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Editorial

On the Special Issue "Limit Theorems of Probability Theory"

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M. Loeve wrote that "the fundamental limit theorems of Probability theory may be classified into two groups. One group deals with the problem of limit laws of sequences of some of random variables, the other deals with the problem of limits of random variables, in the sense of almost sure convergence, of such sequences. These problems will be labeled, respectively, the Central Limit Problem (CLP) and the Strong Central Limit Problem (SCLP). Like all mathematical problems, the CLP and SCLP are not static; as answers to old queries are discovered they experience the usual development and new problems arise".

The papers in this Special Issue present new directions and new advances for limit theorems in probability theory and its applications. The list of topics is extensive, and it includes classical models of sums of both independent and various types of dependent random variables, probabilities of large deviations, functional limit theorems, and limit theorems for random processes, in high-dimensional spaces, for spectra of random matrices and random graphs, and more.

In [1], Xia Wang and Miaomiao Zhang obtain a large deviation principle for the maximum of the absolute value of partial sums of independent, identically distributed, centered, random variables. It is assumed that tail probabilities for "positive" and "negative" tails of the summand have the same exponential decrease.

Estimating the expected value of a random variable via data-driven methods is one of the most fundamental problems in statistics. In [2], Rundong Luo, Yiming Chen, and Shuai Song present an extension of Olivier Catoni's classical M-estimators of the empirical mean, which focus on heavy-tailed data by imposing more precise inequalities on exponential moments of Catoni's estimator. The authors show that their estimators behave better than Catoni's estimators, both in practice and theory. The results obtained are illustrated on modeled and real data.

Paper [3], by Friedrich Götze and Andrei Yu Zaitsev, deals with studying a connection of the Littlewood–Offord problem to estimations of the concentration functions of some symmetric, infinitely divisible distributions. It is shown that the concentration function of a weighted sum of independent, identically distributed, random variables is estimated in terms of the concentration function of a symmetric, infinitely divisible distribution, whose spectral measure is concentrated on the set of plus–minus weights.

There has been a renewed interest in exponential concentration inequalities for stochastic processes in probability and statistics over the last three decades. De la Peña established a good exponential inequality for a discrete time, locally square, integrable martingale. In [4],, Naiqi Liu, Vladimir V. Ulyanov, and Hanchao Wang obtain de la Peña's inequalities for a stochastic integral of multivariate point processes. The proof is primarily based on the Doléans-Dade exponential formula and the optional stopping theorem. As an application, they obtain an exponential inequality for block counting process in the Λ–coalescent.

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In [5], Alexander N. Tikhomirov and Dmitry A. Timushev prove the local Marchenko–Pastur law for sparse sample covariance matrices that corresponded to rectangular observation matrices and sparse probability. The new bounds of the distance between Laplace transforms of the empirical spectral distribution function of the sparse sample covariance matrices and the Marchenko–Pastur law distribution function are obtained in the complex domain. It is assumed that a sparse probability and the moments of the matrix elements satisfy some conditions.

In see [6], Mihailo Jovanović, Vladica Stojanović, Kristijan Kuk, Brankica Popović, and Petar Čisar describe one of the non-linear (and non-stationary) stochastic models, the Gaussian, or Generalized, Split-BREAK (GSB) process, which is used in the analysis of time series with pronounced and accentuated fluctuations. In the beginning, the stochastic structure of the GSB process and its important distributional and asymptotic properties are given. To that end, a method based on characteristic functions (CFs) was used. Various procedures for the estimation of model parameters, asymptotic properties, and numerical simulations of the obtained estimators are also investigated. Finally, as an illustration of the practical application of the GSB process, an analysis of the dynamics and stochastic distribution of the infected and immunized populations in relation to COVID-19 in the Republic of Serbia is presented.

The Poisson Stochastic Index process (PSI-process) represents a special kind of a random process, when the discrete time of a random sequence is replaced by the continuous time of a "counting" process of a Poisson type. In [7], Yuri Yakubovich, Oleg Rusakov, and Alexander Gushchin establish a functional limit theorem for normalized cumulative sums of PSI-processes in the Skorokhod space. This theorem can be used in different ways. The PSI-processes are very simple, and some results can be obtained directly for their sums and imply the corresponding facts of the limiting stationary Gaussian process. On the other hand, the theory of stationary Gaussian processes has been deeply developed in the last few decades, and some results of this theory can have consequences for pre-limiting processes, which model a number of real life phenomena.

In [8], Igor Borisov and Maman Jetpisbaev consider a class of additive functionals of a finite or countable collection of the group frequencies of an empirical point process that corresponds to, at most, a countable partition of the sample space. Under broad conditions, it is shown that the asymptotic behavior of the distributions of such functionals is similar to the behavior of the distributions of the same functionals of the accompanying Poisson point process. However, the Poisson versions of the additive functionals under consideration, unlike the original ones, have the structure of sums (finite or infinite) of independent random variables, which allows them to reduce the asymptotic analysis of the distributions of additive functionals of an empirical point process to classical problems of the theory of summation of independent random variables.

In [9], Shuya Kanagawa investigates asymptotic expansions for U-statistics and V-statistics with degenerate kernels, and finds the order estimates for the remainder terms. It implies the corresponding results for the Cramér–von Mises statistics of a uniform distribution on (0,1). The scheme of the proof is based on three steps. The first one is the almost certain convergence in a Fourier series expansion of the kernel function. The key condition for the convergence is the nuclearity of a linear operator defined by the kernel function. The second one is a representation of U-statistics or V-statistics, by single sums of Hilbert space valued random variables. The third one is the application of asymptotic expansions for single sums of Hilbert space valued random variables.

In [10], Alexander Bulinski and Nikolay Slepov study the convergence rate in the famous Rényi theorem by means of the Stein method refinement. Namely, it is demonstrated that the new estimate of the convergence rate of the normalized geometric sums to exponential laws involving the ideal probability metric of the second order is sharp. Some recent results concerning the convergence rates in Kolmogorov and Kantorovich metrics are extended as well. In contrast to many previous works, there are no assumptions that the summands of geometric sums are positive and have the same distribution. For the first time, an analogue of the Rényi theorem is established for the model of exchangeable random variables. Furthermore, within this model, a sharp estimate of convergence rate to a specified mixture of distributions is provided. The convergence rate of the appropriately normalized random sums of random summands to the generalized gamma distribution is estimated. Here, the number of summands follows the generalized negative binomial law. The sharp estimates of the proximity of random sums of random summand distributions to the limit law are established both for independent summands and for the model of exchangeable ones. The inverse to the equilibrium transformation of the probability measures is introduced and, in this way, a new approximation of the Pareto distributions by exponential laws is proposed. The integral probability metrics, and the techniques of integration with respect to sign measures, are essentially employed.

In [11], Yasunori Fujikoshi and Tetsuro Sakurai consider the high-dimensional consistencies of KOO methods for selecting response variables in multivariate linear regression with some covariance structures. The method, which was named the knock-one-out (KOO) method, determines "selection" or "no selection" for each variable by comparing the model that removes that variable and the full model. It is assumed that the covariance structure is one of three covariance structures: (1) an independent covariance structure with the same variance, (2) an independent covariance structure with different variances, and (3) a uniform covariance structure. A sufficient condition for model selection consistency is obtained using a KOO method under a high-dimensional asymptotic framework, such that sample size, the number of response variables, and the number of explanatory variables are large.

In [12], Alexander N. Tikhomirov considers the limit of the empirical spectral distribution of Laplace matrices of generalized random graphs. Applying the Stieltjes transform method, the author proves under general conditions that the limit spectral distribution of Laplace matrices converges with the free convolution of the semicircular law and the normal law.

In [13], Gerd Christoph and Vladimir V. Ulyanov complete their studies on the formal construction of asymptotic approximations for statistics based on a random number of observations. Second-order Chebyshev–Edgeworth expansions of asymptotically normally or chi-squared distributed statistics from samples with negative binomial or Pareto-like distributed random sample sizes are obtained. The results can have applications for a wide spectrum of asymptotically normally or chi-square distributed statistics. Random, non-random, and mixed scaling factors for each of the studied statistics produce three different limit distributions. In addition to the expected normal or chi-squared distributions, Student's t-, Laplace, Fisher, gamma, and weighted sums of generalized gamma distributions also occur.

The Kolmogorov and total variation distance between the laws of random variables have upper bounds are represented by the L^1-norm of densities when random variables have densities. In [14], Yoon-Tae Kim and Hyun-Suk Park derive an upper bound, in terms of densities such as the Kolmogorov and total variation distance, for several probabilistic distances (e.g., Kolmogorov distance, total variation distance, Wasserstein distance, Forter–Mourier distance, etc.) between the laws of F and G in the case where a random variable F follows the invariant measure that admits a density and a differentiable random variable G, in the sense of Malliavin calculus, and also allows a density function.

In [15], Manuel L. Esquível and Nadezhda P. Krasii describe the structure of the random matrices by deterministic matrices, forming the skeletons of the random matrices. The authors propose to use an algorithm of matrix substitutions with entries in a finite field of integers that modulo some prime number, akin to the algorithm of one dimensional automatic sequences. A random matrix has the structure of a given skeleton if, to the same number of an entry of the skeleton in the finite field, it corresponds a random variable having, at least, as its expected value, the correspondent value of the number in the finite field. Affine matrix substitutions are introduced, and fixed-point theorems that allow for the consideration of steady states of the structure, which are essential for an efficient observation, are proven. For some more restricted classes of structured random matrices, the parameter estimation of the entries is addressed, as well as the convergence in law, and also some aspects of the spectral analysis of the random operators associated with the random matrix. Finally, aiming at possible applications, it is shown that there is a procedure to associate a canonical random surface to every random structured matrix of a certain class.

In summary, this Special Issue proposes and develops new mathematical methods and approaches, new algorithms and research frameworks, and their applications to solve various nontrivial practical problems. We strongly believe that the selected topics and results will be attractive and useful to the international scientific community, and will contribute to further research in the field of limit theorems in probability theory.

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