

6.5.1. Quadratic Unitary Cayley Graphs

The symmetric subset of the involutory elements of a ring is also called the *quadratic units* modulo n, as the square of an element becomes the unit of the ring $\mathbb{Z}_n$ with integers modulo n. As such, the involutory Cayley graphs of $\mathbb{Z}_n$ were also studied independently under the name *quadratic unitary Cayley graphs* for the ring $\mathbb{Z}_n$ in [290]. For the values of n such that $n \equiv 1 \mod 4$ and is prime, these graphs were found to coincide with a class of graphs called the *Paley graphs* on n vertices (refer to [291] for more details on Paley graphs). Some structural properties of the quadratic unitary Cayley graphs of $\mathbb{Z}_n$ were presented in [290], where the diameter of the graph was determined for odd and even values of n by analysing the paths of different lengths in the graph. This analysis led to the examination of self-complementary quadratic unitary Cayley graphs on $\mathbb{Z}_n$, from which the following characterisation of perfect quadratic unitary Cayley graphs was obtained, as described in [290].

Theorem 121 ([290])**.** *The quadratic unitary Cayley graph on $\mathbb{Z}_n$ is perfect if and only if n is even or $n = p^k$, for prime $p \equiv 3 \mod 4$.*

The structural analysis of the graph also led to the characterisation of the quadratic unitary Cayley graph on $\mathbb{Z}_n$ that decompose into direct products of graphs (see Definition 7) over relatively prime factors of n. Based on the proof techniques used to prove the results, a linear operator, called the *sympletic operator*, was defined in [290] as a $2k \times 2k$ matrix, called the *sympletic form* (modulo n),

$$\sigma_{2k} = \begin{pmatrix} 0_k & -I_k \\ I_k & 0_k \end{pmatrix},$$

where I_k and 0_k denote the identity matrix and the zero matrix of order k, respectively. It was proven in [290] that the set of all these sympletic operators with coefficients in $\mathbb{Z}_n$ form the *sympletic group modulo n*. These sympletic operators were examined in [290], and a corollary regarding the decomposition of sympletic matrices in terms of these row operations was obtained. This led to the final result that gave a bound on the complexity of decompositions of these sympletic operators modulo n, which followed from the bounds on the diameter of the quadratic unitary Cayley graph on $\mathbb{Z}_n$ that was obtained in the same article.

This notion of quadratic unitary Cayley graphs was extended to all finite commutative rings R in [292] as the graph with the vertices set as the elements of the ring R, and two vertices are adjacent if their difference is an element of the set S, where $S^* = \{x^2 : x \in R - \{0\}\}$ and $S = S^* \cup -S^*$. In fact, it can be seen that, when the ring is a finite field of prime order k such that $k \equiv 1 \mod 4$, the quadratic unitary Cayley graph of that field is a Paley graph, which, by definition, is the graph with the vertices set as the elements of the field, such that the vertices u and v are adjacent if and only if $u - v$ is a non-zero square of the field.

For a finite commutative ring R that is decomposed as $R = R_1 \times R_2 \times \ldots \times R_t$, where each R_i, $1 \leq i \leq t$ is a local ring with the maximal ideal M_i and, for a local ring R_0 with the maximal ideal M_0 such that $\frac{|R_0|}{|M_0|} \equiv 3 \mod 4$, the spectra of the quadratic unitary Cayley graphs of the ring R_0 and $R_0 \times R$, with the condition that $\frac{|R_i|}{|M_i|} \equiv 1 \mod 4$, $1 \leq i \leq t$, were determined, along with their energies. The spectral moments of the quadratic unitary Cayley graphs of the above-mentioned rings were also computed, and the conditions under which these graphs were hyperenergetic or Ramanujan graphs were determined. A prefatory study on the same graphs was performed in [293], where only a very few results on the structure of the graph and its eigenvalues were obtained.

6.5.2. Quadratic Residue Cayley Graphs

Another variant of Cayley graphs similar to the involutory Cayley graphs are quadratic residue Cayley graphs. These can be seen as an extension of the quadratic residue property to a prime number. Accordingly, these graphs are defined on the rings $\mathbb{Z}_n$, where n is an odd prime. If p is an odd prime and $n \in \mathbb{N}$, such that p divides n and the quadratic congruence $x^2 \equiv n \mod p$ has a solution, then n is called a *quadratic residue mod p*, and the set of all quadratic residues mod p, along with their inverse, is a symmetric subset of $\mathbb{Z}_p$. With respect to this symmetric subset, the *quadratic residue Cayley graph* was defined in [294], exclusively for the rings $\mathbb{Z}_p$, where p is an odd prime, as given in Definition 33, which is followed by an example of a quadratic residue Cayley graph of a ring in Figure 21.

Definition 33 ([294])**.** *For an odd prime integer p, the* quadratic residue Cayley graph *of* $\mathbb{Z}_p$*, denoted by* $\xi_n^{qrcg} = Cay(\mathbb{Z}_p, S)$*, is a graph with the vertices set as the elements of the ring* $\mathbb{Z}_p$*,* $0, 1, 2, \ldots, p$*, and two vertices u and v are adjacent if their difference* $u - v \in S$*, where S the set of all quadratic residues mod p, along with their inverse elements.*

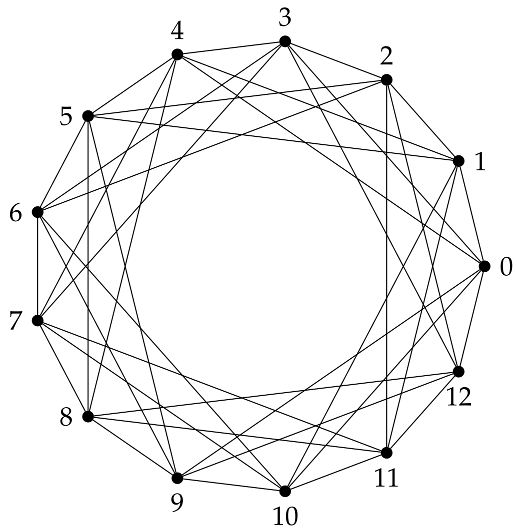


Figure 21. The quadratic residue Cayley graph of the ring $\mathbb{Z}_{10}$.

The studies on the quadratic residue Cayley graph of the ring $\mathbb{Z}_p$ have mainly been focused on finding dominating functions and some of their variants for the graph. The graph was defined, along with the basic invariants and properties such as the degree, regularity, number of triangles, disjoint Hamiltonian cycles, in [294]. Following this, all the investigations were on different dominating functions on the graph.

A function $f : V(G) \to [0,1]$ is a *dominating function* of a graph G if $f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1$ for every vertex $v \in V(G)$, and the dominating function f is minimal if $f(v) \geq g(v)$ for all $v \in V(G)$, where g is also a dominating function. A minimal dominating function f is a *basic minimal dominating function* if it cannot be expressed as a proper convex combination of two distinct minimal dominating functions (see [295]). These definitions related to replacing the vertex with an edge give the corresponding definitions of edge dominating functions.

The edge dominating functions, basic minimal edge dominating functions, and basic minimal dominating functions of the quadratic residue Cayley graphs were computed in [295–297], respectively. Different functions were proven to be the corresponding dominating functions for the graph, and several examples to convey the significance of these functions were also given. Following this, the variations in the total dominating functions for the graph were explored in [298,299] in a similar way.

In [300], the quadratic residue Cayley graph of the ring $\mathbb{Z}_{2^k}$ was studied exclusively. Only for integers of the form 2^k, the quadratic residue Cayley graph was constructed and investigated. This was the earliest known attempt to define a Cayley graph based on quadratic residues. In this article, it was shown that the diameter of these quadratic residue Cayley graphs defined on $\mathbb{Z}_{2^k}$ was two, following which a recursive formula to determine the number of triangles in the graph was obtained. In addition, a small discussion on the number of k residue modulo p^r (prime p) was also given in [300], to extend the defined quadratic residue Cayley graphs on $\mathbb{Z}_{2^k}$.

6.6. Zero-Divisor Cayley Graphs

A symmetric subset of a ring which is highly significant in order to understand the structure of the ring is the set of all zero-divisors. The Cayley graph defined with respect to this symmetric subset of zero-divisors is called the *zero-divisor Cayley graphs*. This graph was first defined on the finite commutative rings in [161], after which it was defined on the rings of integer modulo n, $\mathbb{Z}_n$ in [301]. Illustrations of zero-divisor Cayley graphs of the integer modulo ring and that of a ring R are given in Figure 22.

Definition 34 ([301])**.** *The* zero-divisor Cayley graph *of a ring R, denoted by* $\zeta_R^{zdcg} = Cay(R, Z(R))$*, is defined as the graph whose vertex set is the set of all elements of the ring, and two distinct vertices are adjacent if their difference is a non-zero zero-divisor, that is, for all* $u, v \in V(\zeta_R^{Z(R)})$, $uv \in E(\zeta_R^{Z(R)})$*, when* $u - v \in Z(R)$*, where* $Z(R)$ *is the set of all non-zero zero-divisors of the ring R. The zero-divisor Cayley graph of the ring* $\mathbb{Z}_n$ *is denoted by* ζ_n^{zdcg}*.*

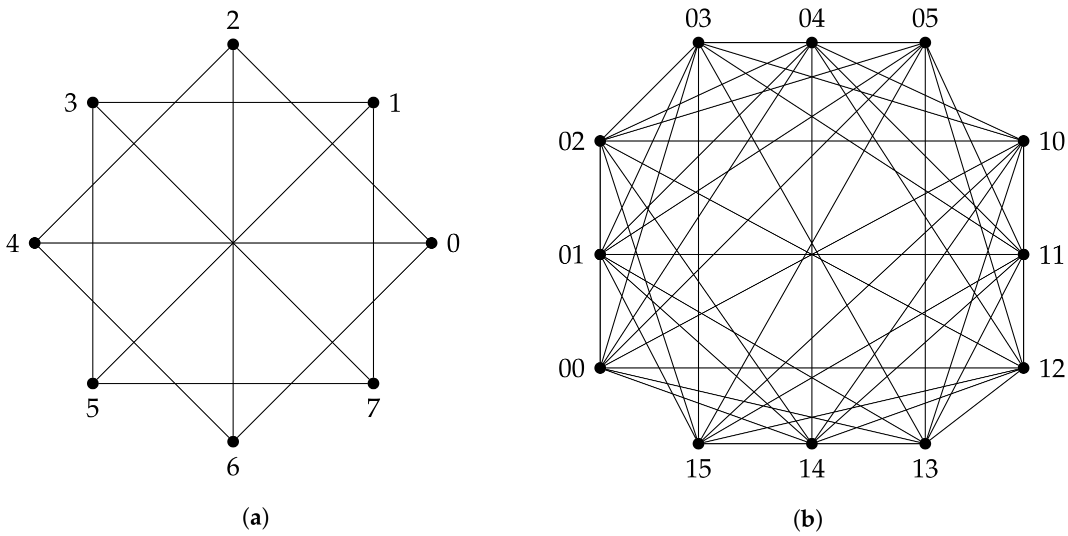


Figure 22. Examples of zero-divisor Cayley graphs of rings. (**a**) Zero-divisor Cayley graph ζ_8^{zdcg}. (**b**) The zero-divisor Cayley graph of $\mathbb{Z}_2 \times \mathbb{Z}_6$.

In [161], the graph parameters such as the clique number, chromatic number, edge chromatic number, domination number, and girth of the graph $\zeta_R^{Z(R)}$ were computed, and the rings for which the zero-divisor Cayley graphs were strongly regular and planar were characterised. By restricting this definition to the ring $\mathbb{Z}_n$, more properties such as the enumeration of triangles, connectivity, etc. were explored in [302].

We know that any element in a ring is either a zero-divisor or a unit, and the set of all non-coprime integers of n are the zero-divisors in the ring $\mathbb{Z}_n$. Hence, in the zero-divisor Cayley graphs of $\mathbb{Z}_n$, two vertices are adjacent if and only if their difference is not relatively prime to n; more precisely, it can be seen as the complement of the unitary Cayley graphs X_n defined on $\mathbb{Z}_n$. As many properties of the unitary Cayley graphs and their complements have already been studied in the literature, only the basic invariants and the basic properties of the graph were studied in [301,302]. The number of triangles in the graph, along with the traversal properties, were studied in [301], and the connectedness of the graph and the properties of the components when the zero-divisor Cayley graphs were disconnected were investigated in [301].

Note that, by modifying the adjacency condition of the zero-divisor Cayley graphs defined on a ring R from the difference to the sum of two elements to be a zero divisor, the definition of a *total graph* of a ring is obtained. As the literature on total graphs is hugely growing, along with several exclusive and detailed surveys and review papers (for examples, see [17,22]), we have not included them in this review.

It can be noted that, for all the variations in Cayley graphs that have been discussed in this section, only a cursory investigation has taken place in the literature. This can be because of two reasons: one is that, while investigating the structure of the newly defined graph, a high similarity with the properties of an already defined, existing Cayley graph were observed and, sometimes, the graphs may also coincide with them, leaving no scope for further study; the other reason to not proceed further is because of the ambiguous structure of the symmetric subset that is considered to define the Cayley graph, or the realisation that the graph structure might not reflect the important properties or the structure of the ring, failing to serve the main purpose of the study.

7. Conclusions

It can be seen that the introduction of the unitary Cayley graphs of the ring $\mathbb{Z}_n$ provided a new direction for research in algebraic graph theory, using the number-theoretic properties of the rings to define variants of Cayley graphs with respect to different symmetric subsets of the group by considering the operations of both sum and difference, giving rise to twin-type variants of such graphs. Apart from some specific open problems that were discussed in their respective sections , there are several other open problems that can be investigated with respect to these algebraic graphs defined on rings that are discussed in the review, among which a few are presented in this section.

It can be observed that there is an overall pattern in the investigations performed on a particular graph when reviewing the literature, as well as while reading this article. Before moving to the open problems, it is important that this pattern is explicitly mentioned, for a better understanding. As a new variant of Cayley graph is defined, its first properties that are determined are the regularity, the degree of the vertices, and the size. Following these, the other parameters of diameter, girth, chromatic number, clique number, etc. are computed. Connectedness, traversability, planarity, and perfection are significant properties through which characterisations of rings are obtained. Investigating different matrices associated with the graphs and their spectra, especially the adjacency spectrum, the eigenvalues and energy of the graph are an inevitable problem. From these spectra, different properties such as hyperenergecity, realising the given graphs as Ramanujan graphs, etc. are discussed.

Furthermore, several matrices are associated, corresponding to which analogous investigations have been made. Realisation of the graph based on isomorphism and structural characterisations of the graph are important problems to address. Apart from these, different chromatic numbers, domination numbers, topological indices, centrality measures, covering numbers, vulnerability parameters, etc. can be computed for the graph, and the possibility of characterisations of the graphs and the rings based on these parameters are also examined. All possible studies are extended to the complements of these graphs, as they are also regular, in most cases.

Moving on to further areas of exploration with respect to the graphs discussed in the review, in most of the graphs that are given, not many studies on different types of domination and colouring parameters exist, except for the unitary Cayley graphs of $\mathbb{Z}_n$. Computation of different topological indices and centrality measures, and associating different matrices to these graphs and computing their energies and colour energies, are also open, especially for the graphs defined in Section 6; different types of vertex-partitioning of the algebraic graphs are also promising problems to work on.

Similarly, several parameters such as covering numbers, metric dimension, resolving sets, etc. have not been computed so far for the graphs; computing them and checking the feasibility of obtaining Nordhaus–Gaddum-type inequalities is also an open avenue to explore. In terms of signed graphs, the signed graph varieties have not been introduced for many Cayley graph variations, and even for the ones that have been introduced, properties apart from balance, clusterability, sign-compatibilty, and canonical consistency can be studied. Furthermore, induced sign graphs based on other properties of the ring elements can also be introduced, instead of introducing modified definitions based on the existence of the end vertices of an edge in a considered subset.

As seen in the literature, several Cayley graphs form an increasing sequence of graphs, where each graph can be related to the previous graph structure, from which various constructions of graph classes with different properties can be obtained. One other such type of study is constructing and realising sequences of *self-similar graphs*, where a self-similar graph is defined as follows.

For any graph G on n vertices and for any symmetric subgraph H of $K_{n,n}$, we construct an infinite sequence of graphs based on the pair (G, H). The first graph in the sequence is G, then, at each stage replacing every vertex and edge of the previous graph by a copy of G, a copy of H, the new graph in the sequence, is constructed. We call these graphs *self-similar graphs* (see [303,304]). As certain Cayley graphs on groups are realised as ones that generate self-similar graphs, an investigation of a similar kind can be taken up for research in Cayley graphs defined on rings.

Based on the definition of the variants in Cayley graphs presented in this review, it can be seen that they are related to each other in some aspects. Hence, chain-like inequalities of these graphs can be identified for certain rings and characterisations of rings when the graphs are equal or when one is a subgraph of another. On the other hand, a similar type of investigation can be performed exclusively with respect to the complements of these graphs or by considering both the graphs defined, as well as their complements, as the complement of some variants of Cayley graphs discussed in this article coincide with some graphs. Based on the huge literature available on Cayley graphs of groups, power graphs, zero-divisor graphs, and other graphs derived from them, certain analogous studies can also be introduced to these types of graphs.

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