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Article

A Representation of the Drazin Inverse for the Sum of Two Matrices and the Anti-Triangular Block Matrices

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Abstract: In this paper, a new formula for the Drazin inverse for the Sum of Two Matrices is given under conditions weaker than those used in some current literature. Further, we apply our results to obtain new representations for the Drazin inverse of an anti-triangular block matrix under some conditions, which also extend some existing results.

Keywords: Drazin inverse; index; anti-triangular block matrix; additive formula

MSC: 15A09; 15B35

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1. Introduction

Let $\mathbb{C}^{n\times n}$ denote the set of all the $n \times n$ matrices over the complex field $\mathbb{C}$. For $A \in \mathbb{C}^{n\times n}$, if $X \in \mathbb{C}^{n\times n}$ satisfies the following conditions

$$AX = XA, \;\; XAX = X, \;\; A^k XA = A^k,$$

then X is called the Drazin inverse [1] of A, denoted by A^d; A^d exists and is unique [2]. The smallest nonnegative integer k that satisfies $rank(A^{k+1}) = rank(A^k)$ is called the index of A, denoted by $k = ind(A)$. Let $A^\pi = I - AA^d$, where I is the identity matrix. For $p \in \mathbb{Z}$, $[\frac{p}{2}]$ represents the largest integer not exceeding $\frac{p}{2}$.

Since the middle of last century, the Drazin inverse of a matrix has become a very important research field. Up to now, it is still one of the most active research branches in the world. The Drazin inverse of a matrix has been widely used in many fields, such as differential equations, integral equations, operator theory, statistics, cybernetics, Markov chains, optimization, etc. (see [2,3]).

For $P, Q \in \mathbb{C}^{n\times n}$, Drazin first gave the explicit formula of $(P + Q)^d$ in the case of $PQ = QP = 0$ (see [1]), which led to later research on the Drazin inverse representation of the sum. In 2001, Hartwig et al. gave the formula $(P + Q)^d$ under the condition $PQ = 0$ (see [4]). In 2009, the formula $(P + Q)^d$ was established under the conditions $P^2Q = 0, Q^2 = 0$ (see [5]). In 2011, an additive formula was given under the conditions $PQ^2 = 0, PQP = 0$ (see [6]). In 2016, Sun et al. (see [7]) derived a formula for $(P + Q)^d$ under the conditions:

(1) $PQ^2 = 0, P^2QP = 0$ and $(QP)^2 = 0$;
(2) $P^2Q = 0, QPQ^2 = 0$ and $(QP)^2 = 0$;
(3) $P^2QP = 0, P^3Q = 0, Q^2 = 0$.

Other representations of the Drazin inverse for $P + Q$ were developed in Refs. [8–12].

One of the reasons for the study on the representations for the Drazin inverse of block matrices essentially originated from finding the general expressions for the solutions to singular systems of differential equations [13–15]. Then, Campbell and Meyer [3] posed a problem: establishing an explicit representation for the Drazin inverse of 2×2 block complex matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in terms of the blocks of the partition, where the blocks A and D are assumed to be square matrices. This problem has not been solved so far but, in recent decades, some results are presented in Refs. [16–19].

In addition, the representation of the Drazin inverse of anti-triangular block matrix $M(D = 0)$ has always concerned many scholars, which can be applied to the study of solutions of second-order differential equations, graph theory, saddle point problems, optimization, and other problems [20–23]. In recent years, the problem has been widely studied, and some results have been obtained under some conditions [22,24–28], but it still remains open.

This objective of this paper is to present the representations for the Drazin inverse of the sum of two matrices and anti-triangular block matrix $M(D = 0)$. In Section 2, we first present some preliminary lemmas which are used in the subsequent proof. In Section 3, we give the explicit formula of $(P + Q)^d$ when $P^2QP = 0, PQ^2P = 0, PQ^3 = 0,$ $P^2Q^2 = 0, (QP)^2 = 0$ which also extends some existing results in Ref. [7]. In Section 4, we give the representation for the Drazin inverse of $M(D = 0)$ under the following conditions:

(1) $ABCAA^\pi = 0, ABCA^\pi B = 0, A^dBC = 0;$
(2) $BCAA^\pi = 0, (BC)^dA^\pi B = 0, CBCA^d = 0, A^dBC = 0;$
(3) $ABCAA^\pi = 0, ABCA^\pi B = 0, CBCAA^\pi = 0, CBCA^\pi B = 0, A^dBCA = 0.$

These can be regarded as the generalizations of some results given in Refs. [5,22,28].

2. Some Lemmas

Lemma 1 ([3])**.** *Let $B, C \in \mathbb{C}^{n\times n}$. For any positive integer i, we obtain $((BC)^d)^i = B((CB)^d)^{i+1}C$.*

Lemma 2 ([6])**.** *Let $P, Q \in \mathbb{C}^{n\times n}$, $ind(P) = r$ and $ind(Q) = s$. If $PQP = 0$ and $PQ^2 = 0$, then*

$$(P+Q)^d = \sum_{i=0}^{r-1}(Q^d)^{i+1}P^iP^\pi + Q^\pi \sum_{i=0}^{s-1} Q^i(P^d)^{i+1} + Q^\pi \sum_{i=0}^{s-1} Q^i(P^d)^{i+2}Q + \sum_{i=0}^{r-2}(Q^d)^{i+3}P^{i+1}P^\pi Q - Q^dP^dQ - (Q^d)^2PP^dQ.$$

Lemma 3 ([18])**.** *Let $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is 2×2 block complex matrix, $A \in \mathbb{C}^{m\times m}$, $D \in \mathbb{C}^{n\times n}$, $\mathrm{ind}(A) = r$ and $\mathrm{ind}(D) = t$. Then*

$$M^d = \begin{pmatrix} A^d & X \\ 0 & D^d \end{pmatrix},$$

where $X = \sum_{i=0}^{t-1}(A^d)^{i+2}BD^iD^\pi + A^\pi \sum_{i=0}^{r-1} A^iB(D^d)^{i+2} - A^dBD^d$.

Lemma 4 ([22])**.** *Let $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ is 2×2 block matrix, $A \in \mathbb{C}^{m\times m}$, $0 \in \mathbb{C}^{n\times n}$, $\mathrm{ind}(A) = r$. If $BCA^\pi = 0$ and $(I - A^\pi)BC = 0$, then*

$$M^d = \begin{pmatrix} A^d + V & (A^d)^2B + VA^dB \\ C(A^d)^2 + CVA^d & C(A^d)^3B + CV(A^d)^2B \end{pmatrix},$$

where $V = \sum_{n=0}^{r-1} A^nBC(A^d)^{n+3}$.

3. The Drazin Inverse for the Sum of Two Matrices

Theorem 1. *Let $P, Q \in \mathbb{C}^{n\times n}$. If $P^2QP = 0$, $PQ^2P = 0$, $PQ^3 = 0$, $P^2Q^2 = 0$, and $(QP)^2 = 0$, then*

$$\begin{aligned}(P+Q)^d =& \big[\big(Q^\pi - Q^dP + PQ(P^d)^2\big)(P^d)^2 + (Q^d)^3(P+Q)P^\pi\big](P+Q)\\ &+\sum_{i=0}^{l-1}(Q^d)^{2i+5}(P+Q)P^{2i+1}(P+Q)P^\pi(P+Q)\\ &+\sum_{i=0}^{s-1}Q^\pi Q^{2i+1}(P+Q)(P^d)^{2i+5}(P+Q)^2,\end{aligned}$$

where $l = \operatorname{ind}(P^2+PQ)$, $s = \operatorname{ind}(Q^2+QP)$.

Proof. Using the definition of the Drazin inverse, we have that

$$(P+Q)^d = (P+Q)(M+N)^d,$$

where $M := P^2 + PQ$, $N := QP + Q^2$. Since $P^2Q^2 = 0$, $P^2QP = 0$, $PQ^3 = 0$, and $PQ^2P = 0$, by Lemma 2, we obtain

$$(M+N)^d = \sum_{i=0}^{l-1}(N^d)^{i+1}M^iM^\pi + \sum_{i=0}^{s-1}N^\pi N^i(M^d)^{i+1},$$

where $l = \operatorname{ind}(M)$, $s = \operatorname{ind}(N)$. Since $PQ^3 = 0$ and $(QP)^2 = 0$, by Lemma 2, we obtain

$$\begin{aligned}N^d =& \sum_{i=0}^{r-1}(Q^d)^{2(i+1)}(QP)^i + \sum_{i=0}^{r-2}(Q^d)^{2(i+3)}(QP)^{i+1}Q^2\\ =& (Q^d)^2 + (Q^d)^3P + (Q^d)^5PQ^2,\end{aligned}$$

where $r = \operatorname{ind}(QP)$. Since $PQ^d = 0$, for any positive integer i, we obtain

$$(N^d)^i = (Q^d)^{2(i-1)}((Q^d)^2 + (Q^d)^3P + (Q^d)^5PQ^2), \tag{1}$$

$$\begin{aligned}N^\pi =& I - (Q^2+QP)\big((Q^d)^2 + (Q^d)^3P + (Q^d)^5PQ^2\big)\\ =& Q^\pi - Q^dP - (Q^d)^3PQ^2.\end{aligned} \tag{2}$$

From $PQ^3 = 0$, $PQ^2P = 0$, for any positive integer i, we obtain

$$N^i = Q^{2i} + Q^{2i-1}P + Q^{2i-3}PQ^2. \tag{3}$$

Since $P^2QP = 0$, $(QP)^2 = 0$, by Lemma 1, we obtain that

$$M^d = P(P^4 + QP^3)^d(P+Q).$$

Noting that $P^2QP = 0$, by Lemma 2, we obtain $(P^4 + QP^3)^d = (P^d)^4 + Q(P^d)^5$. Hence,

$$M^d = (P^d)^2 + (P^d)^3Q + PQ(P^d)^4 + PQ(P^d)^5Q.$$

By $P^dQP = 0$, for any positive integer i, we have

$$(M^d)^i = (P^d)^{2i} + (P^d)^{2i+1}Q + PQ(P^d)^{2i+2} + PQ(P^d)^{2i+3}Q. \tag{4}$$

From $P^2QP = 0$, $QP(QP)^d = 0$, we obtain

$$M^\pi = P^\pi - P^dQ - PQ(P^d)^2 - PQ(P^d)^3Q.$$

For any positive integer i, we can prove by induction that

$$((P+Q)P)^i = (P+Q)P^{2i-1}, \tag{5}$$

$$M^iM^\pi = M^i(P^\pi - P^dQ) = (P(P+Q))^i(P^\pi - P^dQ). \tag{6}$$

By substituting Equations (1), (5) and (6) into $\sum_{i=1}^{l-1}(N^d)^{i+1}M^iM^\pi$, it yields that

$$\begin{aligned}
&\sum_{i=1}^{l-1}(N^d)^{i+1}M^iM^\pi \\
=&\sum_{i=1}^{l-1}(Q^d)^{2i}\big((Q^d)^2 + (Q^d)^3P + (Q^d)^5PQ^2\big)(P(P+Q))^i(P^\pi - P^dQ) \\
=&\sum_{i=1}^{l-1}(Q^d)^{2i+3}(P+Q)(P(P+Q))^i(P^\pi - P^dQ) \\
=&\sum_{i=1}^{l-1}(Q^d)^{2i+3}((P+Q)P)^i(P+Q)(P^\pi - P^dQ) \\
=&\sum_{i=1}^{l-1}(Q^d)^{2i+3}(P+Q)P^{2i-1}(P+Q)(P^\pi - P^dQ).
\end{aligned}$$

Moreover,

$$N^dM^\pi = (Q^d)^2P^\pi - (Q^d)^2P^dQ + (Q^d)^3PP^\pi - (Q^d)^3PP^dQ + (Q^d)^5PQ^2.$$

Since $P^2QP = 0$, $P^2Q^2 = 0$, $PQ^3 = 0$, we obtain

$$N^dM^\pi(P+Q) = \big((Q^d)^2P^\pi + (Q^d)^3PP^\pi\big)(P+Q) = (Q^d)^3(P+Q)P^\pi(P+Q). \tag{7}$$

Using $P^2QP = 0$, $Q^2 = 0$, we obtain

$$\sum_{i=1}^{l-1}(N^d)^{i+1}M^iM^\pi(P+Q) = \sum_{i=1}^{l-1}(Q^d)^{2i+3}(P+Q)P^{2i-1}(P+Q)P^\pi(P+Q). \tag{8}$$

Combining Equations (7) and (8), we have

$$\begin{aligned}
\sum_{i=0}^{l-1}(N^d)^{i+1}M^iM^\pi(P+Q) =& \sum_{i=0}^{l-1}(Q^d)^{2i+5}(P+Q)P^{2i+1}(P+Q)P^\pi(P+Q) \\
&+ \big((Q^d)^3(P+Q)\big)P^\pi(P+Q).
\end{aligned} \tag{9}$$

Substituting Equations (2)–(4) into $\sum_{i=1}^{s-1}N^\pi N^i(M^d)^{i+1}$, we have

$$\sum_{i=1}^{s-1}N^\pi N^i(M^d)^{i+1} = \sum_{i=1}^{s-1}Q^\pi Q^{2i-1}(P+Q)(P^d)^{2i+3}(P+Q). \tag{10}$$

Further,

$$\begin{aligned}
N^\pi M^d =& Q^\pi(P^d)^2 + Q^\pi(P^d)^3Q + Q^\pi PQ(P^d)^4 + Q^\pi PQ(P^d)^5Q \\
&- Q^dP^d - Q^d(P^d)^2Q.
\end{aligned}$$

From $P^2QP = 0$, $P^2Q^2 = 0$, $(QP)^2 = 0$, we have

$$N^\pi M^d(P+Q) = \left(Q^\pi(P^d)^2 + PQ(P^d)^4 - Q^dP^d\right)(P+Q). \tag{11}$$

Combining Equations (10) and (11), we obtain

$$\begin{aligned}\sum_{i=0}^{s-1} N^\pi N^i (M^d)^{i+1}(P+Q) = &\sum_{i=0}^{s-1} Q^\pi Q^{2i+1}(P+Q)(P^d)^{2i+5}(P+Q)^2 \\ &+ \left(Q^\pi - Q^dP + PQ(P^d)^2\right)(P^d)^2(P+Q).\end{aligned} \tag{12}$$

Finally, combining with Equations (9) and (12), we conclude the representation for $(P+Q)^d$. □

Now, we state the symmetrical formulation of Theorem 1.

Theorem 2. *Let $P, Q \in \mathbb{C}^{n\times n}$. If $QPQ^2 = 0$, $QP^2Q = 0$, $P^3Q = 0$, $P^2Q^2 = 0$, and $(QP)^2 = 0$, then*

$$\begin{aligned}(P+Q)^d =&(P+Q)\left[Q^\pi(P+Q)(P^d)^3 + (Q^d)^2(P^\pi - QP^d + (Q^d)^2PQ)\right] \\ &+ \sum_{i=0}^{l-1}(P+Q)Q^\pi(P+Q)Q^{2i+1}(P+Q)(P^d)^{2i+5} \\ &+ \sum_{i=0}^{s-1}(P+Q)^2(Q^d)^{2i+5}(P+Q)P^{2i+1}P^\pi,\end{aligned}$$

where $l = \mathrm{ind}(P^2 + PQ)$, $s = \mathrm{ind}(Q^2 + QP)$.

The following result is a direct corollary of Theorem 1, the conditions of which were considered in [7] (Theorem 1).

Corollary 1. *Let $P, Q \in \mathbb{C}^{n\times n}$. If $PQ^2 = 0$, $P^2QP = 0$, and $(QP)^2 = 0$, then*

$$\begin{aligned}(P+Q)^d =&\left[\left(Q^\pi - Q^dP + PQ(P^d)^2\right)(P^d)^2 + (Q^d)^3(P+Q)P^\pi\right](P+Q) \\ &+ \sum_{i=0}^{m_2-1} Q^{2i+1}Q^\pi(P+Q)(P^d)^{2i+4}(P+Q) \\ &+ \sum_{i=0}^{m_1-1} (Q^d)^{2i+5}(P+Q)P^{2i+2}P^\pi(P+Q),\end{aligned}$$

where $m_1 = ind(P^2)$, $m_2 = ind(Q^2)$.

Proof. It follows from $(QP)^2 = 0$ and $P^2QP = 0$ that $\sum_{i=0}^{m_1-1}(Q^d)^{2i+5}(P+Q)P^{2i+1}P^\pi(P+Q) = 0$, thus we obtain the representation. □

Theorem 3. *Let $P, Q \in \mathbb{C}^{n\times n}$. If $P^2QP = 0$, $Q^2 = 0$, then*

$$\begin{aligned}(P+Q)^d =&\left(\sum_{i=0}^{t-1}\left(((PQ)^d)^{i+1} + ((QP)^d)^{i+1}\right)P^{2i}P^\pi\right. \\ &\left.+\sum_{i=0}^{s-1}\left((PQ)^\pi(PQ)^i + (QP)^\pi(QP)^i\right)(P^d)^{2(i+1)} - (P^d)^2\right)(P+Q),\end{aligned}$$

where $t = \mathrm{ind}(P^2)$, $s = \max\{\mathrm{ind}(PQ), \mathrm{ind}(QP)\}$.

Proof. It follows that

$$\begin{aligned}(P+Q)^d &= (P^2+QP+PQ)^d(P+Q)\\ &:= (P_1+Q_1)^d(P+Q),\end{aligned} \tag{13}$$

where $P_1 = P^2 + QP$, $Q_1 = PQ$. Since $P^2QP = 0$, we have $P_1Q_1 = 0$. Lemma 2 shows that

$$\begin{aligned}P_1^d &= (P^2+QP)^d\\ &= (QP)^\pi \sum_{i=0}^{l-1}(QP)^i(P^d)^{2i+2} + \sum_{i=0}^{t-1}((QP)^d)^{i+1}P^{2i}P^\pi,\end{aligned} \tag{14}$$

where $l = \mathrm{ind}(QP), t = \mathrm{ind}(P^2)$. Since $P_1Q_1P_1 = 0, P_1Q_1^2 = 0$, by Lemma 2, we obtain

$$\begin{aligned}(P_1+Q_1)^d =&(PQ)^\pi \sum_{i=0}^{s_1-1}(PQ)^i((P^2+QP)^d)^{i+1} - (PQ)^d(P^2+QP)^dPQ\\ &+ \sum_{i=0}^{r-2}((PQ)^d)^{i+3}(P^2+QP)^{i+1}(P^2+QP)^\pi PQ\\ &+ \sum_{i=0}^{r-1}((PQ)^d)^{i+1}(P^2+QP)^i(P^2+QP)^\pi\\ &+ (PQ)^\pi \sum_{i=0}^{s_1-1}(PQ)^i((P^2+QP)^d)^{i+2}PQ\\ &- ((PQ)^d)^2(P^2+QP)(P^2+QP)^dPQ,\end{aligned} \tag{15}$$

where $s_1 = \mathrm{ind}(PQ), r = \mathrm{ind}(P^2+QP)$. Combining Equation (14) and $Q^2 = 0, Q(QP)^d = 0$, we have

$$\begin{aligned}&(PQ)^\pi \sum_{i=0}^{t-1}(PQ)^i((P^2+QP)^d)^{i+1}\\ &= (PQ)^\pi \sum_{i=1}^{t-1}(PQ)^i(P^d)^{2i+2} + (PQ)^\pi(P^2+QP)^d.\end{aligned} \tag{16}$$

For any positive integer i, we can prove by induction that

$$\begin{aligned}&\sum_{i=0}^{r-1}((PQ)^d)^{i+1}(P^2+QP)^i = \sum_{i=0}^{r-1}((PQ)^d)^{i+1}P^{2i},\\ &(P^2+QP)(P^2+QP)^d = PP^d + (QP)^\pi \sum_{i=0}^{l-1}(QP)^{i+1}(P^d)^{2i+2} + QP\sum_{i=0}^{s-1}((QP)^d)^{i+1}P^{2i}P^\pi,\\ &\sum_{i=0}^{r-1}((PQ)^d)^{i+1}(P^2+QP)^i(P^2+QP)^\pi = \sum_{i=0}^{r-1}((PQ)^d)^{i+1}P^{2i}P^\pi.\end{aligned} \tag{17}$$

From $(PQ)^dQ = 0, Q(QP)^d = 0$, we obtain

$$\begin{aligned}(PQ)^\pi(P^2+QP)^d =&(PQ)^\pi\left[(QP)^\pi \sum_{i=0}^{l-1}(QP)^i(P^d)^{2i+2} + \sum_{i=0}^{s-1}((QP)^d)^{i+1}P^{2i}P^\pi\right]\\ =&(QP)^\pi \sum_{i=0}^{l-1}(QP)^i(P^d)^{2i+2} - PQ(PQ)^d(P^d)^2 + \sum_{i=0}^{s-1}((QP)^d)^{i+1}P^{2i}P^\pi.\end{aligned} \tag{18}$$

Since $(P^2+QP)PQP = 0, Q^2 = 0$, substituting Equation (15)–(18) into Equation (13), we obtain

$$
\begin{aligned}
(P+Q)^d =& \Big[\sum_{i=0}^{r-1}((PQ)^d)^{i+1}(P^2+QP)^i(P^2+QP)^\pi \\
&+ (PQ)^\pi \sum_{i=0}^{s_1-1}(PQ)^i((P^2+QP)^d)^{i+1}\Big](P+Q) \\
=& \Big[\sum_{i=0}^{r-1}((PQ)^d)^{i+1}P^{2i}P^\pi + (PQ)^\pi \sum_{i=1}^{s_1-1}(PQ)^i(P^d)^{2i+2} - PQ(PQ)^d(P^d)^2 \\
&+ (QP)^\pi \sum_{i=0}^{l-1}(QP)^i(P^d)^{2i+2} + \sum_{i=0}^{t-1}((QP)^d)^{i+1}P^{2i}P^\pi\Big](P+Q) \\
=& \Big[\sum_{i=0}^{r-1}((PQ)^d)^{i+1}P^{2i}P^\pi + (PQ)^\pi \sum_{i=0}^{s_1-1}(PQ)^i(P^d)^{2i+2} - (P^d)^2 \\
&+ (QP)^\pi \sum_{i=0}^{l-1}(QP)^i(P^d)^{2i+2} + \sum_{i=0}^{t-1}((QP)^d)^{i+1}P^{2i}P^\pi\Big](P+Q).
\end{aligned}
$$

Let $s = \max\{s_1, l\}$ and $t \geq r$; we thus conclude the representation for $(P+Q)^d$. □

Theorem 4. *Let $P, Q \in \mathbb{C}^{n\times n}$. If $PQP^2 = 0$, $Q^2 = 0$, then*

$$
\begin{aligned}
(P+Q)^d =&(P+Q)\Big(\sum_{i=0}^{t-1}P^\pi P^{2i}\big(((PQ)^d)^{i+1} + ((QP)^d)^{i+1}\big) - (P^d)^2 \\
&+ \sum_{i=0}^{s-1}(P^d)^{2(i+1)}\big((PQ)^i(PQ)^\pi + (QP)^i(QP)^\pi\big)\Big),
\end{aligned}
$$

where $t = \mathrm{ind}(P^2)$, $s = \max\{\mathrm{ind}(PQ), \mathrm{ind}(QP)\}$.

Applying Theorem 3, we can easily deduce the formula of $(P+Q)^d$ under the conditions $P^2Q = 0$ and $Q^2 = 0$ in Ref. [5].

4. Representations for the Drazin Inverse of Anti-Triangular Block Matrices

This section is devoted to the Drazin inverse of 2×2 anti-triangular block matrix

$$
M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \tag{19}
$$

where $A \in \mathbb{C}^{m\times m}$, $0 \in \mathbb{C}^{n\times n}$.

First, as the application of Theorem 3, we give some representations for the Drazin inverse of M.

Theorem 5. *Let M be the form as in (19). If $ABCAA^\pi = 0$, $ABCA^\pi B = 0$, and $A^dBC = 0$, then*

$$
M^d = \begin{pmatrix} A^d + \Sigma & (A^d)^2B + \Lambda \\ \Omega & \Phi \end{pmatrix}, \tag{20}
$$

where

$$\Sigma = \sum_{i=0}^{l-1}(BC)^i(BC)^\pi\Gamma(A^d)^{2i+1} + \sum_{i=0}^{r-1}((BC)^d)^{i+1}(A^{2i}\Upsilon + (BC)^d A^{2i+1}BCA^\pi),$$

$$\Lambda = \sum_{i=0}^{l-1}(BC)^\pi(BC)^i\Gamma(A^d)^{2i+2}B + \sum_{i=0}^{r-1}((BC)^d)^{i+1}A^{2i}(A^\pi - \Gamma)B,$$

$$\Omega = -C(BC)^d A\Gamma A^d + \delta + \sum_{i=0}^{r-1}C((BC)^d)^{i+2}A^{2i}(\Upsilon A + BC),$$

$$\Phi = -C(BC)^d A\Gamma(A^d)^2B + \delta A^d B + \sum_{i=0}^{r-1}C((BC)^d)^{i+2}A^{2i}\Upsilon B,$$

$$\delta = \sum_{i=0}^{l-1}C(BC)^\pi(BC)^i(A^d)^{2i+2} + C(BC)^\pi(BC)^i A\Gamma(A^d)^{2i+3},$$

$$\Upsilon = AA^\pi - \Gamma A, \quad \Gamma = \sum_{n=0}^{p-1}A^nBC(A^d)^{n+2},$$

and $\mathrm{ind}(A) = p$, $\mathrm{ind}(BC) = l$, $r = [\frac{p}{2}]$.

Proof. We split the matrix $M = P + Q$, where

$$P = \begin{pmatrix} A & B \\ CAA^d & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix}.$$

Obviously, $Q^2 = 0$. Since $ABCAA^\pi = 0$, $ABCA^\pi B = 0$, we have $P^2QP = 0$. Applying Theorem 3, it yields that

$$(P+Q)^d = \Bigg(\sum_{i=0}^{s-1}\big((PQ)^\pi(PQ)^i + (QP)^\pi(QP)^i\big)(P^d)^{2(i+1)} - (P^d)^2 + \sum_{i=0}^{t-1}\big(((PQ)^d)^{i+1} + ((QP)^d)^{i+1}\big)P^{2i}P^\pi\Bigg)(P+Q), \quad (21)$$

where $t = \mathrm{ind}(P^2)$, $s = \max\{\mathrm{ind}(PQ), \mathrm{ind}(QP)\}$.

From $A^dBC = 0$, we obtain $A^d(BC)^d = 0$, $A^\pi BC = BC$, $A^\pi(BC)^d = (BC)^d$. Since $A^\pi BC = BC$, by Lemma 1, we have

$$(CA^\pi B)^d = C((A^\pi BC)^d)^2A^\pi B = C((BC)^d)^2A^\pi B,$$

$$(BCA^\pi)^d = BC\big((A^\pi BC)^d\big)^2A^\pi = (BC)^dA^\pi.$$

From Lemma 3, it follows that

$$(QP)^d = \begin{pmatrix} 0 & 0 \\ C((BC)^d)^2A^\pi A & C((BC)^d)^2A^\pi B \end{pmatrix},$$

$$(QP)^\pi = \begin{pmatrix} I & 0 \\ -C(BC)^dA^\pi A & I - C(BC)^dA^\pi B \end{pmatrix},$$

and for any positive integer i, we can verify that

$$(QP)^i = \begin{pmatrix} 0 & 0 \\ C(BC)^{i-1}A^\pi A & C(BC)^{i-1}A^\pi B \end{pmatrix},$$

$$((QP)^d)^i = \begin{pmatrix} 0 & 0 \\ C((BC)^d)^{i+1}A^\pi A & C((BC)^d)^{i+1}A^\pi B \end{pmatrix}.$$

Similarly, by Lemma 3, for any positive integer i, we have

$$((PQ)^d)^i = \begin{pmatrix} ((BC)^d)^i A^\pi & 0 \\ 0 & 0 \end{pmatrix}.$$

From $(BCA^\pi)^\pi = I - BC(BC)^d A^\pi$, for any positive integer j, we can verify that

$$(PQ)^\pi = \begin{pmatrix} I - BC(BC)^d A^\pi & 0 \\ 0 & I \end{pmatrix},$$

$$(PQ)^j (PQ)^\pi = \begin{pmatrix} (BC)^j (BC)^\pi A^\pi & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $BCAA^dA^\pi = 0$, $AA^dBCAA^d = 0$, matrix P satisfies the condition of Lemma 4 and therefore

$$P^d = \begin{pmatrix} (I+\Gamma)A^d & (I+\Gamma)(A^d)^2 B \\ C(A^d)^2 & C(A^d)^3 B \end{pmatrix},$$

where $\Gamma = \sum_{n=0}^{p-1} A^n BC(A^d)^{n+2}$. From $A^dBC = 0$, for any positive integer j, we obtain

$$(P^d)^j = \begin{pmatrix} (I+\Gamma)(A^d)^j & (I+\Gamma)(A^d)^{j+1}B \\ C(A^d)^{j+1} & C(A^d)^{j+2}B \end{pmatrix}.$$

Since $\Gamma AA^d = \Gamma$, for any positive integer k, we obtain

$$P^\pi = \begin{pmatrix} A^\pi - \Gamma & -(I+\Gamma)A^dB \\ -CA^d & I - C(A^d)^2B \end{pmatrix},$$

$$P^{2k}P^\pi = \begin{pmatrix} A^{2k-1}(AA^\pi - \Gamma A) & A^{2k-2}(AA^\pi - \Gamma A)B \\ 0 & 0 \end{pmatrix}.$$

For any nonnegative integer i, we have

$$((PQ)^d)^{i+1} + ((QP)^d)^{i+1} = \begin{pmatrix} ((BC)^d)^{i+1}A^\pi & 0 \\ C((BC)^d)^{i+2}AA^\pi & C((BC)^d)^{i+2}A^\pi B \end{pmatrix}.$$

Therefore, we obtain

$$\begin{aligned}&\sum_{i=1}^{t-1} \left(((PQ)^d)^{i+1} + ((QP)^d)^{i+1}\right) P^{2i}P^\pi \\ =&\sum_{i=1}^{r-1} \begin{pmatrix} ((BC)^d)^{i+1}A^\pi A^{2i-1}\Upsilon & ((BC)^d)^{i+1}A^\pi A^{2i-2}\Upsilon B \\ C((BC)^d)^{i+2}AA^\pi A^{2i-1}\Upsilon & C((BC)^d)^{i+2}AA^\pi A^{2i-2}\Upsilon B \end{pmatrix},\end{aligned}$$

where $r = [\frac{p}{2}]$, $\Upsilon = AA^\pi - \Gamma A$.

From $A^dBC = 0$, $\Gamma A = BCA^d + A\Gamma$, we obtain $\Gamma BC = 0$, $A^\pi BC = BC$, $\Gamma ABC = 0$, $A^\pi\Gamma = \Gamma$, $A\Gamma A = \Gamma A^2 - BCAA^d$, $\Gamma A^2 = BCAA^d + A\Gamma A$, we compute

$$\begin{aligned}&\sum_{i=1}^{t-1} \left(((PQ)^d)^{i+1} + ((QP)^d)^{i+1}\right) P^{2i}P^\pi(P+Q) \\ =&\sum_{i=1}^{r-1} \begin{pmatrix} ((BC)^d)^{i+1}A^{2i}\Upsilon + ((BC)^d)^{i+1}A^{2i-1}BCA^\pi & ((BC)^d)^{i+1}A^{2i-1}\Upsilon B \\ C((BC)^d)^{i+2}A^{2i}\Upsilon A + C((BC)^d)^{i+2}A^{2i}BC & C((BC)^d)^{i+2}A^{2i}\Upsilon B \end{pmatrix},\end{aligned}$$

where $\Upsilon = AA^{\pi} - \Gamma A$. Further,

$$\begin{aligned}&((PQ)^d + (QP)^d)P^{\pi}\\ &= \begin{pmatrix} (BC)^d A^{\pi} - (BC)^d\Gamma & -(BC)^d\Gamma A^d B \\ C((BC)^d)^2 AA^{\pi} - C((BC)^d)^2 A\Gamma - C((BC)^d)^2 BCA^d & \Theta \end{pmatrix},\end{aligned}$$

where

$$\Theta = -C((BC)^d)^2 A\Gamma A^d B + C((BC)^d)^2 A^{\pi} B - C((BC)^d)^2 BC(A^d)^2 B.$$

Therefore,

$$\begin{aligned}&((PQ)^d + (QP)^d)P^{\pi}(P+Q)\\ &= \begin{pmatrix} (BC)^d\Upsilon & ((BC)^d A^{\pi} - (BC)^d\Gamma)B \\ C((BC)^d)^2(A^2A^{\pi} - \Gamma A^2) + C(BC)^d & C((BC)^d)^2(AA^{\pi} - \Gamma A)B \end{pmatrix}.\end{aligned}$$

Then

$$\begin{aligned}&\sum_{i=0}^{t-1} \left(((PQ)^d)^{i+1} + ((QP)^d)^{i+1}\right)P^{2i}P^{\pi}(P+Q)\\ &= \sum_{i=0}^{r-1} \begin{pmatrix} ((BC)^d)^{i+1}(A^{2i}\Upsilon + (BC)^d A^{2i+1}BCA^{\pi}) & ((BC)^d)^{i+1}A^{2i}(A^{\pi} - \Gamma)B \\ C((BC)^d)^{i+2}A^{2i}(\Upsilon A + BC) & C((BC)^d)^{i+2}A^{2i}\Upsilon B \end{pmatrix}.\end{aligned} \tag{22}$$

For any positive integer i, we can prove by induction that

$$\begin{aligned}&(QP)^i(QP)^{\pi}\\ &= \begin{pmatrix} 0 & 0 \\ (CB)^{\pi}C(BC)^{i-1}AA^{\pi} & (CB)^{\pi}C(BC)^{i-1}A^{\pi}B \end{pmatrix},\end{aligned}$$

$$\begin{aligned}&(PQ)^{\pi}(PQ)^i + (QP)^{\pi}(QP)^i\\ &= \begin{pmatrix} (BC)^i(BC)^{\pi}A^{\pi} & 0 \\ (CB)^{\pi}C(BC)^{i-1}AA^{\pi} & (CB)^{\pi}C(BC)^{i-1}A^{\pi}B \end{pmatrix},\end{aligned}$$

let $\Delta_i = (PQ)^{\pi}(PQ)^i + (QP)^{\pi}(QP)^i$, we have

$$\Delta_i(P^d)^{2i} = \begin{pmatrix} (BC)^i(BC)^{\pi}\Gamma(A^d)^{2i} & (BC)^i(BC)^{\pi}\Gamma(A^d)^{2i+1}B \\ \alpha_i & \alpha_i A^d B \end{pmatrix}, \tag{23}$$

where $\alpha_i = C(BC)^{\pi}(BC)^{i-1}A\Gamma(A^d)^{2i} + C(BC)^{\pi}(BC)^i(A^d)^{2i+1}$. Further,

$$(P^d)^2(P+Q) = \begin{pmatrix} (I+\Gamma)A^d & (I+\Gamma)(A^d)^2B \\ C(A^d)^2 & C(A^d)^3B \end{pmatrix},$$

therefore, we have

$$\begin{aligned}&\sum_{i=1}^{s-1} \Delta_i(P^d)^{2(i+1)}(P+Q)\\ &= \sum_{i=1}^{l-1} \begin{pmatrix} (BC)^i(BC)^{\pi}\Gamma(A^d)^{2i+1} & (BC)^i(BC)^{\pi}\Gamma(A^d)^{2i+2}B \\ \alpha_i A^d & \alpha_i(A^d)^2B \end{pmatrix},\end{aligned}$$

where α_i is shown in Equation (23). In addition, by Equation (21), calculate that

$$\begin{aligned}&((PQ)^{\pi} + (QP)^{\pi} - I)(P^d)^2(P+Q)\\ &= \begin{pmatrix} A^d + (BC)^{\pi}\Gamma A^d & (A^d)^2B + (BC)^{\pi}\Gamma(A^d)^2B \\ C(BC)^{\pi}(A^d)^2 - C(BC)^d A\Gamma A^d & -C(BC)^d A\Gamma(A^d)^2B + C(BC)^{\pi}(A^d)^3B \end{pmatrix},\end{aligned}$$

thus, we obtain

$$\left(\sum_{i=0}^{s-1}\left((PQ)^{\pi}(PQ)^{i}+(QP)^{\pi}(QP)^{i}\right)(P^{d})^{2i+2}-(P^{d})^{2}\right)(P+Q)$$
$$=\begin{pmatrix} A^{d}+\sum_{i=0}^{l-1}(BC)^{i}(BC)^{\pi}\Gamma(A^{d})^{2i+1} & (A^{d})^{2}B+\sum_{i=0}^{l-1}(BC)^{\pi}(BC)^{i}\Gamma(A^{d})^{2i+2}B \\ -C(BC)^{d}A\Gamma A^{d}+\delta & -C(BC)^{d}A\Gamma(A^{d})^{2}B+\delta A^{d}B \end{pmatrix}, \tag{24}$$

where $\delta=\sum_{i=0}^{l-1}C(BC)^{\pi}(BC)^{i}(A^{d})^{2i+2}+C(BC)^{\pi}(BC)^{i}A\Gamma(A^{d})^{2i+3}$.

Finally, substituting Equations (22) and (24) into Equation (21), we conclude the representation for M^{d}. □

The following conditions are discussed in Ref. [28].

Corollary 2. *Let M be the form as in (19). If $ABCA^{\pi}=0$, $A^{d}BC=0$, then*

$$M^{d}=\begin{pmatrix} A^{d}+\widetilde{\Sigma} & (A^{d})^{2}B+\Lambda \\ \Omega & \Phi \end{pmatrix},$$

where $\Upsilon,\Gamma,\Lambda,\Omega,\Phi,\delta$ are defined by Equation (20), $\widetilde{\Sigma}=\sum_{i=0}^{l-1}(BC)^{i}(BC)^{\pi}\Gamma(A^{d})^{2i+1}+\sum_{i=0}^{r-1}((BC)^{d})^{i+1}A^{2i}\Upsilon$, and $\mathrm{ind}(A)=p$, $r=[\frac{p}{2}]$, $\mathrm{ind}(BC)=l$.

As corollary of Theorem 5, the following results were discussed in Refs. [5,22], respectively.

Corollary 3. *Let M be the form as in (19). If $ABC=0$, then*

$$M^{d}=\begin{pmatrix} \widetilde{\Phi}A & \widetilde{\Phi} \\ C\widetilde{\Phi} & C\widetilde{\Phi}^{2}AB \end{pmatrix},$$

where $\widetilde{\Phi}=\sum_{j=0}^{l-1}(BC)^{j}(BC)^{\pi}(A^{d})^{2j+2}+\Phi$, $\Phi=\sum_{k=0}^{r}((BC)^{d})^{k+1}A^{2k}A^{\pi}$, and $\mathrm{ind}(A)=p$, $r=[\frac{p}{2}]$, $\mathrm{ind}(BC)=l$.

In particular, if $BC=0$, we obtain $\widetilde{\Phi}=(A^{d})^{2}$.

Corollary 4. *Let M be the form as in (19). If BCA^{π} is nilpotent matrix, $ABCA^{\pi}=0$, $AA^{d}BC=0$, then*

$$M^{d}=\begin{pmatrix} \Psi A & \Psi B \\ C\Psi & C\Psi A^{d}B \end{pmatrix},$$

where $\Psi=(A^{d})^{2}+\sum_{j=0}^{l-1}(BC)^{j}\Gamma(A^{d})^{2j+2}$, $\Gamma=\sum_{n=0}^{p-1}A^{n}BC(A^{d})^{n+2}$, and $\mathrm{ind}(BC)=l$, $\mathrm{ind}(A)=p$.

The following result is obtained by applying Theorem 4.

Theorem 6. *Let M be the form as in (19). If $BCAA^{\pi}=0$, $(BC)^{d}A^{\pi}B=0$, and $CBCA^{d}=0$, $A^{d}BC=0$, then*

$$M^{d}=\begin{pmatrix} \sum_{i=0}^{r-1}A^{2i+1}((BC)^{d})^{i+1}A^{\pi}+\widetilde{\Psi} & \widetilde{\Psi}A^{d}B \\ \sum_{i=0}^{r-1}CA^{2i}((BC)^{d})^{i+1}A^{\pi}+C\widetilde{\Psi}(A^{d})^{2} & C\widetilde{\Psi}(A^{d})^{2}B \end{pmatrix},$$

where $\widetilde{\Psi}=A^{d}+\sum_{n=0}^{p}A^{n}BC(A^{d})^{n+3}$, and $\mathrm{ind}(A)=p$, $r=[\frac{p}{2}]$.

Proof. Consider the splitting of matrix M,

$$M = \begin{pmatrix} A & B \\ CAA^d & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix} := P + Q.$$

Obviously, $Q^2 = 0$. From $A^dBC = 0$, $BCAA^\pi = 0$, $CBCA^d = 0$, we obtain $PQP^2 = 0$. Since $A^dBC = 0$, by Theorem 5, we have

$$(QP)^d = \begin{pmatrix} 0 & 0 \\ C((BC)^d)^2A^\pi A & C((BC)^d)^2A^\pi B \end{pmatrix}.$$

From $BCAA^\pi = 0$, $(BC)^dA^\pi B = 0$, we obtain $(QP)^d = 0$.

Applying Theorem 4, we obtain

$$\begin{aligned}(P+Q)^d =&(P+Q)\Bigg(\sum_{i=0}^{s-1}(P^d)^{2(i+1)}\big((PQ)^i(PQ)^\pi + (QP)^i\big) \\ &+ \sum_{i=0}^{t-1} P^\pi P^{2i}((PQ)^d)^{i+1} - (P^d)^2\Bigg),\end{aligned} \tag{25}$$

where $t = \text{ind}(P^2)$, $s = \max\{\text{ind}(PQ), \text{ind}(QP)\}$.

Similar to the proof of Theorem 5, we can obtain

$$(P+Q)\sum_{i=0}^{t-1} P^\pi P^{2i}((PQ)^d)^{i+1} = \begin{pmatrix} \sum_{i=0}^{r-1} A^{2i+1}((BC)^d)^{i+1}A^\pi & 0 \\ \sum_{i=0}^{r-1} CA^{2i}((BC)^d)^{i+1}A^\pi & 0 \end{pmatrix}. \tag{26}$$

From $A^dBC = 0$, we compute

$$(P^d)^2\big((PQ)^\pi + I\big) - (P^d)^2 = \begin{pmatrix} (I+\Gamma)(A^d)^2 & (I+\Gamma)(A^d)^3B \\ C(A^d)^3 & C(A^d)^4B \end{pmatrix},$$

$$\sum_{i=1}^{s-1}(P^d)^{2i+2}\big((PQ)^i(PQ)^\pi + (QP)^i\big) = 0,$$

and then

$$\begin{aligned}&(P+Q)(P^d)^2(PQ)^\pi \\ =&\begin{pmatrix} A(I+\Gamma)(A^d)^2 + BC(A^d)^3 & A(I+\Gamma)(A^d)^3B + BC(A^d)^4B \\ C(I+\Gamma)(A^d)^3 & C(I+\Gamma)(A^d)^3B \end{pmatrix}.\end{aligned}$$

Denoting by $\widetilde{\Psi} = A(I+\Gamma)(A^d)^2 + BC(A^d)^3$ and substituting the above three equations and Equation (26) into Equation (25), we obtain

$$M^d = \begin{pmatrix} \sum_{i=0}^{r-1} A^{2i+1}((BC)^d)^{i+1}A^\pi + \widetilde{\Psi} & \widetilde{\Psi}A^dB \\ \sum_{i=0}^{r-1} CA^{2i}((BC)^d)^{i+1}A^\pi + C(I+\Gamma)(A^d)^3 & C(I+\Gamma)(A^d)^3B \end{pmatrix},$$

since $CA(I+\Gamma) = C(I+\Gamma)A$, we have

$$M^d = \begin{pmatrix} \sum_{i=0}^{r-1} A^{2i+1}((BC)^d)^{i+1}A^\pi + \widetilde{\Psi} & \widetilde{\Psi}A^dB \\ \sum_{i=0}^{r-1} CA^{2i}((BC)^d)^{i+1}A^\pi + C\widetilde{\Psi}(A^d)^2 & C\widetilde{\Psi}(A^d)^2B \end{pmatrix},$$

where $\widetilde{\Psi} = A^d + \sum_{n=0}^{p} A^nBC(A^d)^{n+3}$. □

Next, we give another result by using the additive result of the Drazin inverse of Corollary 1.

Theorem 7. *Let M be the form as in (19). If $ABCAA^{\pi}=0$, $ABCA^{\pi}B=0$, and $CBCAA^{\pi}=0$, $CBCA^{\pi}B=0$, $A^dBCA=0$, then*

$$M^d=\begin{pmatrix} (I+BCA^{\pi}\widehat{\Psi})(\widehat{\Psi}A+\widehat{\Psi}A^dBC) & (I+BCA^{\pi}\widehat{\Psi})\widehat{\Psi}B \\ C\widehat{\Psi}(I+(A^d)^2BC) & C\widehat{\Psi}A^dB \end{pmatrix},$$

where $\widehat{\Psi}=(A^d)^2+\sum_{n=0}^{p}A^nBC(A^d)^{n+4}$, and $\mathrm{ind}(A)=p$.

Proof. Decompose the matrix M as follows:

$$M=\begin{pmatrix} A & B \\ CAA^d & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix}:=P+Q.$$

From $ABCAA^{\pi}=0$, $ABCA^{\pi}B=0$, $A^{\pi}BC=BC$, $CBCAA^{\pi}=0$, $CBCA^{\pi}B=0$, we obtain

$$P^2QP=\begin{pmatrix} ABCA^{\pi}A & ABCA^{\pi}B \\ CAA^dBCA^{\pi}A & CAA^dBCA^{\pi}B \end{pmatrix}=0,$$

$$(QP)^2=\begin{pmatrix} 0 & 0 \\ CA^{\pi}BCAA^{\pi} & CA^{\pi}BCA^{\pi}B \end{pmatrix}=0.$$

Using Corollary 2, we obtain

$$(P+Q)^d=\big(I+PQ(P^d)^2\big)(P^d)^2(P+Q)+Q(P+Q)(P^d)^4(P+Q). \tag{27}$$

Using Lemma 4, for any positive integer j, we obtain

$$(P^d)^j=\begin{pmatrix} \widehat{\Psi}A(A^d)^{j-1} & \widehat{\Psi}(A^d)^{j-1}B \\ C(A^d)^{j+1} & C(A^d)^{j+2}B \end{pmatrix},$$

where $\widehat{\Psi}=(A^d)^2+\sum_{n=0}^{p}A^nBC(A^d)^{n+4}$.

Since $\widehat{\Psi}AA^d=\widehat{\Psi}$, $A^dBCA^d=0$, $\widehat{\Psi}^2=\widehat{\Psi}(A^d)^2$, then

$$\begin{aligned}&\big(I+PQ(P^d)^2\big)(P^d)^2(P+Q)\\ =&\begin{pmatrix} (I+BCA^{\pi}\widehat{\Psi})(\widehat{\Psi}A+\widehat{\Psi}A^dBC) & (I+BCA^{\pi}\widehat{\Psi})\widehat{\Psi}B \\ C(A^d)^2+C(A^d)^4BC & C(A^d)^3B \end{pmatrix},\end{aligned} \tag{28}$$

and

$$\begin{aligned}&Q(P+Q)(P^d)^4(P+Q)\\ =&\begin{pmatrix} 0 & 0 \\ (CAA^{\pi}\widehat{\Psi}A^d+CBC(A^d)^4)(I+(A^d)^2BC) & CAA^{\pi}\widehat{\Psi}(A^d)^2B+CBC(A^d)^5B \end{pmatrix}.\end{aligned} \tag{29}$$

Substituting Equations (28) and (29) into Equation (27) and noting that $CAA^{\pi}\widehat{\Psi}A^d+CBC(A^d)^4+C(A^d)^2=C\widehat{\Psi}$, we obtain

$$M^d=\begin{pmatrix} (I+BCA^{\pi}\widehat{\Psi})(\widehat{\Psi}A+\widehat{\Psi}A^dBC) & (I+BCA^{\pi}\widehat{\Psi})\widehat{\Psi}B \\ C\widehat{\Psi}(I+(A^d)^2BC) & C\widehat{\Psi}A^dB \end{pmatrix}.$$

Thus, the statement of the theorem is valid. □

The following Corollary 5 was discussed in Ref. [28].

Corollary 5. *Let M be the form as in (19). If $BCAA^{\pi} = 0$, $BCA^{\pi}B = 0$, $A^{d}BCA = 0$, then*

$$M^{d} = \begin{pmatrix} \widehat{\Psi}A\big(I + (A^{d})^{2}BC\big) & \widehat{\Psi}B \\ C\widehat{\Psi}\big(I + (A^{d})^{2}BC\big) & C\widehat{\Psi}iA^{d}B \end{pmatrix},$$

where $\widehat{\Psi} = (A^{d})^{2} + \sum_{n=0}^{p} A^{n}BC(A^{d})^{n+4}$, and $\mathrm{ind}(A) = p$.

Proof. This result follows from Theorem 7 by noting that $BCA^{\pi}\widehat{\Psi} = 0$. □

5. Examples

In this section, we give some applications of the theorems in Sections 3 and 4.

Example 1. *Let*

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It can be checked that

$$PQ^{2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

therefore $PQ^{2} \neq 0$. One of the conditions from Corollary 1 is not be satisfied, but it is easy to verify that P and Q satisfy the conditions in Theorems 1 and 2.

Example 2. *Let*

$$P = Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It can be checked that

$$PQ^{2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

therefore $PQ^{2} \neq 0$. One of the conditions from Corollary 1 is not be satisfied, but it is easy to verify that P and Q satisfy the conditions in Theorems 1 and 2.

Example 3. *Let*

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, Q = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since

$$P^{2}Q = \begin{pmatrix} 0 & -3 & 3 \\ 0 & -3 & 3 \\ 0 & -3 & 3 \end{pmatrix}$$

and $Q^{2} = 0$, therefore $P^{2}QP = 0, PQP^{2} = 0, P^{2}Q \neq 0$. The matrix P and Q do not satisfy the conditions given in Corollary 1, but satisfies that in Theorems 3 and 4.

Example 4. *Let* $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, *where* $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

$$B = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 2 & 2 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $ABCA^{\pi} \neq 0$, *the conditions given in Corollaries 2–4 are not satisfied. On the other hand, we can check that M satisfy the conditions of Theorem 5.*

Example 5. *Let* $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, *where* $A = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Since $BCAA^{\pi} \neq 0, BCA^{\pi}B \neq 0$, *the conditions mentioned in Corollary 5 are not satisfied. However, we can check that M satisfies the conditions of Theorem 7.*

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