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Pareto Efficiency Criteria and Duality for Multiobjective Fractional Programming Problems with Equilibrium Constraints on Hadamard Manifolds

Arnav Ghosh [1,†], Balendu Bhooshan Upadhyay [1,†] and I. M. Stancu-Minasian [2,*,†]

[1] Department of Mathematics, Indian Institute of Technology Patna, Patna 801103, India; arnav_2021ma09@iitp.ac.in (A.G.); bhooshan@iitp.ac.in (B.B.U.)
[2] "Gheorghe Mihoc-Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, 050711 Bucharest, Romania
* Correspondence: stancu_minasian@yahoo.com
† These authors contributed equally to this work.

Abstract: This article deals with multiobjective fractional programming problems with equilibrium constraints in the setting of Hadamard manifolds (abbreviated as MFPPEC). The generalized Guignard constraint qualification (abbreviated as GGCQ) for MFPPEC is presented. Furthermore, the Karush–Kuhn–Tucker (abbreviated as KKT) type necessary criteria of Pareto efficiency for MFPPEC are derived using GGCQ. Sufficient criteria of Pareto efficiency for MFPPEC are deduced under some geodesic convexity hypotheses. Subsequently, Mond–Weir and Wolfe type dual models related to MFPPEC are formulated. The weak, strong, and strict converse duality results are derived relating MFPPEC and the respective dual models. Suitable nontrivial examples have been furnished to demonstrate the significance of the results established in this article. The results derived in the article extend and generalize several notable results previously existing in the literature. To the best of our knowledge, optimality conditions and duality for MFPPEC have not yet been studied in the framework of manifolds.

Keywords: fractional programming; optimality conditions; duality; Hadamard manifolds

MSC: 90C46; 90C48; 90C29; 90C32

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1. Introduction

In the theory of mathematical programming, any optimization problem which is characterized by some constraints, involving either certain complementarity or some variational inequality, is referred to as a mathematical programming problem with equilibrium constraints (MPEC). One of the first to attempt investigating such optimization problems were Harker and Pang [1], who explored the existence of efficient solutions for MPECs. MPECs have been extensively used to model various real-life problems appearing in several fields of science and technology, for instance, the hydro-economic river basin model [2], process engineering [3], traffic and telecommunications networks [4], cyber attacks in electricity market [5], etc. For further details and updated surveys of MPEC and its applications, we refer the readers to [6–10] and the references cited therein.

In the last few decades, it has been noted that numerous real-life problems emerging in various areas related to engineering, technology, and science can be formulated more effectively on Riemannian and Hadamard manifold frameworks, rather than formulating them on Euclidean space (see [11,12]). Furthermore, extending and generalizing different techniques involved in the optimization theory from the setting of Euclidean spaces to the setting of manifolds result in several crucial advantages. By appropriately using the notions of Riemannian geometry, several constrained mathematical optimization problems can be conveniently converted into unconstrained problems. Apart from that, numerous problems in optimization, which are nonconvex, can be converted into convex problems by utilizing the Riemannian geometry viewpoint (see [13,14]). Furthermore, it is a common observation that numerous important constraints which naturally arise in certain mathematical programming problems have a relative interior, which can be viewed as Hadamard manifolds. For instance, the hypercube $(0,1)^n$ endowed with the metric $\operatorname{diag}\left(z_1^{-2}(1-z_1)^{-2},\ldots,z_n^{-2}(1-z_n)^{-2}\right)$, where $z_1, z_2, \ldots, z_n \in (0,1)^n$, and the set consisting of every symmetric positive definite matrix $\mathcal{S}^n_{++}$ with the metric $-\log\det X$, where $X \in \mathcal{S}^n_{++}$, are Hadamard manifolds (see, for instance, [15]). In view of the above advantages, various researchers have investigated optimization problems in several areas of modern research in the setting of Riemannian and Hadamard manifolds; see, for instance, [16–24].

It is worthwhile to note that constraint qualifications, optimality criteria, and duality results for MPEC have been studied by several researchers in the Euclidean space setting. For instance, several regularity and optimality criteria for MPEC were investigated by Chen and Florian [25]. Furthermore, the Abadie constraint qualification for MPEC was discussed by Flegel and Kanzow [26]. Moreover, necessary and sufficient criteria of optimality for MPEC were explored by Ye [8]. GGCQ as well as criteria of optimality for MPEC were explored by Flegel and Kanzow [27]. KKT-type criteria for optimality, as well as some duality models for multiobjective MPEC, were deduced by Singh and Mishra [6]. Besides this, optimality conditions for multiobjective MPEC on Hadamard manifolds have been studied by Treanţă et al. [7]. However, multiobjective fractional programming problems with equilibrium constraints have not yet been studied in the framework of manifolds. In this article, our primary objective is to address this research gap by investigating GGCQ, Pareto efficiency criteria, and duality results for multiobjective fractional programming problems with equilibrium constraints on Hadamard manifolds.

Motivated by the results presented in [6–8,23,25], a class of multiobjective fractional programming problems with equilibrium constraints MFPPEC is studied in this article, in the setting of Hadamard manifolds. First, we present GGCQ for MFPPEC in the setting of Hadamard manifolds. Next, we employ GGCQ to derive the necessary criteria of Pareto efficiency for MFPPEC. Moreover, by using certain geodesic convexity assumptions, we derive sufficient criteria of Pareto efficiency for MFPPEC. Subsequently, we formulate a couple of dual models related to MFPPEC, namely, the Mond–Weir type and Wolfe type dual models. Several interesting duality results, such as weak, strong, and strict converse duality results, are derived that relate our primal problem (MFPPEC) and the corresponding dual models. We have provided some nontrivial examples of MFPPEC on Hadamard manifolds to illustrate the importance of the results presented in this paper.

The novelty and the contributions of the paper are two-fold. Firstly, the results explored in this article generalize the corresponding results presented by [7] for a broader category of optimization problems, that is, MFPPEC. Secondly, the results investigated by [6] are extended for MFPPEC class in the Hadamard manifolds setting by the results presented in this article. Furthermore, the Pareto efficiency criteria and duality results studied in this article extend various corresponding results of [8,23,28] from the setting of Euclidean spaces to Hadamard manifolds, as well as generalize them for a broader category of problems, namely, MFPPEC. To the best of our knowledge, GGCQ, Pareto efficiency criteria, as well as duality models for MFPPEC, have not been explored before in a Hadamard manifold setting. As a result, the results derived in this article can be applied to study a more general class of mathematical programming problems, as compared to the existing results available in the literature.

The remaining part of the article unfolds in the following manner. Some elementary definitions and mathematical preliminaries are discussed in Section 2. We define MFPPEC in manifold setting and introduce GGCQ for MFPPEC in Section 3. Furthermore, we derive KKT-type necessary criteria of Pareto efficiency employing GGCQ. In Section 4, we use the notions of M-stationary element and geodesic convexity to establish sufficient Pareto efficiency criteria for MFPPEC. Subsequently, in Sections 5 and 6, Mond–Weir and

Wolfe type dual models related to MFPPEC are formulated, respectively. The weak, strong, and strict converse duality results are derived relating MFPPEC and the respective dual models. In Section 7, we provide an interesting practical application of the work presented in this paper in the field of information theory. Finally, in Section 8, we draw conclusions and discuss some future course of our research.

2. Notation and Mathematical Preliminaries

The standard symbols $\mathbb{R}^n$ and $\mathbb{N}$ are employed to signify the Euclidean space having dimension n and the set of all natural numbers, respectively. We use the notation $\mathbb{R}^n_+$ to signify the following set:

$$\mathbb{R}^n_+ := \{(z_1, z_2, \ldots, z_n) : z_k \geq 0,\ \forall k = 1, 2, \ldots, n\}.$$

We use the symbol $\langle \cdot, \cdot \rangle$ to signify the usual Euclidean inner product on $\mathbb{R}^n$. For arbitrary $\alpha, \beta \in \mathbb{R}^n$, we adopt the following notations:

$$\alpha \prec \beta \iff \alpha_k < \beta_k, \quad \forall k = 1, 2, \ldots, n.$$

$$\alpha \preceq \beta \iff \begin{cases} \alpha_k \leq \beta_k, & \forall k \in \{1, \ldots, n\}; \\ \alpha_s < \beta_s, & \text{for at least one } s \in \{1, \ldots, n\}. \end{cases}$$

We will be employing the notation $\mathscr{M}$ to signify a smooth manifold having dimension n. Let $y^* \in \mathscr{M}$ be arbitrary. The set that contains every tangent vector at the element $y^* \in \mathscr{M}$ is known as the tangent space at y^*, and is signified by $T_{y^*}\mathscr{M}$. For any element $y^* \in \mathscr{M}$, $T_{y^*}\mathscr{M}$ is a real linear space, having a dimension n, where $n \in \mathbb{N}$. In case we are restricted to real manifolds, $T_{y^*}\mathscr{M}$ is isomorphic to the n-dimensional Euclidean space $\mathbb{R}^n$. For any arbitrary subset $\mathcal{W} \subset T_{y^*}\mathscr{M}$, the closure and convex hull of $\mathcal{W}$ in $T_{y^*}\mathscr{M}$ are denoted by $\mathrm{cl}(\mathcal{W})$ and $\mathrm{co}(\mathcal{W})$, respectively.

A Riemannian metric, denoted by $\mathscr{G}$, on the set $\mathscr{M}$ is a two-tensor field that is symmetric as well as positive-definite. For every pair of elements $w_1, w_2 \in T_{y^*}\mathscr{M}$, the inner product of w_1 and w_2 is given by:

$$\langle w_1, w_2 \rangle_{y^*} = \mathscr{G}_{y^*}(w_1, w_2),$$

where $\mathscr{G}_{y^*}$ denotes the Riemannian metric at the element $y^* \in \mathscr{M}$. The norm corresponding to the inner product $\langle w_1, w_2 \rangle_{y^*}$ is denoted by $\| \cdot \|_{y^*}$ (or simply, $\| \cdot \|$, when there is no ambiguity regarding the subscript).

Let $a, b \in \mathbb{R}$, $a < b$ and $\nu : [a, b] \to \mathscr{M}$ be any piecewise differentiable curve that joins the elements y^* and $\hat{z}$ in $\mathscr{M}$. That is, we have:

$$\nu(a) = y^*,\ \nu(b) = \hat{z}.$$

For any differentiable curve ν, a vector field Y is referred to be parallel along the curve ν, provided that the following condition is satisfied:

$$\nabla_{\nu'} Y = 0.$$

If $\nabla_{\nu'} \nu' = 0$, then ν is termed as a geodesic. If $\|\nu\| = 1$, then the curve ν is said to be normalized. For any $y^* \in \mathscr{M}$, the exponential function $\exp_{y^*} : T_{y^*}\mathscr{M} \to \mathscr{M}$ is given by:

$$\exp_{y^*}(\hat{w}) = \nu(1),$$

where ν is a geodesic which satisfies $\nu(0) = y^*$ and $\nu'(0) = \hat{w}$. A Riemannian manifold $\mathscr{M}$ is referred to as geodesic complete, provided that the exponential function $\exp_u(v)$ is defined for every arbitrary $v \in T_p\mathscr{M}$ and $u \in \mathscr{M}$.

A Riemannian manifold is referred to as a Hadamard manifold (or a Cartan–Hadamard manifold), provided that $\mathscr{M}$ is simply connected and geodesic complete, as well as having a nonpositive sectional curvature throughout. Henceforth, in our discussions, the notation $\mathscr{M}$ refers to a Hadamard manifold of dimension n, unless mentioned otherwise.

Let $y^* \in \mathscr{M}$ be an arbitrary element lying in the Hadamard manifold $\mathscr{M}$. Then, the exponential function on the tangent space $\exp_{y^*} : T_{y^*}\mathscr{M} \to \mathscr{M}$ is a globally diffeomorphic function. Furthermore, the inverse of the exponential function $\exp_{y^*}^{-1} : \mathscr{M} \to T_{y^*}\mathscr{M}$ satisfies $\exp_{y^*}^{-1}(y^*) = 0$. Furthermore, for every pair of arbitrary elements $y_1^*, y_2^* \in \mathscr{M}$, there exists some unique normalized minimal geodesic $\nu_{y_1^*,y_2^*} : [0,1] \to \mathscr{M}$, such that the geodesic ν satisfies the following:

$$\nu_{y_1^*,y_2^*}(\tau) = \exp_{y_1^*}(\tau \exp_{y_1^*}^{-1}(y_2^*)), \quad \forall \tau \in [0,1].$$

Thus, every Hadamard manifold $\mathscr{M}$ of dimension n is diffeomorphic to the corresponding n-dimensional Euclidean space $\mathbb{R}^n$. The following definition is from Udrişte [12].

Definition 1. *Any subset $\mathcal{G}$ of $\mathscr{M}$ is termed as geodesic convex, provided that for every pair of distinct elements $z_1, z_2 \in \mathcal{G}$ and for any geodesic $\gamma_{z_1,z_2} : [0,1] \to \mathscr{M}$ connecting the elements z_1 and z_2, we have:*

$$\gamma_{z_1,z_2}(\sigma) \in \mathcal{G}, \quad \forall \sigma \in [0,1],$$

where, $\gamma_{z_1,z_2}(\sigma) = \exp_{z_1}\left(\sigma \exp_{z_1}^{-1}(z_2)\right)$.

The following definition from [12] is an extension of the notion of convex functions in the setting of Hadamard manifolds.

Definition 2. *Let $\Psi : \mathcal{G} \to \mathbb{R}$ be a smooth real-valued function defined on a geodesic convex subset $\mathcal{G}$ of $\mathscr{M}$. The function Ψ is termed as a geodesic convex function at the element y, provided that for each $x \in \mathcal{G}$, the following inequality holds:*

$$\Psi(x) - \Psi(y) \geq \left\langle \operatorname{grad} \Psi(y), \exp_y^{-1} x \right\rangle_y.$$

Similarly, the function Ψ is termed as a strictly geodesic convex function at the element y, provided that for every $x \in \mathcal{G}$, $x \neq y$, the following inequality holds:

$$\Psi(x) - \Psi(y) > \left\langle \operatorname{grad} \Psi(y), \exp_y^{-1} x \right\rangle_y.$$

For more detailed discussions on Riemannian and Hadamard manifolds, we refer the readers to [29,30] and the references cited therein.

3. Necessary Criteria of Pareto Efficiency for MFPPEC

In this section, a particular class of MFPPEC in the setting of Hadamard manifolds is considered. We deduce KKT-type necessary criteria of Pareto efficiency for MFPPEC by employing the generalized Guignard constraint qualification.

Let us consider the following MFPPEC in the setting of Hadamard manifolds:

$$\text{MFPPEC} \quad \text{Minimize} \quad \frac{\mathcal{A}(y)}{\mathcal{B}(y)} := \left(\frac{\mathcal{A}_1(y)}{\mathcal{B}_1(y)}, \frac{\mathcal{A}_2(y)}{\mathcal{B}_2(y)}, \dots, \frac{\mathcal{A}_r(y)}{\mathcal{B}_r(y)} \right),$$

$$\begin{aligned} \text{subject to} \quad & \Psi_j(y) \leq 0, \quad \forall j \in \mathcal{I}^{\Psi} := \{1, 2, \ldots, l\}, \\ & \theta_j(y) = 0, \quad \forall j \in \mathcal{I}^{\theta} := \{1, 2, \ldots, p\}, \\ & \mathcal{C}_j(y) \geq 0, \quad \forall j \in \mathcal{T} := \{1, 2, \ldots, m\}, \\ & \mathcal{D}_j(y) \geq 0, \quad \forall j \in \mathcal{T}, \\ & \mathcal{D}_j(y)\mathcal{C}_j(y) = 0, \quad \forall j \in \mathcal{T}. \end{aligned}$$

where each of the functions $\mathcal{A}_j, \mathcal{B}_j : \mathscr{M} \to \mathbb{R}$ $(j \in \mathcal{I} := \{1, 2, \ldots, r\})$, $\Psi_j : \mathscr{M} \to \mathbb{R}$, $(j \in \mathcal{I}^{\Psi})$, $\theta_j : \mathscr{M} \to \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\mathcal{C}_j : \mathscr{M} \to \mathbb{R}$, $\mathcal{D}_j : \mathscr{M} \to \mathbb{R}$ $(j \in \mathcal{T})$ are assumed to be smooth and are defined on some Hadamard manifold $\mathscr{M}$ having a dimension n, where $n \in \mathbb{N}$.

We use the symbol $\mathcal{F}$ to signify the set containing every feasible solution of the considered problem MFPPEC. Without any loss of generality, we suppose that $\mathcal{A}_i(y) \geq 0$ and $\mathcal{B}_i(y) > 0$, for every $y \in \mathcal{F}$ and $i \in \mathcal{I}$. Throughout the remaining part of the article, the following notation will be used:

$$\Phi_i(y) := \frac{\mathcal{A}_i(y)}{\mathcal{B}_i(y)}, \ \forall i \in \mathcal{I}, \text{ and,}$$
$$\Pi_j(y) := \mathcal{D}_j(y)\mathcal{C}_j(y), \ \forall j \in \mathcal{T},$$

for every $y \in \mathscr{M}$.

We recall the concepts of Pareto efficiency and weak Pareto efficiency in the following Definitions 3 and 4, which will be used in the paper (for reference, see [28] for instance).

Definition 3. *Let $\hat{z} \in \mathcal{F}$ be an arbitrary feasible solution of MFPPEC. Then, $\hat{z}$ is termed as a Pareto efficient solution of MFPPEC, provided that there does not exist any other feasible element $\tilde{z} \in \mathcal{F}$, which satisfies the following inequality:*

$$\Phi(\tilde{z}) \preceq \Phi(\hat{z}),$$

that is:

$$\frac{\mathcal{A}(\tilde{z})}{\mathcal{B}(\tilde{z})} \preceq \frac{\mathcal{A}(\hat{z})}{\mathcal{B}(\hat{z})}.$$

Definition 4. *Let $\hat{z} \in \mathcal{F}$ be an arbitrary feasible solution of MFPPEC. Then, $\hat{z}$ is termed as a weak Pareto efficient solution of MFPPEC, provided that there does not exist any other feasible element $\tilde{z} \in \mathcal{F}$, which satisfies the following inequality:*

$$\Phi(\tilde{z}) \prec \Phi(\hat{z}),$$

that is:

$$\frac{\mathcal{A}(\tilde{z})}{\mathcal{B}(\tilde{z})} \prec \frac{\mathcal{A}(\hat{z})}{\mathcal{B}(\hat{z})}.$$

Let $\hat{z} \in \mathcal{F}$ be any arbitrary feasible solution of MFPPEC. The index sets defined below will be crucial in the remaining part of the article:

$$\begin{aligned} \mathscr{A}^{\Psi}(\hat{z}) &:= \{j \in \mathcal{I}^{\Psi} : \Psi_j(\hat{z}) = 0\}, \\ \mathcal{R}_{+0}(\hat{z}) &:= \{j \in \mathcal{T} : \mathcal{C}_j(\hat{z}) > 0, \ \mathcal{D}_j(\hat{z}) = 0\}, \\ \mathcal{R}_{0+}(\hat{z}) &:= \{j \in \mathcal{T} : \mathcal{C}_j(\hat{z}) = 0, \ \mathcal{D}_j(\hat{z}) > 0\}, \\ \mathcal{R}_{00}(\hat{z}) &:= \{j \in \mathcal{T} : \mathcal{C}_j(\hat{z}) = 0, \ \mathcal{D}_j(\hat{z}) = 0\}. \end{aligned}$$

The following may be observed:

Remark 1. *(a) The set $\mathscr{A}(\hat{z})$ is termed as the set of all active inequality indices for the function Ψ at the point $\hat{z}$.*

(b) *The index set $\mathcal{R}_{00}(\hat{z})$ is termed as the degenerate index set at the point $\hat{z}$. The strict complementarity condition is said to be satisfied at $\hat{z}$ provided that $\mathcal{R}_{00}(\hat{z}) = \emptyset$.*

(c) *One may notice that every index set that is defined above is dependent on the particular choice of $\hat{z} \in \mathcal{F}$. Nevertheless, in the remaining part of the article, we shall not indicate such dependence explicitly when it is easily perceivable from the context.*

Let $\tilde{y} \in \mathcal{F}$ be arbitrary. The sets $\mathscr{B}^k$ (for every $k \in \mathcal{I}$) and $\mathscr{B}$ as defined below will be crucial to discuss Guignard constraint qualification and Pareto efficiency conditions for MFPPEC:

$$\mathscr{B}^k := \left\{ y \in \mathcal{F} : \Phi_j(y) \leq \Phi_j(\tilde{y}),\ \forall j \in \mathcal{I},\ j \neq k \right\},$$
$$\mathscr{B} := \left\{ y \in \mathcal{F} : \Phi_j(y) \leq \Phi_j(\tilde{y}),\ \forall j \in \mathcal{I} \right\}.$$

Remark 2. *We may observe the following:*

(a) *From the above definitions of the sets $\mathscr{B}^k$ and $\mathscr{B}$, it is clear that:*

$$\bigcap_{k \in \mathcal{I}} \mathscr{B}^k = \mathscr{B}.$$

(b) *In the case when $\mathcal{I} = \{1\}$, MFPPEC reduces to a single-objective fractional optimization problem with equilibrium constraints. In such a case, we have:*

$$\mathscr{B}^1 = \mathcal{F}.$$

In the next definition, we recall the notion of the contingent cone for any subset of $\mathscr{M}$ (see [31]).

Definition 5. *Let $\mathcal{H} \subseteq \mathscr{M}$ and $\hat{y}$ be some arbitrary element in the closure of the set $\mathcal{H}$. Then, the contingent cone (in other terms, Bouligand tangent cone) of the set $\mathcal{H}$ at the element $\hat{y}$ is symbolized by the notation $\mathscr{C}^{\mathrm{Tan}}(\mathcal{H}, \hat{y})$, and is given by:*

$$\mathscr{C}^{\mathrm{Tan}}(\mathcal{H}, \hat{y}) := \{\xi \in T_{\hat{y}}\mathscr{M} : \exists \sigma_n \downarrow 0, \exists\ \{\xi_n\}_{n=1}^{\infty} \subset T_{\hat{y}}\mathscr{M},\ \xi_n \to \xi,\ exp_{\hat{y}}(\sigma_n \xi_n) \in \mathcal{H}\ \forall n \in \mathbb{N}\}.$$

The following notion of linearizing cone is an extension of Definition 6 from Treanţă et al. [7] for MFPPEC in the setting of Hadamard manifolds.

Definition 6. *Let $\hat{y} \in \mathcal{F}$ be arbitrary. The linearizing cone to the set $\mathscr{B}$ at the element $\hat{y}$ is the set defined as follows:*

$$\begin{aligned}
\mathscr{C}^{\mathrm{Lin}}(\mathscr{B}, \hat{y}) := \Big\{ \overline{u} \in T_{\hat{y}}\mathscr{M} : & \langle \operatorname{grad} \Phi_j(\hat{y}), \overline{u} \rangle_{\hat{y}} \leq 0 \quad \forall j \in \mathcal{I}, \\
& \langle \operatorname{grad} \Psi_j(\hat{y}), \overline{u} \rangle_{\hat{y}} \leq 0, \quad \forall j \in \mathscr{A}^{\Psi}, \\
& \langle \operatorname{grad} \theta_j(\hat{y}), \overline{u} \rangle_{\hat{y}} = 0, \quad \forall j \in \mathcal{I}^{\theta}, \\
& \langle \operatorname{grad} \mathcal{C}_j(\hat{y}), \overline{u} \rangle_{\hat{y}} = 0, \quad \forall j \in \mathcal{R}_{0+}, \\
& \langle \operatorname{grad} \mathcal{D}_j(\hat{y}), \overline{u} \rangle_{\hat{y}} = 0, \quad \forall j \in \mathcal{R}_{+0}, \\
& \langle \operatorname{grad} \mathcal{C}_j(\hat{y}), \overline{u} \rangle_{\hat{y}} \geq 0, \quad \forall j \in \mathcal{R}_{00}, \\
& \langle \operatorname{grad} \mathcal{D}_j(\hat{y}), \overline{u} \rangle_{\hat{y}} \geq 0, \quad \forall j \in \mathcal{R}_{00} \Big\}.
\end{aligned}$$

Remark 3. *We may observe the following:*

(a) *If $\mathscr{M}$ is considered to be the n-dimensional Euclidean space, then Definition 6 is an extension of Definition 3.1 presented by Maeda [28] from the setting of Euclidean spaces to the setting of Hadamard manifolds. Furthermore, Definition 6 generalizes Definition 3.1 of [28] from* *nonlinear optimization problems to MFPPEC, the latter being a wider category of optimization problems.*

(b) *If $\mathscr{M} = \mathbb{R}^n$, then Definition 6 generalizes the notion of linearizing cone provided in [6] from smooth multiobjective MPEC to multiobjective fractional MPEC.*

To introduce GGCQ for our considered problem MFPPEC, we now provide the following definition, which is an extension of the notion of the modified linearizing cone from [6] in the context of our problem MFPPEC.

Definition 7. *Let $\hat{y} \in \mathcal{F}$. The modified linearizing cone to the set $\mathscr{B}$ at the element $\hat{y}$ is the set defined as follows:*

$$\begin{aligned}\mathscr{C}^{\mathrm{Lin}}_{\mathrm{MFPPEC}}(\mathscr{B},\hat{y}) := \Big\{\overline{u} \in T_{\hat{y}}\mathscr{M} :& \langle \operatorname{grad} \Phi_j(\hat{y}), \overline{u}\rangle_{\hat{y}} \leq 0 \quad \forall j \in \mathcal{I}, \\ & \langle \operatorname{grad} \Psi_j(\hat{y}), \overline{u}\rangle_{\hat{y}} \leq 0, \ \ \forall j \in \mathscr{A}^{\Psi}, \\ & \langle \operatorname{grad} \theta_j(\hat{y}), \overline{u}\rangle_{\hat{y}} = 0, \ \ \forall j \in \mathcal{I}^{\theta}, \\ & \langle \operatorname{grad} \mathcal{C}_j(\hat{y}), \overline{u}\rangle_{\hat{y}} = 0, \ \ \forall j \in \mathcal{R}_{0+}, \\ & \langle \operatorname{grad} \mathcal{D}_j(\hat{y}), \overline{u}\rangle_{\hat{y}} = 0, \ \ \forall j \in \mathcal{R}_{+0}, \\ & \langle \operatorname{grad} \mathcal{C}_j(\hat{y}), \overline{u}\rangle_{\hat{y}} \geq 0, \ \ \forall j \in \mathcal{R}_{00}, \\ & \langle \operatorname{grad} \mathcal{D}_j(\hat{y}), \overline{u}\rangle_{\hat{y}} \geq 0, \ \ \forall j \in \mathcal{R}_{00}, \\ \langle \operatorname{grad} \mathcal{C}_j(\hat{y}), \overline{u}\rangle_{\hat{y}} & \langle \operatorname{grad} \mathcal{D}_j(\hat{y}), \overline{u}\rangle_{\hat{y}} = 0, \ \ \forall j \in \mathcal{R}_{00}\Big\}.\end{aligned}$$

Remark 4. *We may observe the following:*

(a) *If $\mathscr{M}$ is considered to be the n-dimensional Euclidean space, then Definition 7 is a generalization of the similar notion presented in [6] for a wider category of optimization problems, that is, MFPPEC.*

(b) *It is significant to note that from Definitions 6 and 7, the following inclusion relation readily follows:*

$$\mathscr{C}^{\mathrm{Lin}}_{\mathrm{MFPPEC}}(\mathscr{B},\hat{y}) \subseteq \mathscr{C}^{\mathrm{Lin}}(\mathscr{B},\hat{y}).$$

Maeda [28] introduced the generalized Guignard constraint qualification for multiobjective optimization problems with inequality constraints in Euclidean space setting. Furthermore, it has been established in [28] that GGCQ is the weakest constraint qualification, as compared to other well-known constraint qualifications for nonlinear multiobjective programming problems (such as Abadie constraint qualification, linearly independent constraint qualification, Slater's constraint qualification, and Cottle constraint qualification). As a result, to derive KKT-type necessary criteria of Pareto efficiency for MFPPEC, we now extend the notion of GGCQ from [28] for our considered problem MFPPEC in the framework of Hadamard manifolds.

Definition 8. *Let $\overline{y} \in \mathcal{F}$ be any arbitrary feasible element. The generalized Guignard constraint qualification GGCQ is said to be satisfied at the point $\overline{y}$, provided that the following inclusion relation is satisfied:*

$$\mathscr{C}^{\mathrm{Lin}}_{\mathrm{MFPPEC}}(\mathscr{B},\overline{y}) \subseteq \bigcap_{t\in\mathcal{I}} \operatorname{cl}\operatorname{co}\mathscr{C}^{\mathrm{Tan}}(\mathscr{B}^t,\overline{y}).$$

Remark 5. *Definition 8 generalizes the notion of GGCQ of [28] from nonlinear programming problems in Euclidean space setting to MFPPEC in Hadamard manifold setting, which belongs to a more general category of mathematical programming problems.*

The following lemma is a variant of Theorem 4 from Treanţă et al. [7] and will be helpful to deduce KKT-type Pareto efficiency criteria for MFPPEC in the sequel.

Lemma 1. *Let $\hat{y} \in \mathcal{F}$ be any Pareto efficient solution of MFPPEC such that GGCQ holds at $\hat{y}$. Then, there always exist real numbers $\alpha_j > 0$ $(j \in \mathcal{I})$, σ_j^{Ψ} $(j \in \mathcal{I}^{\Psi})$, σ_j^{θ} $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}}$ $(j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}}$ $(j \in \mathcal{T})$, which satisfy the following:*

$$\sum_{j\in\mathcal{I}} \sigma_j^{\Phi} \operatorname{grad} \Phi_j(\hat{y}) + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi} \operatorname{grad} \Psi_j(\hat{y}) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta} \operatorname{grad} \theta_j(\hat{y}) \\ - \sum_{j\in\mathcal{T}} \left[\sigma_j^{\mathcal{C}} \operatorname{grad} \mathcal{C}_j(\hat{y}) + \sigma_j^{\mathcal{D}} \operatorname{grad} \mathcal{D}_j(\hat{y})\right] = 0,$$

and:

$$\begin{aligned} &\sigma_j^{\Psi} \geq 0, \quad \sigma_j^{\Psi} \Psi_j(\hat{y}) = 0, \quad \forall j \in \mathcal{I}^{\Psi}, \\ &\sigma_j^{\mathcal{C}} \textit{ free}, \forall j \in \mathcal{R}_{0+}, \quad \sigma_j^{\mathcal{C}} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^{\mathcal{C}} = 0, \forall j \in \mathcal{R}_{+0}, \\ &\sigma_j^{\mathcal{D}} \textit{ free}, \forall j \in \mathcal{R}_{+0}, \quad \sigma_j^{\mathcal{D}} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^{\mathcal{D}} = 0, \forall j \in \mathcal{R}_{0+}, \\ &\sigma_j^{\mathcal{C}} \mathcal{C}_j(\hat{y}) = 0, \quad \sigma_j^{\mathcal{D}} \mathcal{D}_j(\hat{y}) = 0, \quad \forall j \in \mathcal{T}. \end{aligned}$$

Remark 6. *We may observe the following:*

1. *If $\mathcal{B}_j(y) = 1$ for every $j \in \mathcal{I}$ and $y \in \mathcal{F}$, then Lemma 1 reduces to Theorem 4 established by Treanţă et al. [7].*
2. *Lemma 1 generalizes Theorem 3.2 of Maeda [28] from smooth multiobjective programming problems to MFPPEC and extends it from $\mathbb{R}^n$ to the framework of Hadamard manifolds.*

Now, we arrive at the main result of this section. In the next theorem, strong KKT-type necessary criteria of Pareto efficiency for MFPPEC is established by employing GGCQ.

Theorem 1. *Let $\bar{y} \in \mathcal{F}$ be any Pareto efficient solution of MFPPEC. Suppose that GGCQ holds at $\bar{y}$. Then, we can obtain some real numbers $\alpha_j \in \mathbb{R}$ $(\alpha_j > 0, j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$ and $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$, which satisfy the following:*

$$\sum_{j\in\mathcal{I}} \alpha_j \left[\operatorname{grad} \mathcal{A}_j(\bar{y}) - \chi_j \operatorname{grad} \mathcal{B}_j(\bar{y})\right] + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi} \operatorname{grad} \Psi_j(\bar{y}) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta} \operatorname{grad} \theta_j(\bar{y}) \\ - \sum_{j\in\mathcal{T}} \left[\sigma_j^{\mathcal{C}} \operatorname{grad} \mathcal{C}_j(\bar{y}) + \sigma_j^{\mathcal{D}} \operatorname{grad} \mathcal{D}_j(\bar{y})\right] = 0,$$

$$\mathcal{A}_j(\bar{y}) - \chi_j \mathcal{B}_j(\bar{y}) = 0, \quad \forall j \in \mathcal{I},$$

$$\begin{aligned} &\sigma_j^{\Psi} \geq 0, \quad \sigma_j^{\Psi} \Psi_j(\bar{y}) = 0, \quad \forall j \in \mathcal{I}^{\Psi}, \\ &\sigma_j^{\mathcal{C}} \textit{ free}, \forall j \in \mathcal{R}_{0+}, \quad \sigma_j^{\mathcal{C}} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^{\mathcal{C}} = 0, \quad \forall j \in \mathcal{R}_{+0}, \\ &\sigma_j^{\mathcal{D}} \textit{ free}, \forall j \in \mathcal{R}_{+0}, \quad \sigma_j^{\mathcal{D}} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^{\mathcal{D}} = 0, \forall j \in \mathcal{R}_{0+}. \end{aligned}$$

Proof. According to the provided hypotheses, $\bar{y} \in \mathcal{F}$ is any Pareto efficient solution of MFPPEC and GGCQ holds at $\bar{y}$. Let us now define:

$$\chi_j = \frac{\mathcal{A}_j(\bar{y})}{\mathcal{B}_j(\bar{y})}, \quad \forall j \in \mathcal{I}.$$

Consequently, it follows from Lemma 1 that there exists some $\beta_j \in \mathbb{R}$ ($\beta_j > 0, j \in \mathcal{I}$), satisfying the following:

$$\sum_{j\in\mathcal{I}} \frac{\beta_j}{\mathcal{B}_j(\bar{y})}\Big[\operatorname{grad}\mathcal{A}_j(\bar{y}) - \chi_j \operatorname{grad}\mathcal{B}_j(\bar{y})\Big] + \sum_{j\in\mathcal{I}^\Psi} \tilde{\sigma}_j^\Psi \operatorname{grad}\Psi_j(\bar{y}) + \sum_{j\in\mathcal{I}^\theta} \tilde{\sigma}_j^\theta \operatorname{grad}\theta_j(\bar{y}) - \sum_{j\in\mathcal{T}}\Big[\tilde{\sigma}_j^\mathcal{C} \operatorname{grad}\mathcal{C}_j(\bar{y}) + \tilde{\sigma}_j^\mathcal{D} \operatorname{grad}\mathcal{D}_j(\bar{y})\Big] = 0, \tag{1}$$

$$\begin{aligned}
&\tilde{\sigma}_j^\Psi \geq 0, \quad \tilde{\sigma}_j^\Psi \Psi_j(\bar{y}) = 0, \quad \forall j \in \mathcal{I}^\Psi,\\
&\tilde{\sigma}_j^\mathcal{C} \text{ free}, \forall j \in \mathcal{R}_{0+}, \quad \tilde{\sigma}_j^\mathcal{C} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \tilde{\sigma}_j^\mathcal{C} = 0, \quad \forall j \in \mathcal{R}_{+0},\\
&\tilde{\sigma}_j^\mathcal{D} \text{ free}, \forall j \in \mathcal{R}_{+0}, \quad \tilde{\sigma}_j^\mathcal{D} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \tilde{\sigma}_j^\mathcal{D} = 0, \forall j \in \mathcal{R}_{0+}.
\end{aligned}$$

Let us now define the following:

$$\begin{aligned}
\alpha_j &:= \frac{\frac{\beta_j}{\mathcal{B}_j(\bar{y})}}{\sum_{j\in\mathcal{I}} \frac{\beta_j}{\mathcal{B}_j(\bar{y})}}, && \forall j \in \mathcal{I},\\
\sigma_j^\Psi &:= \frac{\tilde{\sigma}_j^\Psi}{\sum_{j\in\mathcal{I}} \frac{\beta_j}{\mathcal{B}_j(\bar{y})}}, && \forall j \in \mathcal{I}^\Psi,\\
\sigma_j^\theta &:= \frac{\tilde{\sigma}_j^\theta}{\sum_{j\in\mathcal{I}} \frac{\beta_j}{\mathcal{B}_j(\bar{y})}}, && \forall j \in \mathcal{I}^\theta,\\
\sigma_j^\mathcal{C} &:= \frac{\tilde{\sigma}_j^\mathcal{C}}{\sum_{j\in\mathcal{I}} \frac{\beta_j}{\mathcal{B}_j(\bar{y})}}, && \forall j \in \mathcal{T},\\
\sigma_j^\mathcal{D} &:= \frac{\tilde{\sigma}_j^\mathcal{D}}{\sum_{j\in\mathcal{I}} \frac{\beta_j}{\mathcal{B}_j(\bar{y})}}, && \forall j \in \mathcal{T}.
\end{aligned}$$

Then, it follows that:

$$\sum_{j\in\mathcal{I}} \alpha_j\Big[\operatorname{grad}\mathcal{A}_j(\bar{y}) - \chi_j \operatorname{grad}\mathcal{B}_j(\bar{y})\Big] + \sum_{j\in\mathcal{I}^\Psi} \sigma_j^\Psi \operatorname{grad}\Psi_j(\bar{y}) + \sum_{j\in\mathcal{I}^\theta} \sigma_j^\theta \operatorname{grad}\theta_j(\bar{y}) - \sum_{j\in\mathcal{T}}\Big[\sigma_j^\mathcal{C} \operatorname{grad}\mathcal{C}_j(\bar{y}) + \sigma_j^\mathcal{D} \operatorname{grad}\mathcal{D}_j(\bar{y})\Big] = 0,$$

$$\begin{aligned}
&\sigma_j^\Psi \geq 0, \quad \sigma_j^\Psi \Psi_j(\bar{y}) = 0, \quad \forall j \in \mathcal{I}^\Psi,\\
&\sigma_j^\mathcal{C} \text{ free}, \forall j \in \mathcal{R}_{0+}, \quad \sigma_j^\mathcal{C} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^\mathcal{C} = 0, \quad \forall j \in \mathcal{R}_{+0},\\
&\sigma_j^\mathcal{D} \text{ free}, \forall j \in \mathcal{R}_{+0}, \quad \sigma_j^\mathcal{D} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^\mathcal{D} = 0, \forall j \in \mathcal{R}_{0+}.
\end{aligned}$$

Thus, the proof is complete. □

Remark 7. *We may observe the following:*

(a) *If $\mathscr{M} = \mathbb{R}^n$, then Theorem 1 is a generalization of Theorem 1 presented by Singh and Mishra [6] from multiobjective MPEC to MFPPEC, which belongs to a more general category of optimization problems.*

(b) *If $\mathscr{B}_j(y) = 1$ ($j \in \mathcal{I}$) for every $y \in \mathcal{F}$, then Theorem 1 reduces to Theorem 4 established by Treanţă et al. [7].*

(c) *Theorem 1 generalizes Theorem 3.2 of Maeda [28] from nonlinear optimization problems with inequality constraints to MFPPEC and further extends it from $\mathbb{R}^n$ to the setting of Hadamard manifolds.*

We now furnish the following numerical example of MFPPEC in the framework of a Hadamard manifold to demonstrate the importance of Theorem 1. In this example, we formulate an MFPPEC in Hadamard manifold setting and illustrate that GGCQ is satisfied at a Pareto efficient solution of the problem. Moreover, we illustrate that KKT-type necessary conditions of Pareto efficiency are satisfied at the Pareto efficient solution.

Example 1. *Let us consider the set $\mathscr{M} \subset \mathbb{R}^2$ as defined below:*

$$\mathscr{M} := \{z = (z_1, z_2) \in \mathbb{R}^2, z_1, z_2 > 0\}.$$

One can verify that the set $\mathscr{M}$ as defined above is a two-dimensional Hadamard manifold (see [15]). Let $y = (y_1, y_2) \in \mathscr{M}$ be arbitrary. The Riemannian metric associated with $\mathscr{M}$ is given by $\langle \hat{u}, \hat{w} \rangle_y = \langle \mathscr{G}(y)\hat{u}, \hat{w} \rangle$, $\forall \hat{u}, \hat{w} \in T_y\mathscr{M} = \mathbb{R}^2$. Furthermore, we have:

$$\mathscr{G}(z) = \begin{pmatrix} \frac{1}{z_1^2} & 0 \\ 0 & \frac{1}{z_2^2} \end{pmatrix}.$$

Moreover, the exponential function denoted by $\exp_y : T_y\mathscr{M} \to \mathscr{M}$ for any arbitrary choice of $\hat{w} \in T_y\mathscr{M}$ is defined as $\exp_y(\hat{w}) := (x_1 e^{\frac{\hat{w}_1}{y_1}}, x_2 e^{\frac{\hat{w}_2}{y_2}})$. Similarly, $\exp_y^{-1} : \mathscr{M} \to T_y\mathscr{M}$ is the inverse of the exponential function for any $y, z \in \mathscr{M}$ and is defined as $\exp_y^{-1}(z) = \left(y_1 \ln \frac{z_1}{y_1}, y_2 \ln \frac{z_2}{y_2} \right)$. Let us consider the following MFPPEC on the manifold $\mathscr{M}$:

$$(P) \quad \textit{Minimize} \left(\frac{\mathcal{A}_1(z)}{\mathcal{B}_1(z)}, \frac{\mathcal{A}_1(z)}{\mathcal{B}_1(z)} \right) := \left(\frac{z_1 - e}{z_1}, \frac{\log z_2}{2} \right),$$

subject to:

$$\begin{aligned} \Psi(z) &:= 1 - \ln z_1 - \ln z_2 \leq 0, \\ \mathcal{C}(z) &:= \ln z_1 - 1 \geq 0, \\ \mathcal{D}(z) &:= \ln z_2 - 1 \geq 0, \\ \mathcal{C}(z)\mathcal{D}(z) &:= (\ln z_1 - 1)(\ln z_2 - 1) = 0. \end{aligned}$$

Clearly, the functions $\mathcal{A}_j, \mathcal{B}_j : \mathscr{M} \to \mathbb{R}, (j = 1, 2)$, $\Psi : \mathscr{M} \to \mathbb{R}$, $\mathcal{C} : \mathscr{M} \to \mathbb{R}$, $\mathcal{D} : \mathscr{M} \to \mathbb{R}$ are smooth functions. The feasible set $\mathcal{F}$ for the problem (P) is given by:

$$\mathcal{F} := \{z \in \mathscr{M}, z_1 = e, z_2 \geq e, \text{ or}, z_1 \geq e, z_2 = e\}.$$

Let us choose the feasible solution $\hat{y} = (e, e)$. Consequently, we obtain the following:

$$\begin{aligned} &\operatorname{grad} \mathcal{A}_1(\hat{y}) = \left(e^2, 0\right)^T, \operatorname{grad} \mathcal{B}_1(\hat{y}) = \left(e^2, 0\right)^T, \\ &\operatorname{grad} \mathcal{A}_2(\hat{y}) = (0, e)^T, \operatorname{grad} \mathcal{B}_2(\hat{y}) = (0, 0)^T, \\ &\operatorname{grad} \Psi(\hat{y}) = (-e, -e)^T, \operatorname{grad} \mathcal{C}(\hat{y}) = (e, 0)^T, \operatorname{grad} \mathcal{D}(\hat{y}) = (0, e)^T. \end{aligned}$$

It can be verified that GGCQ holds at $\hat{y}$. Let us now pick some real numbers $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}$, $\sigma^\Psi = 0$, $\sigma^\mathcal{C} = \frac{e}{2}$, $\sigma^\mathcal{D} = \frac{1}{2}$. Then, we can verify that the following relation is satisfied:

$$\sum_{j=1}^{2} \alpha_j \left[\operatorname{grad} \mathcal{A}_j(\hat{y}) - \chi_j \operatorname{grad} \mathcal{B}_j(\hat{y}) \right] + \sigma^\Psi \operatorname{grad} \Psi(\hat{y}) - \left[\sigma^\mathcal{C} \operatorname{grad} \mathcal{C}(\hat{y}) + \sigma^\mathcal{D} \operatorname{grad} \mathcal{D}(\hat{y}) \right] = (0, 0)^T,$$

$$\mathcal{A}_j(\hat{y}) - \chi_j \mathcal{B}_j(\hat{y}) = 0, \quad \forall j \in \{1, 2\}.$$

Hence, every assumption and conclusion of Theorem 1 for the problem (P) is justified.

4. Sufficient Criteria of Pareto Efficiency for MFPPEC

In this section, we introduce the notion of an M-stationary point in the manifold setting for our considered problem (MFPPEC). We employ certain hypotheses of geodesic convexity to derive the sufficient conditions of Pareto efficiency for MFPPEC.

To begin with, we introduce the notion of an M-stationary element for our considered problem MFPPEC in the setting of Hadamard manifolds (see, for instance, [8]).

Definition 9. *Let $\overline{y} \in \mathcal{F}$ be an arbitrary feasible element of MFPPEC. The feasible element $\overline{y}$ is referred to as Mordukhovich stationary element (abbreviated as, M-stationary element), provided that some real numbers $\alpha_j \in \mathbb{R}$ ($\alpha_j > 0$, $j \in \mathcal{I}$), $\sigma_j^{\Psi} \in \mathbb{R}$ ($j \in \mathcal{I}^{\Psi}$), $\sigma_j^{\theta} \in \mathbb{R}$ ($j \in \mathcal{I}^{\theta}$), $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ ($j \in \mathcal{T}$) and $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ ($j \in \mathcal{T}$) exist, satisfying the following:*

$$\begin{aligned}\sum_{j\in\mathcal{I}} \alpha_j \Big[\operatorname{grad} \mathcal{A}_j(\overline{y}) - \chi_j \operatorname{grad} \mathcal{B}_j(\overline{y})\Big] + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi} \operatorname{grad} \Psi_j(\overline{y}) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta} \operatorname{grad} \theta_j(\overline{y}) \\ - \sum_{j\in\mathcal{T}} \Big[\sigma_j^{\mathcal{C}} \operatorname{grad} \mathcal{C}_j(\overline{y}) + \sigma_j^{\mathcal{D}} \operatorname{grad} \mathcal{D}_j(\overline{y})\Big] = 0, \\ \mathcal{A}_j(\overline{y}) - \chi_j \mathcal{B}_j(\overline{y}) = 0, \quad \forall j \in \mathcal{I},\end{aligned} \tag{2}$$

and:

$$\begin{aligned}&\sigma_j^{\Psi} \geq 0,\ \forall j \in \mathscr{A}^{\Psi}, \\ &\sigma_j^{\mathcal{C}} = 0,\ \forall j \in \mathcal{R}_{+0}, \\ &\sigma_j^{\mathcal{D}} = 0,\ \forall j \in \mathcal{R}_{0+}, \\ &\forall j \in \mathcal{R}_{00},\ \text{either},\ \sigma_j^{\mathcal{C}} > 0,\ \sigma_j^{\mathcal{D}} > 0\ \text{or},\ \sigma_j^{\mathcal{C}}\sigma_j^{\mathcal{D}} = 0.\end{aligned} \tag{3}$$

Now, we define a few index sets as given below that will be useful in the subsequent discussions:

$$\begin{aligned}\mathcal{R}_{00}^{++} &:= \Big\{j \in \mathcal{R}_{00} : \sigma_j^{\mathcal{C}} > 0,\ \sigma_j^{\mathcal{D}} > 0\Big\}, \\ \mathcal{R}_{00}^{0+} &:= \Big\{j \in \mathcal{R}_{00} : \sigma_j^{\mathcal{C}} = 0,\ \sigma_j^{\mathcal{D}} > 0\Big\}, \\ \mathcal{R}_{00}^{0-} &:= \Big\{j \in \mathcal{R}_{00} : \sigma_j^{\mathcal{C}} = 0,\ \sigma_j^{\mathcal{D}} < 0\Big\}, \\ \mathcal{R}_{00}^{+0} &:= \Big\{j \in \mathcal{R}_{00} : \sigma_j^{\mathcal{D}} = 0,\ \sigma_j^{\mathcal{C}} > 0\Big\}, \\ \mathcal{R}_{00}^{-0} &:= \Big\{j \in \mathcal{R}_{00} : \sigma_j^{\mathcal{D}} = 0,\ \sigma_j^{\mathcal{C}} < 0\Big\}, \\ \mathcal{R}_{0+}^{+} &:= \Big\{j \in \mathcal{R}_{0+} : \sigma_j^{\mathcal{C}} > 0\Big\}, \\ \mathcal{R}_{0+}^{-} &:= \Big\{j \in \mathcal{R}_{0+} : \sigma_j^{\mathcal{C}} < 0\Big\}, \\ \mathcal{R}_{+0}^{+} &:= \Big\{j \in \mathcal{R}_{+0} : \sigma_j^{\mathcal{D}} > 0\Big\}, \\ \mathcal{R}_{+0}^{-} &:= \Big\{j \in \mathcal{R}_{+0} : \sigma_j^{\mathcal{D}} < 0\Big\}, \\ \mathcal{I}_{+}^{\theta} &:= \Big\{j \in \mathcal{I}^{\theta} : \sigma_j^{\theta} > 0\Big\}, \\ \mathcal{I}_{-}^{\theta} &:= \Big\{j \in \mathcal{I}^{\theta} : \sigma_j^{\theta} < 0\Big\}.\end{aligned}$$

In the next theorem, we use the notions of M-stationary element and geodesic convexity to derive sufficient criteria of weak Pareto efficiency for MFPPEC.

Theorem 2. *Suppose $\bar{z} \in \mathcal{F}$ is such that $\bar{z}$ is the M-stationary element of MFPPEC. Let the functions $\mathscr{P}_j$ $(j \in \mathcal{I})$, as defined below, be geodesic convex at the element $\bar{z}$:*

$$\mathscr{P}_j(z) := \mathcal{A}_j(z) - \chi_j \mathcal{B}_j(z), \quad \textit{where,}$$
$$\chi_j := \frac{\mathcal{A}_j(\bar{z})}{\mathcal{B}_j(\bar{z})}, \quad \forall j \in \mathcal{I},\ z \in \mathscr{M}.$$

Furthermore, let us suppose that each of the functions Ψ_j $\left(j \in \mathscr{A}^{\Psi}\right)$, θ_j $\left(j \in \mathcal{I}_+^{\theta}\right)$, $-\theta_j$ $\left(j \in \mathcal{I}_-^{\theta}\right)$, $-\mathcal{C}_j$ $(j \in \mathcal{R}_{0+}^{+} \cup \mathcal{R}_{00}^{+0} \cup \mathcal{R}_{00}^{++})$, $-\mathcal{D}_j$ $(j \in \mathcal{R}_{+0}^{+} \cup \mathcal{R}_{00}^{0+} \cup \mathcal{R}_{00}^{++})$ are geodesic convex functions at $\bar{z}$. Moreover, we assume that:

$$\mathcal{R}_{0+}^{-} \cup \mathcal{R}_{+0}^{-} \cup \mathcal{R}_{00}^{-0} \cup \mathcal{R}_{00}^{0-} = \emptyset.$$

Then, $\bar{z}$ is a weak Pareto efficient solution for MFPPEC.

Proof. According to the provided hypotheses, we have that $\bar{z} \in \mathcal{F}$ is an M-stationary element of MFPPEC. In the light of Definition 9, we infer that the relations in (2) and (3) are satisfied for the element $\bar{z}$.

By reductio ad absurdum, we suppose that $\bar{z} \in \mathcal{F}$ is not a weak Pareto efficient solution of MFPPEC. Consequently, a feasible element $z \in \mathcal{F}$ exists, which satisfies the following:

$$\frac{\mathcal{A}_j(z)}{\mathcal{B}_j(z)} < \frac{\mathcal{A}_j(\bar{z})}{\mathcal{B}_j(\bar{z})}, \quad \forall j \in \mathcal{I}. \tag{4}$$

From (4), we obtain the following:

$$\mathcal{A}_j(z) - \chi_j \mathcal{B}_j(z) < \mathcal{A}_j(\bar{z}) - \chi_j \mathcal{B}_j(\bar{z}), \quad \forall j \in \mathcal{I}.$$

In view of the definition of the function $\mathscr{P}$ in the hypothesis of the theorem, we infer that:

$$\mathscr{P}_j(z) < \mathscr{P}_j(\bar{z}), \quad \forall j \in \mathcal{I}.$$

Since for every $j \in \mathcal{I}$, the functions $\mathscr{P}_j$ are geodesic convex at $\bar{y}$, we obtain the following:

$$\left\langle \operatorname{grad} \mathscr{P}_j(\bar{z}),\ \exp_{\bar{z}}^{-1}(z) \right\rangle_{\bar{z}} < 0, \quad \forall j \in \mathcal{I}. \tag{5}$$

From the feasibility conditions of the problem MFPPEC, the following inequalities can be obtained:

$$\begin{aligned}
\Psi_j(z) \le 0 = \Psi_j(\bar{z}), &\qquad \forall j \in \mathscr{A}^{\Psi}(\bar{z}),\\
\theta_j(z) \le 0 = \theta_j(\bar{z}), &\qquad \forall j \in \mathcal{I}_+^{\theta},\\
-\theta_j(z) \le 0 = -\theta_j(\bar{z}), &\qquad \forall j \in \mathcal{I}_-^{\theta},\\
-\mathcal{C}_j(z) \le 0 = -\mathcal{C}_j(\bar{z}), &\qquad \forall j \in \mathcal{R}_{0+} \cup \mathcal{R}_{00},\\
-\mathcal{D}_j(z) \le 0 = -\mathcal{D}_j(\bar{z}), &\qquad \forall j \in \mathcal{R}_{+0} \cup \mathcal{R}_{00},
\end{aligned}$$

Since for every $j \in \mathscr{A}^{\Psi}(\bar{z})$, the functions Ψ_j are geodesic convex at $\tilde{y}$, we obtain the following:

$$\left\langle \operatorname{grad} \Psi_j(\bar{z}),\ \exp_{\bar{z}}^{-1}(z) \right\rangle_{\bar{z}} \le 0, \quad \forall j \in \mathscr{A}^{\Psi}. \tag{6}$$

Similarly, in the light of the geodesic convexity hypothesis on the functions $\theta_j \ (j \in \mathcal{I}_+^\theta)$, $-\theta_j \ (j \in \mathcal{I}_-^\theta)$, $-\mathcal{C}_j \ (j \in \mathcal{R}_{0+}^+ \cup \mathcal{R}_{00}^{+0} \cup \mathcal{R}_{00}^{++})$, $-\mathcal{D}_j \ (j \in \mathcal{R}_{+0}^+ \cup \mathcal{R}_{00}^{0+} \cup \mathcal{R}_{00}^{++})$, we have the following:

$$\begin{aligned}
\left\langle \operatorname{grad} \theta_j(\bar{z}),\ \exp_{\bar{z}}^{-1}(z) \right\rangle_{\bar{z}} &\leq 0, \qquad \forall j \in \mathcal{I}_+^\theta, \\
\left\langle \operatorname{grad} \theta_j(\bar{z}),\ \exp_{\bar{z}}^{-1}(z) \right\rangle_{\bar{z}} &\geq 0, \qquad \forall j \in \mathcal{I}_-^\theta, \\
\left\langle \operatorname{grad} \mathcal{C}_j(\bar{z}),\ \exp_{\bar{z}}^{-1}(z) \right\rangle_{\bar{z}} &\geq 0, \qquad \forall j \in \mathcal{R}_{0+}^+ \cup \mathcal{R}_{00}^{+0} \cup \mathcal{R}_{00}^{++}, \\
\left\langle \operatorname{grad} \mathcal{D}_j(\bar{z}),\ \exp_{\bar{z}}^{-1}(z) \right\rangle_{\bar{z}} &\geq 0, \qquad \forall j \in \mathcal{R}_{+0}^+ \cup \mathcal{R}_{00}^{0+} \cup \mathcal{R}_{00}^{++}.
\end{aligned}$$

Moreover, we have $\mathcal{R}_{0+}^- \cup \mathcal{R}_{+0}^- \cup \mathcal{R}_{00}^{-0} \cup \mathcal{R}_{00}^{0-} = \emptyset$. As a result, we arrive at the following inequalities:

$$\begin{aligned}
\left\langle \sum_{j \in \mathscr{A}^\Psi} \sigma_j^\Psi \operatorname{grad} \Psi_j(\bar{z}),\ \exp_{\bar{z}}^{-1}(z) \right\rangle_{\bar{z}} &\leq 0, \\
\left\langle \sum_{j \in \mathcal{I}^\theta} \sigma_j^\theta \operatorname{grad} \theta_j(\bar{z}),\ \exp_{\bar{z}}^{-1}(z) \right\rangle_{\bar{z}} &\leq 0, \\
\left\langle \sum_{j \in \mathcal{T}} \sigma_j^{\mathcal{C}} \operatorname{grad} \mathcal{C}_j(\bar{z}),\ \exp_{\bar{z}}^{-1}(z) \right\rangle_{\bar{z}} &\geq 0, \\
\left\langle \sum_{j \in \mathcal{T}} \sigma_j^{\mathcal{D}} \operatorname{grad} \mathcal{D}_j(\bar{z}),\ \exp_{\bar{z}}^{-1}(z) \right\rangle_{\bar{z}} &\geq 0.
\end{aligned} \tag{7}$$

We have $\alpha_j > 0$ for every $j \in \mathcal{I}$. Then adding each of the inequalities in (5) and (7), the following inequality can be obtained:

$$\begin{aligned}
\Bigg\langle \sum_{j \in \mathcal{I}} \alpha_j \Big[\operatorname{grad} \mathcal{A}_j(\bar{z}) - \chi_j \operatorname{grad} \mathcal{B}_j(\bar{z}) \Big] + \sum_{j \in \mathscr{A}^\Psi} \sigma_j^\Psi \operatorname{grad} \Psi_j(\bar{z}) + \sum_{j \in \mathcal{I}^\theta} \sigma_j^\theta \operatorname{grad} \theta_j(\bar{z}) \\
- \sum_{j \in \mathcal{T}} \Big[\sigma_j^{\mathcal{C}} \operatorname{grad} \mathcal{C}_j(\bar{z}) + \sigma_j^{\mathcal{D}} \operatorname{grad} \mathcal{D}_j(\bar{z}) \Big],\ \exp_{\bar{z}}^{-1}(z) \Bigg\rangle_{\bar{z}} < 0,
\end{aligned}$$

which contradicts the fact that $\bar{z}$ is the M-stationary element of MFPPEC. As a result, we infer that $\bar{z}$ is a weak Pareto efficient solution for MFPPEC. Hence, the proof is complete. □

Remark 8. *We may observe the following:*

(a) *If $\mathscr{B}_j(y) = 1$ $(j \in \mathcal{I})$ for every $y \in \mathcal{F}$, then the sufficient optimality condition derived in Theorem 2 reduces to the corresponding result (Theorem 5) presented by Treanţă et al. [7].*

(b) *Theorem 2 is an extension of Theorem 2.3 provided by [8] from the Euclidean space framework to the framework of Hadamard manifolds, and furthermore, generalizes it in the setting of a wider category of optimization problems, which is MFPPEC.*

In Theorem 3, another sufficient condition of the Pareto efficiency for MFPPEC is presented in the manifold setting using the notion of the M-stationary element and under the hypotheses of geodesic convexity. The proof of the theorem follows in similar lines as the proof of Theorem 2.

Theorem 3. *Let $\overline{z} \in \mathcal{F}$ be any arbitrary feasible element of MFPPEC, such that $\overline{z}$ is an M-stationary element of MFPPEC. Let us suppose that the functions $\mathscr{P}_j$ $(j \in \mathcal{I})$, as defined below, are strictly geodesic convex at the element $\overline{z}$:*

$$\mathscr{P}_j(z) := \mathcal{A}_j(z) - \chi_j \mathcal{B}_j(z), \quad \text{where,}$$
$$\chi_j := \frac{\mathcal{A}_j(\overline{z})}{\mathcal{B}_j(\overline{z})}, \quad \forall j \in \mathcal{I},\, z \in \mathscr{M}.$$

Furthermore, let us suppose that each of the functions Ψ_j $\left(j \in \mathscr{A}^{\Psi}\right)$, θ_j $\left(j \in \mathcal{I}_+^{\theta}\right)$, $-\theta_j$ $\left(j \in \mathcal{I}_-^{\theta}\right)$, $-\mathcal{C}_j$ $(j \in \mathcal{R}_{0+}^{+} \cup \mathcal{R}_{00}^{+0} \cup \mathcal{R}_{00}^{++})$, $-\mathcal{D}_j$ $(j \in \mathcal{R}_{+0}^{+} \cup \mathcal{R}_{00}^{0+} \cup \mathcal{R}_{00}^{++})$ are geodesic convex functions at $\overline{z}$. Moreover, we assume that:

$$\mathcal{R}_{0+}^{-} \cup \mathcal{R}_{+0}^{-} \cup \mathcal{R}_{00}^{-0} \cup \mathcal{R}_{00}^{0-} = \emptyset.$$

Then, $\overline{z}$ is a Pareto efficient solution for MFPPEC.

Now, we furnish a nontrivial numerical example to illustrate and validate the consequences of Theorem 2. In this example, we justify that the geodesic convexity assumptions of Theorem 2 are sufficient criteria for Pareto efficiency for MFPPEC.

Example 2. *Let us consider the MFPPEC (Problem (P)) defined in Example 1 on the manifold $\mathscr{M}$. We use the symbol $\mathcal{F}$ to signify the set containing every feasible solution of the problem (P). That is, we have:*

$$\mathcal{F} = \{z \in \mathscr{M}, z_1 = e, z_2 \geq e, \text{ or, } z_1 \geq e, z_2 = e\}.$$

Choose the feasible solution $\hat{y} = (e,e)$. Let us now pick some real numbers $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}$, $\sigma^{\Psi} = 0$, $\sigma^{\mathcal{C}} = \frac{e}{2}$, $\sigma^{\mathcal{D}} = \frac{1}{2}$. Then, we can verify that the following relation is satisfied:

$$\sum_{j=1}^{2} \alpha_j \left[\operatorname{grad} \mathcal{A}_j(\hat{y}) - \chi_j \operatorname{grad} \mathcal{B}_j(\hat{y}) \right] + \sigma^{\Psi} \operatorname{grad} \Psi(\hat{y})$$
$$- \left[\sigma^{\mathcal{C}} \operatorname{grad} \mathcal{C}(\hat{y}) + \sigma^{\mathcal{D}} \operatorname{grad} \mathcal{D}(\hat{y}) \right] = (0,0),$$
$$\mathcal{A}_j(\hat{y}) - \chi_j \mathcal{B}_j(\hat{y}) = 0, \quad \forall j \in \{1,2\}.$$

It can, thus, be verified that the point $\hat{y}$ is an M-stationary element for (P). Furthermore, one can verify that each of the geodesic convexity assumptions stated in the sufficiency optimality criteria (Theorem 2) are satisfied. Therefore, $\hat{y}$ is a weak Pareto efficient solution of (P).

5. Mond–Weir Type Dual Model for (MFFPEC)

Let $w \in \mathscr{M}$ be an arbitrary element of the Hadamard manifold $\mathscr{M}$. Furthermore, let $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$. Then, related to the primal MFPPEC, the corresponding Mond–Weir type dual model (abbreviated as DP-MW) is formulated as given below:

$$\text{(DP-MW)} \quad \text{Maximize } \mathscr{L}(w) := \left(\frac{\mathcal{A}_1(w)}{\mathcal{B}_1(w)}, \frac{\mathcal{A}_2(w)}{\mathcal{B}_2(w)}, \ldots, \frac{\mathcal{A}_l(w)}{\mathcal{B}_l(w)} \right), \tag{8}$$

subject to:

$$\sum_{j \in \mathcal{I}} \alpha_j \left[\operatorname{grad} \mathcal{A}_j(w) - \chi_j \operatorname{grad} \mathcal{B}_j(w) \right] + \sum_{j \in \mathcal{I}^{\Psi}} \sigma_j^{\Psi} \operatorname{grad} \Psi_j(w) +$$
$$\sum_{j \in \mathcal{I}^{\theta}} \sigma_j^{\theta} \operatorname{grad} \theta_j(w) - \sum_{j \in \mathcal{T}} \left[\sigma_j^{\mathcal{C}} \operatorname{grad} \mathcal{C}_j(w) + \sigma_j^{\mathcal{D}} \operatorname{grad} \mathcal{D}_j(w) \right] = 0,$$

$$\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\Psi_j(w)\geq 0,\ \sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\theta_j(w)\geq 0,\ \sum_{j\in\mathcal{T}}\sigma_j^{\mathcal{C}}\mathcal{C}_j(w)\leq 0,\ \sum_{j\in\mathcal{T}}\sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\leq 0, \tag{9}$$

where:

$$\begin{aligned}
&\mathcal{A}_j(w)-\chi_j\mathcal{B}_j(w)=0, \quad \forall j\in\mathcal{I},\\
&\sigma_j^{\Psi}\geq 0,\ \forall j\in\mathscr{A}^{\Psi},\\
&\sigma_j^{\mathcal{C}}=0,\ \forall j\in\mathcal{R}_{+0},\\
&\sigma_j^{\mathcal{D}}=0,\ \forall j\in\mathcal{R}_{0+},\\
&\forall j\in\mathcal{R}_{00},\ \text{either, } \sigma_j^{\mathcal{C}}>0,\ \sigma_j^{\mathcal{D}}>0 \text{ or, } \sigma_j^{\mathcal{C}}\sigma_j^{\mathcal{D}}=0.
\end{aligned} \tag{10}$$

The set containing every feasible element of (DP-MW) is signified by the symbol $\mathcal{F}_M$.

In the next theorem, the weak duality result relating to our considered primal problem (MPPEC) and (DP-MW) is derived.

Theorem 4. *Let $\overline{z}\in\mathcal{F}$ and $(w,\alpha,\sigma)\in\mathcal{F}_M$ be arbitrary. Suppose that the functions $\mathscr{P}_j$ $(j\in\mathcal{I})$, as defined below, are geodesic convex at the element $\overline{z}$:*

$$\mathscr{P}_j(z):=\mathcal{A}_j(z)-\chi_j\mathcal{B}_j(z), \quad \textit{where,}$$
$$\chi_j:=\frac{\mathcal{A}_j(w)}{\mathcal{B}_j(w)}, \quad \forall j\in\mathcal{I},\ z\in\mathscr{M},$$

Furthermore, let us suppose that each of the functions Ψ_j $\left(j\in\mathscr{A}^{\Psi}\right)$, θ_j $\left(j\in\mathcal{I}_+^{\theta}\right)$, $-\theta_j$ $\left(j\in\mathcal{I}_-^{\theta}\right)$, $-\mathcal{C}_j$ $(j\in\mathcal{R}_{0+}^{+}\cup\mathcal{R}_{00}^{+0}\cup\mathcal{R}_{00}^{++})$, $-\mathcal{D}_j$ $(j\in\mathcal{R}_{+0}^{+}\cup\mathcal{R}_{00}^{0+}\cup\mathcal{R}_{00}^{++})$ are geodesic convex functions at $\overline{z}$: Moreover, assuming that:

$$\mathcal{R}_{0+}^{-}\cup\mathcal{R}_{+0}^{-}\cup\mathcal{R}_{00}^{-0}\cup\mathcal{R}_{00}^{0-}=\emptyset,$$

we have:

$$\mathscr{L}(\overline{z})\nprec\mathscr{L}(w).$$

Proof. Given that $\overline{z}\in\mathcal{F}$ and $(w,\alpha,\sigma)\in\mathcal{F}_M$ are arbitrary feasible elements of MFPPEC and (DP-MW), respectively. By reductio ad absurdum, we suppose that $\mathscr{L}(\overline{z})\prec\mathscr{L}(w)$. Consequently, the following can be obtained:

$$\frac{\mathcal{A}_j(\overline{z})}{\mathcal{B}_j(\overline{z})}<\frac{\mathcal{A}_j(w)}{\mathcal{B}_j(w)}, \quad \forall j\in\mathcal{I}. \tag{11}$$

From (11), we obtain the following:

$$\mathcal{A}_j(\overline{z})-\chi_j\mathcal{B}_j(\overline{z})<\mathcal{A}_j(w)-\chi_j\mathcal{B}_j(w), \quad \forall j\in\mathcal{I}.$$

In the view of the definition of the function $\mathscr{P}$ in the hypothesis of the theorem, we infer that:

$$\mathscr{P}_j(\overline{z})<\mathscr{P}_j(w), \quad \forall j\in\mathcal{I}.$$

Since for every $j\in\mathcal{I}$, the functions $\mathscr{P}_j$ are strictly geodesic convex at $\overline{y}$, we obtain the following:

$$\left\langle \operatorname{grad}\mathscr{P}_j(w),\ \exp_w^{-1}(\overline{z})\right\rangle_w<0, \quad \forall j\in\mathcal{I}. \tag{12}$$

From the feasibility conditions of the problem MFPPEC, the following inequalities can be obtained:

$$\begin{aligned} \Psi_j(\bar{z}) \leq 0 = \Psi_j(w), &\qquad \forall j \in \mathscr{A}^{\Psi}(w), \\ \theta_j(\bar{z}) \leq 0 = \theta_j(w), &\qquad \forall j \in \mathcal{I}_+^{\theta}, \\ -\theta_j(\bar{z}) \leq 0 = -\theta_j(w), &\qquad \forall j \in \mathcal{I}_-^{\theta}, \\ -\mathcal{C}_j(\bar{z}) \leq 0 = -\mathcal{C}_j(w), &\qquad \forall j \in \mathcal{R}_{0+} \cup \mathcal{R}_{00}, \\ -\mathcal{D}_j(\bar{z}) \leq 0 = -\mathcal{D}_j(w), &\qquad \forall j \in \mathcal{R}_{+0} \cup \mathcal{R}_{00}, \end{aligned}$$

Since for every $j \in \mathscr{A}^{\Psi}(w)$, the functions Ψ_j are geodesic convex at the element w, we obtain the following:

$$\left\langle \operatorname{grad} \Psi_j(w),\ \exp_w^{-1}(\bar{z}) \right\rangle_w \leq 0, \quad \forall j \in \mathscr{A}^{\Psi}. \tag{13}$$

Similarly, in the light of the geodesic convexity hypothesis on the functions $\theta_j \ \left(j \in \mathcal{I}_+^{\theta}\right)$, $-\theta_j \ \left(j \in \mathcal{I}_-^{\theta}\right)$, $-\mathcal{C}_j \ (j \in \mathcal{R}_{0+}^{+} \cup \mathcal{R}_{00}^{+0} \cup \mathcal{R}_{00}^{++})$, $-\mathcal{D}_j \ (j \in \mathcal{R}_{+0}^{+} \cup \mathcal{R}_{00}^{0+} \cup \mathcal{R}_{00}^{++})$, we have the following:

$$\begin{aligned} \left\langle \operatorname{grad} \theta_j(w),\ \exp_w^{-1}(\bar{z}) \right\rangle_w \leq 0, &\qquad \forall j \in \mathcal{I}_+^{\theta}, \\ \left\langle \operatorname{grad} \theta_j(w),\ \exp_w^{-1}(\bar{z}) \right\rangle_w \geq 0, &\qquad \forall j \in \mathcal{I}_-^{\theta}, \\ \left\langle \operatorname{grad} \mathcal{C}_j(w),\ \exp_w^{-1}(\bar{z}) \right\rangle_w \geq 0, &\qquad \forall j \in \mathcal{R}_{0+}^{+} \cup \mathcal{R}_{00}^{+0} \cup \mathcal{R}_{00}^{++}, \\ \left\langle \operatorname{grad} \mathcal{D}_j(w),\ \exp_w^{-1}(\bar{z}) \right\rangle_w \geq 0, &\qquad \forall j \in \mathcal{R}_{+0}^{+} \cup \mathcal{R}_{00}^{0+} \cup \mathcal{R}_{00}^{++}. \end{aligned}$$

Moreover, we have $\mathcal{R}_{0+}^{-} \cup \mathcal{R}_{+0}^{-} \cup \mathcal{R}_{00}^{-0} \cup \mathcal{R}_{00}^{0-} = \emptyset$. As a result, we arrive at the following inequalities:

$$\begin{aligned} \left\langle \sum_{j \in \mathscr{A}^{\Psi}} \sigma_j^{\Psi} \operatorname{grad} \Psi_j(w),\ \exp_w^{-1}(\bar{z}) \right\rangle_w &\leq 0, \\ \left\langle \sum_{j \in \mathcal{I}^{\theta}} \sigma_j^{\theta} \operatorname{grad} \theta_j(w),\ \exp_w^{-1}(\bar{z}) \right\rangle_w &\leq 0, \\ \left\langle \sum_{j \in \mathcal{T}} \sigma_j^{\mathcal{C}} \operatorname{grad} \mathcal{C}_j(w),\ \exp_w^{-1}(\bar{z}) \right\rangle_w &\geq 0, \\ \left\langle \sum_{j \in \mathcal{T}} \sigma_j^{\mathcal{D}} \operatorname{grad} \mathcal{D}_j(w),\ \exp_w^{-1}(\bar{z}) \right\rangle_w &\geq 0. \end{aligned} \tag{14}$$

We have $\alpha_j > 0$ for every $j \in \mathcal{I}$. Then, adding each of the inequalities in (12) and (14), the following inequality can be obtained:

$$\begin{aligned} \Bigg\langle \sum_{j \in \mathcal{I}} \alpha_j \Big[\operatorname{grad} \mathcal{A}_j(w) - \chi_j \operatorname{grad} \mathcal{B}_j(w) \Big] + \sum_{j \in \mathscr{A}^{\Psi}} \sigma_j^{\Psi} \operatorname{grad} \Psi_j(w) + \sum_{j \in \mathcal{I}^{\theta}} \sigma_j^{\theta} \operatorname{grad} \theta_j(w) \\ - \sum_{j \in \mathcal{T}} \Big[\sigma_j^{\mathcal{C}} \operatorname{grad} \mathcal{C}_j(w) + \sigma_j^{\mathcal{D}} \operatorname{grad} \mathcal{D}_j(w) \Big],\ \exp_w^{-1}(\bar{z}) \Bigg\rangle_w < 0, \end{aligned}$$

which contradicts the fact that w is a feasible element of (DP-MW). Hence, the proof is complete. □

Next, we derive another weak duality relation referring to our primal problem MF-PPEC and (DP-MW). The proof may be formulated similarly to the proof of Theorem 4.

Theorem 5. *Let $\bar{z} \in \mathcal{F}$ and $(w, \alpha, \sigma) \in \mathcal{F}_M$ be arbitrary feasible elements of MFPPEC and (DP-MW), respectively. Let us suppose that the functions $\mathscr{P}_j$ $(j \in \mathcal{I})$, as defined below, are strictly geodesic convex at the element w:*

$$\mathscr{P}_j(z) := \mathcal{A}_j(z) - \chi_j \mathcal{B}_j(z), \quad \text{where,}$$
$$\chi_j := \frac{\mathcal{A}_j(w)}{\mathcal{B}_j(w)}, \quad \forall j \in \mathcal{I},\ z \in \mathscr{M}.$$

Furthermore, let us suppose that each of the functions Ψ_j $\left(j \in \mathscr{A}^{\Psi}\right)$, θ_j $\left(j \in \mathcal{I}^{\theta}_{+}\right)$, $-\theta_j$ $\left(j \in \mathcal{I}^{\theta}_{-}\right)$, $-\mathcal{C}_j$ $(j \in \mathcal{R}^{+}_{0+} \cup \mathcal{R}^{+0}_{00} \cup \mathcal{R}^{++}_{00})$, $-\mathcal{D}_j$ $(j \in \mathcal{R}^{+}_{+0} \cup \mathcal{R}^{0+}_{00} \cup \mathcal{R}^{++}_{00})$ are geodesic convex functions at w. Moreover, we assume that:

$$\mathcal{R}^{-}_{0+} \cup \mathcal{R}^{-}_{+0} \cup \mathcal{R}^{-0}_{00} \cup \mathcal{R}^{0-}_{00} = \emptyset.$$

Then:

$$\mathscr{L}(\bar{z}) \npreceq \mathscr{L}(w).$$

In the next theorem, the strong duality relation referring to our considered primal problem (MPPEC) and the Mond–Weir dual problem (DP-MW) is deduced.

Theorem 6. *Let $\bar{z} \in \mathcal{F}$ be any arbitrary Pareto efficient solution of MFPPEC at which GGCQ is satisfied. Then, there exist some $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma^{\Psi}_j \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma^{\theta}_j \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma^{\mathcal{C}}_j \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma^{\mathcal{D}}_j \in \mathbb{R}$ $(j \in \mathcal{T})$ such that $(\bar{z}, \alpha, \sigma) \in \mathcal{F}_M$. Moreover, we have:*

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})} = \mathscr{L}(\bar{z}, \alpha, \sigma).$$

Supposing that each of the hypotheses stated in Theorem 4 (respectively, Theorem 5) is satisfied, then $(\bar{z}, \alpha, \sigma)$ is a weak Pareto efficient (respectively, Pareto efficient) solution of (DP-MW).

Proof. According to the provided hypotheses, we have that $\bar{z} \in \mathcal{F}$ is any arbitrary weak Pareto efficient solution of MFPPEC at which GGCQ holds.

In the light of Theorem 1, we obtain some real multipliers: $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma^{\Psi}_j \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma^{\theta}_j \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma^{\mathcal{C}}_j \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma^{\mathcal{D}}_j \in \mathbb{R}$ $(j \in \mathcal{T})$, such that:

$$\begin{aligned} \sum_{j \in \mathcal{I}} \alpha_j \Big[\operatorname{grad} \mathcal{A}_j(\bar{z}) - \chi_j \operatorname{grad} \mathcal{B}_j(\bar{z}) \Big] + \sum_{j \in \mathcal{I}^{\Psi}} \sigma^{\Psi}_j \operatorname{grad} \Psi_j(\bar{z}) + \sum_{j \in \mathcal{I}^{\theta}} \sigma^{\theta}_j \operatorname{grad} \theta_j(\bar{z}) \\ - \sum_{j \in \mathcal{T}} \Big[\sigma^{\mathcal{C}}_j \operatorname{grad} \mathcal{C}_j(\bar{y}) + \sigma^{\mathcal{D}}_j \operatorname{grad} \mathcal{D}_j(\bar{z}) \Big] = 0, \\ \mathcal{A}_j(\bar{z}) - \chi_j \mathcal{B}_j(\bar{z}) = 0, \quad \forall j \in \mathcal{I}, \end{aligned} \tag{15}$$

$$\begin{aligned} &\sigma^{\Psi}_j \geq 0, \quad \sigma^{\Psi}_j \Psi_j(\bar{y}) = 0, \quad \forall j \in \mathcal{I}^{\Psi}, \\ &\sigma^{\mathcal{C}}_j \text{ free}, \forall j \in \mathcal{R}_{0+}, \quad \sigma^{\mathcal{C}}_j \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma^{\mathcal{C}}_j = 0, \ \forall j \in \mathcal{R}_{+0}, \\ &\sigma^{\mathcal{D}}_j \text{ free}, \forall j \in \mathcal{R}_{+0}, \quad \sigma^{\mathcal{D}}_j \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma^{\mathcal{D}}_j = 0, \ \forall j \in \mathcal{R}_{0+}. \end{aligned} \tag{16}$$

Consequently, it follows that:

$$(\bar{z}, \alpha, \sigma) \in \mathcal{F}_M,$$

and:

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})} = \mathscr{L}(\bar{z}, \alpha, \sigma).$$

By reductio ad absurdum, we consider that $(\overline{z}, \alpha, \sigma)$ is not some weak Pareto efficient solution of (DP-MW). As a result, one can find some $(\overline{u}, \overline{\alpha}, \overline{\sigma}) \in \mathcal{F}_M$, which satisfies the following:

$$\mathscr{L}(\overline{z}) \prec \mathscr{L}(\overline{u}),$$

which contradicts the Theorem 4. Hence, $(\overline{z}, \alpha, \sigma)$ is a weak Pareto efficient solution of (DP-MW). Similarly, we can prove that if each of the hypotheses stated in Theorem 5 is satisfied, then $(\overline{z}, \alpha, \sigma)$ Pareto efficient solution of (DP-MW). □

In the next theorem, we present a strict converse duality relation referring to our considered primal problem MFPPEC and (DP-MW).

Theorem 7. *Let us suppose that $\overline{y} \in \mathcal{F}$ is any Pareto efficient solution of MFPPEC at which GGCQ is satisfied. Let $(\overline{w}, \overline{\alpha}, \overline{\sigma})$ be a Pareto efficient solution of (DP-MW). Let us suppose that each of the hypotheses stated in Theorem 5 holds. Then, $\overline{y} = \overline{w}$.*

Proof. According to the provided hypotheses, we have that $\overline{y} \in \mathcal{F}$ is a Pareto efficient solution of MFPPEC at which GGCQ holds.

By reductio ad absurdum, we suppose that $\overline{y} \neq \overline{w}$. As a result, in light of Theorem 6, we obtain $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$, such that $(\overline{y}, \alpha, \sigma) \in \mathcal{F}_M$. Moreover, we have:

$$\frac{\mathcal{A}(\overline{y})}{\mathcal{B}(\overline{y})} = \mathscr{L}(\overline{y}, \alpha, \sigma).$$

On the other hand, in the view of the conclusions of the strong duality theorem (Theorem 6), one can conclude that $(\overline{y}, \alpha, \sigma)$ is a Pareto efficient solution for MFPPEC. Since $\overline{y} \in \mathcal{F}$ and $(\overline{w}, \overline{\alpha}, \overline{\sigma}) \in \mathcal{F}_{MW}$, then from Theorem 6, we obtain:

$$\mathscr{L}(\overline{y}) \not\preceq \mathscr{L}(\overline{w}),$$

which is a contradiction. Hence, the proof is complete. □

Remark 9. *We observe the following:*

1. *If $\mathscr{M} = \mathbb{R}^n$, then the weak and strong duality theorems (Theorems 4 and 6) generalize the corresponding theorems (Theorem 6 and Theorem 7) deduced in [6] from multiobjective MPEC to MFPPEC.*
2. *The weak, strong, and strict converse duality relations (Theorems 4, 6, and 7) extend Theorem 4.1, Theorem 4.2, and Theorem 4.3, respectively, deduced in [23] for a wider category of optimization problems, that is, MFPPEC.*

In the following example, we demonstrate the formulation of Mond–Weir dual problem corresponding to a primal MFPPEC problem. Furthermore, we illustrate the results derived for Mond–Weir duality for MFPPEC in the framework of a Hadamard manifold.

Example 3. *Let us consider the MFPPEC (Problem (P)) defined in Example 1 on the manifold $\mathscr{M}$. We use the symbol $\mathcal{F}$ to signify the set containing every feasible solution of the problem (P). That is, we have:*

$$\mathcal{F} = \{z \in \mathscr{M}, z_1 = e, z_2 \geq e, \text{ or, } z_1 \geq e, z_2 = e\}.$$

Let $w \in \mathscr{M}$ be an arbitrary element of the Hadamard manifold $\mathscr{M}$. Furthermore, let $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \{1,2\})$, $\sigma^{\Psi} \in \mathbb{R}$, $\sigma^{\mathcal{C}} \in \mathbb{R}$, $\sigma^{\mathcal{D}} \in \mathbb{R}$. Then, related to the primal (P), the corresponding Mond–Weir type dual model (abbreviated as DP-MW) is formulated as given below:

$$\text{(DP-MW)} \quad \text{Maximize } \mathscr{L}(w) = \left(\frac{\mathcal{A}_1(w)}{\mathcal{B}_1(w)}, \frac{\mathcal{A}_2(w)}{\mathcal{B}_2(w)} \right) := \left(\frac{|w_1 - e|}{e}, \frac{\log w_2}{2} \right),$$

subject to:

$$\sum_{j=1}^{2}\alpha_j\Big[\operatorname{grad}\mathcal{A}_j(w)-\chi_j\operatorname{grad}\mathcal{B}_j(w)\Big]+\sigma^{\Psi}\operatorname{grad}\Psi(w)+$$
$$-\Big[\sigma^{\mathcal{C}}\operatorname{grad}\mathcal{C}(w)+\sigma^{\mathcal{D}}\operatorname{grad}\mathcal{D}(w)\Big]=0,$$
$$\sigma^{\Psi}\Psi(w)\geq 0,\ \sigma^{\mathcal{C}}\mathcal{C}(w)\leq 0,\ \sigma^{\mathcal{D}}\mathcal{D}(w)\leq 0,$$

where:

$$\mathcal{A}_j(w)-\chi_j\mathcal{B}_j(w)=0,\quad \forall j\in\{1,2\}. \tag{17}$$

Let us choose the feasible solution $\hat{y}=(e,e)$. One can easily verify that the feasible solution $\hat{y}$ is, indeed, a Pareto efficient solution for (P). Furthermore, GGCQ holds at $\hat{y}$. Let us now pick some real numbers $\alpha_1=\frac{1}{2},\alpha_2=\frac{1}{2}$, $\sigma^{\Psi}=0$, $\sigma^{\mathcal{C}}=\frac{e}{2}$, $\sigma^{\mathcal{D}}=\frac{1}{2}$. Then, we can verify that the following relation is satisfied:

$$\sum_{j=1}^{2}\alpha_j\Big[\operatorname{grad}\mathcal{A}_j(\hat{y})-\chi_j\operatorname{grad}\mathcal{B}_j(\hat{y})\Big]+\sigma^{\Psi}\operatorname{grad}\Psi(\hat{y})-\Big[\sigma^{\mathcal{C}}\operatorname{grad}\mathcal{C}(\hat{y})+\sigma^{\mathcal{D}}\operatorname{grad}\mathcal{D}(\hat{y})\Big]=0,$$
$$\mathcal{A}_j(\hat{y})-\chi_j\mathcal{B}_j(\hat{y})=0,\quad \forall j\in\{1,2\}.$$

Thus, we see that $\hat{y}$ is a feasible element of (DP-MW). Furthermore, every assumption of the strong duality theorem is satisfied. As a result, $(\hat{y},\alpha,\sigma)$ is a Pareto efficient solution for (DP-MW).

6. Wolfe Type Dual Model for MFPPEC

In the following theorem, we establish a different variant of the necessary criteria of Pareto efficiency for MFPPEC derived in Theorem 1, which will be helpful to formulate the Wolfe type dual model for MFPPEC.

Theorem 8. *Let $\overline{y}\in\mathcal{F}$ be any Pareto efficient solution of MFPPEC. Suppose that GGCQ holds at $\overline{y}$. Then, we can obtain some real Lagrange multipliers: $\varepsilon_j\in\mathbb{R}$ ($\varepsilon_j>0$, $j\in\mathcal{I}$), $\overline{\sigma}_j^{\Psi}\in\mathbb{R}$ ($j\in\mathcal{I}^{\Psi}$), $\overline{\sigma}_j^{\theta}\in\mathbb{R}$ ($j\in\mathcal{I}^{\theta}$), $\overline{\sigma}_j^{\mathcal{C}}\in\mathbb{R}$ ($j\in\mathcal{T}$) and $\overline{\sigma}_j^{\mathcal{D}}\in\mathbb{R}$ ($j\in\mathcal{T}$), which satisfy the following:*

$$\sum_{i\in\mathcal{I}}\varepsilon_i\mathcal{B}_i(\overline{y})\Big[\operatorname{grad}\mathcal{A}_i(\overline{y})+\sum_{j\in\mathcal{I}^{\Psi}}\overline{\sigma}_j^{\Psi}\operatorname{grad}\Psi_j(\overline{y})+\sum_{j\in\mathcal{I}^{\theta}}\overline{\sigma}_j^{\theta}\operatorname{grad}\theta_j(\overline{y})$$
$$-\sum_{j\in\mathcal{T}}\Big(\overline{\sigma}_j^{\mathcal{C}}\operatorname{grad}\mathcal{C}_j(\overline{y})+\overline{\sigma}_j^{\mathcal{D}}\operatorname{grad}\mathcal{D}_j(\overline{y})\Big)\Big]-\sum_{i\in\mathcal{I}}\varepsilon_i\operatorname{grad}\mathcal{B}_i(\overline{y})\Big[\mathcal{A}_i(\overline{y})$$
$$+\sum_{j\in\mathcal{I}^{\Psi}}\overline{\sigma}_j^{\Psi}\Psi_j(\overline{y})+\sum_{j\in\mathcal{I}^{\theta}}\overline{\sigma}_j^{\theta}\theta_j(\overline{y})-\sum_{j\in\mathcal{T}}\Big(\overline{\sigma}_j^{\mathcal{C}}\mathcal{C}_j(\overline{y})+\overline{\sigma}_j^{\mathcal{D}}\mathcal{D}_j(\overline{y})\Big)\Big]=0,$$

$$\overline{\sigma}_j^{\Psi}\geq 0,\quad \sigma_j^{\Psi}\Psi_j(\overline{y})=0,\quad \forall j\in\mathcal{I}^{\Psi},$$
$$\overline{\sigma}_j^{\mathcal{C}}\ \textit{free},\ \forall j\in\mathcal{R}_{0+},\quad \overline{\sigma}_j^{\mathcal{C}}\geq 0,\ \forall j\in\mathcal{R}_{00},\quad \overline{\sigma}_j^{\mathcal{C}}=0,\ \forall j\in\mathcal{R}_{+0},$$
$$\overline{\sigma}_j^{\mathcal{D}}\ \textit{free},\ \forall j\in\mathcal{R}_{+0},\quad \overline{\sigma}_j^{\mathcal{D}}\geq 0,\ \forall j\in\mathcal{R}_{00},\quad \overline{\sigma}_j^{\mathcal{D}}=0,\ \forall j\in\mathcal{R}_{0+}.$$

Proof. According to the provided hypotheses, $\overline{y}$ is a weak Pareto efficient solution for MFPPEC at which GGCQ holds. In view of Theorem 1, one can obtain some real numbers:

$\alpha_j \in \mathbb{R}$ $(\alpha_j > 0, j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$ and $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$, which satisfy the following:

$$\sum_{j\in\mathcal{I}} \alpha_j \Big[\operatorname{grad} \mathcal{A}_j(\bar{y}) - \chi_j \operatorname{grad} \mathcal{B}_j(\bar{y}) \Big] + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi} \operatorname{grad} \Psi_j(\bar{y}) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta} \operatorname{grad} \theta_j(\bar{y}) - \sum_{j\in\mathcal{T}} \Big[\sigma_j^{\mathcal{C}} \operatorname{grad} \mathcal{C}_j(\bar{y}) + \sigma_j^{\mathcal{D}} \operatorname{grad} \mathcal{D}_j(\bar{y}) \Big] = 0, \tag{18}$$

$$\mathcal{A}_j(\bar{y}) - \chi_j \mathcal{B}_j(\bar{y}) = 0, \quad \forall j \in \mathcal{I},$$

$$\begin{aligned}
&\sigma_j^{\Psi} \geq 0, \quad \sigma_j^{\Psi} \Psi_j(\bar{y}) = 0, \quad \forall j \in \mathcal{I}^{\Psi}, \\
&\sigma_j^{\mathcal{C}} \text{ free}, \forall j \in \mathcal{R}_{0+}, \quad \sigma_j^{\mathcal{C}} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^{\mathcal{C}} = 0, \quad \forall j \in \mathcal{R}_{+0}, \\
&\sigma_j^{\mathcal{D}} \text{ free}, \forall j \in \mathcal{R}_{+0}, \quad \sigma_j^{\mathcal{D}} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^{\mathcal{D}} = 0, \forall j \in \mathcal{R}_{0+}.
\end{aligned}$$

Let us now define ε_j $(j \in \mathcal{I})$ in the following manner:

$$\varepsilon_j := \frac{\alpha_j}{\mathcal{B}_j(\bar{y})}, \quad \forall j \in \mathcal{I}.$$

Consequently, from (18), we can obtain the following:

$$\sum_{j\in\mathcal{I}} \varepsilon_j \mathcal{B}_j(\bar{y}) \left[\operatorname{grad} \mathcal{A}_j(\bar{y}) - \frac{\mathcal{A}_j(\bar{y})}{\mathcal{B}_j(\bar{y})} \operatorname{grad} \mathcal{B}_j(\bar{y}) \right] + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi} \operatorname{grad} \Psi_j(\bar{y}) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta} \operatorname{grad} \theta_j(\bar{y}) - \sum_{j\in\mathcal{T}} \Big[\sigma_j^{\mathcal{C}} \operatorname{grad} \mathcal{C}_j(\bar{y}) + \sigma_j^{\mathcal{D}} \operatorname{grad} \mathcal{D}_j(\bar{y}) \Big] = 0. \tag{19}$$

From (19), the following equation can be obtained:

$$\sum_{j\in\mathcal{I}} \varepsilon_j \mathcal{B}_j(\bar{y}) \operatorname{grad} \mathcal{A}_j(\bar{y}) + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi} \operatorname{grad} \Psi_j(\bar{y}) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta} \operatorname{grad} \theta_j(\bar{y}) - \sum_{j\in\mathcal{T}} \Big(\sigma_j^{\mathcal{C}} \operatorname{grad} \mathcal{C}_j(\bar{y}) + \sigma_j^{\mathcal{D}} \operatorname{grad} \mathcal{D}_j(\bar{y}) \Big) - \sum_{j\in\mathcal{I}} \varepsilon_j \mathcal{A}_j(\bar{y}) \operatorname{grad} \mathcal{B}_j(\bar{y}) = 0. \tag{20}$$

Furthermore, let us define the following:

$$\begin{aligned}
\bar{\sigma}_j^{\Psi} &:= \frac{\sigma_j^{\Psi}}{\sum_{i\in\mathcal{I}} \varepsilon_i \mathcal{B}_i(\bar{y})}, \quad \forall j \in \mathcal{I}^{\Psi}, \\
\bar{\sigma}_j^{\theta} &:= \frac{\sigma_j^{\theta}}{\sum_{i\in\mathcal{I}} \varepsilon_i \mathcal{B}_i(\bar{y})}, \quad \forall j \in \mathcal{I}^{\theta}, \\
\bar{\sigma}_j^{\mathcal{C}} &:= \frac{\sigma_j^{\mathcal{C}}}{\sum_{i\in\mathcal{I}} \varepsilon_i \mathcal{B}_i(\bar{y})}, \quad \forall j \in \mathcal{T}, \\
\bar{\sigma}_j^{\mathcal{D}} &:= \frac{\sigma_j^{\mathcal{D}}}{\sum_{i\in\mathcal{I}} \varepsilon_i \mathcal{B}_i(\bar{y})}, \quad \forall j \in \mathcal{T}.
\end{aligned}$$

Then, from the feasibility conditions of MFPPEC and the definition of indices, the required relations follow. □

Let us suppose that $w \in \mathscr{M}$ is any arbitrary element. Furthermore, let $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$. Let $e = (1, 1, \ldots, 1) \in \mathbb{R}^r$ be the unit vector having r components. Then, related to the primal

MFPPEC, the corresponding Wolfe type dual model (abbreviated as DP-W) is formulated as given below:

$$\text{(DP-W)} \quad \text{Maximize } \mathcal{L}(w,\alpha,\sigma) := (\mathcal{L}_1(w,\alpha,\sigma), \mathcal{L}_2(w,\alpha,\sigma), \ldots, \mathcal{L}_r(w,\alpha,\sigma)),$$

subject to:

$$\sum_{i\in\mathcal{I}}\alpha_i\mathcal{B}_i(w)\Big[\operatorname{grad}\mathcal{A}_i(w) + \sum_{j\in\mathcal{I}^\Psi}\sigma_j^\Psi \operatorname{grad}\Psi_j(w) + \sum_{j\in\mathcal{I}^\theta}\sigma_j^\theta \operatorname{grad}\theta_j(w)$$
$$- \sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\operatorname{grad}\mathcal{C}_j(w) + \sigma_j^{\mathcal{D}}\operatorname{grad}\mathcal{D}_j(w)\Big)\Big] - \sum_{i\in\mathcal{I}}\alpha_i \operatorname{grad}\mathcal{B}_i(w)\Big[\mathcal{A}_i(w)$$
$$+ \sum_{j\in\mathcal{I}^\Psi}\sigma_j^\Psi\Psi_j(w) + \sum_{j\in\mathcal{I}^\theta}\sigma_j^\theta\theta_j(w) - \sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\mathcal{C}_j(w) + \sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\Big)\Big] = 0,$$

where, for every $j \in \mathcal{I}$, the function $\mathcal{L}_j(w,\alpha,\sigma)$ is defined as:

$$\mathcal{L}_j(w,\alpha,\sigma) := \frac{\mathcal{A}_j(w) + \sum_{j\in\mathcal{I}^\Psi}\sigma_j^\Psi\Psi_j(w) + \sum_{j\in\mathcal{I}^\theta}\sigma_j^\theta\theta_j(w) - \sum_{j\in\mathcal{T}}\Big[\sigma_j^{\mathcal{C}}\mathcal{C}_j(w) + \sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\Big]}{\mathcal{B}_j(w)},$$

and:

$$\sigma_j^\Psi \geq 0,\ \forall j \in \mathscr{A}^\Psi,$$
$$\sigma_j^{\mathcal{C}} = 0,\ \forall j \in \mathcal{R}_{+0},$$
$$\sigma_j^{\mathcal{D}} = 0,\ \forall j \in \mathcal{R}_{0+},$$
$$\forall j \in \mathcal{R}_{00},\ \text{either, } \sigma_j^{\mathcal{C}} > 0,\ \sigma_j^{\mathcal{D}} > 0 \text{ or, } \sigma_j^{\mathcal{C}}\sigma_j^{\mathcal{D}} = 0.$$

We use the notation $\mathcal{F}_W$ to signify the set containing every feasible solution of the problem (DP-W). For the sake of convenience, we now construct an auxiliary function $\Omega : \mathscr{M} \to \mathbb{R}$ in the following manner:

$$\Omega(\cdot) := \sum_{i\in\mathcal{I}}\alpha_i\mathcal{B}_i(w)\Big[\mathcal{A}_i(\cdot) + \sum_{j\in\mathcal{I}^\Psi}\sigma_j^\Psi\Psi_j(\cdot) + \sum_{j\in\mathcal{I}^\theta}\sigma_j^\theta\theta_j(\cdot) - \sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\mathcal{C}_j(\cdot) + \sigma_j^{\mathcal{D}}\mathcal{D}_j(\cdot)\Big)\Big]$$
$$- \sum_{i\in\mathcal{I}}\alpha_i\mathcal{B}_i(\cdot)\Big[\mathcal{A}_i(w) + \sum_{j\in\mathcal{I}^\Psi}\sigma_j^\Psi\Psi_j(w) + \sum_{j\in\mathcal{I}^\theta}\sigma_j^\theta\theta_j(w) - \sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\mathcal{C}_j(w) + \sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\Big)\Big],$$

where $w \in \mathcal{F}_w$. Throughout the remaining part of the section, we shall always assume that:

$$\mathcal{A}_j(w) + \sum_{j\in\mathcal{I}^\Psi}\sigma_j^\Psi\Psi_j(w) + \sum_{j\in\mathcal{I}^\theta}\sigma_j^\theta\theta_j(w) - \sum_{j\in\mathcal{T}}\Big[\sigma_j^{\mathcal{C}}\mathcal{C}_j(w) + \sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\Big] \geq 0,$$
$$\mathcal{B}_j(w) > 0,$$

for every $j \in \mathcal{I}$.

The weak duality relation relating our considered primal problem MFPPEC and Wolfe dual model (DP-W) is established in the following theorem.

Theorem 9. *Let us suppose that $\bar{z} \in \mathcal{F}$ and $(w,\alpha,\sigma) \in \mathcal{F}_W$ are arbitrary elements. Furthermore, assume that the function Ω is geodesic convex at w. Then:*

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})} \nprec \mathcal{L}(w,\alpha,\sigma).$$

Proof. From the feasibility conditions of the problem (DP-W), we have that:

$$\sum_{i\in\mathcal{I}}\alpha_i\mathcal{B}_i(w)\Big[\operatorname{grad}\mathcal{A}_i(w)+\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\operatorname{grad}\Psi_j(w)+\sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\operatorname{grad}\theta_j(w)$$
$$-\sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\operatorname{grad}\mathcal{C}_j(w)+\sigma_j^{\mathcal{D}}\operatorname{grad}\mathcal{D}_j(w)\Big)\Big]-\sum_{i\in\mathcal{I}}\alpha_i\operatorname{grad}\mathcal{B}_i(w)\Big[\mathcal{A}_i(w)$$
$$+\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\Psi_j(w)+\sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\theta_j(w)-\sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\mathcal{C}_j(w)+\sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\Big)\Big]=0.$$

By reductio ad absurdum, we suppose that:

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})}\prec\mathcal{L}(w,\alpha,\sigma).$$

Consequently, we have the following inequality for every $j\in\mathcal{I}$:

$$\frac{\mathcal{A}_j(\bar{z})}{\mathcal{B}_j(\bar{z})}<\frac{\mathcal{A}_j(w)+\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\Psi_j(w)+\sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\theta_j(w)-\sum_{j\in\mathcal{T}}\Big[\sigma_j^{\mathcal{C}}\mathcal{C}_j(w)+\sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\Big]}{\mathcal{B}_j(w)}.$$

Given that $\alpha_i>0$ for every $i\in\mathcal{I}$, we obtain the following:

$$\sum_{j\in\mathcal{I}}\alpha_j\big(\mathcal{A}_j(\bar{z})\mathcal{B}_j(w)\big)<\sum_{j\in\mathcal{I}}\alpha_j\mathcal{B}_j(\bar{z})\Big(\mathcal{A}_j(w)+\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\Psi_j(w)+\sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\theta_j(w)-$$
$$\sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\mathcal{C}_j(w)+\sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\Big)\Big).$$

Equivalently, we can rewrite the above inequality in the following manner:

$$\sum_{j\in\mathcal{I}}\alpha_j\mathcal{B}_j(w)\Big[\mathcal{A}_j(\bar{z})+\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\Psi_j(\bar{z})+\sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\theta_j(\bar{z})-\sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\mathcal{C}_j(\bar{z})+\sigma_j^{\mathcal{D}}\mathcal{D}_j(\bar{z})\Big)\Big]$$
$$-\sum_{j\in\mathcal{I}}\alpha_j\mathcal{B}_j(\bar{z})\Big[\mathcal{A}_j(w)+\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\Psi_j(w)+\sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\theta_j(w)-\sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\mathcal{C}_j(w)+\sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\Big)\Big] \quad (21)$$
$$<\sum_{j\in\mathcal{I}}\alpha_j\mathcal{B}_j(w)\Big[\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\Psi_j(\bar{z})+\sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\theta_j(\bar{z})-\sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\mathcal{C}_j(\bar{z})+\sigma_j^{\mathcal{D}}\mathcal{D}_j(\bar{z})\Big)\Big].$$

We now note that $\alpha_j>0$ and $\mathcal{B}_j>0$ for every $j\in\mathcal{I}$. Combining these with the feasibility conditions of MFPPEC, we infer that:

$$\sum_{j\in\mathcal{I}}\alpha_j\mathcal{B}_j(w)\Big[\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\Psi_j(\bar{z})+\sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\theta_j(\bar{z})-\sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\mathcal{C}_j(\bar{z})+\sigma_j^{\mathcal{D}}\mathcal{D}_j(\bar{z})\Big)\Big]\leq 0. \quad (22)$$

From Inequalities (21) and (22) and in view of the definition of function Ω, we have the following:

$$\Omega(\bar{z})<0=\Omega(w).$$

By invoking the hypothesis of geodesic convexity on the function Ω at w, the following inequality arises:

$$\langle\operatorname{grad}\Omega(w),\exp_w^{-1}(\bar{z})\rangle_w<0, \quad (23)$$

which contradicts the fact that $w\in\mathcal{F}_W$. Thus, the proof is complete. □

In the next theorem, we present another weak duality relation referring to our considered primal problem MFPPEC and (DP-W). The proof of the theorem may be obtained along similar lines as the proof of Theorem 9.

Theorem 10. *Let us suppose that* $\bar{z} \in \mathcal{F}$ *and* $(w, \alpha, \sigma) \in \mathcal{F}_W$ *are arbitrary elements. Furthermore, let us assume that the function* Ω *is strictly geodesic convex at z. Then, we have the following:*

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})} \not\preceq \mathcal{L}(w, \alpha, \sigma).$$

In the next theorem, the strong duality relation referring to our considered primal problem MFPPEC and Wolfe dual problem (DP-W) is established.

Theorem 11. *Let* $\bar{z} \in \mathcal{F}$ *be any arbitrary Pareto efficient solution of MFPPEC at which GGCQ holds. Then, some real numbers* $\alpha_j \in \mathbb{R}$, $\alpha_j > 0\ (j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}\ (j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}\ (j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}\ (j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}\ (j \in \mathcal{T})$ *exist, such that* $(\bar{z}, \alpha, \sigma) \in \mathcal{F}_W$. *Furthermore, the corresponding values of the objective functions of MFPPEC and (DP-W) are equal, that is:*

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})} = \mathcal{L}(\bar{z}, \alpha, \sigma).$$

Consequently, the following assertions hold true:

(a) *Let us suppose that each of the hypotheses stated in Theorem 9 are satisfied. Then,* $(\bar{z}, \alpha, \sigma)$ *is a weak Pareto efficient solution of (DP-W).*

(b) *Let us suppose that each of the hypotheses stated in the Theorem 10 are satisfied. Then,* $(\bar{z}, \alpha, \sigma)$ *is a Pareto efficient solution of (DP-W).*

Proof. According to the provided hypotheses, we have that $\bar{z} \in \mathcal{F}$ is any arbitrary Pareto efficient solution of MFPPEC at which GGCQ holds. As a result, in the light of Theorem 1, we obtain some real multipliers $\alpha_j \in \mathbb{R}$, $\alpha_j > 0\ (j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}\ (j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}\ (j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}\ (j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}\ (j \in \mathcal{T})$, satisfying the following:

$$\begin{aligned}
&\sum_{i \in \mathcal{I}} \alpha_i \mathcal{B}_i(w) \Bigg[\operatorname{grad} \mathcal{A}_i(w) + \sum_{j \in \mathcal{I}^{\Psi}} \sigma_j^{\Psi} \operatorname{grad} \Psi_j(w) + \sum_{j \in \mathcal{I}^{\theta}} \sigma_j^{\theta} \operatorname{grad} \theta_j(w) \\
&\quad - \sum_{j \in \mathcal{T}} \Big(\sigma_j^{\mathcal{C}} \operatorname{grad} \mathcal{C}_j(w) + \sigma_j^{\mathcal{D}} \operatorname{grad} \mathcal{D}_j(w) \Big) \Bigg] - \sum_{i \in \mathcal{I}} \alpha_i \operatorname{grad} \mathcal{B}_i(w) \Bigg[\mathcal{A}_i(w) \\
&\quad + \sum_{j \in \mathcal{I}^{\Psi}} \sigma_j^{\Psi} \Psi_j(w) + \sum_{j \in \mathcal{I}^{\theta}} \sigma_j^{\theta} \theta_j(w) - \sum_{j \in \mathcal{T}} \Big(\sigma_j^{\mathcal{C}} \mathcal{C}_j(w) + \sigma_j^{\mathcal{D}} \mathcal{D}_j(w) \Big) \Bigg] = 0,
\end{aligned}$$

and:

$$\begin{aligned}
&\sigma_j^{\Psi} \geq 0, \quad \sigma_j^{\Psi} \Psi_j(\bar{y}) = 0, \quad \forall j \in \mathcal{I}^{\Psi}, \\
&\sigma_j^{\mathcal{C}} \text{ free}, \forall j \in \mathcal{R}_{0+}, \quad \sigma_j^{\mathcal{C}} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^{\mathcal{C}} = 0, \quad \forall j \in \mathcal{R}_{+0}, \\
&\sigma_j^{\mathcal{D}} \text{ free}, \forall j \in \mathcal{R}_{+0}, \quad \sigma_j^{\mathcal{D}} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^{\mathcal{D}} = 0, \forall j \in \mathcal{R}_{0+}.
\end{aligned}$$

Consequently, it follows that:

$$(\bar{z}, \alpha, \sigma) \in \mathcal{F}_W,$$

and:

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})} = \mathcal{L}(\bar{z}, \alpha, \sigma).$$

Now, we consider the following cases:

(a) Let us consider that $(\bar{z}, \alpha, \sigma)$ is not a weak Pareto efficient solution for (DP-W). As a result, one can find some $(\bar{u}, \bar{\alpha}, \bar{\sigma}) \in \mathcal{F}_W$, which satisfies the following:

$$\mathcal{L}(\bar{z}) \prec \mathcal{L}(\bar{u}).$$

This contradicts the consequences of Theorem 9.

(b) Let us consider that $(\bar{z}, \alpha, \sigma)$ is not a Pareto efficient solution for (DP-W). As a result, one can find some $(\bar{u}, \bar{\alpha}, \bar{\sigma}) \in \mathcal{F}_W$, which satisfies the following:

$$\mathcal{L}(\bar{z}) \preceq \mathcal{L}(\bar{u}).$$

This contradicts the consequences of Theorem 10. □

In the next theorem, the strict converse duality relation referring to our considered primal problem (MPPEC) and Wolfe dual problem (DP-W) is established.

Theorem 12. *Let us suppose that $\bar{y} \in \mathcal{F}$ is any Pareto efficient solution of MFPPEC at which GGCQ is satisfied. Let $(\bar{w}, \bar{\alpha}, \bar{\sigma})$ be any Pareto efficient solution for (DP-MW). Let us suppose that each of the hypotheses stated in Corollary 5 holds. Then, $\bar{y} = \bar{w}$.*

Proof. According to the provided hypotheses, we have that $\bar{y} \in \mathcal{F}$ is any Pareto efficient solution of MFPPEC at which GGCQ holds.

By reductio ad absurdum, we suppose that $\bar{y} \neq \bar{w}$. As a result, in the light of Theorem 6, we obtain: $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$ such that $(\bar{y}, \alpha, \sigma) \in \mathcal{F}_M$. Moreover, we have:

$$\frac{\mathcal{A}(\bar{y})}{\mathcal{B}(\bar{y})} = \mathscr{L}(\bar{y}, \alpha, \sigma).$$

On the other hand, in the view of the conclusions of the strong duality theorem (Theorem 6), one can conclude that $(\bar{y}, \alpha, \sigma)$ is a Pareto efficient solution for MFPPEC. Since $\bar{y} \in \mathcal{F}$ and $(\bar{w}, \bar{\alpha}, \bar{\sigma}) \in \mathcal{F}_{MW}$, then from Theorem 6, we obtain:

$$\mathscr{L}(\bar{y}) \npreceq \mathscr{L}(\bar{w}),$$

which is a contradiction. □

Remark 10. *The following may be observed:*

1. *If $\mathscr{M} = \mathbb{R}^n$, then Theorems 9 and 11 generalize Theorem 4 and Theorem 5 deduced in [6] from multiobjective MPEC to MFPPEC.*
2. *Theorems 9, 11 and 12 extend Theorem 3.1, Theorem 3.2, and Theorem 3.3, respectively, deduced in [23] for a wider category of optimization problems, that is, MFPPEC.*

In the following example, we demonstrate the formulation of the Wolfe type dual problem corresponding to a primal MFPPEC problem. Furthermore, we illustrate the duality results for the Wolfe dual problem corresponding to MFPPEC in the framework of a Hadamard manifold.

Example 4. *Let us consider the MFPPEC (Problem (P)) defined in Example 1 on the manifold $\mathscr{M}$. We use the symbol $\mathcal{F}$ to signify the set containing every feasible solution of the problem (P). That is, we have:*

$$\mathcal{F} = \{y \in \mathscr{M}, y_1 = e, y_2 \geq e, \text{ or}, y_1 \geq e, y_2 = e\}.$$

Let $w \in \mathscr{M}$ be arbitrary. Furthermore, let $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \{1,2\})$, $\sigma^{\Psi} \in \mathbb{R}$, $\sigma^{\mathcal{C}} \in \mathbb{R}$, $\sigma^{\mathcal{D}} \in \mathbb{R}$. Then, related to the primal (P), the corresponding Wolfe type dual model (abbreviated as DP-W) is formulated as given below:

$$\text{(DP-W)} \quad \text{Maximize } \mathcal{L}(w) = (\mathcal{L}_1(w), \mathcal{L}_2(w)),$$

subject to:

$$\sum_{i\in\{1,2\}} \alpha_i \mathcal{B}_i(w) \left[\operatorname{grad} \mathcal{A}_i(w) + \sigma^{\Psi} \operatorname{grad} \Psi(w) - \left(\sigma^{\mathcal{C}} \operatorname{grad} \mathcal{C}(w) + \sigma^{\mathcal{D}} \operatorname{grad} \mathcal{D}(w) \right) \right]$$
$$- \sum_{i\in\{1,2\}} \alpha_i \operatorname{grad} \mathcal{B}_i(w) \left[\mathcal{A}_i(w) + \sigma^{\Psi} \Psi_j(w) - \left(\sigma^{\mathcal{C}} \mathcal{C}(w) + \sigma^{\mathcal{D}} \mathcal{D}(w) \right) \right] = 0.$$

Let us choose the feasible solution $\hat{y} = (e, e)$. One can easily verify that the feasible solution $\hat{y}$ is, indeed, a Pareto efficient solution for (P). Furthermore, GGCQ holds at $\hat{y}$. Let us now pick some real numbers $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}$, $\sigma^{\Psi} = 0$, $\sigma^{\mathcal{C}} = \frac{e}{2}$, $\sigma^{\mathcal{D}} = \frac{1}{2}$. Then, we can verify that $\hat{y}$ is a feasible element of (DP-MW).

Furthermore, every assumption of the strong duality theorem is satisfied. Thus, one can verify the fact that $(\hat{y}, \alpha, \sigma)$ is a Pareto efficient solution of (DP-W).

7. Applications

A very interesting practical application of fractional programming problems can be found in information theory. In particular, the problem of calculating the maximum transmission rate in any information channel can be modeled as a fractional programming problem (see, for instance, [32]).

Let us consider a constant and discrete transmission channel consisting of n^1 input symbols and n^2 output symbols. The corresponding transition matrix is given by:

$$\mathscr{B} = (b_{rs}), \quad r = 1, \ldots, n^2; \quad s = 1, \ldots, n^1,$$

where $n^1, n^2 \in \mathbb{N}$, $b_{rs} \geq 0$, and $\sum_r b_{rs} = 1$. It is assumed that the matrix $\mathscr{B}$ does not have any zero lines. In other words, we consider that there does not exist any output symbol which is never received. It can be noted that every element b_{rs} of the transition matrix $\mathscr{B}$ signifies the probability of obtaining the symbol r at the output, subject to the assumption that the input symbol was s.

Let us suppose that corresponding to the transmission of the input symbol s, a certain cost (denoted by $\hat{c}_s$) is associated. Furthermore, let us employ the notation $p = (p_s)$ to signify the probability distribution function (abbreviated as, PDF) of s^{th} input. Consequently, it follows that:

$$p_s \geq 0, \ \forall s = 1, 2, \ldots, n^2; \quad \sum_s p_s = 1.$$

Then, we may define the corresponding transmission rate of the channel in the following manner (see, for instance, [32]):

$$\mathscr{R}(p) := \frac{\sum_r \sum_s p_s b_{rs} \log \frac{b_{rs}}{\sum_t p_t b_{rt}}}{\sum_s \hat{c}_s p_s}.$$

The maximum value of the function $\mathscr{R}(p)$, as defined above, is referred to as the relative capacity of the channel. As a result, to obtain the relative capacity of the channel, we arrive at the following fractional nonlinear programming problem:

$$\underset{x}{\text{Maximize}} \left\{ \mathscr{R}(p) : p_s \geq 0, \ \sum_s p_s = 1 \right\}.$$

It is significant to note that the phenomenon of convexity plays a crucial role in optimization theory. For a convex optimization problem, it is well known that every local minimum is a global minimum. Moreover, the set of all global minima is a convex set for such problems. In this context, we note that several nonconvex functions in the Euclidean space setting can be suitably transformed into geodesic convex functions in the framework of manifolds. As a result, a wider range of optimization problems can be explored by formulating the problems in the framework of manifolds. For instance, consider the nonconvex set $\mathscr{D} \subset \mathbb{R}^2$ defined in the following manner:

$$\mathscr{D} := \left\{ (y_1, y_2) \in \mathbb{R}^2 : y_2 = y_1^2, y_1 \in \left[\frac{1}{2}, 1\right] \right\}.$$

Let us now consider the function $\phi : \mathscr{D} \times \mathscr{D} \to \mathbb{R}$, defined in the following manner:

$$\phi(x, y) = y_1^2 - y_1^4 + y_2 + y_2^2 - x_1^2 - x_2,$$

for every $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathscr{D}$.

One can view the set $\mathscr{D}$ as the image of a geodesic segment on the paraboloid of revolution $P(u_1, u_2) = \left(u_2 \cos u_1, u_2 \sin u_1, u_2^2\right)$, $u_1, u_2 \in \mathscr{D}$, which is endowed with the Riemannian metric $\mathscr{G}$, given by:

$$\mathscr{G}(u_1, u_2) = \begin{pmatrix} u_2^2 & 0 \\ 0 & 1 + 4u_2^2 \end{pmatrix}.$$

It can be verified that the set $\mathscr{D}$ is a geodesic convex set on the Riemannian manifold formed by the image of the paraboloid of revolution (see, for instance, [33]). Furthermore, the function $\phi(x, \cdot)$ is not a convex function in the Euclidean space setting. However, $\phi(x, \cdot)$ is a geodesic convex function on the set $\mathscr{D}$.

As a result, by studying optimization methods in the framework of manifolds, many constrained nonconvex problems can be converted into unconstrained geodesic convex problems. As a result, not only the complexity of the original problem may be reduced, but also the theory and algorithms of convex optimization can be made applicable.

8. Conclusions and Future Research Directions

In this article, we have explored a category of MFPPEC in the setting of Hadamard manifolds. The main features of the results derived in the paper are as follows. KKT-type necessary criteria of Pareto efficiency for MFPPEC have been presented by using GGCQ. Apart from this, the sufficient criteria of Pareto efficiency for MFPPEC have been derived using notions of M-stationary element and geodesic convexity. Mond–Weir and Wolfe type dual models related to MFPPEC have been formulated. The weak, strong, and strict converse duality results have been derived relating MFPPEC and the respective dual models. Some nontrivial examples have been furnished to demonstrate the results presented in this paper.

The various results that are derived in this article extend as well as generalize various noteworthy results available in the literature. In particular, we have extended the corresponding results presented in [7] from multiobjective MPEC to MFPPEC. Moreover, the necessary and sufficient conditions in the present article extend and generalize similar results derived in [6] for a broader category of optimization problems on Hadamard manifolds. Furthermore, the Pareto efficiency conditions of this paper extend various corresponding results of [8,28] from the setting of Euclidean spaces to Hadamard manifolds, as well as generalize them for a broader category of problems, namely, MFPPEC.

The sufficient criteria of Pareto efficiency derived in this article could be further generalized using generalized geodesic convexity assumptions. Furthermore, all the functions involved in our considered problem MFPPEC are assumed to be smooth. As a result, the results of this paper cannot be applied when the corresponding functions involved in the considered problem are nonsmooth. This may be considered as a limitation of this paper. We intend to address this in our future course of study.

For future work, investigating the Pareto efficiency criteria for nonsmooth mathematical programming problems with switching constraints on Hadamard manifolds would be an interesting problem. For such problems, standard constraint qualifications (such as Mangasarian–Fromovitz constraint qualification and linearly independent constraint qualification) are generally not satisfied at any feasible point of the problem (see [34]), which makes the problem intriguing to study. Furthermore, mathematical programming problems with switching constraints have applications in numerous fields of modern research, in particular, in the field of optimal control (see, for instance, [34] and the references cited therein).

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