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Synchronization of Takagi–Sugeno Fuzzy Time-Delayed Stochastic Bidirectional Associative Memory Neural Networks Driven by Brownian Motion in Pre-Assigned Settling Time

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Abstract: We are devoted, in this paper, to the study of the pre-assigned-time drive-response synchronization problem for a class of Takagi–Sugeno fuzzy logic-based stochastic bidirectional associative memory neural networks, driven by Brownian motion, with continuous-time delay and (finitely and infinitely) distributed time delay. To achieve the drive-response synchronization between the neural network systems, concerned in this paper, and the corresponding response neural network systems (identical to our concerned neural network systems), we bring forward, based on the structural properties, a class of control strategies. By meticulously coining an elaborate Lyapunov–Krasovskii functional, we prove a criterion guaranteeing the desired pre-assigned-time drive-response synchronizability: For any given positive time instant, some of our designed controls make sure that our concerned neural network systems and the corresponding response neural network systems achieve synchronization, with the settling times not exceeding the pre-assigned positive time instant. In addition, we equip our theoretical studies with a numerical example, to illustrate that the synchronization controls designed in this paper are indeed effective. Our concerned neural network systems incorporate several types of time delays simultaneously, in particular, they have a continuous-time delay in their leakage terms, are based on Takagi–Sugeno fuzzy logic, and can be synchronized before any pre-given finite-time instant by the suggested control; therefore, our theoretical results in this paper have wide potential applications in the real world. The conservatism is reduced by introducing parameters in our designed Lyapunov–Krasovskii functional and synchronization control.

Keywords: bidirectional associative memory neural networks; pre-assigned-time synchronization; Takagi–Sugeno fuzzy logic; time delays; Lyapunov–Krasovskii functional

MSC: 93E15; 28E10; 34K20; 34K37; 34K50; 60H10

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1. Introduction

In recent years, it was found that neural networks have been widely used in many theoretical and/or application fields; see [1–3] and the vast references cited therein. For example, experts and engineers have already utilized suitable neural networks in vast fields such as optimization theory and the related field applications, associative memories, signal processing, and machine learning. As a result, it is extremely interesting and important to invent neural networks having new structural properties to satisfy specific needs and desires. For instance, in the 1980s, Kosko came up with a class of neural networks, nowadays known as bidirectional associative memory neural networks (BAMNNs), to generalize a single-layer auto-associative Hebbian correlator to two-layer pattern-matched hetero-associative circuits; see References [3–6]. On the other hand, it seems that people are even more interested in quantitatively studying the structural properties of neural networks and in designing the control, based on the obtained structural properties, to improve the properties of neural networks; the related meaningful results can be seen in [1,2,5,7–16], to name just a few of the vast references.

As a typical phenomenon, chaos occurs frequently in complicated nonlinear dynamical systems; see References [3,6,17]. For example, in Reference [17], a nonlinear financial dynamical system was shown, via a numerical approach, to be chaotic. Chaos in the systems could lead to the high sensitivity of trajectories in their initial states. This brings enormous difficulty in applying systems. Therefore, control strategies (synchronization control, for example) should be designed to reduce or even remove the chaos in the systems. For instance, various synchronization problems associated with neural networks have been studied extensively and intensively in recent years; see References [6,7,9,18].

In this paper, we are interested in the synchronization problem for BAMNNs. As with other neural networks, BAMNNs are of wide applicability, for example, they have been frequently exploited in classification, associative memory, signal processing, image processing, parallel computation, combinatorial optimization, and pattern recognition; see References [1,2,4,19]. BAMNNs have their neurons grouped into two layers (the U-layer and the V-layer, as shall be marked in this paper). The neurons of a BAMNN in one layer are fully interconnected to the neurons in the other layer, while there is no interconnection between any two pair of neurons in the same layer; in BAMNNs, the information flows propagate forward and backward between the two layers. Thanks to such a special structure, experts and engineers can realize in BAMNNs a bidirectional associative search for stored bipolar vector pairs; see References [3,4] and some references cited therein for a more detailed explanation on the importance of BAMNNs.

In real-world applications, the switching speed of amplifiers in the electronic implementation of analog neural networks is finite. This leads to the occurrence of a time delay in the communication and response of neurons. And therefore it seems to be more realistic to study the neural networks with time delays. Zhu and Cao [1], Wang and Zhu [2], and Samidurai, Senthilraj et al. [7] studied BAMNNs with various time delays and obtained a criterion guaranteeing the stability of the equilibrium of their concerned BAMNNs. Yuan, Luo et al. [18] investigated a class of time-delayed memristor-based BAMNNs and applied their obtained theoretical results into the field of image hiding. Time delays would cause difficulties in treating problems related to BAMNNs. In recent years, experts have developed many methods to overcome these difficulties; see [9,18,20–24] and the vast references cited therein. For example, Lin and Zhang [20] established several asymptotic synchronization criteria for a class of BAM neural networks with time delays via integrating inequality techniques, Yang, Chen et al. [21] proved their claimed synchronization results concerning BAMNN via convex analysis, and Yang and Zhang [22] applied the quadratic analysis approach to treat a class of delayed BAMNNs.

The realistic neural networks contain unavoidable uncertainty, due to the transmission of information through neurons. It is well-known that fuzzy logic could play an important role in dealing with uncertainty; see References [5,6,25,26]. Wang, Zhao et al. [6] designed, for a class of fuzzy BAMNNs, some intermittent quantized control, and they provided an interesting criterion ensuring that the controlled BAMNNs achieve finite-time drive-response synchronization. Zhou, Zhang et al. [26] considered the finite-time synchronization problem for fuzzy delayed neutral-type inertial BAM neural networks and obtained some novel criteria by applying integral inequality techniques and the figure analysis approach.

Actually, stochastic BAMNNs have also been widely used in many areas and therefore have aroused a large number of experts' interest in studying their dynamics from both mathematical and engineering viewpoints. The synaptic transmission in nervous systems can be considered as a noisy process brought on by random fluctuations from the release of neurotransmitters or other probabilistic factors; this would cause some uncertainty which can not be modeled by fuzzy logic but can be modeled by a special stochastic process, such as general martingales, Lévy processes, Markovian chains (time homogeneous or time inhomogeneous), Brownian motions (Wiener processes), and so on; see References [6,27,28] and the vast references cited therein. For example, the BAMNN concerned in Reference [6] is subject to a Markovian chain. As with fuzzy uncertainty, random (or stochastic) uncertainty causes difficulties in deriving synchronization criteria for BAMNNs.

After reviewing References [1–11,17–24,26–33], we are tempted to further investigate BAMNNs for their synchronizability. In the literature, quite a few interesting results were obtained recently in this direction. For example, the finite-time synchronization problems for BAMNNs were treated systematically in References [34,35], the fixed-time synchronization problems associated with BAMNNs were investigated extensively in [36,37] and the references therein, the pre-assigned-time synchronization problems for BAMNNs were also considered in References [38–44], and some interesting results related to the synchronizability of BAMNNs were presented in References [45–49]. Chen and Zhang [34] as well as Yang and Zhang [35] obtained some finite-time synchronization results for time-delayed BAMNNs via different approaches. As with finite-time synchronizability, fixed-time synchronizability (the synchronization can be realized within a fixed-time instant) seems to have relatively wide applicability but brings on more challenges. Wang, Zhang et al. [36] considered the fixed-time synchronization problem for complex-valued BAMNNs with time-varying delays via (adaptive) pinning control. Duan and Li [37] studied a class of fuzzy neutral-type memristor-based inertial BAMNNs with proportional delays for their fixed-time synchronizability. As mentioned several times above, we consider BAMNNs for their pre-assigned-time synchronizability (the synchronization can be realized within any specified time instant in advance) in this paper. Let us mention here several related results in the literature. Chen, Xiong et al. [44] and Liu, Zhao et al. [43] obtained pre-assigned synchronization results for complex-valued BAMNNs via different approaches. Liu, Zhao et al. [42] applied the pre-assigned synchronization results of complex-valued BAMNNs to image protection. Wang, Zhao et al. [38], Mahemuti and Abdurahman [39], Abdurahman, Abudusaimaiti et al. [40], as well as You, Abdurahman et al. [41] came up with various methods to treat stochastic BAMNNs for their pre-assigned-time synchronizability.

By reviewing the aforementioned references, we conclude that it is interesting to design a pre-assigned-time synchronization control strategy for Takagi–Sugeno logic-based stochastic BAMNNs with continuous-time delay in leakage terms and with continuous-time delay and (finitely/infinitely) distributed-time delay in transmission terms, and it is interesting to provide a criterion ensuring that our concerned BAMNNs (viewed as the drive network systems) and the response BAMNNs, with our proposed control implemented, achieve synchronization within the pre-defined time.

Notational Conventions. We write $\mathbb{R}$ for the totality of real numbers, and $\mathbb{R}_+$, $\mathbb{R}_-$ for the closed interval $[0,+\infty)$, the closed interval $(-\infty,0]$, respectively. D^+f denotes the right upper Dini derivative of the given function f with respect to the independent variable t. $(\mathbb{R},\mathscr{L},dt)$ denotes the usual Lebesgue measure space. We designate by $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$ (or $(\Omega,\mathscr{F},\mathbb{F},d\mathbb{P})$) a complete filtered probability space, in which the filtration $\mathbb{F}=\{\mathcal{F}_t;\, t\in\mathbb{R}_+\}$ is assumed to satisfy the usual conditions; in other words, the σ-algebra $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in the σ-algebra $\mathscr{F}$, and $\mathbb{F}$ is right-continuous in the sense that

$$\bigcap_{s>t}\mathcal{F}_s=\mathcal{F}_t,\quad t\in\mathbb{R}_+.$$

"$\mathbb{P}$ almost surely" is abbreviated as $\mathbb{P}$-a.s.; $\mathbb{E}X$ denotes the mathematical expectation of X, where X is an arbitrarily given random variable on Ω; $(\Omega\times\mathbb{R},\mathscr{L}\otimes\mathscr{F},d\mathbb{P}\times dt)$ denotes the product measure space of $(\mathbb{R},\mathscr{L},dt)$ and $(\Omega,\mathscr{F},d\mathbb{P})$; and $\{W(t);\, t\in\mathbb{R}_+\}$, an $\mathbb{F}$-adapted stochastic process, denotes a one-dimensional standard Brownian motion (Wiener process) defined on the probability space $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$. $A^{\#}$ denotes the cardinality of a set A.

The remainder of this paper is organized as follows. In Section 2, we formulate our concerned synchronization problem for BAMNNs and present some preliminaries necessary for our later description. In Section 3, we state our main result in this paper and provide in detail the proof. In Section 4, we validate, numerically and visually, our theoretical results via coming up with specific example BAMNNs which display the chaos phenomenon and verifying that the example BAMNNs and the corresponding response BAMNNs with our proposed control implemented achieve synchronization within the pre-assigned time. In Section 5, we provide several concluding remarks.

2. Problem Formulation and Preliminaries

In this section, our principal aim is to state our problem and the main mathematical tools to be used to treat our problem. We shall explicitly present our model BAMNNs, explain in some detail the structure of our concerned model BAMNNs, formulate clearly our problem considered in this paper, and prepare some key ingredients to be used in our later treatment of the main problem in this paper.

Let p and r be given positive integers and M_{ij} a fuzzy set, more precisely, M_{ij} a function mapping $\mathbb{R}$ into $[0,1]$, $i = 1,2,...,r$, $j = 1,2,...,p$. In this paper, we assume that our concerned BAMNNs obey the Takagi–Sugeno IF–THEN rule. By the "BAMNNs obey the Takagi–Sugeno IF–THEN rule", we mean IF the premise variable $\xi_j(t)$ is M_{ij}, $j = 1,2,...,p$, THEN the dynamics of our concerned BAMNNs are governed by the following coupled system of forward stochastic differential equations

$$
\begin{cases}
du_\mu(t) = \Bigg[-\sigma_{i\mu} u_\mu(t-\tau_\mu(t)) + \sum\limits_{\nu\in\gimel} b^1_{i\mu\nu} g^1_\nu(v_\nu(t)) \\
\qquad + \sum\limits_{\nu\in\gimel} b^2_{i\mu\nu} g^2_\nu(v_\nu(t-\varrho^1_{\mu\nu}(t))) \\
\qquad + \sum\limits_{\nu\in\gimel} b^3_{i\mu\nu} \int_{t-\varrho^2_{\mu\nu}(t)}^{t} g^3_\nu(v_\nu(s))ds \\
\qquad + \sum\limits_{\nu\in\gimel} b^4_{i\mu\nu} \int_{-\infty}^{t} \Psi^1_{\mu\nu}(t-s) g^4_\nu(v_\nu(s))ds + U^1_{i\mu}(t) \Bigg] dt \\
\qquad + \Bigg[\sum\limits_{\nu\in\gimel} b^5_{i\mu\nu} g^5_\nu(v_\nu(t)) + \sum\limits_{\nu\in\gimel} b^6_{i\mu\nu} g^6_\nu(v_\nu(t-\varrho^3_{\mu\nu}(t))) \\
\qquad + \sum\limits_{\nu\in\gimel} b^7_{i\mu\nu} \int_{t-\varrho^4_{\mu\nu}(t)}^{t} g^7_\nu(v_\nu(s))ds \\
\qquad + \sum\limits_{\nu\in\gimel} b^8_{i\mu\nu} \int_{-\infty}^{t} \Psi^2_{\mu\nu}(t-s) g^8_\nu(v_\nu(s))ds \\
\qquad + U^2_{i\mu}(t) \Bigg] dW(t), \quad t \in \mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\ \mu \in \beth,\ i = 1,2,\dots,r, \\
dv_\nu(t) = \Bigg[-\eta_{i\nu} v_\nu(t-\iota_\nu(t)) + \sum\limits_{\mu\in\beth} a^1_{i\nu\mu} f^1_\mu(u_\mu(t)) \\
\qquad + \sum\limits_{\mu\in\beth} a^2_{i\nu\mu} f^2_\mu(u_\mu(t-\varsigma^1_{\nu\mu}(t))) \\
\qquad + \sum\limits_{\mu\in\beth} a^3_{i\nu\mu} \int_{t-\varsigma^2_{\nu\mu}(t)}^{t} f^3_\mu(u_\mu(s))ds \\
\qquad + \sum\limits_{\mu\in\beth} a^4_{i\nu\mu} \int_{-\infty}^{t} \Phi^1_{\nu\mu}(t-s) f^4_\mu(u_\mu(s))ds + V^1_{i\nu}(t) \Bigg] dt \\
\qquad + \Bigg[\sum\limits_{\mu\in\beth} a^5_{i\nu\mu} f^5_\mu(u_\mu(t)) + \sum\limits_{\mu\in\beth} a^6_{i\nu\mu} f^6_\mu(u_\mu(t-\varsigma^3_{\nu\mu}(t))) \\
\qquad + \sum\limits_{\mu\in\beth} a^7_{i\nu\mu} \int_{t-\varsigma^4_{\nu\mu}(t)}^{t} f^7_\mu(u_\mu(s))ds \\
\qquad + \sum\limits_{\mu\in\beth} a^8_{i\nu\mu} \int_{-\infty}^{t} \Phi^2_{\nu\mu}(t-s) f^8_\mu(u_\mu(s))ds \\
\qquad + V^2_{i\nu}(t) \Bigg] dW(t), \quad t \in \mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\ \nu \in \gimel,\ i = 1,2,\dots,r,
\end{cases}
\tag{1}
$$

supplemented by the initial condition

$$
\left.
\begin{aligned}
u_\mu(t) &= u_{i\mu 0}(t), \quad d\mathbb{P}\times dt\text{-a.e. in } \Omega\times\mathbb{R}_-,\ \mu\in\beth, \\
v_\nu(t) &= v_{i\nu 0}(t), \quad d\mathbb{P}\times dt\text{-a.e. in } \Omega\times\mathbb{R}_-,\ \nu\in\gimel,
\end{aligned}
\right\}
\quad i = 1,2,\dots,r,
\tag{2}
$$

in which the stochastic processes $\{U^1_{i\mu}(t)\}_{t\in\mathbb{R}_+}$, $\{U^2_{i\mu}(t)\}_{t\in\mathbb{R}_+}$, $\{V^1_{i\nu}(t)\}_{t\in\mathbb{R}_+}$, and $\{V^2_{i\nu}(t)\}_{t\in\mathbb{R}_+}$, required to be $\mathbb{F}$-adapted, are given in detail by

$$\left.\begin{aligned}
U^k_{i\mu}(t) &= \sum_{\nu\in\gimel} b^9_{i\mu\nu}\tilde{v}^1_{i\mu\nu}(t) + \sum_{\nu\in\gimel} b^{10}_{i\mu\nu}\tilde{v}^2_{i\mu\nu}(t-\varrho^5_{\mu\nu}(t)) \\
&\quad + \sum_{\nu\in\gimel} b^{11}_{i\mu\nu}\int_{t-\varrho^6_{\mu\nu}(t)}^{t}\tilde{v}^3_{i\mu\nu}(s)ds + \sum_{\nu\in\gimel} b^{12}_{i\mu\nu}\int_{-\infty}^{t}\Psi^3_{\mu\nu}(t-s)\tilde{v}^4_{i\mu\nu}(s)ds \\
&\quad + \tilde{U}^k_{i\mu}(t), \quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\ \mu\in\beth,\ i=1,2,\ldots,r,\ k=1,2, \\
V^k_{i\nu}(t) &= \sum_{\mu\in\beth} a^9_{i\nu\mu}\tilde{u}^1_{i\nu\mu}(t) + \sum_{\mu\in\beth} a^{10}_{i\nu\mu}\tilde{u}^2_{i\nu\mu}(t-\varsigma^5_{\nu\mu}(t)) \\
&\quad + \sum_{\mu\in\beth} a^{11}_{i\nu\mu}\int_{t-\varsigma^6_{\nu\mu}(t)}^{t}\tilde{u}^3_{i\nu\mu}(s)ds + \sum_{\mu\in\beth} a^{12}_{i\nu\mu}\int_{-\infty}^{t}\Phi^3_{\nu\mu}(t-s)\tilde{u}^4_{i\nu\mu}(s)ds \\
&\quad + \tilde{V}^k_{i\nu}(t), \quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\ \nu\in\gimel,\ i=1,2,\ldots,r,\ k=1,2,
\end{aligned}\right\} \tag{3}$$

where $\mathbb{F}$ is the filtration, required to satisfy the usual conditions (see the paragraph of notational conventions in Section 1 for the detailed explanation), of a complete filtered probability space $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$. The stochastic process $\{W(t)\}_{t\in\mathbb{R}_+}$ denotes, throughout this paper, a one-dimensional standard Brownian motion (Wiener process) defined on the probability space $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$. Let us spare here some more lines to explain our model BAMNNs (1)-(2)-(3). $\beth$ and $\gimel$ are two given sets containing finitely many elements (let us remind that $\beth^\#$ and $\gimel^\#$ denote the cardinality of $\beth$ and $\gimel$, respectively). The constants $\sigma_{i\mu}$ and $\eta_{i\nu}$ are the amplification coefficients of the neurons themselves; $f^h_\mu(v)$ and $g^h_\nu(u)$ are activation functions; the real constants $a^\jmath_{i\nu\mu}$ and $b^\jmath_{i\mu\nu}$ are connection coefficients; the functions $\tau_\mu(t)$ and $\iota_\nu(t)$ represent (continuous-)time delays in leakage (also known as forgetting) terms; the functions $\varsigma^1_{\nu\mu}(t)$ and $\varrho^1_{\mu\nu}(t)$, $\varsigma^3_{\nu\mu}(t)$ and $\varrho^3_{\mu\nu}(t)$, as well as $\varsigma^5_{\nu\mu}(t)$ and $\varrho^5_{\mu\nu}(t)$ represent (continuous-)time delays in transmission terms; the functions $\varsigma^2_{\nu\mu}(t)$ and $\varrho^2_{\mu\nu}(t)$, $\varsigma^4_{\nu\mu}(t)$ and $\varrho^4_{\mu\nu}(t)$, as well as $\varsigma^6_{\nu\mu}(t)$ and $\varrho^6_{\mu\nu}(t)$ represent distributed time delays in transmission terms; the functions $\Phi^\ell_{\nu\mu}(t)$ and $\Psi^\ell_{\mu\nu}(t)$ are kernels of the infinitely distributed time delays in transmission terms; the stochastic processes $u_\mu(t)$ and $v_\nu(t)$, required to be $\mathbb{F}$-adapted, are the state trajectories of our concerned BAMNNs (1)-(2)-(3); the stochastic processes $U^1_{i\mu}(t)$, $U^2_{i\mu}(t)$, $V^1_{i\nu}(t)$, and $V^2_{i\nu}(t)$ represent the overall exogenous disturbance; the $\mathbb{F}$-adapted stochastic processes $\tilde{u}^1_{i\nu\mu}(t)$ and $\tilde{v}^1_{i\mu\nu}(t)$ represent the instant exogenous disturbance; the $\mathbb{F}$-adapted stochastic processes $\tilde{u}^2_{i\nu\mu}(t)$ and $\tilde{v}^2_{i\mu\nu}(t)$ represent the exogenous disturbance subject to the continuous-time delay effect; the $\mathbb{F}$-adapted stochastic processes $\tilde{u}^3_{i\nu\mu}(t)$ and $\tilde{v}^3_{i\mu\nu}(t)$ represent the exogenous disturbance subject to the finitely distributed time delay effect; the $\mathbb{F}$-adapted stochastic processes $\tilde{u}^4_{i\nu\mu}(t)$ and $\tilde{v}^4_{i\mu\nu}(t)$ represent the exogenous disturbance subject to the infinitely distributed time delay effect; the $\mathbb{F}$-adapted stochastic processes $\tilde{U}^k_{i\mu}(t)$ and $\tilde{V}^k_{i\nu}(t)$ represent the other exogenous disturbance which can not be described as the aforementioned types of exogenous disturbance; and the initial data (stochastic processes) $u_{i\mu0}(t)$ and $v_{i\nu0}(t)$, functions mapping $\Omega\times\mathbb{R}_-$ into $\mathbb{R}$, are $\mathscr{F}\otimes\mathscr{L}$-measurable (see Section 1 for the definition of $\mathscr{F}\otimes\mathscr{L}$). In addition, $u_{i\mu0}(t)$ and $v_{i\nu0}(t)$ are $\mathscr{F}_0$-measurable for all $t\in\mathbb{R}_-$, and have their path essentially bounded in $\mathbb{R}_-$, $\mathbb{P}$-a.s., where $i=1,2,\ldots,r$, $\mu\in\beth$, $\nu\in\gimel$, $h=1,2,\ldots,8$, $\jmath=1,2,\ldots,12$, $\ell=1,2,3,4$.

As usual, to proceed further, we need to defuzzify the Takagi–Sugeno fuzzy BAMNNs (1)-(2)-(3). Let us denote by $M_{ij}(\xi_j(t))$ the grade of membership of the element $\xi_j(t)$ (viewed as premise variable) and now introduce the following weight functions

$$\vartheta_i(\boldsymbol{\xi}(t)) = \frac{\omega_i(\boldsymbol{\xi}(t))}{\sum\limits_{k=1}^{r}\omega_k(\boldsymbol{\xi}(t))}, \quad t\in\mathbb{R}_+,\ i=1,2,\ldots,r, \tag{4}$$

in which $\boldsymbol{\xi}(t) = (\xi_1(t), \ldots, \xi_p(t))^\top$ and $\omega_i(\boldsymbol{\xi}(t)) = \prod_{j=1}^{p} M_{ij}(\xi_j(t))$, $i = 1, 2, \ldots, r$. Aided by $\vartheta_i(\boldsymbol{\xi}(t))$ (see (4)), we can defuzzify the Takagi–Sugeno BAMNNs (1)-(2)-(3) into

$$
\begin{cases}
du_\mu(t) = \Big[-\bar{\sigma}_\mu u_\mu(t-\tau_\mu(t)) + \sum_{\nu\in\daleth} \bar{b}^1_{\mu\nu} g^1_\nu(v_\nu(t)) \\
\qquad + \sum_{\nu\in\daleth} \bar{b}^2_{\mu\nu} g^2_\nu(v_\nu(t-\varrho^1_{\mu\nu}(t))) \\
\qquad + \sum_{\nu\in\daleth} \bar{b}^3_{\mu\nu} \int_{t-\varrho^2_{\mu\nu}(t)}^{t} g^3_\nu(v_\nu(s))ds \\
\qquad + \sum_{\nu\in\daleth} \bar{b}^4_{\mu\nu} \int_{-\infty}^{t} \Psi^1_{\mu\nu}(t-s) g^4_\nu(v_\nu(s))ds + \bar{U}^1_\mu(t)\Big]dt \\
\qquad + \Big[\sum_{\nu\in\daleth} \bar{b}^5_{\mu\nu} g^5_\nu(v_\nu(t)) + \sum_{\nu\in\daleth} \bar{b}^6_{\mu\nu} g^6_\nu(v_\nu(t-\varrho^3_{\mu\nu}(t))) \\
\qquad + \sum_{\nu\in\daleth} \bar{b}^7_{\mu\nu} \int_{t-\varrho^4_{\mu\nu}(t)}^{t} g^7_\nu(v_\nu(s))ds \\
\qquad + \sum_{\nu\in\daleth} \bar{b}^8_{\mu\nu} \int_{-\infty}^{t} \Psi^2_{\mu\nu}(t-s) g^8_\nu(v_\nu(s))ds \\
\qquad + \bar{U}^2_\mu(t)\Big]dW(t), \quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\ \mu\in\beth, \\
dv_\nu(t) = \Big[-\bar{\eta}_\nu v_\nu(t-\iota_\nu(t)) + \sum_{\mu\in\beth} \bar{a}^1_{\nu\mu} f^1_\mu(u_\mu(t)) \\
\qquad + \sum_{\mu\in\beth} \bar{a}^2_{\nu\mu} f^2_\mu(u_\mu(t-\varsigma^1_{\nu\mu}(t))) \\
\qquad + \sum_{\mu\in\beth} \bar{a}^3_{\nu\mu} \int_{t-\varsigma^2_{\nu\mu}(t)}^{t} f^3_\mu(u_\mu(s))ds \\
\qquad + \sum_{\mu\in\beth} \bar{a}^4_{\nu\mu} \int_{-\infty}^{t} \Phi^1_{\nu\mu}(t-s) f^4_\mu(u_\mu(s))ds + \bar{V}^1_\nu(t)\Big]dt \\
\qquad + \Big[\sum_{\mu\in\beth} \bar{a}^5_{\nu\mu} f^5_\mu(u_\mu(t)) + \sum_{\mu\in\beth} \bar{a}^6_{\nu\mu} f^6_\mu(u_\mu(t-\varsigma^3_{\nu\mu}(t))) \\
\qquad + \sum_{\mu\in\beth} \bar{a}^7_{\nu\mu} \int_{t-\varsigma^4_{\nu\mu}(t)}^{t} f^7_\mu(u_\mu(s))ds \\
\qquad + \sum_{\mu\in\beth} \bar{a}^8_{\nu\mu} \int_{-\infty}^{t} \Phi^2_{\nu\mu}(t-s) f^8_\mu(u_\mu(s))ds \\
\qquad + \bar{V}^2_\nu(t)\Big]dW(t), \quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\ \nu\in\daleth, \\
u_\mu(t) = \bar{u}_{\mu0}(t), \quad d\mathbb{P}\times dt\text{-a.e. in } \Omega\times\mathbb{R}_-,\ \mu\in\beth, \\
v_\nu(t) = \bar{v}_{\nu0}(t), \quad d\mathbb{P}\times dt\text{-a.e. in } \Omega\times\mathbb{R}_-,\ \nu\in\daleth,
\end{cases}
\tag{5}
$$

where the initial data stochastic processes $\bar{u}_{\mu0}(t)$ and $\bar{v}_{\nu0}(t)$ are given, respectively, by

$$
\bar{u}_{\mu0}(t) = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t)) u_{i\mu0}(t), \quad d\mathbb{P}\times dt\text{-a.e. in } \Omega\times\mathbb{R}_-,\ \mu\in\beth,
$$

and

$$
\bar{v}_{\nu0}(t) = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t)) v_{i\nu0}(t), \quad d\mathbb{P}\times dt\text{-a.e. in } \Omega\times\mathbb{R}_-,\ \nu\in\daleth,
$$

the coefficients $\bar{\sigma}_\mu$, $\bar{\eta}_\nu$, $\bar{a}^k_{\nu\mu}$, and $\bar{b}^k_{\mu\nu}$, $k = 1, 2, \ldots, 8$, are given by

$$
\begin{aligned}
&\bar{\sigma}_\mu = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))\sigma_{i\mu}, \quad \bar{b}^1_{\mu\nu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))b^1_{i\mu\nu},\\
&\bar{b}^2_{\mu\nu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))b^2_{i\mu\nu}, \quad \bar{b}^3_{\mu\nu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))b^3_{i\mu\nu},\\
&\bar{b}^4_{\mu\nu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))b^4_{i\mu\nu}, \quad \bar{b}^5_{\mu\nu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))b^5_{i\mu\nu},\\
&\bar{b}^6_{\mu\nu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))b^6_{i\mu\nu}, \quad \bar{b}^7_{\mu\nu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))b^7_{i\mu\nu},\\
&\bar{b}^8_{\mu\nu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))b^8_{i\mu\nu}, \quad \bar{\eta}_\nu = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))\eta_{i\nu},\\
&\bar{a}^1_{\nu\mu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))a^1_{i\nu\mu}, \quad \bar{a}^2_{\nu\mu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))a^2_{i\nu\mu},\\
&\bar{a}^3_{\nu\mu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))a^3_{i\nu\mu}, \quad \bar{a}^4_{\nu\mu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))a^4_{i\nu\mu},\\
&\bar{a}^5_{\nu\mu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))a^5_{i\nu\mu}, \quad \bar{a}^6_{\nu\mu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))a^6_{i\nu\mu},\\
&\bar{a}^7_{\nu\mu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))a^7_{i\nu\mu}, \quad \bar{a}^8_{\nu\mu} = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))a^8_{i\nu\mu}, \quad \mu \in \daleth, \ \nu \in \gimel,
\end{aligned}
$$

the exogenous disturbances $\bar{U}^k_\mu(t)$ and $\bar{V}^k_\nu(t)$ are given, respectively, by

$$
\begin{aligned}
\bar{U}^k_\mu(t) = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))U^k_{i\mu}(t) &= \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t)) \sum_{\nu\in\gimel} b^9_{i\mu\nu}\tilde{v}^1_{i\mu\nu}(t) + \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t)) \sum_{\nu\in\gimel} b^{10}_{i\mu\nu}\tilde{v}^2_{i\mu\nu}(t-\varrho^5_{\mu\nu}(t))\\
&+ \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t)) \sum_{\nu\in\gimel} b^{11}_{i\mu\nu} \int_{t-\varrho^6_{\mu\nu}(t)}^{t} \tilde{v}^3_{i\mu\nu}(s)ds\\
&+ \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t)) \sum_{\nu\in\gimel} b^{12}_{i\mu\nu} \int_{-\infty}^{t} \Psi^3_{\mu\nu}(t-s)\tilde{v}^4_{i\mu\nu}(s)ds\\
&+ \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))\tilde{U}^k_{i\mu}(t), \quad t \in \mathbb{R}_+, \ \mathbb{P}\text{-a.s.}, \ \mu \in \daleth, \ k = 1,2,
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
\bar{V}^k_\nu(t) = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))V^k_{i\nu}(t) &= \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t)) \sum_{\mu\in\daleth} a^9_{i\nu\mu}\tilde{u}^1_{i\nu\mu}(t) + \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t)) \sum_{\mu\in\daleth} a^{10}_{i\nu\mu}\tilde{u}^2_{i\nu\mu}(t-\varsigma^5_{\nu\mu}(t))\\
&+ \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t)) \sum_{\mu\in\daleth} a^{11}_{i\nu\mu} \int_{t-\varsigma^6_{\nu\mu}(t)}^{t} \tilde{u}^3_{i\nu\mu}(s)ds\\
&+ \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t)) \sum_{\mu\in\daleth} a^{12}_{i\nu\mu} \int_{-\infty}^{t} \Phi^3_{\nu\mu}(t-s)\tilde{u}^4_{i\nu\mu}(s)ds\\
&+ \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t))\tilde{V}^k_{i\nu}(t), \quad t \in \mathbb{R}_+, \ \mathbb{P}\text{-a.s.}, \ \nu \in \gimel, \ k = 1,2.
\end{aligned} \tag{7}
$$

Now, we are in a position to introduce the response BAMNNs

$$
\begin{cases}
\begin{aligned}
d\hat{u}_\mu(t) = \Bigg[& -\bar{\sigma}_\mu \hat{u}_\mu(t-\tau_\mu(t)) + \sum_{\nu\in\gimel} \bar{b}^1_{\mu\nu} g^1_\nu(\hat{v}_\nu(t)) \\
& + \sum_{\nu\in\gimel} \bar{b}^2_{\mu\nu} g^2_\nu(\hat{v}_\nu(t-\varrho^1_{\mu\nu}(t))) \\
& + \sum_{\nu\in\gimel} \bar{b}^3_{\mu\nu} \int_{t-\varrho^2_{\mu\nu}(t)}^{t} g^3_\nu(\hat{v}_\nu(s))ds \\
& + \sum_{\nu\in\gimel} \bar{b}^4_{\mu\nu} \int_{-\infty}^{t} \Psi^1_{\mu\nu}(t-s) g^4_\nu(\hat{v}_\nu(s))ds + \bar{U}^1_\mu(t) + \mathscr{U}_\mu(t) \Bigg] dt \\
& + \Bigg[\sum_{\nu\in\gimel} \bar{b}^5_{\mu\nu} g^5_\nu(\hat{v}_\nu(t)) + \sum_{\nu\in\gimel} \bar{b}^6_{\mu\nu} g^6_\nu(\hat{v}_\nu(t-\varrho^3_{\mu\nu}(t))) \\
& + \sum_{\nu\in\gimel} \bar{b}^7_{\mu\nu} \int_{t-\varrho^4_{\mu\nu}(t)}^{t} g^7_\nu(\hat{v}_\nu(s))ds \\
& + \sum_{\nu\in\gimel} \bar{b}^8_{\mu\nu} \int_{-\infty}^{t} \Psi^2_{\mu\nu}(t-s) g^8_\nu(\hat{v}_\nu(s))ds \\
& + \bar{U}^2_\mu(t) \Bigg] dW(t), \quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\ \mu\in\beth, \\
d\hat{v}_\nu(t) = \Bigg[& -\bar{\eta}_\nu \hat{v}_\nu(t-\iota_\nu(t)) + \sum_{\mu\in\beth} \bar{a}^1_{\nu\mu} f^1_\mu(\hat{u}_\mu(t)) \\
& + \sum_{\mu\in\beth} \bar{a}^2_{\nu\mu} f^2_\mu(\hat{u}_\mu(t-\varsigma^1_{\nu\mu}(t))) \\
& + \sum_{\mu\in\beth} \bar{a}^3_{\nu\mu} \int_{t-\varsigma^2_{\nu\mu}(t)}^{t} f^3_\mu(\hat{u}_\mu(s))ds \\
& + \sum_{\mu\in\beth} \bar{a}^4_{\nu\mu} \int_{-\infty}^{t} \Phi^1_{\nu\mu}(t-s) f^4_\mu(\hat{u}_\mu(s))ds + \bar{V}^1_\nu(t) + \mathscr{V}_\nu(t) \Bigg] dt \\
& + \Bigg[\sum_{\mu\in\beth} \bar{a}^5_{\nu\mu} f^5_\mu(\hat{u}_\mu(t)) + \sum_{\mu\in\beth} \bar{a}^6_{\nu\mu} f^6_\mu(\hat{u}_\mu(t-\varsigma^3_{\nu\mu}(t))) \\
& + \sum_{\mu\in\beth} \bar{a}^7_{\nu\mu} \int_{t-\varsigma^4_{\nu\mu}(t)}^{t} f^7_\mu(\hat{u}_\mu(s))ds \\
& + \sum_{\mu\in\beth} \bar{a}^8_{\nu\mu} \int_{-\infty}^{t} \Phi^2_{\nu\mu}(t-s) f^8_\mu(\hat{u}_\mu(s))ds \\
& + \bar{V}^2_\nu(t) \Bigg] dW(t), \quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\ \nu\in\gimel, \\
\hat{u}_\mu(t) = {} & \bar{\hat{u}}_{\mu 0}(t), \quad d\mathbb{P}\times dt\text{-a.e. in } \Omega\times\mathbb{R}_-,\ \mu\in\beth, \\
\hat{v}_\nu(t) = {} & \bar{\hat{v}}_{\nu 0}(t), \quad d\mathbb{P}\times dt\text{-a.e. in } \Omega\times\mathbb{R}_-,\ \nu\in\gimel,
\end{aligned}
\end{cases}
\tag{8}
$$

where the initial data stochastic processes $\bar{\hat{u}}_{\mu 0}(t)$ and $\bar{\hat{v}}_{\nu 0}(t)$ are given, respectively, by

$$\bar{\hat{u}}_{\mu 0}(t) = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t)) \hat{u}_{i\mu 0}(t), \quad d\mathbb{P}\times dt\text{-a.e. in } \Omega\times\mathbb{R}_-,\ \mu\in\beth, \tag{9}$$

and

$$\bar{\hat{v}}_{\nu 0}(t) = \sum_{i=1}^{r} \vartheta_i(\boldsymbol{\xi}(t)) \hat{v}_{i\nu 0}(t), \quad d\mathbb{P}\times dt\text{-a.e. in } \Omega\times\mathbb{R}_-,\ \nu\in\gimel, \tag{10}$$

with the given initial data stochastic processes $\hat{u}_{i\mu 0}(t)$ and $\hat{v}_{i\nu 0}(t)$ being $\mathscr{F} \otimes \mathscr{L}$-measurable and being $\mathscr{F}_0$-measurable for all $t \in \mathbb{R}_-$, the exogenous disturbances $\bar{U}^k_\mu(t)$ and $\bar{V}^k_\nu(t)$ are given as in (6) and (7), respectively, and the $\mathbb{F}$-adapted stochastic processes $\mathscr{U}_\mu(t)$ and $\mathscr{V}_\nu(t)$ are the control inputs. To study the claimed identical synchronization problem in the pre-assigned time, we need to introduce the error BAMNNs

$$\begin{cases} dx_\mu(t) = \Big[-\bar{\sigma}_\mu x_\mu(t-\tau_\mu(t)) + \sum\limits_{\nu \in \daleth} \bar{b}^1_{\mu\nu} \check{g}^1_\nu(y_\nu(t)) \\ \quad + \sum\limits_{\nu \in \daleth} \bar{b}^2_{\mu\nu} \check{g}^2_\nu(y_\nu(t-\varrho^1_{\mu\nu}(t))) \\ \quad + \sum\limits_{\nu \in \daleth} \bar{b}^3_{\mu\nu} \int_{t-\varrho^2_{\mu\nu}(t)}^t \check{g}^3_\nu(y_\nu(s))ds \\ \quad + \sum\limits_{\nu \in \daleth} \bar{b}^4_{\mu\nu} \int_{-\infty}^t \Psi^1_{\mu\nu}(t-s) \check{g}^4_\nu(y_\nu(s))ds + \mathscr{U}_\mu(t) \Big] dt \\ \quad + \Big[\sum\limits_{\nu \in \daleth} \bar{b}^5_{\mu\nu} \check{g}^5_\nu(y_\nu(t)) + \sum\limits_{\nu \in \daleth} \bar{b}^6_{\mu\nu} \check{g}^6_\nu(y_\nu(t-\varrho^3_{\mu\nu}(t))) \\ \quad + \sum\limits_{\nu \in \daleth} \bar{b}^7_{\mu\nu} \int_{t-\varrho^4_{\mu\nu}(t)}^t \check{g}^7_\nu(y_\nu(s))ds \\ \quad + \sum\limits_{\nu \in \daleth} \bar{b}^8_{\mu\nu} \int_{-\infty}^t \Psi^2_{\mu\nu}(t-s) \check{g}^8_\nu(y_\nu(s))ds \Big] dW(t), \quad t \in \mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\ \mu \in \beth, \\ dy_\nu(t) = \Big[-\bar{\eta}_\nu y_\nu(t-\iota_\nu(t)) + \sum\limits_{\mu \in \beth} \bar{a}^1_{\nu\mu} \check{f}^1_\mu(x_\mu(t)) \\ \quad + \sum\limits_{\mu \in \beth} \bar{a}^2_{\nu\mu} \check{f}^2_\mu(x_\mu(t-\varsigma^1_{\nu\mu}(t))) \\ \quad + \sum\limits_{\mu \in \beth} \bar{a}^3_{\nu\mu} \int_{t-\varsigma^2_{\nu\mu}(t)}^t \check{f}^3_\mu(x_\mu(s))ds \\ \quad + \sum\limits_{\mu \in \beth} \bar{a}^4_{\nu\mu} \int_{-\infty}^t \Phi^1_{\nu\mu}(t-s) \check{f}^4_\mu(x_\mu(s))ds + \mathscr{V}_\nu(t) \Big] dt \\ \quad + \Big[\sum\limits_{\mu \in \beth} \bar{a}^5_{\nu\mu} \check{f}^5_\mu(x_\mu(t)) + \sum\limits_{\mu \in \beth} \bar{a}^6_{\nu\mu} \check{f}^6_\mu(x_\mu(t-\varsigma^3_{\nu\mu}(t))) \\ \quad + \sum\limits_{\mu \in \beth} \bar{a}^7_{\nu\mu} \int_{t-\varsigma^4_{\nu\mu}(t)}^t \check{f}^7_\mu(x_\mu(s))ds \\ \quad + \sum\limits_{\mu \in \beth} \bar{a}^8_{\nu\mu} \int_{-\infty}^t \Phi^2_{\nu\mu}(t-s) \check{f}^8_\mu(x_\mu(s))ds \Big] dW(t), \quad t \in \mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\ \nu \in \daleth, \\ x_\mu(t) = \bar{\hat{u}}_{\mu 0}(t) - \bar{u}_{\mu 0}(t) \\ \quad = \sum\limits_{i=1}^r \vartheta_i(\xi(t))(\hat{u}_{i\mu 0}(t) - u_{i\mu 0}(t)), \quad d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-,\ \mu \in \beth, \\ y_\nu(t) = \bar{\hat{v}}_{\nu 0}(t) - \bar{v}_{\nu 0}(t) \\ \quad = \sum\limits_{i=1}^r \vartheta_i(\xi(t))(\hat{v}_{i\nu 0}(t) - v_{i\nu 0}(t)), \quad d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-,\ \nu \in \daleth, \end{cases} \tag{11}$$

where the stochastic processes, as in (8), $\mathscr{U}_\mu(t)$ and $\mathscr{V}_\nu(t)$, are the control inputs, the state trajectories $x_\mu(t)$ and $y_\nu(t)$ of the error BAMNNs (11) are given, respectively, by

$$x_\mu(t) = \hat{u}_\mu(t) - u_\mu(t), \quad t \in \mathbb{R},\ \mathbb{P}\text{-a.s.},\ \mu \in \beth, \tag{12}$$

and

$$y_\nu(t) = \hat{v}_\nu(t) - v_\nu(t), \quad t \in \mathbb{R},\ \mathbb{P}\text{-a.s.},\ \nu \in \daleth, \tag{13}$$

and the functions $\check{f}^i_\mu(x_\mu(t))$ and $\check{g}^i_\nu(y_\nu(t))$ are given, respectively, by

$$\begin{aligned}\check{f}^i_\mu(x_\mu(t)) =& f^i_\mu(\hat{u}_\mu(t)) - f^i_\mu(x_\mu(t))\\ =& f^i_\mu(u_\mu(t) + x_\mu(t)) - f^i_\mu(u_\mu(t)), \quad t \in \mathbb{R},\ \mathbb{P}\text{-a.s.},\ \mu \in \beth,\ i = 1,2,\ldots,8,\end{aligned}$$

and

$$\begin{aligned}\check{g}^i_\nu(y_\nu(t)) =& g^i_\nu(\hat{v}_\nu(t)) - g^i_\nu(v_\nu(t))\\ =& g^i_\nu(v_\nu(t) + y_\nu(t)) - g^i_\nu(v_\nu(t)), \quad t \in \mathbb{R},\ \mathbb{P}\text{-a.s.},\ \nu \in \daleth,\ i = 1,2,\ldots,8.\end{aligned}$$

Throughout this paper, we assume that the activation functions f^i_μ and g^i_ν are Lipschitz continuous and have positive constants as the lower bounds of their difference quotients, that the functions $\tau_\mu(t)$, $\varsigma^k_{\nu\mu}(t)$, $\iota_\nu(t)$, and $\varrho^k_{\mu\nu}(t)$ satisfy some regularity and growth conditions, and that the kernels $\Phi^\ell_{\nu\mu}(t)$ and $\Psi^\ell_{\mu\nu}(t)$ are Lebesgue integrable, $\mu \in \beth$, $\nu \in \daleth$, $i = 1,2,\ldots,8$, $k = 1,2,\ldots,6$, $\ell = 1,2,3$; see Assumptions 1–3 for the details.

Assumption 1. *There exist positive constants $\underline{Ł}_{f^i_\mu}$, $\bar{Ł}_{f^i_\mu}$, $\underline{Ł}_{g^i_\nu}$, and $\bar{Ł}_{g^i_\nu}$, satisfying the inequality condition $\underline{Ł}_{f^i_\mu} < \bar{Ł}_{f^i_\mu}$ and $\underline{Ł}_{g^i_\nu} < \bar{Ł}_{g^i_\nu}$, such that*

$$\underline{Ł}_{f^i_\mu} \leqslant \frac{f^i_\mu(u) - f^i_\mu(v)}{u - v} \leqslant \bar{Ł}_{f^i_\mu}, \quad u, v \in \mathbb{R} \text{ with } u \neq v,\ \mu \in \beth,\ i = 1,2,\ldots,8,$$

and

$$\underline{Ł}_{g^i_\nu} \leqslant \frac{g^i_\nu(u) - g^i_\nu(v)}{u - v} \leqslant \bar{Ł}_{g^i_\nu}, \quad u, v \in \mathbb{R} \text{ with } u \neq v,\ \nu \in \daleth,\ i = 1,2,\ldots,8.$$

Assumption 2. *The continuous functions $\tau_\mu(t)$, $\varsigma^k_{\nu\mu}(t)$, $\iota_\nu(t)$, and $\varrho^k_{\mu\nu}(t)$, mapping $\mathbb{R}_+$ into itself, satisfy $\tau_\mu(t) > 0$, $\varsigma^k_{\nu\mu}(t) > 0$, $\iota_\nu(t) > 0$, and $\varrho^k_{\mu\nu}(t) > 0$ for all $t \in (0, +\infty)$, $k = 1,2,\ldots,6$, $\mu \in \beth$, $\nu \in \daleth$. The functions $\tau_\mu(t)$, $\varsigma^\jmath_{\nu\mu}(t)$, $\iota_\nu(t)$, and $\varrho^\jmath_{\mu\nu}(t)$ are differentiable in $\mathbb{R}_+$ and are Lipschitz continuous in $\mathbb{R}_+$, so it holds that $\bar{\dot{\tau}}_\mu < 1$, $\bar{\dot{\varsigma}}^\jmath_{\nu\mu} < 1$, $\bar{\dot{\iota}}_\nu < 1$, and $\bar{\dot{\varrho}}^\jmath_{\mu\nu} < 1$, where the constants $\bar{\dot{\tau}}_\mu$, $\bar{\dot{\varsigma}}^\jmath_{\nu\mu}$, $\bar{\dot{\iota}}_\nu$, and $\bar{\dot{\varrho}}^\jmath_{\mu\nu}$ are given, respectively, by*

$$\begin{aligned}&\bar{\dot{\tau}}_\mu = \operatorname*{ess\,sup}_{t \in \mathbb{R}_+} \dot{\tau}_\mu(t), \quad \bar{\dot{\varsigma}}^\jmath_{\nu\mu} = \operatorname*{ess\,sup}_{t \in \mathbb{R}_+} \dot{\varsigma}^\jmath_{\nu\mu}(t),\\ &\bar{\dot{\iota}}_\nu = \operatorname*{ess\,sup}_{t \in \mathbb{R}_+} \dot{\iota}_\nu(t), \quad \text{and } \bar{\dot{\varrho}}^\jmath_{\mu\nu} = \operatorname*{ess\,sup}_{t \in \mathbb{R}_+} \dot{\varrho}^\jmath_{\mu\nu}(t), \quad \jmath = 1,3,\ \mu \in \beth,\ \nu \in \daleth,\end{aligned} \tag{14}$$

and in addition, it holds that $\bar{\varsigma}^\ell_{\nu\mu} \in (0, +\infty)$, as well as $\bar{\varrho}^\ell_{\mu\nu} \in (0, +\infty)$, in which the constants $\bar{\varsigma}^\ell_{\nu\mu}$ and $\bar{\varrho}^\ell_{\mu\nu}$ are given, respectively, by

$$\bar{\varsigma}^\ell_{\nu\mu} = \sup_{t \in \mathbb{R}_+} \varsigma^\ell_{\nu\mu}(t), \quad \text{and } \bar{\varrho}^\ell_{\mu\nu} = \sup_{t \in \mathbb{R}_+} \varrho^\ell_{\mu\nu}(t), \quad \ell = 2,4,\ \mu \in \beth,\ \nu \in \daleth. \tag{15}$$

Assumption 3. *The kernels $\Phi^k_{\nu\mu}(t)$ and $\Psi^k_{\mu\nu}(t)$ are continuous functions mapping $\mathbb{R}_+$ into itself, and they are Lebesgue integrable in every compact subinterval of $\mathbb{R}_+$, $k = 1,2,3$, $\mu \in \daleth$, $\nu \in \gimel$. Moreover, the kernels $\Phi^{\jmath}_{\nu\mu}(t)$ and $\Psi^{\jmath}_{\mu\nu}(t)$ satisfy*

$$\left.\begin{array}{l} \displaystyle\int_0^{+\infty} |\Phi^{\jmath}_{\nu\mu}(t)|dt = \int_0^{+\infty} \Phi^{\jmath}_{\nu\mu}(t)dt < +\infty, \\ \displaystyle\int_0^{+\infty} |\Psi^{\jmath}_{\mu\nu}(t)|dt = \int_0^{+\infty} \Psi^{\jmath}_{\mu\nu}(t)dt < +\infty, \end{array}\right\} \quad \jmath = 1,2,\ \mu \in \daleth,\ \nu \in \gimel.$$

For the sake of convenience, we introduce here the constants $\text{\L}_{f^i_\mu}$ and $\text{\L}_{g^i_\nu}$ by

$$\text{\L}_{f^i_\mu} = \sup_{\substack{u,v\in\mathbb{R}, \\ u\neq v}} \left| \frac{f^i_\mu(u) - f^i_\mu(v)}{u - v} \right|, \quad \mu \in \daleth,\ i = 1,2,\dots,8, \tag{16}$$

and

$$\text{\L}_{g^i_\nu} = \sup_{\substack{u,v\in\mathbb{R}, \\ u\neq v}} \left| \frac{g^i_\nu(u) - g^i_\nu(v)}{u - v} \right|, \quad \nu \in \gimel,\ i = 1,2,\dots,8. \tag{17}$$

Remark 1. *By Assumption 1, the constants $\text{\L}_{f^i_\mu}$ and $\text{\L}_{g^i_\nu}$, given, respectively, by (16) and (17), are well-defined and are indeed positive constants, $\mu \in \daleth$, $\nu \in \gimel$, $i = 1,2,\dots,8$.*

Suppose that the stochastic processes $\tilde{u}^1_{i\nu\mu}(t)$, $\tilde{v}^1_{i\mu\nu}(t)$, $\tilde{u}^2_{i\nu\mu}(t)$, $\tilde{v}^2_{i\mu\nu}(t)$, $\tilde{u}^3_{i\nu\mu}(t)$, $\tilde{v}^3_{i\mu\nu}(t)$, $\tilde{u}^4_{i\nu\mu}(t)$, $\tilde{v}^4_{i\mu\nu}(t)$, $\tilde{U}^k_{i\mu}(t)$, $\tilde{V}^k_{i\nu}(t)$, $\mathscr{U}_\mu(t)$, and $\mathscr{V}_\nu(t)$ are all $\mathbb{F}$-adapted, $i = 1,2,\dots,r$, $\mu \in \daleth$, $\nu \in \gimel$. Under Assumptions 1–3, for any initial data stochastic processes $u_{i\mu 0}(t)$ and $v_{i\nu 0}(t)$, as well as $\hat{u}_{i\mu 0}(t)$ and $\hat{v}_{i\nu 0}(t)$, functions mapping $\Omega \times \mathbb{R}_-$ into $\mathbb{R}$, being $\mathscr{F} \otimes \mathscr{L}$-measurable, being $\mathscr{F}_0$-measurable for all $t \in \mathbb{R}_-$, being $\mathbb{P}$ almost surely bounded in $\mathbb{R}_-$, and satisfying

$$\begin{array}{ll} \operatorname*{ess\,sup}_{t\in\mathbb{R}_-} \mathbb{E}|u_{i\mu 0}(t)| < +\infty, & \operatorname*{ess\,sup}_{t\in\mathbb{R}_-} \mathbb{E}|v_{i\nu 0}(t)| < +\infty, \\ \operatorname*{ess\,sup}_{t\in\mathbb{R}_-} \mathbb{E}|\hat{u}_{i\mu 0}(t)| < +\infty, & \text{and } \operatorname*{ess\,sup}_{t\in\mathbb{R}_-} \mathbb{E}|\hat{v}_{i\nu 0}(t)| < +\infty, \end{array}$$

$i = 1,2,\dots,r$, $\mu \in \daleth$, $\nu \in \gimel$, the BAMNNs (1)-(2)-(3), BAMNNs (5)-(6)-(7), BAMNNs (8)-(6)-(7), and error BAMNNs (11) admit a unique state trajectory, respectively.

Definition 1 ([11])**.** *The drive BAMNNs (5)-(6)-(7) and the response BAMNNs (8)-(6)-(7) are said to achieve synchronization in a pre-assigned settling time, or to achieve pre-assigned-time synchronization, provided that for any given positive time instant T, there exists a collection of control inputs $\mathscr{U}_\mu(t)$, $\mathscr{V}_\nu(t)$, $\mu \in \daleth$, $\nu \in \gimel$, such that for any state trajectory $\{u_\mu(t)\}_{\mu\in\daleth}$, $\{v_\nu(t)\}_{\mu\in\daleth}$ of BAMNNs (5)-(6)-(7) (the drive network systems), and any state trajectory $\{\hat{u}_\mu(t)\}_{\mu\in\daleth}$, $\{\hat{v}_\nu(t)\}_{\mu\in\daleth}$ of BAMNNs (8)-(6)-(7) (the response network systems), with our designed control implemented, it holds always that*

$$\begin{cases} \left.\begin{array}{l} \displaystyle\lim_{t\to T^-} \mathbb{E}|u_\mu(t) - \hat{u}_\mu(t)|^2 \\ \displaystyle= \lim_{t\to T^-} \mathbb{E}|x_\mu(t)|^2 = 0, \\ u_\mu(t) = \hat{u}_\mu(t),\ t \in [T, +\infty),\ \mathbb{P}\text{-a.s.}, \end{array}\right\} \quad \mu \in \daleth, \\ \left.\begin{array}{l} \displaystyle\lim_{t\to T^-} \mathbb{E}|v_\nu(t) - \hat{v}_\nu(t)|^2 \\ \displaystyle= \lim_{t\to T^-} \mathbb{E}|y_\nu(t)|^2 = 0, \\ v_\nu(t) = \hat{v}_\nu(t),\ t \in [T, +\infty),\ \mathbb{P}\text{-a.s.}, \end{array}\right\} \quad \nu \in \gimel, \end{cases} \tag{18}$$

where $\{x_\mu(t)\}_{\mu\in\beth}$*, together with* $\{y_\nu(t)\}_{\mu\in\beth}$*, given by* (12)*, together with* (13)*, denotes the state trajectory of the error BAMNNs* (11)*.*

To put it concisely, observing that the functions $\mathbb{E}|x_\mu(t)|^2$ and $\mathbb{E}|y_\nu(t)|^2$ are continuous in time t, $\mu\in\beth$, $\nu\in\daleth$, we conclude, by Definition 1, that proving BAMNNs (5)-(6)-(7) and BAMNNs (8)-(6)-(7) achieve drive-response synchronization within the pre-assigned time T boils down to proving $\mathbb{E}|x_\mu(t)|^2=\mathbb{E}|y_\nu(t)|^2=0$ for all $t\in[T,+\infty)$, $\mu\in\beth$, $\nu\in\daleth$.

Illuminated by References [11,44], we define an important auxiliary function:

$$\mathcal{T}(k,\alpha,\beta,\delta,\theta)=\begin{cases}\dfrac{\pi}{\beta(\delta-\theta)}(\dfrac{\beta}{\alpha})^{\frac{1-\theta}{\delta-\theta}}\csc(\pi\dfrac{1-\theta}{\delta-\theta}), & k\leqslant 0,\ \alpha>0,\ \beta>0,\ \delta>1,\ 0\leqslant\theta<1,\\ \dfrac{\pi}{\alpha(\delta-\theta)}(\dfrac{\alpha}{\beta-k})^{\frac{\delta-1}{\delta-\theta}}\dfrac{\flat(\frac{\alpha}{\alpha+\beta-k},\frac{1-\theta}{\delta-\theta},\frac{\delta-1}{\delta-\theta})}{\mathrm{B}(\frac{1-\theta}{\delta-\theta},\frac{\delta-1}{\delta-\theta})} & \\ +\dfrac{\pi}{\beta(\delta-\theta)}(\dfrac{\beta}{\alpha-k})^{\frac{1-\theta}{\delta-\theta}}\dfrac{\flat(\frac{\beta}{\alpha+\beta-k},\frac{\delta-1}{\delta-\theta},\frac{1-\theta}{\delta-\theta})}{\mathrm{B}(\frac{\delta-1}{\delta-\theta},\frac{1-\theta}{\delta-\theta})}, & 0<k<\min(\alpha,\beta),\ \delta>1,\ 0\leqslant\theta<1,\\ \dfrac{2}{(\delta-1)\sqrt{4\alpha\beta-k^2}}(\dfrac{\pi}{2}+\arctan\dfrac{k}{\sqrt{4\alpha\beta-k^2}}), & 0<k<2\sqrt{\alpha\beta},\ \delta+\theta=2,\ \alpha>0,\ 0\leqslant\theta<1,\end{cases}\tag{19}$$

in which $\mathrm{B}(p,q)$ is the celebrated Euler's Beta function, which is explicitly given by

$$\mathrm{B}(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt,\quad p,q\in\mathbb{C}\text{ with }\mathrm{Re}\,p>0,\ \mathrm{Re}\,q>0,$$

and $\flat(\theta,p,q)$ is the so-called incomplete Beta function, which is defined by

$$\flat(\theta,p,q)=\int_0^\theta t^{p-1}(1-t)^{q-1}dt,\quad \theta\in[0,1],\ p,q\in\mathbb{C}\text{ with }\mathrm{Re}\,p>0,\ \mathrm{Re}\,q>0.$$

Lemma 1 ([11])**.** *Let* $a_0\in\mathbb{R}$*,* $a_k\in(0,+\infty)$ *(*$k=1,2$*),* $\gamma_1\in(1,+\infty)$*,* $\gamma_2\in[0,1)$*, and* $T_\natural\in(0,+\infty)$ *be given. For any decreasing function* $V(t)$ *(mapping of* $\mathbb{R}_+$ *into itself) satisfying*

$$D^+V(t)\leqslant\frac{\mathcal{T}(a_0,a_1,a_2,\gamma_1,\gamma_2)}{T_\natural}\Big(a_0V(t)-a_1(V(t))^{\gamma_1}-a_2(V(t))^{\gamma_2}\Big),\quad a.e.\ t\in\mathbb{R}_+,\tag{20}$$

it holds that $V(t)=0$ *for all* $t\geqslant T_\natural$*, where* $\mathcal{T}(a_0,a_1,a_2,\gamma_1,\gamma_2)$ *is given as in* (19)*.*

3. Main Results and Their Proofs

In this section, our main aim is to state our main results in this paper and to provide detailed proofs of our claimed theoretical results. We shall first come up with a class of synchronization control inputs for the response BAMNNs (8)-(6)-(7), construct secondly a suitable Lyapunov–Krasovskii functional along the state trajectories of the error BAMNNs (11), and finally establish, with our cleverly developed Lyapunov–Krasovskii functional as the key ingredient, a criterion ensuring that BAMNNs (5)-(6)-(7) and BAMNNs (8)-(6)-(7), with our designed control implemented, achieve pre-assigned-time synchronization.

To obtain our desired pre-assigned-time synchronization, we put forward, after some basic analysis, the following synchronization control

$$\begin{aligned}\mathscr{U}_\mu(t)=&-\frac{\mathfrak{m}_\mu^1(\hat{u}_\mu(t)-u_\mu(t))}{\varepsilon+|\hat{u}_\mu(t)-u_\mu(t)|^2}\mathit{\Pi}_{11}(t)-\frac{\mathfrak{m}_\mu^2(\hat{u}_\mu(t)-u_\mu(t))}{\varepsilon+|\hat{u}_\mu(t)-u_\mu(t)|^2}(\mathit{\Pi}_{12}(t))^{\gamma_1}\\&-\frac{\mathfrak{m}_\mu^3(\hat{u}_\mu(t)-u_\mu(t))}{\varepsilon+|\hat{u}_\mu(t)-u_\mu(t)|^2}(\mathbb{E}\mathit{\Pi}_{13}(t))^{\gamma_2}\\=&-\frac{\mathfrak{m}_\mu^1x_\mu(t)}{\varepsilon+|x_\mu(t)|^2}\mathit{\Pi}_{11}(t)-\frac{\mathfrak{m}_\mu^2x_\mu(t)}{\varepsilon+|x_\mu(t)|^2}(\mathit{\Pi}_{12}(t))^{\gamma_1}\\&-\frac{\mathfrak{m}_\mu^3x_\mu(t)}{\varepsilon+|x_\mu(t)|^2}(\mathbb{E}\mathit{\Pi}_{13}(t))^{\gamma_2},\quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\ \mu\in\beth,\end{aligned}\tag{21}$$

and

$$\begin{aligned}\mathscr{V}_\nu(t) =& -\frac{\mathfrak{n}_\nu^1(\hat{v}_\nu(t)-v_\nu(t))}{\varepsilon+|\hat{v}_\nu(t)-v_\nu(t)|^2}\Pi_{21}(t)-\frac{\mathfrak{n}_\nu^2(\hat{v}_\nu(t)-v_\nu(t))}{\varepsilon+|\hat{v}_\nu(t)-v_\nu(t)|^2}(\Pi_{22}(t))^{\gamma_1}\\ &-\frac{\mathfrak{n}_\nu^3(\hat{v}_\nu(t)-v_\nu(t))}{\varepsilon+|\hat{v}_\nu(t)-v_\nu(t)|^2}(\mathbb{E}\Pi_{23}(t))^{\gamma_2}\\ =&-\frac{\mathfrak{n}_\nu^1 y_\nu(t)}{\varepsilon+|y_\nu(t)|^2}\Pi_{21}(t)-\frac{\mathfrak{n}_\nu^2 y_\nu(t)}{\varepsilon+|y_\nu(t)|^2}(\Pi_{22}(t))^{\gamma_1}\\ &-\frac{\mathfrak{n}_\nu^3 y_\nu(t)}{\varepsilon+|y_\nu(t)|^2}(\mathbb{E}\Pi_{23}(t))^{\gamma_2},\quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\ \nu\in\beth,\end{aligned}\tag{22}$$

in which the positive constant ε is fixed arbitrarily small, the positive constants $\mathfrak{m}_\mu^\jmath$ and $\mathfrak{n}_\nu^\jmath$ are yet to be determined, $\jmath=1,2,3$, and the stochastic process $\Pi_{hk}(t)$ is given by

$$\begin{aligned}\Pi_{hk}(t) =&\tilde{\Pi}_{hk}(\hat{u}_\mu(t)-u_\mu(t),\hat{v}_\nu(t)-v_\nu(t))\\ =&\sum_{\mu\in\beth}\mathfrak{m}_{hk1\mu}|\hat{u}_\mu(t)-u_\mu(t)|^2+\sum_{\nu\in\beth}\mathfrak{n}_{hk1\nu}|\hat{v}_\nu(t)-v_\nu(t)|^2\\ &+\sum_{\mu\in\beth}\mathfrak{m}_{hk2\mu}\max_{1\leqslant i\leqslant r}\sigma_{i\mu}\int_{t-\tau_\mu(t)}^t|\hat{u}_\mu(s)-u_\mu(s)|^2ds\\ &+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{m}_{hk3\nu\mu}\max_{1\leqslant i\leqslant r}|a_{i\nu\mu}^2|\text{Ł}_{f_\mu^2}\int_{t-\varsigma_{\nu\mu}^1(t)}^t|\hat{u}_\mu(s)-u_\mu(s)|^2ds\\ &+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{m}_{hk4\nu\mu}\max_{1\leqslant i\leqslant r}|a_{i\nu\mu}^6|^2(\text{Ł}_{f_\mu^6})^2\int_{t-\varsigma_{\nu\mu}^3(t)}^t|\hat{u}_\mu(s)-u_\mu(s)|^2ds\\ &+\sum_{\nu\in\beth}\mathfrak{n}_{hk2\nu}\max_{1\leqslant i\leqslant r}\eta_{i\nu}\int_{t-\iota_\nu(t)}^t|\hat{v}_\nu(s)-v_\nu(s)|^2ds\\ &+\sum_{\nu\in\beth}\sum_{\mu\in\beth}\mathfrak{n}_{hk3\mu\nu}\max_{1\leqslant i\leqslant r}|b_{i\mu\nu}^2|\text{Ł}_{g_\nu^2}\int_{t-\varrho_{\mu\nu}^1(t)}^t|\hat{v}_\nu(s)-v_\nu(s)|^2ds\\ &+\sum_{\nu\in\beth}\sum_{\mu\in\beth}\mathfrak{n}_{hk4\mu\nu}\max_{1\leqslant i\leqslant r}|b_{i\mu\nu}^6|^2(\text{Ł}_{g_\nu^6})^2\int_{t-\varrho_{\mu\nu}^3(t)}^t|\hat{v}_\nu(s)-v_\nu(s)|^2ds\\ &+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{m}_{hk5\nu\mu}\max_{1\leqslant i\leqslant r}|a_{i\nu\mu}^3|\text{Ł}_{f_\mu^3}\int_{t-\varsigma_{\nu\mu}^2(t)}^t\int_s^t|\hat{u}_\mu(\zeta)-u_\mu(\zeta)|^2d\zeta ds\\ &+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{m}_{hk6\nu\mu}\max_{1\leqslant i\leqslant r}|a_{i\nu\mu}^7|^2(\text{Ł}_{f_\mu^7})^2\bar{\varsigma}_{\nu\mu}^4\int_{t-\varsigma_{\nu\mu}^4(t)}^t\int_s^t|\hat{u}_\mu(\zeta)-u_\mu(\zeta)|^2d\zeta ds\\ &+\sum_{\nu\in\beth}\sum_{\mu\in\beth}\mathfrak{n}_{hk5\mu\nu}\max_{1\leqslant i\leqslant r}|b_{i\mu\nu}^3|\text{Ł}_{g_\nu^3}\int_{t-\varrho_{\mu\nu}^2(t)}^t\int_s^t|\hat{v}_\nu(\zeta)-v_\nu(\zeta)|^2d\zeta ds\\ &+\sum_{\nu\in\beth}\sum_{\mu\in\beth}\mathfrak{n}_{hk6\mu\nu}\max_{1\leqslant i\leqslant r}|b_{i\mu\nu}^7|^2(\text{Ł}_{g_\nu^7})^2\bar{\varrho}_{\mu\nu}^4\int_{t-\varrho_{\mu\nu}^4(t)}^t\int_s^t|\hat{v}_\nu(\zeta)-v_\nu(\zeta)|^2d\zeta ds\\ &+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{m}_{hk7\nu\mu}\max_{1\leqslant i\leqslant r}|a_{i\nu\mu}^4|\text{Ł}_{f_\mu^4}\int_0^{+\infty}\Phi_{\nu\mu}^1(s)\int_{t-s}^t|\hat{u}_\mu(\zeta)-u_\mu(\zeta)|^2d\zeta ds\\ &+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{m}_{hk8\nu\mu}\max_{1\leqslant i\leqslant r}|a_{i\nu\mu}^8|^2(\text{Ł}_{f_\mu^8})^2\int_0^{+\infty}\Phi_{\nu\mu}^2(s)ds\int_0^{+\infty}\Phi_{\nu\mu}^2(s)\int_{t-s}^t|\hat{u}_\mu(\zeta)-u_\mu(\zeta)|^2d\zeta ds\\ &+\sum_{\nu\in\beth}\sum_{\mu\in\beth}\mathfrak{n}_{hk7\mu\nu}\max_{1\leqslant i\leqslant r}|b_{i\mu\nu}^4|\text{Ł}_{g_\nu^4}\int_0^{+\infty}\Psi_{\mu\nu}^1(s)\int_{t-s}^t|\hat{v}_\nu(\zeta)-v_\nu(\zeta)|^2d\zeta ds\\ &+\sum_{\nu\in\beth}\sum_{\mu\in\beth}\mathfrak{n}_{hk8\mu\nu}\max_{1\leqslant i\leqslant r}|b_{i\mu\nu}^8|^2(\text{Ł}_{g_\nu^8})^2\int_0^{+\infty}\Psi_{\mu\nu}^2(s)ds\int_0^{+\infty}\Psi_{\mu\nu}^2(s)\int_{t-s}^t|\hat{v}_\nu(\zeta)-v_\nu(\zeta)|^2d\zeta ds,\end{aligned}\tag{23}$$

in which the constants $\mathfrak{m}_{hk1\mu}$, $\mathfrak{m}_{hk2\mu}$, $\mathfrak{m}_{hk3\nu\mu}$, $\mathfrak{m}_{hk4\nu\mu}$, $\mathfrak{m}_{hk5\nu\mu}$, $\mathfrak{m}_{hk6\nu\mu}$, $\mathfrak{m}_{hk7\nu\mu}$, $\mathfrak{m}_{hk8\nu\mu}$, $\mathfrak{n}_{hk1\nu}$, $\mathfrak{n}_{hk2\nu}$, $\mathfrak{n}_{hk3\mu\nu}$, $\mathfrak{n}_{hk4\mu\nu}$, $\mathfrak{n}_{hk5\mu\nu}$, $\mathfrak{n}_{hk6\mu\nu}$, $\mathfrak{n}_{hk7\mu\nu}$, and $\mathfrak{n}_{hk8\mu\nu}$ are to be determined suitably, where

$h = 1,2$, $k = 1,2,3$, $\mu \in \beth$, $\nu \in \gimel$. Following the basic idea to obtain a pre-assigned-time synchronization criterion, we introduce the auxiliary functional

$$V(t) = \mathbb{E}\mathcal{V}(t), \quad t \in \mathbb{R}_+, \tag{24}$$

where $\mathcal{V}(t)$ is the Lyapunov–Krasovskii functional candidate that is given by

$$\mathcal{V}(t) = \sum_{k=1}^{4} \mathcal{V}_k(t), \quad t \in \mathbb{R}_+,\ \mathbb{P}\text{-a.s.}, \tag{25}$$

in which $\mathcal{V}_1(t)$, $\mathcal{V}_2(t)$, $\mathcal{V}_3(t)$ and $\mathcal{V}_4(t)$ are defined, respectively, by

$$\begin{aligned}\mathcal{V}_1(t) &= \sum_{\mu\in\beth} \mathfrak{p}_\mu |\hat{u}_\mu(t) - u_\mu(t)|^2 + \sum_{\nu\in\gimel} \mathfrak{q}_\nu |\hat{v}_\nu(t) - v_\nu(t)|^2 \\ &= \sum_{\mu\in\beth} \mathfrak{p}_\mu |x_\mu(t)|^2 + \sum_{\nu\in\gimel} \mathfrak{q}_\nu |y_\nu(t)|^2, \quad t \in \mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\end{aligned} \tag{26}$$

$$\begin{aligned}\mathcal{V}_2(t) &= \sum_{\mu\in\beth} \frac{\mathfrak{p}_\mu \max\limits_{1\leqslant i\leqslant r} \sigma_{i\mu}}{1-\bar{\tau}_\mu} \int_{t-\tau_\mu(t)}^{t} |\hat{u}_\mu(s) - u_\mu(s)|^2 ds \\
&\quad + \sum_{\mu\in\beth}\sum_{\nu\in\gimel} \frac{\mathfrak{q}_\nu \max\limits_{1\leqslant i\leqslant r} |a^2_{i\nu\mu}| \text{Ł}_{f^2_\mu}}{1-\bar{\varsigma}^1_{\nu\mu}} \int_{t-\varsigma^1_{\nu\mu}(t)}^{t} |\hat{u}_\mu(s) - u_\mu(s)|^2 ds \\
&\quad + 4\beth^\# \sum_{\mu\in\beth}\sum_{\nu\in\gimel} \frac{\mathfrak{q}_\nu \max\limits_{1\leqslant i\leqslant r} |a^6_{i\nu\mu}|^2 (\text{Ł}_{f^6_\mu})^2}{1-\bar{\varsigma}^3_{\nu\mu}} \int_{t-\varsigma^3_{\nu\mu}(t)}^{t} |\hat{u}_\mu(s) - u_\mu(s)|^2 ds \\
&\quad + \sum_{\nu\in\gimel} \frac{\mathfrak{q}_\nu \max\limits_{1\leqslant i\leqslant r} \eta_{i\nu}}{1-\bar{\iota}_\nu} \int_{t-\iota_\nu(t)}^{t} |\hat{v}_\nu(s) - v_\nu(s)|^2 ds \\
&\quad + \sum_{\nu\in\gimel}\sum_{\mu\in\beth} \frac{\mathfrak{p}_\mu \max\limits_{1\leqslant i\leqslant r} |b^2_{i\mu\nu}| \text{Ł}_{g^2_\nu}}{1-\bar{\varrho}^1_{\mu\nu}} \int_{t-\varrho^1_{\mu\nu}(t)}^{t} |\hat{v}_\nu(s) - v_\nu(s)|^2 ds \\
&\quad + 4\gimel^\# \sum_{\nu\in\gimel}\sum_{\mu\in\beth} \frac{\mathfrak{p}_\mu \max\limits_{1\leqslant i\leqslant r} |b^6_{i\mu\nu}|^2 (\text{Ł}_{g^6_\nu})^2}{1-\bar{\varrho}^3_{\mu\nu}} \int_{t-\varrho^3_{\mu\nu}(t)}^{t} |\hat{v}_\nu(s) - v_\nu(s)|^2 ds \\
&= \sum_{\mu\in\beth} \frac{\mathfrak{p}_\mu \max\limits_{1\leqslant i\leqslant r} \sigma_{i\mu}}{1-\bar{\tau}_\mu} \int_{t-\tau_\mu(t)}^{t} |x_\mu(s)|^2 ds \\
&\quad + \sum_{\mu\in\beth}\sum_{\nu\in\gimel} \frac{\mathfrak{q}_\nu \max\limits_{1\leqslant i\leqslant r} |a^2_{i\nu\mu}| \text{Ł}_{f^2_\mu}}{1-\bar{\varsigma}^1_{\nu\mu}} \int_{t-\varsigma^1_{\nu\mu}(t)}^{t} |x_\mu(s)|^2 ds \\
&\quad + 4\beth^\# \sum_{\mu\in\beth}\sum_{\nu\in\gimel} \frac{\mathfrak{q}_\nu \max\limits_{1\leqslant i\leqslant r} |a^6_{i\nu\mu}|^2 (\text{Ł}_{f^6_\mu})^2}{1-\bar{\varsigma}^3_{\nu\mu}} \int_{t-\varsigma^3_{\nu\mu}(t)}^{t} |x_\mu(s)|^2 ds \\
&\quad + \sum_{\nu\in\gimel} \frac{\mathfrak{q}_\nu \max\limits_{1\leqslant i\leqslant r} \eta_{i\nu}}{1-\bar{\iota}_\nu} \int_{t-\iota_\nu(t)}^{t} |y_\nu(s)|^2 ds \\
&\quad + \sum_{\nu\in\gimel}\sum_{\mu\in\beth} \frac{\mathfrak{p}_\mu \max\limits_{1\leqslant i\leqslant r} |b^2_{i\mu\nu}| \text{Ł}_{g^2_\nu}}{1-\bar{\varrho}^1_{\mu\nu}} \int_{t-\varrho^1_{\mu\nu}(t)}^{t} |y_\nu(s)|^2 ds \\
&\quad + 4\gimel^\# \sum_{\nu\in\gimel}\sum_{\mu\in\beth} \frac{\mathfrak{p}_\mu \max\limits_{1\leqslant i\leqslant r} |b^6_{i\mu\nu}|^2 (\text{Ł}_{g^6_\nu})^2}{1-\bar{\varrho}^3_{\mu\nu}} \int_{t-\varrho^3_{\mu\nu}(t)}^{t} |y_\nu(s)|^2 ds, \quad t \in \mathbb{R}_+,\ \mathbb{P}\text{-a.s.},\end{aligned}$$

$$
\begin{aligned}
\mathcal{V}_3(t) = {} & \sum_{\mu\in\beth}\sum_{\nu\in\gimel} \frac{\mathfrak{q}_\nu \max\limits_{1\leqslant i\leqslant r} |a^3_{i\nu\mu}| \text{Ł}_{f^3_\mu}}{1-\bar{\dot{\varsigma}}^2_{\nu\mu}} \int_{t-\varsigma^2_{\nu\mu}(t)}^{t} \int_s^t |\hat{u}_\mu(\zeta)-u_\mu(\zeta)|^2 d\zeta ds \\
& + 4\beth^{\#} \sum_{\mu\in\beth}\sum_{\nu\in\gimel} \frac{\mathfrak{q}_\nu \max\limits_{1\leqslant i\leqslant r} |a^7_{i\nu\mu}|^2 (\text{Ł}_{f^7_\mu})^2 \bar{\varsigma}^4_{\nu\mu}}{1-\bar{\dot{\varsigma}}^4_{\nu\mu}} \int_{t-\varsigma^4_{\nu\mu}(t)}^{t} \int_s^t |\hat{u}_\mu(\zeta)-u_\mu(\zeta)|^2 d\zeta ds \\
& + \sum_{\nu\in\gimel}\sum_{\mu\in\beth} \frac{\mathfrak{p}_\mu \max\limits_{1\leqslant i\leqslant r} |b^3_{i\mu\nu}| \text{Ł}_{g^3_\nu}}{1-\bar{\dot{\varrho}}^2_{\mu\nu}} \int_{t-\varrho^2_{\mu\nu}(t)}^{t} \int_s^t |\hat{v}_\nu(\zeta)-v_\nu(\zeta)|^2 d\zeta ds \\
& + 4\gimel^{\#} \sum_{\nu\in\gimel}\sum_{\mu\in\beth} \frac{\mathfrak{p}_\mu \max\limits_{1\leqslant i\leqslant r} |b^7_{i\mu\nu}|^2 (\text{Ł}_{g^7_\nu})^2 \bar{\varrho}^4_{\mu\nu}}{1-\bar{\dot{\varrho}}^4_{\mu\nu}} \int_{t-\varrho^4_{\mu\nu}(t)}^{t} \int_s^t |\hat{v}_\nu(\zeta)-v_\nu(\zeta)|^2 d\zeta ds \\
= {} & \sum_{\mu\in\beth}\sum_{\nu\in\gimel} \frac{\mathfrak{q}_\nu \max\limits_{1\leqslant i\leqslant r} |a^3_{i\nu\mu}| \text{Ł}_{f^3_\mu}}{1-\bar{\dot{\varsigma}}^2_{\nu\mu}} \int_{t-\varsigma^2_{\nu\mu}(t)}^{t} \int_s^t |x_\mu(\zeta)|^2 d\zeta ds \\
& + 4\beth^{\#} \sum_{\mu\in\beth}\sum_{\nu\in\gimel} \frac{\mathfrak{q}_\nu \max\limits_{1\leqslant i\leqslant r} |a^7_{i\nu\mu}|^2 (\text{Ł}_{f^7_\mu})^2 \bar{\varsigma}^4_{\nu\mu}}{1-\bar{\dot{\varsigma}}^4_{\nu\mu}} \int_{t-\varsigma^4_{\nu\mu}(t)}^{t} \int_s^t |x_\mu(\zeta)|^2 d\zeta ds \\
& + \sum_{\nu\in\gimel}\sum_{\mu\in\beth} \frac{\mathfrak{p}_\mu \max\limits_{1\leqslant i\leqslant r} |b^3_{i\mu\nu}| \text{Ł}_{g^3_\nu}}{1-\bar{\dot{\varrho}}^2_{\mu\nu}} \int_{t-\varrho^2_{\mu\nu}(t)}^{t} \int_s^t |y_\nu(\zeta)|^2 d\zeta ds \\
& + 4\gimel^{\#} \sum_{\nu\in\gimel}\sum_{\mu\in\beth} \frac{\mathfrak{p}_\mu \max\limits_{1\leqslant i\leqslant r} |b^7_{i\mu\nu}|^2 (\text{Ł}_{g^7_\nu})^2 \bar{\varrho}^4_{\mu\nu}}{1-\bar{\dot{\varrho}}^4_{\mu\nu}} \int_{t-\varrho^4_{\mu\nu}(t)}^{t} \int_s^t |y_\nu(\zeta)|^2 d\zeta ds, \quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.,}
\end{aligned} \tag{27}
$$

and

$$
\begin{aligned}
\mathcal{V}_4(t) = {} & \sum_{\mu\in\beth}\sum_{\nu\in\gimel} \mathfrak{q}_\nu \max_{1\leqslant i\leqslant r} |a^4_{i\nu\mu}| \text{Ł}_{f^4_\mu} \int_0^{+\infty} \Phi^1_{\nu\mu}(s) \int_{t-s}^t |\hat{u}_\mu(\zeta)-u_\mu(\zeta)|^2 d\zeta ds \\
& + 4\beth^{\#} \sum_{\mu\in\beth}\sum_{\nu\in\gimel} \mathfrak{q}_\nu \max_{1\leqslant i\leqslant r} |a^8_{i\nu\mu}|^2 (\text{Ł}_{f^8_\mu})^2 \int_0^{+\infty} \Phi^2_{\nu\mu}(s) ds \int_0^{+\infty} \Phi^2_{\nu\mu}(s) \int_{t-s}^t |\hat{u}_\mu(\zeta)-u_\mu(\zeta)|^2 d\zeta ds \\
& + \sum_{\nu\in\gimel}\sum_{\mu\in\beth} \mathfrak{p}_\mu \max_{1\leqslant i\leqslant r} |b^4_{i\mu\nu}| \text{Ł}_{g^4_\nu} \int_0^{+\infty} \Psi^1_{\mu\nu}(s) \int_{t-s}^t |\hat{v}_\nu(\zeta)-v_\nu(\zeta)|^2 d\zeta ds \\
& + 4\gimel^{\#} \sum_{\nu\in\gimel}\sum_{\mu\in\beth} \mathfrak{p}_\mu \max_{1\leqslant i\leqslant r} |b^8_{i\mu\nu}|^2 (\text{Ł}_{g^8_\nu})^2 \int_0^{+\infty} \Psi^2_{\mu\nu}(s) ds \int_0^{+\infty} \Psi^2_{\mu\nu}(s) \int_{t-s}^t |\hat{v}_\nu(\zeta)-v_\nu(\zeta)|^2 d\zeta ds \\
= {} & \sum_{\mu\in\beth}\sum_{\nu\in\gimel} \mathfrak{q}_\nu \max_{1\leqslant i\leqslant r} |a^4_{i\nu\mu}| \text{Ł}_{f^4_\mu} \int_0^{+\infty} \Phi^1_{\nu\mu}(s) \int_{t-s}^t |x_\mu(\zeta)|^2 d\zeta ds \\
& + 4\beth^{\#} \sum_{\mu\in\beth}\sum_{\nu\in\gimel} \mathfrak{q}_\nu \max_{1\leqslant i\leqslant r} |a^8_{i\nu\mu}|^2 (\text{Ł}_{f^8_\mu})^2 \int_0^{+\infty} \Phi^2_{\nu\mu}(s) ds \int_0^{+\infty} \Phi^2_{\nu\mu}(s) \int_{t-s}^t |x_\mu(\zeta)|^2 d\zeta ds \\
& + \sum_{\nu\in\gimel}\sum_{\mu\in\beth} \mathfrak{p}_\mu \max_{1\leqslant i\leqslant r} |b^4_{i\mu\nu}| \text{Ł}_{g^4_\nu} \int_0^{+\infty} \Psi^1_{\mu\nu}(s) \int_{t-s}^t |y_\nu(\zeta)|^2 d\zeta ds \\
& + 4\gimel^{\#} \sum_{\nu\in\gimel}\sum_{\mu\in\beth} \mathfrak{p}_\mu \max_{1\leqslant i\leqslant r} |b^8_{i\mu\nu}|^2 (\text{Ł}_{g^8_\nu})^2 \int_0^{+\infty} \Psi^2_{\mu\nu}(s) ds \int_0^{+\infty} \Psi^2_{\mu\nu}(s) \int_{t-s}^t |y_\nu(\zeta)|^2 d\zeta ds, \quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.}
\end{aligned} \tag{28}
$$

To $\Pi_{hk}(t)$ (see (23)) and $V(t)$ (see (25)), we associate the following constants

$$
b_0 = \max\Big(\max_{\mu\in\beth} \frac{\Sigma^u_\mu}{\mathfrak{p}_\mu}, \max_{\nu\in\gimel} \frac{\Sigma^v_\nu}{\mathfrak{q}_\nu} \Big) - 2\min\Big(\ni_{11} \min_{\mu\in\beth} \mathfrak{p}_\mu \mathfrak{m}^1_\mu, \ni_{21} \min_{\nu\in\gimel} \mathfrak{q}_\nu \mathfrak{n}^1_\nu \Big), \tag{29}
$$

$$b_1 = 2\min\Big((\ni_{12})^{\gamma_1}\min_{\mu\in\beth}\mathfrak{p}_\mu\mathfrak{m}_\mu^2,\ (\ni_{22})^{\gamma_1}\min_{\nu\in\daleth}\mathfrak{q}_\nu\mathfrak{n}_\nu^2\Big), \tag{30}$$

$$b_2 = 2\min\Big((\ni_{13})^{\gamma_2}\min_{\mu\in\beth}\mathfrak{p}_\mu\mathfrak{m}_\mu^3,\ (\ni_{23})^{\gamma_2}\min_{\nu\in\daleth}\mathfrak{q}_\nu\mathfrak{n}_\nu^3\Big), \tag{31}$$

$$\begin{aligned}
\varSigma_\mu^u =&\mathfrak{p}_\mu\Bigg[\bar\sigma_\mu + \frac{\max\limits_{1\leqslant i\leqslant r}\sigma_{i\mu}}{1-\bar\tau_\mu} + \sum_{\nu\in\daleth}\Big(|\bar b_{\mu\nu}^1|Ł_{g_\nu^1} + |\bar b_{\mu\nu}^2|Ł_{g_\nu^2}\\
&+ |\bar b_{\mu\nu}^3|Ł_{g_\nu^3}\bar\varrho_{\mu\nu}^2 + |\bar b_{\mu\nu}^4|Ł_{g_\nu^4}\int_0^{+\infty}\Psi_{\mu\nu}^1(s)ds\Big)\Bigg]\\
&+ \sum_{\nu\in\daleth}\mathfrak{q}_\nu\Bigg(|\bar a_{\nu\mu}^1|Ł_{f_\mu^1} + \frac{\max\limits_{1\leqslant i\leqslant r}|a_{i\nu\mu}^2|Ł_{f_\mu^2}}{1-\bar\varsigma_{\nu\mu}^1} + \frac{\max\limits_{1\leqslant i\leqslant r}|a_{i\nu\mu}^3|Ł_{f_\mu^3}\bar\varsigma_{\nu\mu}^2}{1-\bar\varsigma_{\nu\mu}^2}\\
&+ \max_{1\leqslant i\leqslant r}|a_{i\nu\mu}^4|Ł_{f_\mu^4}\int_0^{+\infty}\Phi_{\nu\mu}^1(s)ds\Bigg)\\
&+ 4\beth^\#\sum_{\nu\in\daleth}\mathfrak{q}_\nu\Bigg(|\bar a_{\nu\mu}^5|^2(Ł_{f_\mu^5})^2 + \frac{\max\limits_{1\leqslant i\leqslant r}|a_{i\nu\mu}^6|^2(Ł_{f_\mu^6})^2}{1-\bar\varsigma_{\nu\mu}^3} + \frac{\max\limits_{1\leqslant i\leqslant r}|a_{i\nu\mu}^7|^2(Ł_{f_\mu^7})^2(\bar\varsigma_{\nu\mu}^4)^2}{1-\bar\varsigma_{\nu\mu}^4}\\
&+ \max_{1\leqslant i\leqslant r}|a_{i\nu\mu}^8|^2(Ł_{f_\mu^8}\int_0^{+\infty}\Phi_{\nu\mu}^2(s)ds)^2\Bigg),\quad \mu\in\beth,\\
\varSigma_\nu^v =&\mathfrak{q}_\nu\Bigg[\bar\eta_\nu + \frac{\max\limits_{1\leqslant i\leqslant r}\eta_{i\nu}}{1-\bar\iota_\nu} + \sum_{\mu\in\beth}\Big(|\bar a_{\nu\mu}^1|Ł_{f_\mu^1} + \sum_{\mu\in\beth}|\bar a_{\nu\mu}^2|Ł_{f_\mu^2}\\
&+ \sum_{\mu\in\beth}|\bar a_{\nu\mu}^3|Ł_{f_\mu^3}\bar\varsigma_{\nu\mu}^2 + \sum_{\mu\in\beth}|\bar a_{\nu\mu}^4|Ł_{f_\mu^4}\int_0^{+\infty}\Phi_{\nu\mu}^1(s)ds\Big)\Bigg]\\
&+ \sum_{\mu\in\beth}\mathfrak{p}_\mu\Bigg(|\bar b_{\mu\nu}^1|Ł_{g_\nu^1} + \frac{\max\limits_{1\leqslant i\leqslant r}|b_{i\mu\nu}^2|Ł_{g_\nu^2}}{1-\bar\varrho_{\mu\nu}^1} + \frac{\max\limits_{1\leqslant i\leqslant r}|b_{i\mu\nu}^3|Ł_{g_\nu^3}\bar\varrho_{\mu\nu}^2}{1-\bar\varrho_{\mu\nu}^2}\\
&+ \max_{1\leqslant i\leqslant r}|b_{i\mu\nu}^4|Ł_{g_\nu^4}\int_0^{+\infty}\Psi_{\mu\nu}^1(s)ds\Bigg)\\
&+ 4\daleth^\#\sum_{\mu\in\beth}\mathfrak{p}_\mu\Bigg(|\bar b_{\mu\nu}^5|^2(Ł_{g_\nu^5})^2 + \frac{\max\limits_{1\leqslant i\leqslant r}|b_{i\mu\nu}^6|^2(Ł_{g_\nu^6})^2}{1-\bar\varrho_{\mu\nu}^3} + \frac{\max\limits_{1\leqslant i\leqslant r}|b_{i\mu\nu}^7|^2(Ł_{g_\nu^7})^2(\bar\varrho_{\mu\nu}^4)^2}{1-\bar\varrho_{\mu\nu}^4}\\
&+ \max_{1\leqslant i\leqslant r}|b_{i\mu\nu}^8|^2(Ł_{g_\nu^8}\int_0^{+\infty}\Psi_{\mu\nu}^2(s)ds)^2\Bigg),\quad \nu\in\daleth,
\end{aligned}$$

and

$$\begin{aligned}
\ni_{hk}= \min\Bigg(&\min_{\mu\in\beth}\frac{\mathfrak{m}_{hk1\mu}}{\mathfrak{p}_\mu},\ \min_{\nu\in\daleth}\frac{\mathfrak{n}_{hk1\nu}}{\mathfrak{q}_\nu},\ \min_{\mu\in\beth}\frac{\mathfrak{m}_{hk2\mu}(1-\bar\tau_\mu)}{\mathfrak{p}_\mu},\ \min_{\substack{\nu\in\daleth,\\ \mu\in\beth}}\frac{\mathfrak{m}_{hk3\nu\mu}(1-\bar\varsigma_{\nu\mu}^1)}{\mathfrak{q}_\nu},\\
&\min_{\substack{\nu\in\daleth,\\ \mu\in\beth}}\frac{\mathfrak{m}_{hk4\nu\mu}(1-\bar\varsigma_{\nu\mu}^3)}{4\mathfrak{q}_\nu\beth^\#},\ \min_{\nu\in\daleth}\frac{\mathfrak{n}_{hk2\nu}(1-\bar\iota_\nu)}{\mathfrak{q}_\nu},\ \min_{\substack{\mu\in\beth,\\ \nu\in\daleth}}\frac{\mathfrak{n}_{hk3\mu\nu}(1-\bar\varrho_{\mu\nu}^1)}{\mathfrak{p}_\mu},\\
&\min_{\substack{\mu\in\beth,\\ \nu\in\daleth}}\frac{\mathfrak{n}_{hk4\mu\nu}(1-\bar\varrho_{\mu\nu}^3)}{4\mathfrak{p}_\mu\daleth^\#},\ \min_{\substack{\nu\in\daleth,\\ \mu\in\beth}}\frac{\mathfrak{m}_{hk5\nu\mu}(1-\bar\varsigma_{\nu\mu}^2)}{\mathfrak{q}_\nu},\ \min_{\substack{\nu\in\daleth,\\ \mu\in\beth}}\frac{\mathfrak{m}_{hk6\nu\mu}(1-\bar\varsigma_{\nu\mu}^4)}{4\mathfrak{q}_\nu\beth^\#},\\
&\min_{\substack{\mu\in\beth,\\ \nu\in\daleth}}\frac{\mathfrak{n}_{hk5\mu\nu}(1-\bar\varrho_{\mu\nu}^2)}{\mathfrak{p}_\mu},\ \min_{\substack{\mu\in\beth,\\ \nu\in\daleth}}\frac{\mathfrak{n}_{hk6\mu\nu}(1-\bar\varrho_{\mu\nu}^4)}{4\mathfrak{p}_\mu\daleth^\#},\ \min_{\substack{\nu\in\daleth,\\ \mu\in\beth}}\frac{\mathfrak{m}_{hk7\nu\mu}}{\mathfrak{q}_\nu},\\
&\min_{\substack{\nu\in\daleth,\\ \mu\in\beth}}\frac{\mathfrak{m}_{hk8\nu\mu}}{4\mathfrak{q}_\nu\beth^\#},\ \min_{\substack{\mu\in\beth,\\ \nu\in\daleth}}\frac{\mathfrak{n}_{hk7\mu\nu}}{\mathfrak{p}_\mu},\ \min_{\substack{\mu\in\beth,\\ \nu\in\daleth}}\frac{\mathfrak{n}_{hk8\mu\nu}}{4\mathfrak{p}_\mu\daleth^\#}\Bigg),\quad h=1,2,\ k=1,2,3.
\end{aligned} \tag{32}$$

Theorem 1. *Suppose that Assumptions 1–3 hold true. Let* $T_\natural \in (0, +\infty)$ *be given. If there exist some positive constants* $\mathfrak{m}_\mu^1$, $\mathfrak{m}_\mu^2$, $\mathfrak{m}_\mu^3$, $\mathfrak{m}_{hk1\mu}$, $\mathfrak{m}_{hk2\mu}$, $\mathfrak{m}_{hk3\nu\mu}$, $\mathfrak{m}_{hk4\nu\mu}$, $\mathfrak{m}_{hk5\nu\mu}$, $\mathfrak{m}_{hk6\nu\mu}$, $\mathfrak{m}_{hk7\nu\mu}$, $\mathfrak{m}_{hk8\nu\mu}$, $\mathfrak{n}_\nu^1$, $\mathfrak{n}_\nu^2$, $\mathfrak{n}_\nu^3$, $\mathfrak{n}_{hk1\nu}$, $\mathfrak{n}_{hk2\nu}$, $\mathfrak{n}_{hk3\mu\nu}$, $\mathfrak{n}_{hk4\mu\nu}$, $\mathfrak{n}_{hk5\mu\nu}$, $\mathfrak{n}_{hk6\mu\nu}$, $\mathfrak{n}_{hk7\mu\nu}$, $\mathfrak{n}_{hk8\mu\nu}$, $\mathfrak{p}_\mu$, $\mathfrak{q}_\nu$ $(\mu \in \beth, \nu \in \daleth,$ $h = 1, 2, 3, k = 1, 2, 3)$*, and the constants* $\gamma_1 \in (1, +\infty)$, $\gamma_2 \in [0, 1)$*, and* $T_\wr \in (0, +\infty)$ *such that the function* $V(t)$ *given by* (24) *satisfies the differential inequality* (20) *in Lemma 1 and it holds that* $T_\wr = \mathcal{T}(a_0, a_1, a_2, \gamma_1, \gamma_2)$ *where* $\mathcal{T}$ *is defined as in* (19) *and the constant* a_k *is given by*

$$a_k = \frac{b_k T_\natural}{T_\wr}, \quad k = 0, 1, 2, \tag{33}$$

with the constants b_0, b_1*, and* b_2 *given, in turn, by* (29), (30)*, and* (31)*, respectively, then the drive BAMNNs* (5)-(6)-(7) *and the response BAMNNs* (8)-(6)-(7)*, with the control* (21)–(22) *implemented, achieve synchronization within the pre-assigned time* $T_\natural$*.*

Proof. From the analysis conducted in Section 2, we realize that to prove Theorem 1, it boils down to proving a stability criterion for the error BAMNNs (11). To establish the desired stability criterion, we have to analyze in some detail the Lyapunov–Krasovskii functional $\mathcal{V}(t)$, defined as in (25), and its mathematical expectation $V(t)$, given by (24). Apply Itô's differentiation rule to $\mathcal{V}_1(t)$ given by (26) to obtain

$$\begin{aligned}
d\mathcal{V}_1(t) =& 2\sum_{\mu\in\beth} \mathfrak{p}_\mu x_\mu(t)\Bigg[-\bar{\sigma}_\mu x_\mu(t-\tau_\mu(t)) + \sum_{\nu\in\daleth} \bar{b}^1_{\mu\nu}\check{g}^1_\nu(y_\nu(t)) + \sum_{\nu\in\daleth} \bar{b}^2_{\mu\nu}\check{g}^2_\nu(y_\nu(t-\varrho^1_{\mu\nu}(t))) \\
&+ \sum_{\nu\in\daleth} \bar{b}^3_{\mu\nu}\int_{t-\varrho^2_{\mu\nu}(t)}^t \check{g}^3_\nu(y_\nu(s))ds + \sum_{\nu\in\daleth} \bar{b}^4_{\mu\nu}\int_{-\infty}^t \Psi^1_{\mu\nu}(t-s)\check{g}^4_\nu(y_\nu(s))ds + \mathscr{U}_\mu(t)\Bigg]dt \\
&+ 2\sum_{\mu\in\beth} \mathfrak{p}_\mu x_\mu(t)\Bigg[\sum_{\nu\in\daleth} \bar{b}^5_{\mu\nu}\check{g}^5_\nu(y_\nu(t)) + \sum_{\nu\in\daleth} \bar{b}^6_{\mu\nu}\check{g}^6_\nu(y_\nu(t-\varrho^3_{\mu\nu}(t))) \\
&+ \sum_{\nu\in\daleth} \bar{b}^7_{\mu\nu}\int_{t-\varrho^4_{\mu\nu}(t)}^t \check{g}^7_\nu(y_\nu(s))ds + \sum_{\nu\in\daleth} \bar{b}^8_{\mu\nu}\int_{-\infty}^t \Psi^2_{\mu\nu}(t-s)\check{g}^8_\nu(y_\nu(s))ds\Bigg]dW(t) \\
&+ \sum_{\mu\in\beth} \mathfrak{p}_\mu\Bigg[\sum_{\nu\in\daleth} \bar{b}^5_{\mu\nu}\check{g}^5_\nu(y_\nu(t)) + \sum_{\nu\in\daleth} \bar{b}^6_{\mu\nu}\check{g}^6_\nu(y_\nu(t-\varrho^3_{\mu\nu}(t))) \\
&+ \sum_{\nu\in\daleth} \bar{b}^7_{\mu\nu}\int_{t-\varrho^4_{\mu\nu}(t)}^t \check{g}^7_\nu(y_\nu(s))ds + \sum_{\nu\in\daleth} \bar{b}^8_{\mu\nu}\int_{-\infty}^t \Psi^2_{\mu\nu}(t-s)\check{g}^8_\nu(y_\nu(s))ds\Bigg]^2 dt \\
&+ 2\sum_{\nu\in\daleth} \mathfrak{q}_\nu y_\nu(t)\Bigg[-\bar{\eta}_\nu y_\nu(t-\iota_\nu(t)) + \sum_{\mu\in\beth} \bar{a}^1_{\nu\mu}\check{f}^1_\mu(x_\mu(t)) + \sum_{\mu\in\beth} \bar{a}^2_{\nu\mu}\check{f}^2_\mu(x_\mu(t-\varsigma^1_{\nu\mu}(t))) \\
&+ \sum_{\mu\in\beth} \bar{a}^3_{\nu\mu}\int_{t-\varsigma^2_{\nu\mu}(t)}^t \check{f}^3_\mu(x_\mu(s))ds + \sum_{\mu\in\beth} \bar{a}^4_{\nu\mu}\int_{-\infty}^t \Phi^1_{\nu\mu}(t-s)\check{f}^4_\mu(x_\mu(s))ds + \mathscr{V}_\nu(t)\Bigg]dt \\
&+ 2\sum_{\nu\in\daleth} \mathfrak{q}_\nu y_\nu(t)\Bigg[\sum_{\mu\in\beth} \bar{a}^5_{\nu\mu}\check{f}^5_\mu(x_\mu(t)) + \sum_{\mu\in\beth} \bar{a}^6_{\nu\mu}\check{f}^6_\mu(x_\mu(t-\varsigma^3_{\nu\mu}(t))) \\
&+ \sum_{\mu\in\beth} \bar{a}^7_{\nu\mu}\int_{t-\varsigma^4_{\nu\mu}(t)}^t \check{f}^7_\mu(x_\mu(s))ds + \sum_{\mu\in\beth} \bar{a}^8_{\nu\mu}\int_{-\infty}^t \Phi^2_{\nu\mu}(t-s)\check{f}^8_\mu(x_\mu(s))ds\Bigg]dW(t) \\
&+ \sum_{\nu\in\daleth} \mathfrak{q}_\nu\Bigg[\sum_{\mu\in\beth} \bar{a}^5_{\nu\mu}\check{f}^5_\mu(x_\mu(t)) + \sum_{\mu\in\beth} \bar{a}^6_{\nu\mu}\check{f}^6_\mu(x_\mu(t-\varsigma^3_{\nu\mu}(t))) \\
&+ \sum_{\mu\in\beth} \bar{a}^7_{\nu\mu}\int_{t-\varsigma^4_{\nu\mu}(t)}^t \check{f}^7_\mu(x_\mu(s))ds + \sum_{\mu\in\beth} \bar{a}^8_{\nu\mu}\int_{-\infty}^t \Phi^2_{\nu\mu}(t-s)\check{f}^8_\mu(x_\mu(s))ds\Bigg]^2 dt, \quad t \in \mathbb{R}_+, \ \mathbb{P}\text{-a.s.}
\end{aligned} \tag{34}$$

By some routine calculations, we have

$$
\begin{aligned}
&2\sum_{\mu\in\beth}\mathfrak{p}_\mu x_\mu(t)\bigg[-\bar{\sigma}_\mu x_\mu(t-\tau_\mu(t))+\sum_{\nu\in\beth}\bar{b}^1_{\mu\nu}\check{g}^1_\nu(y_\nu(t))+\sum_{\nu\in\beth}\bar{b}^2_{\mu\nu}\check{g}^2_\nu(y_\nu(t-\varrho^1_{\mu\nu}(t)))\\
&+\sum_{\nu\in\beth}\bar{b}^3_{\mu\nu}\int_{t-\varrho^2_{\mu\nu}(t)}^{t}\check{g}^3_\nu(y_\nu(s))ds+\sum_{\nu\in\beth}\bar{b}^4_{\mu\nu}\int_{-\infty}^{t}\Psi^1_{\mu\nu}(t-s)\check{g}^4_\nu(y_\nu(s))ds\bigg]\\
\leqslant&2\sum_{\mu\in\beth}\mathfrak{p}_\mu x_\mu(t)\mathscr{U}_\mu(t)+\sum_{\mu\in\beth}\mathfrak{p}_\mu\bar{\sigma}_\mu\big(|x_\mu(t)|^2+|x_\mu(t-\tau_\mu(t))|^2\big)\\
&+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^1_{\mu\nu}|\text{Ł}_{g^1_\nu}\big(|x_\mu(t)|^2+|y_\nu(t)|^2\big)+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^2_{\mu\nu}|\text{Ł}_{g^2_\nu}\big(|x_\mu(t)|^2+|y_\nu(t-\varrho^1_{\mu\nu}(t))|^2\big)\\
&+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^3_{\mu\nu}|\text{Ł}_{g^3_\nu}\int_{t-\varrho^2_{\mu\nu}(t)}^{t}\big(|x_\mu(t)|^2+|y_\nu(s)|^2\big)ds\\
&+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^4_{\mu\nu}|\text{Ł}_{g^4_\nu}\int_{-\infty}^{t}\Psi^1_{\mu\nu}(t-s)\big(|x_\mu(t)|^2+|y_\nu(s)|^2\big)ds\\
=&2\sum_{\mu\in\beth}\mathfrak{p}_\mu x_\mu(t)\mathscr{U}_\mu(t)+\sum_{\nu\in\beth}\sum_{\mu\in\beth}\mathfrak{p}_\mu|\bar{b}^1_{\mu\nu}|\text{Ł}_{g^1_\nu}|y_\nu(t)|^2\\
&+\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\bar{\sigma}_\mu}{1-\dot{\tau}_\mu(t)}\Big(|x_\mu(t)|^2-\frac{d}{dt}\int_{t-\tau_\mu(t)}^{t}|x_\mu(s)|^2ds\Big)\\
&+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\frac{\mathfrak{p}_\mu|\bar{b}^2_{\mu\nu}|\text{Ł}_{g^2_\nu}}{1-\dot{\varrho}^1_{\mu\nu}(t)}\Big(|y_\nu(t)|^2-\frac{d}{dt}\int_{t-\varrho^1_{\mu\nu}(t)}^{t}|y_\nu(s)|^2ds\Big)\\
&+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\frac{\mathfrak{p}_\mu|\bar{b}^3_{\mu\nu}|\text{Ł}_{g^3_\nu}}{1-\dot{\varrho}^2_{\mu\nu}(t)}\Big(\varrho^2_{\mu\nu}(t)|y_\nu(t)|^2-\frac{d}{dt}\int_{t-\varrho^2_{\mu\nu}(t)}^{t}\int_{s}^{t}|y_\nu(\zeta)|^2d\zeta ds\Big)\\
&+\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^4_{\mu\nu}|\text{Ł}_{g^4_\nu}\Big(\int_{0}^{+\infty}\Psi^1_{\mu\nu}(s)ds|y_\nu(t)|^2-\frac{d}{dt}\int_{0}^{+\infty}\Psi^1_{\mu\nu}(s)\int_{t-s}^{t}|y_\nu(\zeta)|^2d\zeta ds\Big)\\
&+\sum_{\mu\in\beth}\Big(\mathfrak{p}_\mu\bar{\sigma}_\mu+\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^1_{\mu\nu}|\text{Ł}_{g^1_\nu}+\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^2_{\mu\nu}|\text{Ł}_{g^2_\nu}\\
&+\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^3_{\mu\nu}|\text{Ł}_{g^3_\nu}\varrho^2_{\mu\nu}(t)+\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^4_{\mu\nu}|\text{Ł}_{g^4_\nu}\int_{0}^{+\infty}\Psi^1_{\mu\nu}(s)ds\Big)|x_\mu(t)|^2\\
\leqslant&2\sum_{\mu\in\beth}\mathfrak{p}_\mu x_\mu(t)\mathscr{U}_\mu(t)+\sum_{\mu\in\beth}\Big(\mathfrak{p}_\mu\bar{\sigma}_\mu+\frac{\mathfrak{p}_\mu\max\limits_{1\leqslant i\leqslant r}\sigma_{i\mu}}{1-\bar{\tau}_\mu}+\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^1_{\mu\nu}|\text{Ł}_{g^1_\nu}+\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^2_{\mu\nu}|\text{Ł}_{g^2_\nu}\\
&+\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^3_{\mu\nu}|\text{Ł}_{g^3_\nu}\varrho^2_{\mu\nu}(t)+\sum_{\nu\in\beth}\mathfrak{p}_\mu|\bar{b}^4_{\mu\nu}|\text{Ł}_{g^4_\nu}\int_{0}^{+\infty}\Psi^1_{\mu\nu}(s)ds\Big)|x_\mu(t)|^2\\
&-\frac{d}{dt}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\max\limits_{1\leqslant i\leqslant r}\sigma_{i\mu}}{1-\bar{\tau}_\mu}\int_{t-\tau_\mu(t)}^{t}|x_\mu(s)|^2ds+\sum_{\nu\in\beth}\Big(\sum_{\mu\in\beth}\mathfrak{p}_\mu|\bar{b}^1_{\mu\nu}|\text{Ł}_{g^1_\nu}+\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\max\limits_{1\leqslant i\leqslant r}|b^2_{i\mu\nu}|\text{Ł}_{g^2_\nu}}{1-\bar{\varrho}^1_{\mu\nu}}\\
&+\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\max\limits_{1\leqslant i\leqslant r}|b^3_{i\mu\nu}|\text{Ł}_{g^3_\nu}\varrho^2_{\mu\nu}(t)}{1-\bar{\varrho}^2_{\mu\nu}}+\sum_{\mu\in\beth}\mathfrak{p}_\mu\max_{1\leqslant i\leqslant r}|b^4_{i\mu\nu}|\text{Ł}_{g^4_\nu}\int_{0}^{+\infty}\Psi^1_{\mu\nu}(s)ds\Big)|y_\nu(t)|^2\\
&-\frac{d}{dt}\sum_{\nu\in\beth}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\max\limits_{1\leqslant i\leqslant r}|b^2_{i\mu\nu}|\text{Ł}_{g^2_\nu}}{1-\bar{\varrho}^1_{\mu\nu}}\int_{t-\varrho^1_{\mu\nu}(t)}^{t}|y_\nu(s)|^2ds\\
&-\frac{d}{dt}\sum_{\nu\in\beth}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\max\limits_{1\leqslant i\leqslant r}|b^3_{i\mu\nu}|\text{Ł}_{g^3_\nu}}{1-\bar{\varrho}^2_{\mu\nu}}\int_{t-\varrho^2_{\mu\nu}(t)}^{t}\int_{s}^{t}|y_\nu(\zeta)|^2d\zeta ds\\
&-\frac{d}{dt}\sum_{\nu\in\beth}\sum_{\mu\in\beth}\mathfrak{p}_\mu\max_{1\leqslant i\leqslant r}|b^4_{i\mu\nu}|\text{Ł}_{g^4_\nu}\int_{0}^{+\infty}\Psi^1_{\mu\nu}(s)\int_{t-s}^{t}|y_\nu(\zeta)|^2d\zeta ds,\quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.}
\end{aligned}
\tag{35}
$$

By applying mainly

$$(a_1+a_2+a_3+a_4)^2 \leqslant 4(|a_1|^2+|a_2|^2+|a_3|^2+|a_4|^2), \quad a_k\in\mathbb{R},\ k=1,2,3,4 \tag{36}$$

as well as Leibniz's integral rule, we have

$$\begin{aligned}
&\sum_{\mu\in\beth}\mathfrak{p}_\mu\Big[\sum_{\nu\in\beth}\bar{b}^5_{\mu\nu}\breve{g}^5_\nu(y_\nu(t))+\sum_{\nu\in\beth}\bar{b}^6_{\mu\nu}\breve{g}^6_\nu(y_\nu(t-\varrho^3_{\mu\nu}(t)))\\
&+\sum_{\nu\in\beth}\bar{b}^7_{\mu\nu}\int_{t-\varrho^4_{\mu\nu}(t)}^t\breve{g}^7_\nu(y_\nu(s))ds+\sum_{\nu\in\beth}\bar{b}^8_{\mu\nu}\int_{-\infty}^t\Psi^2_{\mu\nu}(t-s)\breve{g}^8_\nu(y_\nu(s))ds\Big]^2\\
\leqslant&4\beth^\#\sum_{\mu\in\beth}\mathfrak{p}_\mu\sum_{\nu\in\beth}\Big(|\bar{b}^5_{\mu\nu}\breve{g}^5_\nu(y_\nu(t))|^2+|\bar{b}^6_{\mu\nu}\breve{g}^6_\nu(y_\nu(t-\varrho^3_{\mu\nu}(t)))|^2\\
&+|\bar{b}^7_{\mu\nu}\int_{t-\varrho^4_{\mu\nu}(t)}^t\breve{g}^7_\nu(y_\nu(s))ds|^2+|\bar{b}^8_{\mu\nu}\int_{-\infty}^t\Psi^2_{\mu\nu}(t-s)\breve{g}^8_\nu(y_\nu(s))ds|^2\Big)\\
=&4\beth^\#\sum_{\mu\in\beth}\mathfrak{p}_\mu\sum_{\nu\in\beth}\Big(|\bar{b}^5_{\mu\nu}|^2(\text{\L}_{g^5_\nu})^2|y_\nu(t)|^2+|\bar{b}^6_{\mu\nu}|^2(\text{\L}_{g^6_\nu})^2|y_\nu(t-\varrho^3_{\mu\nu}(t))|^2\\
&-|\bar{b}^7_{\mu\nu}|^2\int_{t-\varrho^4_{\mu\nu}(t)}^t\Big|\sqrt{\varrho^4_{\mu\nu}(t)}\breve{g}^7_\nu(y_\nu(s))-\frac{1}{\sqrt{\varrho^4_{\mu\nu}(t)}}\int_{t-\varrho^4_{\mu\nu}(t)}^t\breve{g}^7_\nu(y_\nu(\zeta))d\zeta\Big|^2ds\\
&+|\bar{b}^7_{\mu\nu}|^2\varrho^4_{\mu\nu}(t)\int_{t-\varrho^4_{\mu\nu}(t)}^t|\breve{g}^7_\nu(y_\nu(s))|^2ds+|\bar{b}^8_{\mu\nu}|^2\int_0^{+\infty}\Psi^2_{\mu\nu}(s)ds\int_{-\infty}^t\Psi^2_{\mu\nu}(t-s)|\breve{g}^8_\nu(y_\nu(s))|^2ds\\
&-|\bar{b}^8_{\mu\nu}|^2\int_{-\infty}^t\Psi^2_{\mu\nu}(t-s)\Big|\sqrt{\int_{-\infty}^t\Psi^2_{\mu\nu}(t-s)ds}\breve{g}^8_\nu(y_\nu(s))-\frac{1}{\sqrt{\int_{-\infty}^t\Psi^2_{\mu\nu}(t-s)ds}}\int_{-\infty}^t\Psi^2_{\mu\nu}(t-\zeta)\breve{g}^8_\nu(y_\nu(\zeta))d\zeta\Big|^2ds\Big)\\
\leqslant&4\beth^\#\sum_{\mu\in\beth}\mathfrak{p}_\mu\sum_{\nu\in\beth}\Big(|\bar{b}^5_{\mu\nu}|^2(\text{\L}_{g^5_\nu})^2|y_\nu(t)|^2+|\bar{b}^6_{\mu\nu}|^2(\text{\L}_{g^6_\nu})^2|y_\nu(t-\varrho^3_{\mu\nu}(t))|^2+|\bar{b}^7_{\mu\nu}|^2(\text{\L}_{g^7_\nu})^2\varrho^4_{\mu\nu}(t)\int_{t-\varrho^4_{\mu\nu}(t)}^t|y_\nu(s)|^2ds\\
&+|\bar{b}^8_{\mu\nu}|^2(\text{\L}_{g^8_\nu})^2\int_0^{+\infty}\Psi^2_{\mu\nu}(s)ds\int_{-\infty}^t\Psi^2_{\mu\nu}(t-s)|y_\nu(s))|^2ds\Big)\\
\leqslant&4\beth^\#\sum_{\mu\in\beth}\mathfrak{p}_\mu\sum_{\nu\in\beth}\Big[|\bar{b}^5_{\mu\nu}|^2(\text{\L}_{g^5_\nu})^2|y_\nu(t)|^2+\frac{|\bar{b}^6_{\mu\nu}|^2(\text{\L}_{g^6_\nu})^2}{1-\bar{\varrho}^3_{\mu\nu}}\Big(|y_\nu(t)|^2-\frac{d}{dt}\int_{t-\varrho^3_{\mu\nu}(t)}^t|y_\nu(s)|^2ds\Big)\\
&+\frac{|\bar{b}^7_{\mu\nu}|^2(\text{\L}_{g^7_\nu})^2\bar{\varrho}^4_{\mu\nu}}{1-\bar{\varrho}^4_{\mu\nu}}\Big(\varrho^4_{\mu\nu}(t)|y_\nu(t)|^2-\frac{d}{dt}\int_{t-\varrho^4_{\mu\nu}(t)}^t\int_s^t|y_\nu(\zeta)|^2d\zeta ds\Big)\\
&+|\bar{b}^8_{\mu\nu}|^2(\text{\L}_{g^8_\nu})^2\int_0^{+\infty}\Psi^2_{\mu\nu}(s)ds\Big(\int_0^{+\infty}\Psi^2_{\mu\nu}(s)ds|y_\nu(t)|^2-\frac{d}{dt}\int_0^{+\infty}\Psi^2_{\mu\nu}(s)\int_{t-s}^t|y_\nu(\zeta)|^2d\zeta ds\Big)\Big]\\
=&4\beth^\#\sum_{\nu\in\beth}\sum_{\mu\in\beth}\mathfrak{p}_\mu\Big(|\bar{b}^5_{\mu\nu}|^2(\text{\L}_{g^5_\nu})^2+\frac{\max\limits_{1\leqslant i\leqslant r}|b^6_{i\mu\nu}|^2(\text{\L}_{g^6_\nu})^2}{1-\bar{\varrho}^3_{\mu\nu}}+\frac{\max\limits_{1\leqslant i\leqslant r}|b^7_{i\mu\nu}|^2(\text{\L}_{g^7_\nu})^2\bar{\varrho}^4_{\mu\nu}\varrho^4_{\mu\nu}(t)}{1-\bar{\varrho}^4_{\mu\nu}}\\
&+\max_{1\leqslant i\leqslant r}|b^8_{i\mu\nu}|^2(\text{\L}_{g^8_\nu}\int_0^{+\infty}\Psi^2_{\mu\nu}(s)ds)^2\Big)|y_\nu(t)|^2\\
&-4\beth^\#\frac{d}{dt}\sum_{\nu\in\beth}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\max\limits_{1\leqslant i\leqslant r}|b^6_{i\mu\nu}|^2(\text{\L}_{g^6_\nu})^2}{1-\bar{\varrho}^3_{\mu\nu}}\int_{t-\varrho^3_{\mu\nu}(t)}^t|y_\nu(s)|^2ds\\
&-4\beth^\#\frac{d}{dt}\sum_{\nu\in\beth}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\max\limits_{1\leqslant i\leqslant r}|b^7_{i\mu\nu}|^2(\text{\L}_{g^7_\nu})^2\bar{\varrho}^4_{\mu\nu}}{1-\bar{\varrho}^4_{\mu\nu}}\int_{t-\varrho^4_{\mu\nu}(t)}^t\int_s^t|y_\nu(\zeta)|^2d\zeta ds\\
&-4\beth^\#\frac{d}{dt}\sum_{\nu\in\beth}\sum_{\mu\in\beth}\mathfrak{p}_\mu\max_{1\leqslant i\leqslant r}|b^8_{i\mu\nu}|^2(\text{\L}_{g^8_\nu})^2\int_0^{+\infty}\Psi^2_{\mu\nu}(s)ds\int_0^{+\infty}\Psi^2_{\mu\nu}(s)\int_{t-s}^t|y_\nu(\zeta)|^2d\zeta ds,\quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.}
\end{aligned}\tag{37}$$

Mimick steps to obtain (35), to arrive at

$$
\begin{aligned}
&2\sum_{\nu\in\beth}\mathfrak{q}_\nu y_\nu(t)\Big[-\bar{\eta}_\nu y_\nu(t-\iota_\nu(t))+\sum_{\mu\in\beth}\bar{a}^1_{\nu\mu}\check{f}^1_\mu(x_\mu(t))\\
&+\sum_{\mu\in\beth}\bar{a}^2_{\nu\mu}\check{f}^2_\mu(x_\mu(t-\varsigma^1_{\nu\mu}(t)))+\sum_{\mu\in\beth}\bar{a}^3_{\nu\mu}\int_{t-\varsigma^2_{\nu\mu}(t)}^t\check{f}^3_\mu(x_\mu(s))ds\\
&+\sum_{\mu\in\beth}\bar{a}^4_{\nu\mu}\int_{-\infty}^t\Phi^1_{\nu\mu}(t-s)\check{f}^4_\mu(x_\mu(s))ds+\mathscr{V}_\nu(t)\Big]\\
\leqslant&2\sum_{\nu\in\beth}\mathfrak{q}_\nu y_\nu(t)\mathscr{V}_\nu(t)+\sum_{\nu\in\beth}\Big(\mathfrak{q}_\nu\bar{\eta}_\nu+\frac{\mathfrak{q}_\nu\max\limits_{1\leqslant i\leqslant r}\eta_{i\nu}}{1-\bar{\iota}_\nu}+\sum_{\mu\in\beth}\mathfrak{q}_\nu|\bar{a}^1_{\nu\mu}|\text{Ł}_{f^1_\mu}+\sum_{\mu\in\beth}\mathfrak{q}_\nu|\bar{a}^2_{\nu\mu}|\text{Ł}_{f^2_\mu}\\
&+\sum_{\mu\in\beth}\mathfrak{q}_\nu|\bar{a}^3_{\nu\mu}|\text{Ł}_{f^3_\mu}\varsigma^2_{\nu\mu}(t)+\sum_{\mu\in\beth}\mathfrak{q}_\nu|\bar{a}^4_{\nu\mu}|\text{Ł}_{f^4_\mu}\int_0^{+\infty}\Phi^1_{\nu\mu}(s)ds\Big)|y_\nu(t)|^2\\
&-\frac{d}{dt}\sum_{\nu\in\beth}\frac{\mathfrak{q}_\nu\max\limits_{1\leqslant i\leqslant r}\eta_{i\nu}}{1-\bar{\iota}_\nu}\int_{t-\iota_\nu(t)}^t|y_\nu(s)|^2ds+\sum_{\mu\in\beth}\Big(\sum_{\nu\in\beth}\mathfrak{q}_\nu|\bar{a}^1_{\nu\mu}|\text{Ł}_{f^1_\mu}+\sum_{\nu\in\beth}\frac{\mathfrak{q}_\nu\max\limits_{1\leqslant i\leqslant r}|a^2_{i\nu\mu}|\text{Ł}_{f^2_\mu}}{1-\bar{\xi}^1_{\nu\mu}}\\
&+\sum_{\nu\in\beth}\frac{\mathfrak{q}_\nu\max\limits_{1\leqslant i\leqslant r}|a^3_{i\nu\mu}|\text{Ł}_{f^3_\mu}\varsigma^2_{\nu\mu}(t)}{1-\bar{\xi}^2_{\nu\mu}}+\sum_{\nu\in\beth}\mathfrak{q}_\nu\max_{1\leqslant i\leqslant r}|a^4_{i\nu\mu}|\text{Ł}_{f^4_\mu}\int_0^{+\infty}\Phi^1_{\nu\mu}(s)ds\Big)|x_\mu(t)|^2\\
&-\frac{d}{dt}\sum_{\mu\in\beth}\sum_{\nu\in\beth}\frac{\mathfrak{q}_\nu\max\limits_{1\leqslant i\leqslant r}|a^2_{i\nu\mu}|\text{Ł}_{f^2_\mu}}{1-\bar{\xi}^1_{\nu\mu}}\int_{t-\varsigma^1_{\nu\mu}(t)}^t|x_\mu(s)|^2ds\\
&-\frac{d}{dt}\sum_{\mu\in\beth}\sum_{\nu\in\beth}\frac{\mathfrak{q}_\nu\max\limits_{1\leqslant i\leqslant r}|a^3_{i\nu\mu}|\text{Ł}_{f^3_\mu}}{1-\bar{\xi}^2_{\nu\mu}}\int_{t-\varsigma^2_{\nu\mu}(t)}^t\int_s^t|x_\mu(\zeta)|^2d\zeta ds\\
&-\frac{d}{dt}\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{q}_\nu\max_{1\leqslant i\leqslant r}|a^4_{i\nu\mu}|\text{Ł}_{f^4_\mu}\int_0^{+\infty}\Phi^1_{\nu\mu}(s)\int_{t-s}^t|x_\mu(\zeta)|^2d\zeta ds,\quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.}
\end{aligned}\tag{38}
$$

Borrowing an idea from the derivation of (37), we have by some routine calculations

$$
\begin{aligned}
&\sum_{\nu\in\beth}\mathfrak{q}_\nu\Big[\sum_{\mu\in\beth}\bar{a}^5_{\nu\mu}\check{f}^5_\mu(x_\mu(t))+\sum_{\mu\in\beth}\bar{a}^6_{\nu\mu}\check{f}^6_\mu(x_\mu(t-\varsigma^3_{\nu\mu}(t)))\\
&+\sum_{\mu\in\beth}\bar{a}^7_{\nu\mu}\int_{t-\varsigma^4_{\nu\mu}(t)}^t\check{f}^7_\mu(x_\mu(s))ds+\sum_{\mu\in\beth}\bar{a}^8_{\nu\mu}\int_{-\infty}^t\Phi^2_{\nu\mu}(t-s)\check{f}^8_\mu(x_\mu(s))ds\Big]^2\\
\leqslant&4\beth^{\#}\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{q}_\nu\Big(|\bar{a}^5_{\nu\mu}|^2(\text{Ł}_{f^5_\mu})^2+\frac{\max\limits_{1\leqslant i\leqslant r}|a^6_{i\nu\mu}|^2(\text{Ł}_{f^6_\mu})^2}{1-\bar{\xi}^3_{\nu\mu}}+\frac{\max\limits_{1\leqslant i\leqslant r}|a^7_{i\nu\mu}|^2(\text{Ł}_{f^7_\mu})^2\bar{\xi}^4_{\nu\mu}\varsigma^4_{\nu\mu}(t)}{1-\bar{\xi}^4_{\nu\mu}}\\
&+\max_{1\leqslant i\leqslant r}|a^8_{i\nu\mu}|^2(\text{Ł}_{f^8_\mu}\int_0^{+\infty}\Phi^2_{\nu\mu}(s)ds)^2\Big)|x_\mu(t)|^2\\
&-4\beth^{\#}\frac{d}{dt}\sum_{\mu\in\beth}\sum_{\nu\in\beth}\frac{\mathfrak{q}_\nu\max\limits_{1\leqslant i\leqslant r}|a^6_{i\nu\mu}|^2(\text{Ł}_{f^6_\mu})^2}{1-\bar{\xi}^3_{\nu\mu}}\int_{t-\varsigma^3_{\nu\mu}(t)}^t|x_\mu(s)|^2ds\\
&-4\beth^{\#}\frac{d}{dt}\sum_{\mu\in\beth}\sum_{\nu\in\beth}\frac{\mathfrak{q}_\nu\max\limits_{1\leqslant i\leqslant r}|a^7_{i\nu\mu}|^2(\text{Ł}_{f^7_\mu})^2\bar{\xi}^4_{\nu\mu}}{1-\bar{\xi}^4_{\nu\mu}}\int_{t-\varsigma^4_{\nu\mu}(t)}^t\int_s^t|x_\mu(\zeta)|^2d\zeta ds\\
&-4\beth^{\#}\frac{d}{dt}\sum_{\mu\in\beth}\sum_{\nu\in\beth}\mathfrak{q}_\nu\max_{1\leqslant i\leqslant r}|a^8_{i\nu\mu}|^2(\text{Ł}_{f^8_\mu})^2\int_0^{+\infty}\Phi^2_{\nu\mu}(s)ds\int_0^{+\infty}\Phi^2_{\nu\mu}(s)\int_{t-s}^t|x_\mu(\zeta)|^2d\zeta ds,\quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.}
\end{aligned}\tag{39}
$$

Apply Itô's differentiation rule to $\mathcal{V}(t)$ given by (25), to find that there exist two $\mathbb{F}$-adapted stochastic processes $\mho(t)$ and $\amalg(t)$ such that

$$d\mathcal{V}(t) = \mho(t)dt + \amalg(t)dW(t),$$

or equivalently

$$\mathcal{V}(t+\Delta t) - \mathcal{V}(t) = \int_t^{t+\Delta t} \mho(s)ds + \int_t^{t+\Delta t} \amalg(s)dW(s), \quad t,\ \Delta t \in \mathbb{R}_+,\ \mathbb{P}\text{-a.s.,} \tag{40}$$

where the stochastic process $\amalg(t)$ is given by

$$\begin{aligned}
\amalg(t) =& 2\sum_{\mu\in\beth} \mathfrak{p}_\mu x_\mu(t)\Bigg[\sum_{\nu\in\beth} \bar{b}^5_{\mu\nu}\breve{g}^5_\nu(y_\nu(t)) + \sum_{\nu\in\beth} \bar{b}^6_{\mu\nu}\breve{g}^6_\nu(y_\nu(t-\varrho^3_{\mu\nu}(t)))\\
&+ \sum_{\nu\in\beth} \bar{b}^7_{\mu\nu}\int_{t-\varrho^4_{\mu\nu}(t)}^t \breve{g}^7_\nu(y_\nu(s))ds\\
&+ \sum_{\nu\in\beth} \bar{b}^8_{\mu\nu}\int_{-\infty}^t \Psi^2_{\mu\nu}(t-s)\breve{g}^8_\nu(y_\nu(s))ds\Bigg]\\
&+ 2\sum_{\nu\in\beth} \mathfrak{q}_\nu y_\nu(t)\Bigg[\sum_{\mu\in\beth} \bar{a}^5_{\nu\mu}\breve{f}^5_\mu(x_\mu(t)) + \sum_{\mu\in\beth} \bar{a}^6_{\nu\mu}\breve{f}^6_\mu(x_\mu(t-\varsigma^3_{\nu\mu}(t)))\\
&+ \sum_{\mu\in\beth} \bar{a}^7_{\nu\mu}\int_{t-\varsigma^4_{\nu\mu}(t)}^t \breve{f}^7_\mu(x_\mu(s))ds\\
&+ \sum_{\mu\in\beth} \bar{a}^8_{\nu\mu}\int_{-\infty}^t \Phi^2_{\nu\mu}(t-s)\breve{f}^8_\mu(x_\mu(s))ds\Bigg], \quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.}
\end{aligned}$$

Combine (35) and (37)–(39) to obtain

$$\begin{aligned}
\mho(t) \leqslant& 2\sum_{\mu\in\beth} \mathfrak{p}_\mu x_\mu(t)\mathscr{U}_\mu(t) + 2\sum_{\nu\in\beth} \mathfrak{q}_\nu y_\nu(t)\mathscr{V}_\nu(t) + \sum_{\mu\in\beth}\Sigma^u_\mu|x_\mu(t)|^2 + \sum_{\nu\in\beth}\Sigma^v_\nu|y_\nu(t)|^2\\
\leqslant& -2\sum_{\mu\in\beth}\mathfrak{p}_\mu\Big(\frac{\mathfrak{m}^1_\mu|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}\mathit{\Pi}_{11}(t) + \frac{\mathfrak{m}^2_\mu|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}(\mathit{\Pi}_{12}(t))^{\gamma_1} + \frac{\mathfrak{m}^3_\mu|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}(\mathbb{E}\mathit{\Pi}_{13}(t))^{\gamma_2}\Big)\\
&-2\sum_{\nu\in\beth}\mathfrak{q}_\nu\Big(\frac{\mathfrak{n}^1_\nu|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}\mathit{\Pi}_{21}(t) + \frac{\mathfrak{n}^2_\nu|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}(\mathit{\Pi}_{22}(t))^{\gamma_1} + \frac{\mathfrak{n}^3_\nu|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}(\mathbb{E}\mathit{\Pi}_{23}(t))^{\gamma_2}\Big)\\
&+\sum_{\mu\in\beth}\Sigma^u_\mu|x_\mu(t)|^2 + \sum_{\nu\in\beth}\Sigma^v_\nu|y_\nu(t)|^2\\
\leqslant& \sum_{\mu\in\beth}\Sigma^u_\mu|x_\mu(t)|^2 - 2\sum_{\mu\in\beth}\mathfrak{p}_\mu\Big(\frac{\backepsilon_{11}\,\mathfrak{m}^1_\mu|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}\mathcal{V}(t) + \frac{\mathfrak{m}^2_\mu|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}(\backepsilon_{12}\,\mathcal{V}(t))^{\gamma_1} + \frac{\mathfrak{m}^3_\mu|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}(\backepsilon_{13}\,V(t))^{\gamma_2}\Big)\\
&+\sum_{\nu\in\beth}\Sigma^v_\nu|y_\nu(t)|^2 - 2\sum_{\nu\in\beth}\mathfrak{q}_\nu\Big(\frac{\backepsilon_{21}\,\mathfrak{n}^1_\nu|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}\mathcal{V}(t) + \frac{\mathfrak{n}^2_\nu|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}(\backepsilon_{22}\,\mathcal{V}(t))^{\gamma_1} + \frac{\mathfrak{n}^3_\nu|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}(\backepsilon_{23}\,V(t))^{\gamma_2}\Big)\\
=& \sum_{\mu\in\beth}\Sigma^u_\mu|x_\mu(t)|^2 - 2\Big(\backepsilon_{11}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}^1_\mu|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2} + \backepsilon_{21}\sum_{\nu\in\beth}\frac{\mathfrak{q}_\nu\mathfrak{n}^1_\nu|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}\Big)\mathcal{V}(t)\\
&+\sum_{\nu\in\beth}\Sigma^v_\nu|y_\nu(t)|^2 - 2\Big((\backepsilon_{12})^{\gamma_1}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}^2_\mu|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2} + (\backepsilon_{22})^{\gamma_1}\sum_{\nu\in\beth}\frac{\mathfrak{q}_\nu\mathfrak{n}^2_\nu|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}\Big)(\mathcal{V}(t))^{\gamma_1}\\
&-2\Big((\backepsilon_{13})^{\gamma_2}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}^3_\mu|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2} + (\backepsilon_{23})^{\gamma_2}\sum_{\nu\in\beth}\frac{\mathfrak{q}_\nu\mathfrak{n}^3_\nu|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}\Big)(V(t))^{\gamma_2}, \quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.,}
\end{aligned} \tag{41}$$

in which the second "$\leqslant$" follows directly from the next inequality

$$
\begin{aligned}
\varPi_{hk}(t) =& \tilde{\varPi}_{hk}(\{x_\mu(t)\}_{\mu\in\beth},\{y_\nu(t)\}_{\nu\in\gimel}) \\
=& \sum_{\mu\in\beth}\mathfrak{m}_{hk1\mu}|x_\mu(t)|^2+\sum_{\nu\in\gimel}\mathfrak{n}_{hk1\nu}|y_\nu(t)|^2 \\
&+\sum_{\mu\in\beth}\mathfrak{m}_{hk2\mu}\max_{1\leqslant i\leqslant r}\sigma_{i\mu}\int_{t-\tau_\mu(t)}^{t}|x_\mu(s)|^2ds \\
&+\sum_{\mu\in\beth}\sum_{\nu\in\gimel}\mathfrak{m}_{hk3\nu\mu}\max_{1\leqslant i\leqslant r}|a^2_{i\nu\mu}|Ł_{f^2_\mu}\int_{t-\varsigma^1_{\nu\mu}(t)}^{t}|x_\mu(s)|^2ds \\
&+\sum_{\mu\in\beth}\sum_{\nu\in\gimel}\mathfrak{m}_{hk4\nu\mu}\max_{1\leqslant i\leqslant r}|a^6_{i\nu\mu}|^2(Ł_{f^6_\mu})^2\int_{t-\varsigma^3_{\nu\mu}(t)}^{t}|x_\mu(s)|^2ds \\
&+\sum_{\nu\in\gimel}\mathfrak{n}_{hk2\nu}\max_{1\leqslant i\leqslant r}\eta_{i\nu}\int_{t-\iota_\nu(t)}^{t}|y_\nu(s)|^2ds \\
&+\sum_{\nu\in\gimel}\sum_{\mu\in\beth}\mathfrak{n}_{hk3\mu\nu}\max_{1\leqslant i\leqslant r}|b^2_{i\mu\nu}|Ł_{g^2_\nu}\int_{t-\varrho^1_{\mu\nu}(t)}^{t}|y_\nu(s)|^2ds \\
&+\sum_{\nu\in\gimel}\sum_{\mu\in\beth}\mathfrak{n}_{hk4\mu\nu}\max_{1\leqslant i\leqslant r}|b^6_{i\mu\nu}|^2(Ł_{g^6_\nu})^2\int_{t-\varrho^3_{\mu\nu}(t)}^{t}|y_\nu(s)|^2ds \\
&+\sum_{\mu\in\beth}\sum_{\nu\in\gimel}\mathfrak{m}_{hk5\nu\mu}\max_{1\leqslant i\leqslant r}|a^3_{i\nu\mu}|Ł_{f^3_\mu}\int_{t-\varsigma^2_{\nu\mu}(t)}^{t}\int_{s}^{t}|x_\mu(\zeta)|^2d\zeta ds \\
&+\sum_{\mu\in\beth}\sum_{\nu\in\gimel}\mathfrak{m}_{hk6\nu\mu}\max_{1\leqslant i\leqslant r}|a^7_{i\nu\mu}|^2(Ł_{f^7_\mu})^2\bar{\varsigma}^4_{\nu\mu}\int_{t-\varsigma^4_{\nu\mu}(t)}^{t}\int_{s}^{t}|x_\mu(\zeta)|^2d\zeta ds \\
&+\sum_{\nu\in\gimel}\sum_{\mu\in\beth}\mathfrak{n}_{hk5\mu\nu}\max_{1\leqslant i\leqslant r}|b^3_{i\mu\nu}|Ł_{g^3_\nu}\int_{t-\varrho^2_{\mu\nu}(t)}^{t}\int_{s}^{t}|y_\nu(\zeta)|^2d\zeta ds \\
&+\sum_{\nu\in\gimel}\sum_{\mu\in\beth}\mathfrak{n}_{hk6\mu\nu}\max_{1\leqslant i\leqslant r}|b^7_{i\mu\nu}|^2(Ł_{g^7_\nu})^2\bar{\varrho}^4_{\mu\nu}\int_{t-\varrho^4_{\mu\nu}(t)}^{t}\int_{s}^{t}|y_\nu(\zeta)|^2d\zeta ds \\
&+\sum_{\mu\in\beth}\sum_{\nu\in\gimel}\mathfrak{m}_{hk7\nu\mu}\max_{1\leqslant i\leqslant r}|a^4_{i\nu\mu}|Ł_{f^4_\mu}\int_{0}^{+\infty}\varPhi^1_{\nu\mu}(s)\int_{t-s}^{t}|x_\mu(\zeta)|^2d\zeta ds \\
&+\sum_{\mu\in\beth}\sum_{\nu\in\gimel}\mathfrak{m}_{hk8\nu\mu}\max_{1\leqslant i\leqslant r}|a^8_{i\nu\mu}|^2(Ł_{f^8_\mu})^2\int_{0}^{+\infty}\varPhi^2_{\nu\mu}(s)ds\int_{0}^{+\infty}\varPhi^2_{\nu\mu}(s)\int_{t-s}^{t}|x_\mu(\zeta)|^2d\zeta ds \\
&+\sum_{\nu\in\gimel}\sum_{\mu\in\beth}\mathfrak{n}_{hk7\mu\nu}\max_{1\leqslant i\leqslant r}|b^4_{i\mu\nu}|Ł_{g^4_\nu}\int_{0}^{+\infty}\varPsi^1_{\mu\nu}(s)\int_{t-s}^{t}|y_\nu(\zeta)|^2d\zeta ds \\
&+\sum_{\nu\in\gimel}\sum_{\mu\in\beth}\mathfrak{n}_{hk8\mu\nu}\max_{1\leqslant i\leqslant r}|b^8_{i\mu\nu}|^2(Ł_{g^8_\nu})^2\int_{0}^{+\infty}\varPsi^2_{\mu\nu}(s)ds\int_{0}^{+\infty}\varPsi^2_{\mu\nu}(s)\int_{t-s}^{t}|y_\nu(\zeta)|^2d\zeta ds \\
\geqslant& \backepsilon_{hk}\,\mathcal{V}(t),\quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.},
\end{aligned}
\tag{42}
$$

where the stochastic processes $\varPi_{hk}(t)$ and $\tilde{\varPi}_{hk}(x_\mu(t),y_\nu(t))$ satisfy

$$
\begin{aligned}
\varPi_{hk}(t) =& \tilde{\varPi}_{hk}(\hat{u}_\mu(t)-u_\mu(t),\hat{v}_\nu(t)-v_\nu(t)) \\
=& \tilde{\varPi}_{hk}(x_\mu(t),y_\nu(t)),\quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.}
\end{aligned}
$$

with $\tilde{\varPi}_{hk}(x_\mu(t),y_\nu(t))$ defined as in (23), the constant $\backepsilon_{hk}$ is given as in (32), and the stochastic process $\mathcal{V}(t)$ is given as in (25). Thanks to (32) (as well as (23) and (42)), the collection of constants $\mathfrak{m}_{hk1\mu}$, $\mathfrak{m}_{hk2\mu}$, $\mathfrak{m}_{hk3\nu\mu}$, $\mathfrak{m}_{hk4\nu\mu}$, $\mathfrak{m}_{hk5\nu\mu}$, $\mathfrak{m}_{hk6\nu\mu}$, $\mathfrak{m}_{hk7\nu\mu}$, $\mathfrak{m}_{hk8\nu\mu}$, $\mathfrak{n}_{hk1\nu}$, $\mathfrak{n}_{hk2\nu}$, $\mathfrak{n}_{hk3\mu\nu}$, $\mathfrak{n}_{hk4\mu\nu}$, $\mathfrak{n}_{hk5\mu\nu}$, $\mathfrak{n}_{hk6\mu\nu}$, $\mathfrak{n}_{hk7\mu\nu}$, $\mathfrak{n}_{hk8\mu\nu}$ and the collection of constants $\mathfrak{p}_\mu$, $\mathfrak{q}_\nu$ influence each other, and both collections, together with the collection of constants, constant $\backepsilon_{hk}$, depend on $f^\imath_\mu(v)$, $g^\imath_\nu(u)$, $\sigma_{i\mu}$, $\eta_{i\nu}$, $a^\jmath_{i\nu\mu}$, $b^\jmath_{i\mu\nu}$, $\tau_\mu(t)$, $\iota_\nu(t)$, $\varsigma^l_{\nu\mu}(t)$, $\varrho^l_{\mu\nu}(t)$, $\varPhi^\ell_{\nu\mu}(t)$, and $\varPsi^\ell_{\mu\nu}(t)$, $h=1,2$, $k=1,2,3$, $i=1,2,\ldots,r$, $\mu\in\beth$, $\nu\in\gimel$, $\imath,\jmath=1,2,\ldots,8$, $l=1,2,3,4$, $\ell=1,2$.

Recalling Itô's integral identity (40) and the inequality (41), we have

$$
\begin{aligned}
&V(t+\Delta t)-V(t)\\
=&\mathbb{E}\mathcal{V}(t+\Delta t)-\mathbb{E}\mathcal{V}(t)\\
=&\mathbb{E}\int_t^{t+\Delta t}\mho(s)ds+\mathbb{E}\int_t^{t+\Delta t}\Pi(s)dW(s)\\
=&\mathbb{E}\int_t^{t+\Delta t}\mho(s)ds\\
\leqslant&\mathbb{E}\int_t^{t+\Delta t}\Big(\sum_{\mu\in\beth}\Sigma_\mu^u(s)|x_\mu(s)|^2+\sum_{\nu\in\gimel}\Sigma_\nu^v(s)|y_\nu(s)|^2\Big)ds\\
&-2\mathbb{E}\int_t^{t+\Delta t}\Big(\backepsilon_{11}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^1|x_\mu(s)|^2}{\varepsilon+|x_\mu(s)|^2}+\backepsilon_{21}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^1|y_\nu(s)|^2}{\varepsilon+|y_\nu(s)|^2}\Big)\mathcal{V}(s)ds\\
&-2\mathbb{E}\int_t^{t+\Delta t}\Big((\backepsilon_{12})^{\gamma_1}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^2|x_\mu(s)|^2}{\varepsilon+|x_\mu(s)|^2}+(\backepsilon_{22})^{\gamma_1}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^2|y_\nu(s)|^2}{\varepsilon+|y_\nu(s)|^2}\Big)(\mathcal{V}(s))^{\gamma_1}ds\\
&-2\mathbb{E}\int_t^{t+\Delta t}\Big((\backepsilon_{13})^{\gamma_2}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^3|x_\mu(s)|^2}{\varepsilon+|x_\mu(s)|^2}+(\backepsilon_{23})^{\gamma_2}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^3|y_\nu(s)|^2}{\varepsilon+|y_\nu(s)|^2}\Big)(V(s))^{\gamma_2}ds,\quad t,\ \Delta t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.}
\end{aligned}
\tag{43}
$$

To facilitate our later presentation, we would like to treat our problems from two different perspectives. We consider first the following situation:

$$
\begin{aligned}
&\backepsilon_{11}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^1|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}+\backepsilon_{21}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^1|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}=0,\\
&(\backepsilon_{12})^{\gamma_1}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^2|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}+(\backepsilon_{22})^{\gamma_1}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^2|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}=0,\ \text{or}\\
&(\backepsilon_{13})^{\gamma_2}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^3|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}+(\backepsilon_{23})^{\gamma_2}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^3|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}=0,\quad\mathbb{P}\text{-a.s.}
\end{aligned}
$$

In this case, the state of the error BAMNNs (11) arrives at the null state or, equivalently, our concerned drive BAMNNs (5)-(6)-(7) and response BAMNNs (8)-(6)-(7) are already synchronized. Now, we are in a position to consider the following situation:

$$
\begin{aligned}
&\backepsilon_{11}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^1|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}+\backepsilon_{21}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^1|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}\neq0,\\
&(\backepsilon_{12})^{\gamma_1}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^2|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}+(\backepsilon_{22})^{\gamma_1}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^2|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}\neq0,\ \text{and}\\
&(\backepsilon_{13})^{\gamma_2}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^3|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}+(\backepsilon_{23})^{\gamma_2}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^3|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}\neq0,\quad\mathbb{P}\text{-a.s.}
\end{aligned}
$$

The analysis in the above paragraph, together with (41), implies

$$
\begin{aligned}
\mho(t)=\lim_{\varepsilon\to0^+}\mho(t)\leqslant&\lim_{\varepsilon\to0^+}\sum_{\mu\in\beth}\Sigma_\mu^u(t)|x_\mu(t)|^2-2\lim_{\varepsilon\to0^+}\Big(\backepsilon_{11}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^1|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}+\backepsilon_{21}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^1|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}\Big)\mathcal{V}(t)\\
&+\lim_{\varepsilon\to0^+}\sum_{\nu\in\gimel}\Sigma_\nu^v(t)|y_\nu(t)|^2-2\lim_{\varepsilon\to0^+}\Big((\backepsilon_{12})^{\gamma_1}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^2|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}+(\backepsilon_{22})^{\gamma_1}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^2|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}\Big)(\mathcal{V}(t))^{\gamma_1}\\
&-2\lim_{\varepsilon\to0^+}\Big((\backepsilon_{13})^{\gamma_2}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^3|x_\mu(t)|^2}{\varepsilon+|x_\mu(t)|^2}+(\backepsilon_{23})^{\gamma_2}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^3|y_\nu(t)|^2}{\varepsilon+|y_\nu(t)|^2}\Big)(V(t))^{\gamma_2}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant -2\min\Big(\backepsilon_{11}\min_{\mu\in\beth}\mathfrak{p}_\mu\mathfrak{m}_\mu^1,\ \backepsilon_{21}\min_{\nu\in\gimel}\mathfrak{q}_\nu\mathfrak{n}_\nu^1\Big)\mathcal{V}(t)\\
&\quad -2\min\Big((\backepsilon_{12})^{\gamma_1}\min_{\mu\in\beth}\mathfrak{p}_\mu\mathfrak{m}_\mu^2,\ (\backepsilon_{22})^{\gamma_1}\min_{\nu\in\gimel}\mathfrak{q}_\nu\mathfrak{n}_\nu^2\Big)(\mathcal{V}(t))^{\gamma_1}\\
&\quad -2\min\Big((\backepsilon_{13})^{\gamma_2}\min_{\mu\in\beth}\mathfrak{p}_\mu\mathfrak{m}_\mu^3,\ (\backepsilon_{23})^{\gamma_2}\min_{\nu\in\gimel}\mathfrak{q}_\nu\mathfrak{n}_\nu^3\Big)(V(t))^{\gamma_2}\\
&\quad +\sum_{\mu\in\beth}\Sigma_\mu^u(t)|x_\mu(t)|^2+\sum_{\nu\in\gimel}\Sigma_\nu^v(t)|y_\nu(t)|^2,\quad t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.}
\end{aligned}
\tag{44}
$$

By Lebesgue's dominated convergence theorem, we derive from (43) and (44) that

$$
\begin{aligned}
V(t+\Delta t)-V(t) &= \lim_{\varepsilon\to 0^+}\big(V(t+\Delta t)-V(t)\big)\\
&\leqslant \mathbb{E}\int_t^{t+\Delta t}\lim_{\varepsilon\to 0^+}\Big(\sum_{\mu\in\beth}\Sigma_\mu^u(s)|x_\mu(s)|^2+\sum_{\nu\in\gimel}\Sigma_\nu^v(s)|y_\nu(s)|^2\Big)ds\\
&\quad -2\mathbb{E}\int_t^{t+\Delta t}\lim_{\varepsilon\to 0^+}\Big(\backepsilon_{11}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^1|x_\mu(s)|^2}{\varepsilon+|x_\mu(s)|^2}+\backepsilon_{21}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^1|y_\nu(s)|^2}{\varepsilon+|y_\nu(s)|^2}\Big)\mathcal{V}(s)ds\\
&\quad -2\mathbb{E}\int_t^{t+\Delta t}\lim_{\varepsilon\to 0^+}\Big((\backepsilon_{12})^{\gamma_1}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^2|x_\mu(s)|^2}{\varepsilon+|x_\mu(s)|^2}+(\backepsilon_{22})^{\gamma_1}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^2|y_\nu(s)|^2}{\varepsilon+|y_\nu(s)|^2}\Big)(\mathcal{V}(s))^{\gamma_1}ds\\
&\quad -2\mathbb{E}\int_t^{t+\Delta t}\lim_{\varepsilon\to 0^+}\Big((\backepsilon_{13})^{\gamma_2}\sum_{\mu\in\beth}\frac{\mathfrak{p}_\mu\mathfrak{m}_\mu^3|x_\mu(s)|^2}{\varepsilon+|x_\mu(s)|^2}+(\backepsilon_{23})^{\gamma_2}\sum_{\nu\in\gimel}\frac{\mathfrak{q}_\nu\mathfrak{n}_\nu^3|y_\nu(s)|^2}{\varepsilon+|y_\nu(s)|^2}\Big)(V(s))^{\gamma_2}ds\\
&\leqslant \mathbb{E}\int_t^{t+\Delta t}\sum_{\mu\in\beth}\Sigma_\mu^u(s)|x_\mu(s)|^2ds-2\int_t^{t+\Delta t}\min\Big(\backepsilon_{11}\min_{\mu\in\beth}\mathfrak{p}_\mu\mathfrak{m}_\mu^1,\ \backepsilon_{21}\min_{\nu\in\gimel}\mathfrak{q}_\nu\mathfrak{n}_\nu^1\Big)V(s)ds\\
&\quad +\mathbb{E}\int_t^{t+\Delta t}\sum_{\nu\in\gimel}\Sigma_\nu^v(s)|y_\nu(s)|^2ds-2\int_t^{t+\Delta t}\min\Big((\backepsilon_{12})^{\gamma_1}\min_{\mu\in\beth}\mathfrak{p}_\mu\mathfrak{m}_\mu^2,\ (\backepsilon_{22})^{\gamma_1}\min_{\nu\in\gimel}\mathfrak{q}_\nu\mathfrak{n}_\nu^2\Big)\mathbb{E}(\mathcal{V}(s))^{\gamma_1}ds\\
&\quad -2\int_t^{t+\Delta t}\min\Big((\backepsilon_{13})^{\gamma_2}\min_{\mu\in\beth}\mathfrak{p}_\mu\mathfrak{m}_\mu^3,\ (\backepsilon_{23})^{\gamma_2}\min_{\nu\in\gimel}\mathfrak{q}_\nu\mathfrak{n}_\nu^3\Big)(V(s))^{\gamma_2}ds\\
&\leqslant \frac{1}{T_\natural}\int_t^{t+\Delta t}\Big(b_0V(s)-b_1(V(s))^{\gamma_1}-b_2(V(s))^{\gamma_2}\Big)ds\\
&= \frac{\mathcal{T}(a_0,a_1,a_2,\gamma_1,\gamma_2)}{T_\natural}\int_t^{t+\Delta t}\Big(a_0V(s)-a_1(V(s))^{\gamma_1}-a_2(V(s))^{\gamma_2}\Big)ds,\quad t,\ \Delta t\in\mathbb{R}_+,\ \mathbb{P}\text{-a.s.},
\end{aligned}
\tag{45}
$$

where the constant a_k is defined as in (33), and the function $\mathcal{T}(a_0,a_1,a_2,\gamma_1,\gamma_2)$ is defined by (19). To obtain the "$\leqslant$" next to the last line of (45), we used the assumption that $\gamma_1>1$ and the following inequality (can be deduced by Jensen's inequality):

$$
\mathbb{E}(\mathcal{V}(t))^{\gamma_1}\geqslant(\mathbb{E}\mathcal{V}(t))^{\gamma_1}=(V(t))^{\gamma_1},\quad t\in\mathbb{R}_+.
$$

Recalling (45) and passing to the limit, we have immediately

$$
\begin{aligned}
D^+V(t) &= \limsup_{\Delta t\to 0^+}\frac{V(t+\Delta t)-V(t)}{\Delta t}\\
&\leqslant \frac{\mathcal{T}(a_0,a_1,a_2,\gamma_1,\gamma_2)}{T_\natural}\limsup_{\Delta t\to 0^+}\frac{1}{\Delta t}\int_t^{t+\Delta t}\Big(a_0V(s)-a_1(V(s))^{\gamma_1}-a_2(V(s))^{\gamma_2}\Big)ds\\
&= \frac{\mathcal{T}(a_0,a_1,a_2,\gamma_1,\gamma_2)}{T_\natural}\lim_{\Delta t\to 0^+}\frac{1}{\Delta t}\int_t^{t+\Delta t}\Big(a_0V(s)-a_1(V(s))^{\gamma_1}-a_2(V(s))^{\gamma_2}\Big)ds\\
&= \frac{\mathcal{T}(a_0,a_1,a_2,\gamma_1,\gamma_2)}{T_\natural}\Big(a_0V(t)-a_1(V(t))^{\gamma_1}-a_2(V(t))^{\gamma_2}\Big),\quad t\in\mathbb{R}_+.
\end{aligned}
$$

By Lemma 1, this implies that the proof of Theorem 1 is complete. □

4. Numerical Validation of Our Theoretical Results

In this section, we are devoted to the numerical simulations of the validity of our aforementioned synchronization criterion (see Theorem 1). We assume that the defuzzified network system of our concerned multiplied time-delayed BAM based on the Takagi–Sugeno IF–THEN logic is of the form (5), in which we assume basically throughout this section that $\beth = \{1\}$, $\gimel = \{1, 2\}$, $p = 1$ and $r = 2$. We assume in this example that

$$\omega_1(\xi(t)) = M_{11}(\xi(t)) = \frac{e^t}{1 + 2e^t}, \quad t \in \mathbb{R}_+,$$

and

$$\omega_2(\xi(t)) = M_{21}(\xi(t)) = \frac{e^t}{2 + 4e^t}, \quad t \in \mathbb{R}_+,$$

and, as a consequence, we have

$$\begin{aligned}\vartheta_1(\xi(t)) =& \frac{\omega_1(\xi(t))}{\omega_1(\xi(t)) + \omega_2(\xi(t))} \\ =& \frac{\frac{e^t}{1+2e^t}}{\frac{e^t}{1+2e^t} + \frac{e^t}{2+4e^t}} = \frac{2}{3}, \qquad t \in \mathbb{R}_+,\end{aligned}$$

and

$$\vartheta_2(\xi(t)) = \frac{\omega_2(\xi(t))}{\omega_1(\xi(t)) + \omega_2(\xi(t))} = \frac{1}{3}, \quad t \in \mathbb{R}_+.$$

We assume that the time delays $\tau_1(t)$, $\iota_1(t)$, and $\iota_2(t)$ in the leakage terms of our concerned example BAM are given, respectively, by

$$\tau_1(t) = \frac{e^t}{1 + 2e^t}, \quad \iota_1(t) = \frac{1 + e^t}{1 + 2e^t}, \quad \text{and } \iota_2(t) = \frac{2e^t}{1 + 3e^t}, \quad t \in \mathbb{R}_+.$$

In the meantime, we assume that the time delays $\varsigma_{11}^1(t)$, $\varsigma_{21}^1(t)$, $\varsigma_{11}^2(t)$, $\varsigma_{21}^2(t)$, $\varsigma_{11}^3(t)$, $\varsigma_{21}^3(t)$, $\varsigma_{11}^4(t)$, $\varsigma_{21}^4(t)$, $\varrho_{11}^1(t)$, $\varrho_{12}^1(t)$, $\varrho_{11}^2(t)$, $\varrho_{12}^2(t)$, $\varrho_{11}^3(t)$, $\varrho_{12}^3(t)$, $\varrho_{11}^4(t)$, and $\varrho_{12}^4(t)$ in the transmission terms of our concerned example BAM are given, respectively, by

$$\begin{aligned}&\varsigma_{11}^1(t) = \frac{e^t}{1 + 4e^t}, \quad \varsigma_{21}^1(t) = \frac{2e^t}{1 + 4e^t}, \quad \varsigma_{11}^2(t) = \frac{3e^t}{1 + 4e^t}, \quad \varsigma_{21}^2(t) = \frac{4e^t}{1 + 4e^t}, \\ &\varsigma_{11}^3(t) = \frac{e^t}{1 + 5e^t}, \quad \varsigma_{21}^3(t) = \frac{2e^t}{1 + 5e^t}, \quad \varsigma_{11}^4(t) = \frac{3e^t}{1 + 5e^t}, \quad \varsigma_{21}^4(t) = \frac{4e^t}{1 + 5e^t}, \\ &\varrho_{11}^1(t) = \frac{5e^t}{1 + 5e^t}, \quad \varrho_{12}^1(t) = \frac{e^t}{1 + 6e^t}, \quad \varrho_{11}^2(t) = \frac{2e^t}{1 + 6e^t}, \quad \varrho_{12}^2(t) = \frac{3e^t}{1 + 6e^t}, \\ &\varrho_{11}^3(t) = \frac{4e^t}{1 + 6e^t}, \quad \varrho_{12}^3(t) = \frac{5e^t}{1 + 6e^t}, \quad \varrho_{11}^4(t) = \frac{6e^t}{1 + 6e^t}, \quad \text{and } \varrho_{12}^4(t) = \frac{7e^t}{1 + 6e^t}, \quad t \in \mathbb{R}_+.\end{aligned}$$

Suppose also in our concerned example BAM that $\sigma_{11} = 1$, $\sigma_{21} = 2$, $\eta_{11} = 2$, $\eta_{12} = 3$, $\eta_{21} = 4$, and $\eta_{22} = 1$. We assume throughout this section that the kernels $\varPhi_{11}^1(t)$, $\varPhi_{21}^1(t)$, $\varPhi_{11}^2(t)$, $\varPhi_{21}^2(t)$, $\varPsi_{11}^1(t)$, $\varPsi_{12}^1(t)$, $\varPsi_{11}^2(t)$, and $\varPsi_{12}^2(t)$ are defined, respectively, by

$$\begin{aligned}&\varPhi_{11}^1(t) = e^{-5t}, \quad \varPhi_{21}^1(t) = e^{-15t}, \quad \varPhi_{11}^2(t) = e^{-25t}, \quad \varPhi_{21}^2(t) = e^{-35t}, \\ &\varPsi_{11}^1(t) = e^{-45t}, \quad \varPsi_{12}^1(t) = e^{-55t}, \quad \varPsi_{11}^2(t) = e^{-65t}, \quad \text{and } \varPsi_{12}^2(t) = e^{-75t}, \quad t \in \mathbb{R}_+.\end{aligned}$$

For the sake of the convenience of our later numerical simulations, we assume that

$$\begin{aligned}&U_{11}^1(t) = U_{21}^1(t) = U_{11}^2(t) = U_{21}^2(t) \\ =& V_{11}^1(t) = V_{12}^1(t) = V_{21}^1(t) = V_{22}^1(t) \\ =& V_{11}^2(t) = V_{12}^2(t) = V_{21}^2(t) = V_{22}^2(t) = 0, \quad t \in \mathbb{R}_+, \ \mathbb{P}\text{-a.s.}\end{aligned}$$

We assume that the transmission connection weight coefficients satisfy

$$
\begin{aligned}
&(a^1_{iv1}) = \begin{pmatrix} -5 & 7 \\ 8 & 3 \end{pmatrix}, \quad (a^2_{iv1}) = \begin{pmatrix} 1 & 8 \\ 9 & -7 \end{pmatrix}, \quad (a^3_{iv1}) = \begin{pmatrix} 7 & -1 \\ 3 & 2 \end{pmatrix}, \quad (a^4_{iv1}) = \begin{pmatrix} 8 & 1 \\ -4 & 3 \end{pmatrix}, \\
&(a^5_{iv1}) = \begin{pmatrix} 2 & 8 \\ 7 & -4 \end{pmatrix}, \quad (a^6_{iv1}) = \begin{pmatrix} -2 & 7 \\ 5 & 3 \end{pmatrix}, \quad (a^7_{iv1}) = \begin{pmatrix} 5 & 4 \\ -6 & 2 \end{pmatrix}, \quad (a^8_{iv1}) = \begin{pmatrix} 7 & -9 \\ 5 & 3 \end{pmatrix}, \\
&(b^1_{i1v}) = \begin{pmatrix} -3 & 7 \\ 8 & 5 \end{pmatrix}, \quad (b^2_{i1v}) = \begin{pmatrix} 1 & 9 \\ -8 & 7 \end{pmatrix}, \quad (b^3_{i1v}) = \begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix}, \quad (b^4_{i1v}) = \begin{pmatrix} 3 & 1 \\ 4 & -9 \end{pmatrix}, \\
&(b^5_{i1v}) = \begin{pmatrix} 4 & -8 \\ 7 & 2 \end{pmatrix}, \quad (b^6_{i1v}) = \begin{pmatrix} -4 & 7 \\ 5 & 2 \end{pmatrix}, \quad (b^7_{i1v}) = \begin{pmatrix} 3 & 4 \\ -6 & 5 \end{pmatrix}, \quad (b^8_{i1v}) = \begin{pmatrix} 4 & -7 \\ 5 & 2 \end{pmatrix}.
\end{aligned}
$$

We assume that the activation functions $f^1_1(u)$, $f^2_1(u)$, $f^3_1(u)$, $f^4_1(u)$, $f^5_1(u)$, $f^6_1(u)$, $f^7_1(u)$, $f^8_1(u)$, $g^1_1(v)$, $g^2_1(v)$, $g^3_1(v)$, $g^4_1(v)$, $g^5_1(v)$, $g^6_1(v)$, $g^7_1(v)$, $g^8_1(v)$, $g^1_2(v)$, $g^2_2(v)$, $g^3_2(v)$, $g^4_2(v)$, $g^5_2(v)$, $g^6_2(v)$, $g^7_2(v)$, and $g^8_2(v)$ are given, respectively, by

$$
\begin{aligned}
&f^1_1(u) = \grave{F}(u), \quad f^2_1(u) = \grave{F}(2u), \quad f^3_1(u) = \grave{F}(3u), \quad f^4_1(u) = \grave{F}(4u), \\
&f^5_1(u) = \grave{F}(5u), \quad f^6_1(u) = \grave{F}(6u), \quad f^7_1(u) = \grave{F}(7u), \quad f^8_1(u) = \grave{F}(8u), \\
&g^1_1(v) = \acute{F}(v), \quad g^2_1(v) = \acute{F}(2v), \quad g^3_1(v) = \acute{F}(3v), \quad g^4_1(v) = \acute{F}(4v), \\
&g^5_1(v) = \acute{F}(5v), \quad g^6_1(v) = \acute{F}(6v), \quad g^7_1(v) = \acute{F}(7v), \quad g^8_1(v) = \acute{F}(8v), \\
&g^1_2(v) = \breve{F}(v), \quad g^2_2(v) = \breve{F}(2v), \quad g^3_2(v) = \breve{F}(3v), \quad g^4_2(v) = \breve{F}(4v), \\
&g^5_2(v) = \breve{F}(5v), \quad g^6_2(v) = \breve{F}(6v), \quad g^7_2(v) = \breve{F}(7v), \quad \text{and } g^8_2(v) = \breve{F}(8v), \quad u, v \in \mathbb{R},
\end{aligned}
$$

in which the functions $\grave{F}(x)$, $\acute{F}(x)$, and $\breve{F}(x)$ are, respectively, defined by

$$\grave{F}(x) = \int_0^x \frac{e^{t^2}}{1 + e^{t^2}} dt, \quad x \in \mathbb{R},$$

$$\acute{F}(x) = x - \arctan \frac{x}{2}, \quad x \in \mathbb{R},$$

and

$$\breve{F}(x) = 2x - \sin x, \quad x \in \mathbb{R}.$$

The chaos phenomenon occurs frequently in many complicated nonlinear differential dynamical systems. Chaos could prevent some pairs of different state trajectories of the concerned dynamical system from approaching each other as time escapes to infinity. That is, chaotic dynamical systems do not achieve identical synchronization automatically. And, therefore, experts have been attracted to designing suitable controls to synchronize chaotic dynamical systems; see [6,17] and the vast references mentioned therein.

To "demonstrate" that our proposed synchronization control essentially improves the structural property of the example BAMNN concerned in this section, we first show via MATLAB software that our concerned example BAMNN could be "chaotic". To this end, we solve first numerically the solution, denoted by $(u_1(t), v_1(t), v_2(t))^\top$ throughout this section, to our concerned example BAMNN, of which the initial data are composed of data in two modes, namely, $(u_{110}(t), v_{110}(t), v_{120}(t))^\top$ and $(u_{210}(t), v_{210}(t), v_{220}(t))^\top$, where

$$
\begin{cases}
u_{110}(t) = 3\sin x - 3, & d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-, \\
v_{110}(t) = 6 - 3\sin 2x, & d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-, \\
v_{120}(t) = 3 - 6\sin 3x, & d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-,
\end{cases}
$$

and

$$
\begin{cases}
u_{210}(t) = 3 - 6\sin x, & d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-, \\
v_{210}(t) = 6\sin 2x - 6, & d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-, \\
v_{220}(t) = 12\sin 3x - 3, & d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-.
\end{cases}
$$

See Figure 1 (including subfigures (a), (b), (c), and (d)) for the detailed description of the graph of the state trajectory $(u_1(t), v_1(t), v_2(t))^\top$, $t \in [0, 100]$. And, similarly, we solve numerically the solution, denoted by $(\hat{u}_1(t), \hat{v}_1(t), \hat{v}_2(t))^\top$, to the response BAMNN associated with our concerned drive BAMNN, of which the initial data are composed of data in two modes, namely, $(\hat{u}_{110}(t), \hat{v}_{110}(t), \hat{v}_{120}(t))^\top$ and $(\hat{u}_{210}(t), \hat{v}_{210}(t), \hat{v}_{220}(t))^\top$, where

$$\begin{cases} \hat{u}_{110}(t) = 3\cos x - 3, & d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-, \\ \hat{v}_{110}(t) = 6 - 3\cos 2x, & d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-, \\ \hat{v}_{120}(t) = 9 - 6\cos 3x, & d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-, \end{cases}$$

and

$$\begin{cases} \hat{u}_{210}(t) = 6 - 6\cos x, & d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-, \\ \hat{v}_{210}(t) = 6\cos 2x - 3, & d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-, \\ \hat{v}_{220}(t) = 12\cos 3x - 9, & d\mathbb{P} \times dt\text{-a.e. in } \Omega \times \mathbb{R}_-. \end{cases}$$

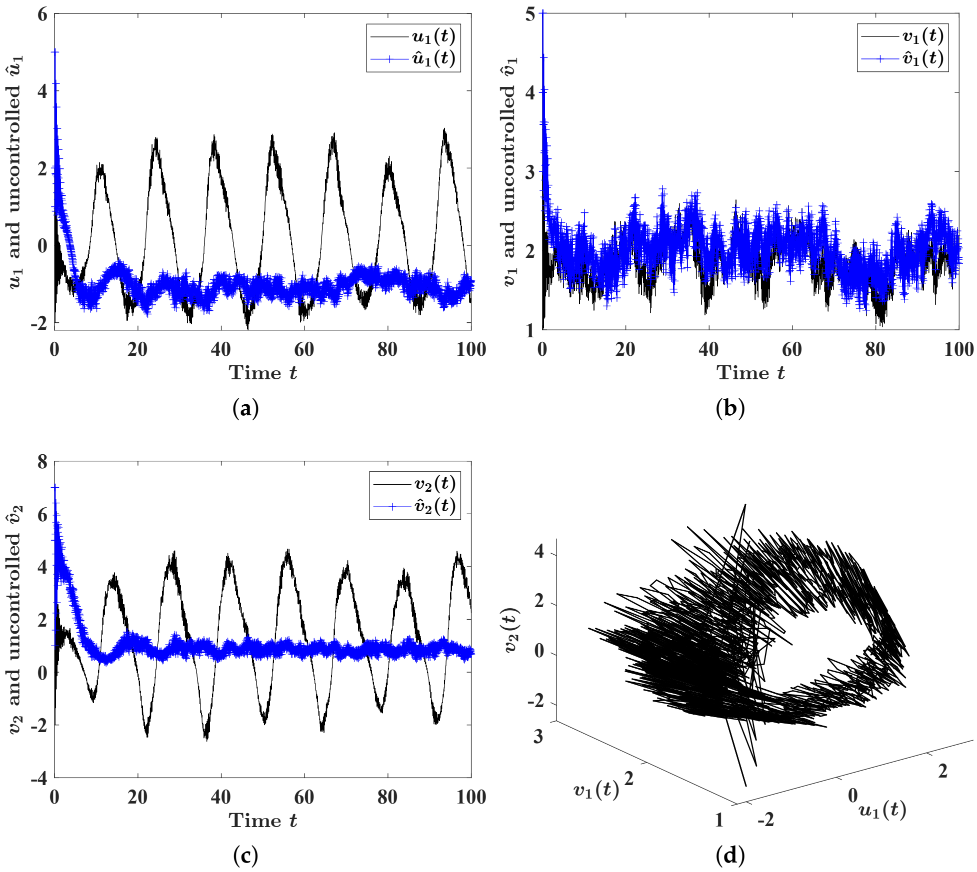
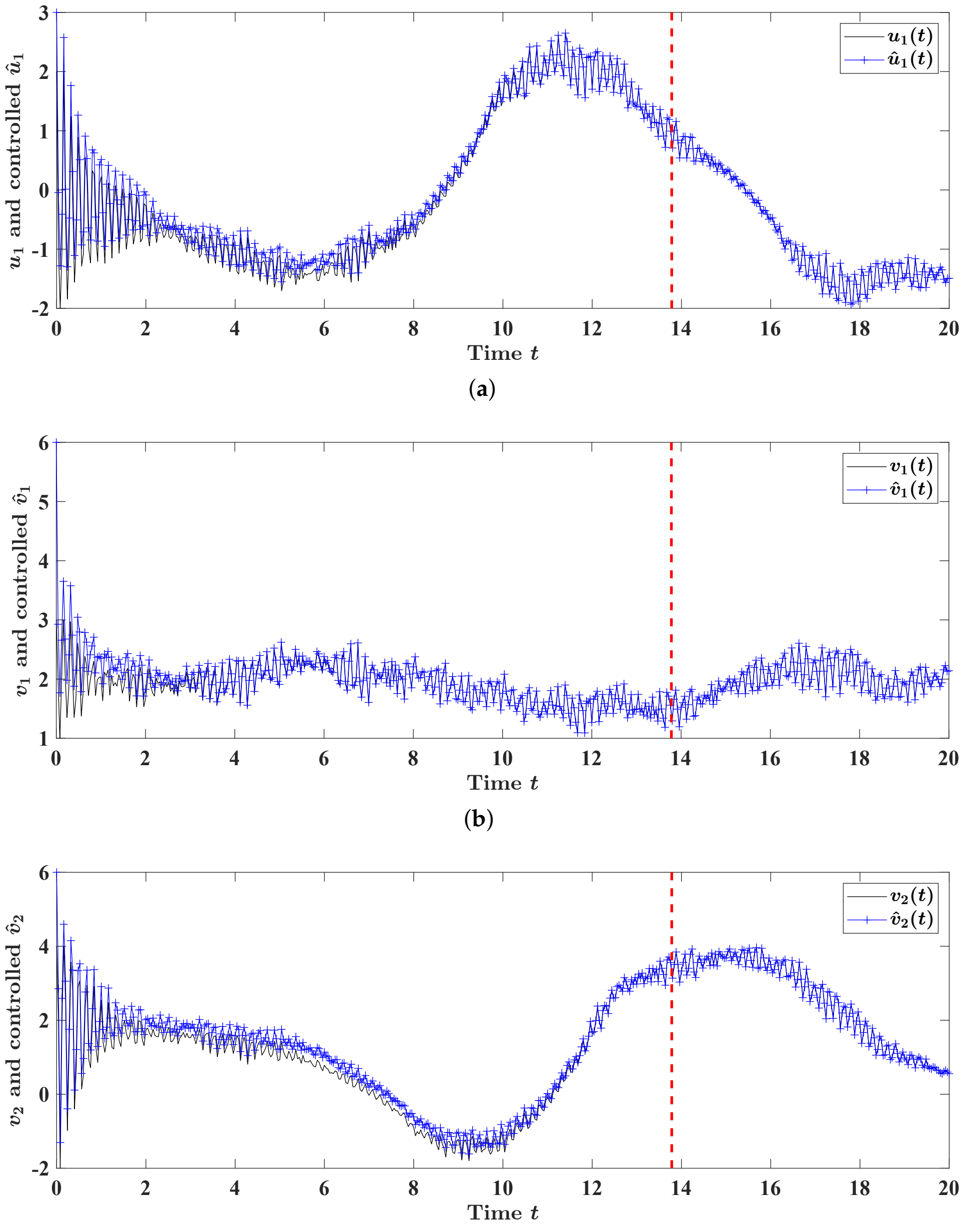


Figure 1. Numerical and graphical illustration of the occurrence of chaos phenomenon in the example BAMNN concerned in this section. $(u_1(t), v_1(t), v_2(t))^\top$, $t \in [0, 100]$, is the state trajectory triple of our concerned example BAMNN with $u_1(t) \equiv -1 = \frac{2}{3}(3\sin x - 3) + \frac{1}{3}(3 - 6\sin x)$, $v_1(t) \equiv 2 = \frac{2}{3}(6 - 3\sin 2x) + \frac{1}{3}(6\sin 2x - 6)$, and $v_2(t) \equiv 1 = \frac{2}{3}(3 - 6\sin 3x) + \frac{1}{3}(6\cos 2x + 6)$, $t \in \mathbb{R}_-$, $\mathbb{P}$-a.s.; see (**a**–**c**) for the graph (the solid curves) of the functions $u_1(t)$, $v_1(t)$, $v_2(t)$ in the interval $[0, 100]$. The graph of the parametric curve $(u_1(t), v_1(t), v_2(t))^\top$ is visualized in the phase space (state space); see (**d**). $(\hat{u}_1(t), \hat{v}_1(t), \hat{v}_2(t))^\top$, $t \in [0, 100]$, is the state trajectory triple of the response BAMNN with no controls implemented, associated with our concerned example BAMNN, with $\hat{u}_1(t) \equiv 0 = \frac{2}{3}(3\cos x - 3) + \frac{1}{3}(6 - 6\cos x)$, $\hat{v}_1(t) \equiv 5 = \frac{2}{3}(6 - 3\cos 2x) + \frac{1}{3}(6\cos 2x - 3)$, and $\hat{v}_2(t) \equiv 3 = \frac{2}{3}(9 - 6\cos 3x) + \frac{1}{3}(12\cos 3x - 9)$, $t \in \mathbb{R}_-$, $\mathbb{P}$-a.s.; see also (**a**–**c**) for the graph (the curves composed of "+") of the functions $\hat{u}_1(t)$, $\hat{v}_1(t)$, $\hat{v}_2(t)$ in $[0, 100]$.

The detailed description of the graph of the state trajectory $(\hat{u}_1(t), \hat{v}_1(t), \hat{v}_2(t))^\top$, $t \in [0, 100]$, can also be seen in Figure 1 (including subfigures (a), (b), and (c)). To summarize, we "demonstrate", by Figure 1, in a visual way, that our concerned example BAMNN is "chaotic", in particular, some of the trajectories are sensitive to their initial states. We next show numerically that our proposed control law (21)-(22) could effectively synchronize our concerned example BAMNN in any pre-assigned time: For any given positive time instant T, our example BAMNN and the corresponding controlled response system achieve synchronization before $\min(T, T_0)$ with $T_0 = 13.7825$; see Figures 2 and 3 for the details.



(**a**)

(**b**)

(**c**)

Figure 2. Numerical and graphical validation of our theoretical synchronization results; see Theorem 1 for the details. As in Figure 1, $(u_1(t), v_1(t), v_2(t))^\top$, $[0, 20]$ (see the solid curves in (**a**–**c**)), is the state trajectory triple of our concerned example BAMNN with $u_1(t) \equiv -1$, $v_1(t) \equiv 2$, and $v_2(t) \equiv 1$, $t \in \mathbb{R}_-$, $\mathbb{P}$-a.s. $(\hat{u}_1(t), \hat{v}_1(t), \hat{v}_2(t))^\top$, $[0, 20]$ (see the curves composed of "+" in (**a**–**c**)), is the state trajectory triple of the controlled response BAMNN associated with our concerned example BAMNN with $\hat{u}_1(t) \equiv 0$, $\hat{v}_1(t) \equiv 5$, and $\hat{v}_2(t) \equiv 3$, $t \in \mathbb{R}_-$, $\mathbb{P}$-a.s. The dashed straight vertical line segments are the graphs of $t = 13.7825$.

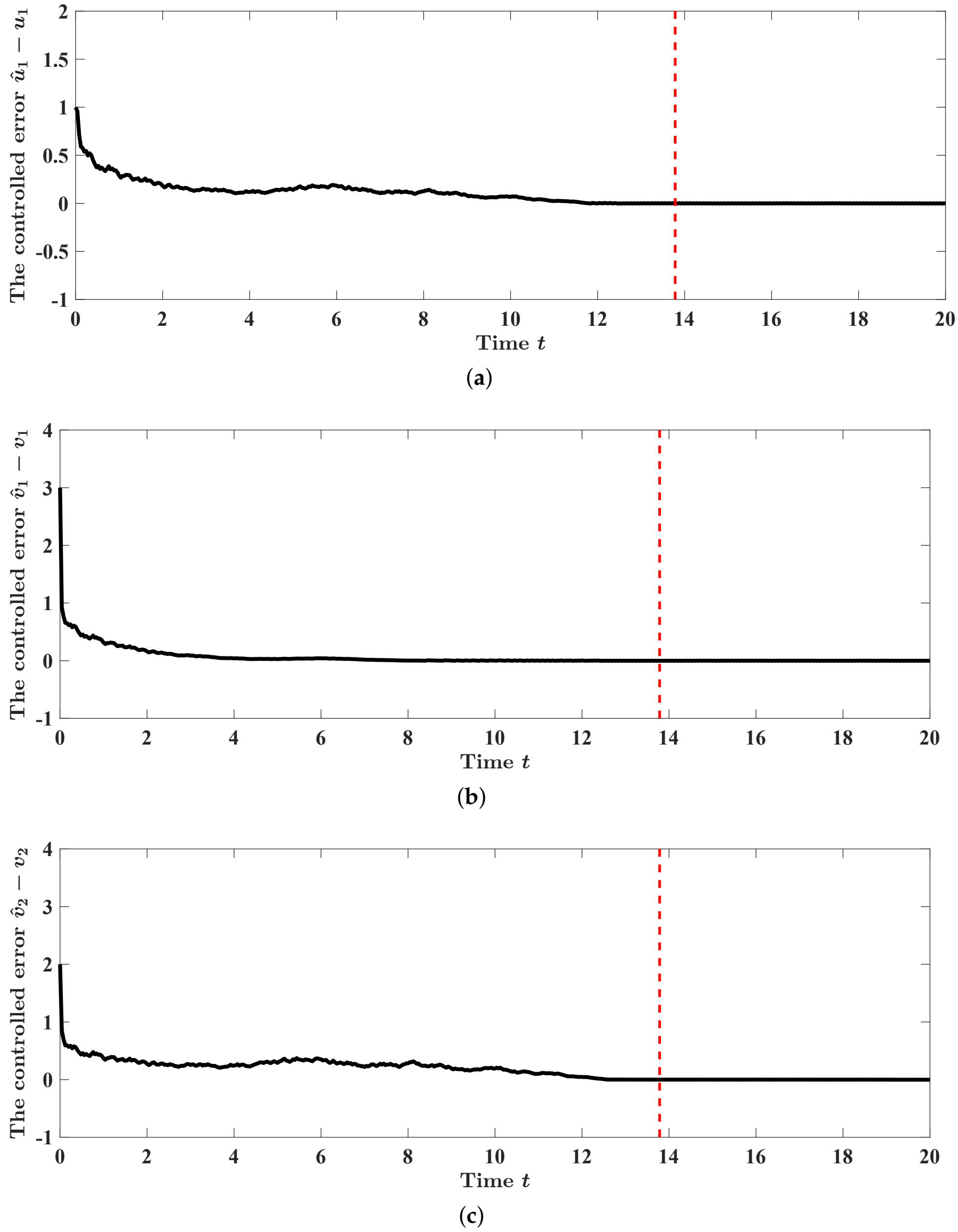


Figure 3. Numerical and graphical validation of our theoretical synchronization results; see Theorem 1 for the details. As in Figures 1 and 2, $(\hat{u}_1(t) - u_1(t), \hat{v}_1(t) - v_1(t), \hat{v}_2(t) - v_2(t))^\top$, $[0, 20]$, is the state trajectory triple of the error system, in which $(u_1(t), v_1(t), v_2(t))^\top$, $[0, 20]$, is the state trajectory triple of our concerned example BAMNN with $u_1(t) \equiv -1$, $v_1(t) \equiv 2$, and $v_2(t) \equiv 1$, $t \in \mathbb{R}_-$, $\mathbb{P}$-a.s. $(\hat{u}_1(t), \hat{v}_1(t), \hat{v}_2(t))^\top$, $[0, 20]$, is the state trajectory triple of the controlled response BAMNN associated with our concerned example BAMNN with $\hat{u}_1(t) \equiv 0$, $\hat{v}_1(t) \equiv 5$, and $\hat{v}_2(t) \equiv 3$, $t \in \mathbb{R}_-$, $\mathbb{P}$-a.s. The dashed straight vertical line segments are the graphs of $t = 13.7825$. The graphs of the tracking error $\hat{u}_1(t) - u_1(t)$, $\hat{v}_1(t) - v_1(t)$ and $\hat{v}_2(t) - v_2(t)$ can be seen in (**a**–**c**).

5. Concluding Remarks

In this paper, we studied a class of time-delayed stochastic BAMNNs, namely, BAMNNs (1)-(2)-(3), based on the Takagi–Sugeno IF–THEN logic and driven by a one-dimensional standard Brownian motion (also termed the Wiener process). Our concerned BAMNNs include a continuous-time delay in leakage terms and a continuous-time delay and (finitely as well as infinitely) a time-distributed delay in transmission terms. Our study, in this paper, is inspired considerably by References [1–7,11,36–38,40,43,44], but we are confronted with quite a few new challenges. For example, different from References [1,3,5,7,11,36,38,43,44], we have to apply a technique to overcome the difficulty brought on by the infinitely time-distributed delay in transmission terms of our concerned BAMNNs, or as opposed to References [5,6], we have to find a new clue to cope with the difficulty caused by the Takagi–Sugeno fuzzy logic in the concerned BAMNNs.

In this paper, we designed a class of control for our concerned BAMNNs and provided a criterion to ensure that our concerned BAMNNs and their response BAMNNs, with our designed control implemented, achieve synchronization within the pre-assigned time. In more detail: (i) We followed the common idea utilized to deal with Takagi–Sugeno fuzzy dynamical systems, to defuzzify the Takagi–Sugeno fuzzy BAMNNs (1)-(2)-(3) into BAMNNs (5)-(6)-(7); (ii) we designed, based on the structure of the response BAMNNs (8)-(6)-(7) of BAMNNs (5)-(6)-(7), the synchronization control (21)–(22); (iii) for any pre-specified time instant (T, say), we established a criterion, meticulously constructed the Lyapunov–Krasovskii functional $\mathcal{V}(t)$ (see (25)), and proved that the BAMNNs (5)-(6)-(7) and the response BAMNNs (8)-(6)-(7), with the control (21)–(22) implemented, achieve synchronization within the pre-assigned time T (see Theorem 1 for the details); and (iv) based on the careful and complicated mathematical derivations in Section 3, we came up with an example which validates our main theoretical results in this paper.

One of the merits of our designed control (21)–(22) is that we only render the control (21)–(22) to be implemented in the drift terms of the response BAMNNs (8)-(6)-(7). Another merit of our designed control (21)–(22) is that a collection of parameters are included so as to reduce the conservatism of the synchronization criterion (see Theorem 1). On the other hand, our designed control (21)–(22) has a disadvantage: The aftereffect in our designed control seems to be strong; see (21)–(22) for the details. To remove or attenuate the aftereffect in synchronization control for BAMNNs (1)-(2)-(3) is one of our primary research directions in the near future. And inspired by the research experience of this paper and the references cited in this paper, we shall work in the direction of improving, in a certain sense, synchronization control for BAMNNs (1)-(2)-(3). For example, we shall try to come up with impulsive control, intermittent control, quantized control, adaptive control, pinning control, sliding mode control, event-triggered control, and so forth, to synchronize BAMNNs (1)-(2)-(3) asymptotically, in finite time, in fixed time, or in pre-assigned time.

As mentioned above, and by observing BAMNNs (1)-(2)-(3) (or, equivalently, BAMNNs (5)-(6)-(7), BAMNNs (8)-(6)-(7), and the BAMNN (11)), it is not difficult to confirm that our concerned model BAMNNs include merely a one-dimensional Brownian motion (Wiener process). By reviewing mathematical derivations throughout this paper, we find that our methods can be adapted to treat the multi-dimensional Brownian motion (Wiener process) case. From both the theoretical and applied viewpoint, it is interesting to consider stochastic Takagi–Sugeno fuzzy BAMNNs, including Markovian jumps, reaction–diffusion terms, and/or proportional time delay. Aided by the experience of this paper, we shall study the synchronization problem for BAMNNs with their dynamics influenced by fuzzy logic, randomness described by Brownian motions, randomness described by Markovian jumps, reaction–diffusion terms, and/or proportional time delay in the near future.

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