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Bivariate Unit-Weibull Distribution: Properties and Inference

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Abstract: In this article, we introduce a novel bivariate probability distribution that is absolutely continuous. Considering the Farlie–Gumbel–Morgenstern (FGM) copula and the unit-Weibull distribution, we can obtain a bivariate unit-Weibull distribution. We evaluate the main properties of the new proposal and use two estimation methods to estimate the parameter for the bivariate probability distribution. A brief Monte Carlo simulation study is conducted to assess the behavior of the employed estimation method and the characteristics of the estimators. Ultimately, as an illustration, a real-life application is presented, demonstrating the utility of the proposal.

Keywords: bivariate probability distribution; distribution for bounded data; proportion data; two-step estimation; copula

MSC: 60E05; 62H05

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1. Introduction

The probability distribution of proportion or rate data has been extensively studied as it is a prevalent finding in research across all domains of knowledge. The beta distribution is perhaps the most widely recognized. Furthermore, many extensions have been made, such as the beta regression model proposed by Ferrari and Cribari-Neto [1], studied later by Branscum et al. [2], and the beta distribution with an excess of zeros and/or ones of Ospina and Ferrari [3], which was extended to the class of inflated beta regression models, see [4], among others. Other outstanding works are those of Paolino [5], Cribari-Neto and Vasconcellos [6], Kieschnick and Mccullough [7] and Vasconcellos and Cribari-Neto [8].

Alternatives to the beta distribution for analysis of bounded data have been considered by other authors. Martínez-Flórez et al. [9] introduced a mixture model between a Bernoulli process and the power normal distribution to model proportions or rates with inflation at zero and/or one values. Given the flexibility of the power normal distribution to fit high degrees of kurtosis and skewness, this proposal is a viable alternative to the beta model. On the other hand, from the logarithmic transformation of the type $X = \exp(-Y)$, where Y is an absolutely continuous random variable with positive support, it is possible to define new families of adequate distributions to fit data in the interval $(0, 1)$. Such obtained distributions are generically called unit-f, where f is the density to which the transformation is applied. Based on this idea, the unit-Gamma distribution, see [10], the log-Lindley distribution, see [11], and unit-Weibull, unit-Birnbaum–Saunders and unit-Lindley distributions, see work by Mazucheli et al. [12–14], have been developed. In addition, Martínez-Flórez et al. [15] introduced a class of distributions to model rates and proportions through an extension of the alpha-power extension of the skew normal distribution. The authors proposed a regression model based on this distribution for bounded responses.

An extension of the unit-Birnbaum–Saunders distribution and its associated regression model was introduced by Martínez-Flórez and Tovar-Falón [16], while another proposed method capable of fitting data between zeros and/or ones (inclusive) with inflation on the zeros/ones values was considered by Martínez-Flórez et al. [17].

Although there are many proposals developed to model proportion or rate data, there are few works that address the problem of this type of response from a multivariate perspective, that is, models capable of jointly modeling two or more responses in the interval $(0,1)$ in such a way that the possible correlations existing in the responses can be considered. In relation to this, some authors have recently made some contributions, among which the works of Lemonte and Moreno-Arenas [18] and more recently Martínez-Flórez et al. [19] stand out.

This paper aims to introduce a novel multivariate distribution to handle responses in the region $(0,1) \times (0.1)$ jointly. The new distribution, which is called the bivariate unit-Weibull distribution is obtained from an extension of the univariate unit-Weibull distribution introduced by Mazucheli et al. [12] by using the Farlie–Gumbel–Morgenstern (FGM) copula, see [20].

In all areas of knowledge, the proposed model can be used to model proportion, rate, or index data. Particularly in the field of engineering, it is common to come across this type of information as a result of investigations, for example, to model the percentage of carbon dioxide and the concentration of ozone in the atmosphere, see [21,22], among others. The proposal can also prove to be advantageous for analyzing data sets from other domains of study, such as the social sciences and civic culture. For instance, it can be utilized to analyze data on laws pertaining to drunk driving and traffic fatalities in 48 states in the United States of America, see [23], or data on the Human Development Index and illiteracy rate, see [18,24].

The main benefits of using this proposal are related to the applicability of a usual regression model, which is the prediction or estimate of the mean response, given the values of the covariates considered in the model. Studying the relationship between the dependent variables and the covariates in such a way means that that this can be used in making decisions that help solve real-life problems.

The univariate unit-Weibull distribution has the follow cumulative distribution function (cdf) and probability density function (pdf), respectively.

$$F(y;\alpha,\beta) = \exp\left(-\alpha(-\log y)^{\beta}\right); \quad 0 < y < 1, \tag{1}$$

where $\alpha, \beta > 0$, and

$$f(y;\alpha,\beta) = (1/y) \times \alpha\beta(-\log y)^{\beta-1} \exp\left(-\alpha(-\log y)^{\beta}\right); \quad 0 < y < 1, \tag{2}$$

where α, β are the scale and shape parameters, respectively. More details and properties of the univariate unit-Weibull distribution can be found in [12].

The following describes the remainder of the article. Section 2 presents the bivariate UW distribution and some of its structural properties are also derived. In Section 3, the statistical inference process is carried out considering two estimation methods. The expected and observed information matrices are also derived. The outcomes of a simulation study and its corresponding discourse are presented in Section 4. Finally, application to real data is provided in Section 5.

2. Bivariate Unit-Weibull Distribution

In this section, a bivariate extension of the unit-Weibull (UW) distribution [12] is proposed. This extension is obtained from the Farlie–Gumbel–Morgenstern (FGM) copula discussed by Gumbel [20] and is called the bivariate unit-Weibull (BVUW) distribution. This new distribution is very attractive, since it allows modeling bivariate data whose response lies on the plane $(0,1)^2$.

To introduce the BVUW distribution, we follow the idea of Almetwally et al. [25] using the FMG copula. Thus, according to the theorem of Sklar [26], Definition 1 follows.

Definition 1. *A random vector $X = (X_1, X_2)^\top$ is said to have a BVUW distribution, if its joint pdf is given by,*

$$f(x_1, x_2) = \frac{1}{x_1}\alpha_1\beta_1(-\ln x_1)^{\beta_1-1}e^{-\alpha_1(-\ln x_1)^{\beta_1}}\frac{1}{x_2}\alpha_2\beta_2(-\ln x_2)^{\beta_2-1}e^{-\alpha_2(-\ln x_2)^{\beta_2}}$$
$$\times\left(1+\theta\left(1-2e^{-\alpha_1(-\ln x_1)^{\beta_1}}\right)\left(1-2e^{-\alpha_2(-\ln x_2)^{\beta_2}}\right)\right), \quad 0 < x_1, x_2 < 1, \quad (3)$$

where $-1 < \theta < 1$ *and* $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$*.*

For the joint pdf given in (3), the notation BVUW$(\alpha_1, \alpha_2, \beta_1, \beta_2, \theta)$ is used. The picture in Figure 1 shows how the BVUW distribution looks in three dimensions for certain values of α_1, α_2, β_1, β_2 and θ. In the figure, one can observe the different shapes that the UW density can fit, for example, symmetrical and asymmetrical shapes in the shape of a bell or a bathtub.

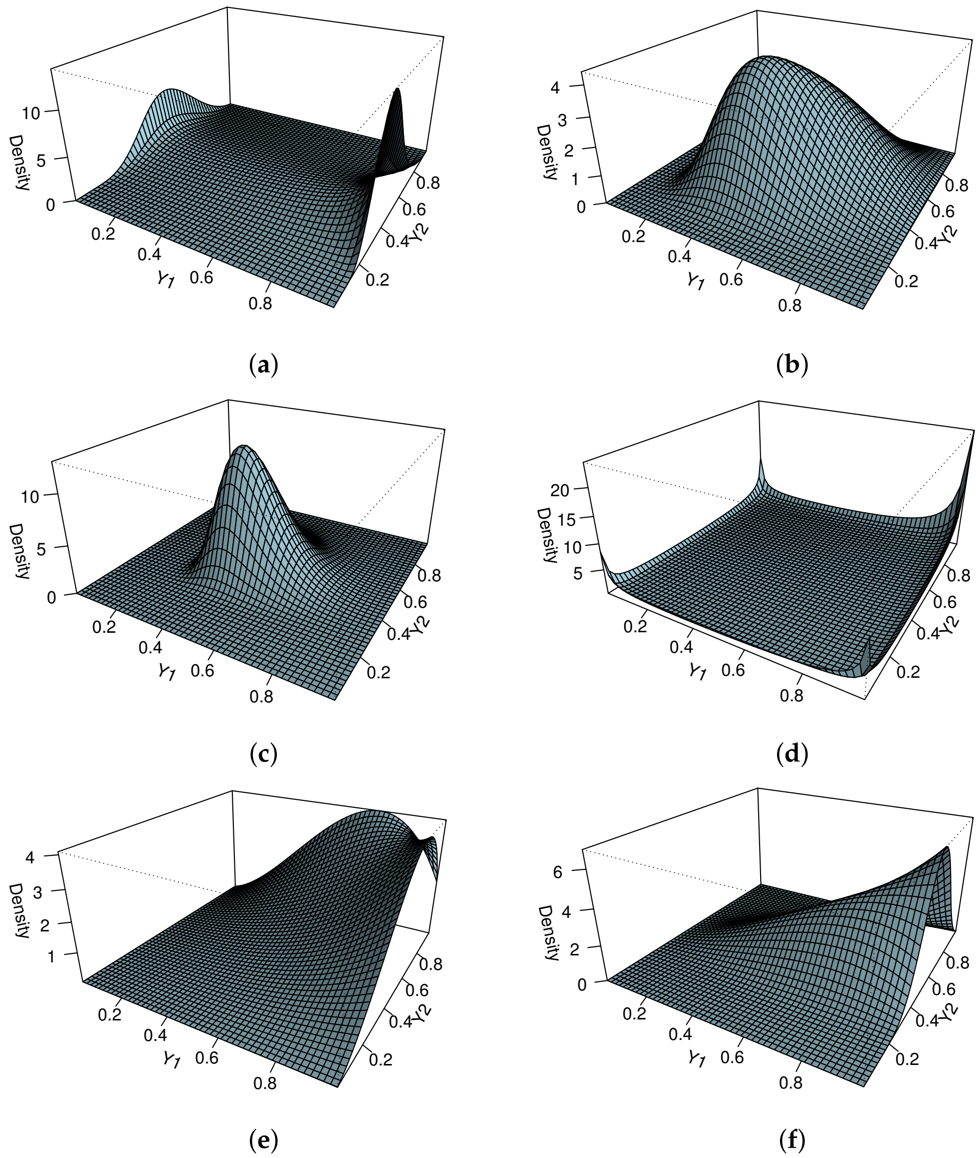

Figure 1. Joint probability density function of the BVUW distribution with parameter vector $\varphi = (\alpha_1, \alpha_2, \beta_1, \beta_2, \theta)$ for **(a)** $\varphi = (1.0, 1.5, 0.5, 2.5, -0.75)$ **(b)** $\varphi = (1.5, 1.5, 2.0, 2.0, 0.75)$, **(c)** $\varphi = (2.0, 2.0, 3.5, 3.5, 0.25)$, **(d)** $\varphi = (1.0, 1.0, 0.5, 0.5, 0.25)$, **(e)** $\varphi = (2.5, 2.5, 1.0, 1.0, -0.75)$, **(f)** $\varphi = (2.5, 2.5, 1.0, 2.5, 0.75)$.

The joint cdf of the BUW distribution can be written as

$$F(x_1,x_2) = e^{-\alpha_1(-\ln x_1)^{\beta_1}} e^{-\alpha_2(-\ln x_2)^{\beta_2}} \left[1+\theta\left(1-e^{-\alpha_1(-\ln x_1)^{\beta_1}}\right)\left(1-e^{-\alpha_2(-\ln x_2)^{\beta_2}}\right)\right] \quad (4)$$

where $-1<\theta<1; 0<x_1,x_2<1; \alpha_1,\alpha_2,\beta_1,\beta_2>0$.

From Definition 1, we have the following theorem.

Theorem 1. *Let $(X_1,X_2) \sim BVUW(\alpha_1,\alpha_2,\beta_1,\beta_2,\rho)$, then*

1. $X_j \sim UW(\alpha_j,\beta_j)$ *for* $j=1,2$.
2. *The pdf of X_1, given $X_2=x_2$, is*

$$f(x_1 \mid x_2) = \frac{1}{x_1}\alpha_1\beta_1(-\ln x_1)^{\beta_1-1}u(x_1)\left[1+\theta(1-2u(x_1))(1-2u(x_2))\right]$$

where $u(x_j)=e^{-\alpha_j(-\ln x_j)^{\beta_j}}$, *for* $j=1,2$,

3. *The pdf of X_i, given $X_2=x_2$, is*

$$f(x_i \mid x_j) = \frac{1}{x_i}\alpha_i\beta_i(-\ln x_i)^{\beta_i-1}e^{-\alpha_i(-\ln x_i)^{\beta_i}}\left[1+\theta(1-2u(x_i))(1-2u(x_j))\right]$$

where $u(x_i)=e^{-\alpha_i(-\ln x_i)^{\beta_i}}$, *for* $i=1,2$.

4. *The cdf X_i, given $X_2=x_2$, is*

$$F(x_i \mid x_j) = (1+\theta)u(x_i) - \theta e^{-2\alpha_i(-\ln y_i)^{\beta_i}} - 2\theta u(x_i)u(x_j) + 2\theta e^{-\alpha_j(-\ln y_j)^{\beta_j}-2\alpha_i(-\ln y_i)^{\beta_i}}.$$

2.1. Generating Random Variables

The conditional distribution method proposed by Nelsen [27] is employed to generate a random sample from a joint unit-Weibull distribution. It is noteworthy that the joint distribution function can be formulated as follows,

$$f(x_1,x_2) = f(x_1)f(x_2 \mid x_1)$$

By utilizing the subsequent procedures, we can generate a bivariate sample by utilizing the conditional approach.

1. From a uniform $(0,1)$ distribution, generate U_1 and U_2 independently of each other.
2. Let $X_1 = \exp\left(-(-\ln U_1/\alpha_1)^{1/\beta_1}\right)$.
3. Allow $F(x_2 \mid x_1) = U_2$ to determine X_2 through numerical simulation.
4. Repeat Steps 1 to 3 n times to get the result (x_{1i},x_{2i}), $i=1,2,\ldots,n$.

2.2. Product Moments

Proposition 1. *Let $\boldsymbol{X}=(X_1,X_2)^\top$ be a random vector with a BVUW($\boldsymbol{\varphi}$) distribution, with $\boldsymbol{\varphi} = (\alpha_1,\alpha_2,\beta_1,\beta_1,\theta)$; then, the rth and sth moments about zero can be computed in the following way*

$$\begin{aligned}\mu_{rs} &= E(X_1^r X_2^s) \\ &= \int_0^1\int_0^1 x_1^r x_2^s \frac{1}{x_1}\alpha_1\beta_1(-\ln x_1)^{\beta_1-1}e^{-\alpha_1(-\ln x_1)^{\beta_1}} \frac{1}{x_2}\alpha_2\beta_2(-\ln x_2)^{\beta_2-1}e^{-\alpha_2(-\ln x_2)^{\beta_2}} \\ &\quad \times \left(1+\theta\left(1-2e^{-\alpha_1(-\ln x_1)^{\beta_1}}\right)\left(1-2e^{-\alpha_2(-\ln x_2)^{\beta_2}}\right)\right)dx_1dx_2 \end{aligned} \quad (5)$$

2.3. Correlation Coefficient

Let $\boldsymbol{X} = (X_1, X_2)^\top$ be a random vector with a BVUW($\boldsymbol{\varphi}$) distribution, then the correlation coefficient between X_1 and X_2 is given by:

$$\rho = cor(Y_1, Y_2) = \frac{E(Y_1Y_2) - E(Y_1)E(Y_2)}{\sigma_1\sigma_2} \tag{6}$$

where

$$\sigma_j = \sqrt{E\big(Y_j - E(Y_j)\big)^2} \quad \text{for} \quad j = 1, 2.$$

By means of a numerical simulation carried out in the R Development Core Team [28] software, it was found that the range of possible values for the correlation coefficient is:

$$-\frac{1}{3} \le \rho \le \frac{1}{3}$$

Table 1 shows some values of the correlation coefficient for certain parameters.

Table 1. Correlation coefficient values for some selected parameter values.

α_1	α_2	β_1	β_2	θ	ρ
3.0	2.0	1.0	1.0	−0.75	−0.235
3.0	2.0	1.0	1.0	−0.25	−0.078
3.0	2.0	1.0	1.0	0.25	0.078
3.0	2.0	1.0	1.0	0.75	0.235
1.5	2.5	1.0	1.0	−0.99	−0.317
1.5	2.5	1.0	1.0	0.50	0.160
4.5	1.5	1.0	1.0	−0.99	−0.307
4.5	1.5	1.0	1.0	0.99	0.307
7.0	2.0	1.0	1.0	−0.90	−0.269
7.0	2.0	1.0	1.0	−0.50	−0.150
7.0	2.0	1.0	1.0	−0.10	−0.030
3.5	1.8	1.0	1.0	0.90	0.281
2.0	12	1.0	2.5	−0.80	−0.258
3.5	1.8	1.8	1.5	0.99	0.325
0.9	3.0	1.0	1.0	−0.990	−0.315

2.4. Reliability Function

The joint survival function can be expressed as a copula of its marginal survival functions using the reliability function, where X_1 and X_2 are random variables with survival functions $\bar{F}(x_1)$ and $\bar{F}(x_2)$, see Almetwally et al. [25].

Proposition 2. *The reliability function of the marginal unit-Weibull distributions for the univariate case is given as follows:*

$$R(x_j) = 1 - F(x_j) = 1 - e^{-\alpha_j(-\ln x_j)^{\beta_j}}; \quad 0 < x_j < 1, \quad \alpha, \beta > 0, \quad j = 1, 2.$$

Proposition 3. *The joint survival function for the FGM copula Nelsen [27] is*

$$H(x_1, x_2) = 1 - F(x_1) - F(x_2) + C(F(x_1), F(x_2)).$$

It then follows that the reliability function of the joint FGM distribution for the bivariate UW distribution is

$$\begin{aligned} H(x_1, x_2) = 1 &- e^{-\alpha_1(-\ln x_1)^{\beta_1}} - e^{-\alpha_2(-\ln x_2)^{\beta_2}} \\ &+ e^{-\alpha_1(-\ln x_1)^{\beta_1}} e^{-\alpha_2(-\ln x_2)^{\beta_2}} \left[1 + \theta\left(1 - e^{-\alpha_1(-\ln x_1)^{\beta_1}}\right)\left(1 - e^{-\alpha_2(-\ln x_2)^{\beta_2}}\right)\right]. \end{aligned}$$

Proposition 4. *Basu [29] defines the hazard function for the bivariate case as*

$$h(x_1, x_2) = \frac{f(x_1, x_2)}{H(x_1, x_2)}.$$

Then, the hazard function of the FGM joint BUW distribution is,

$$h(x_1, x_2) = \frac{\frac{1}{x_1}\alpha_1\beta_1(-\ln x_1)^{\beta_1-1}e^{-\alpha_1(-\ln x_1)^{\beta_1}}\,\frac{1}{y_2}\alpha_2\beta_2(-\ln x_2)^{\beta_2-1}e^{-\alpha_2(-\ln x_2)^{\beta_2}}}{1 - F(x_1) - F(x_2) + e^{-\alpha_1(-\ln x_1)^{\beta_1}}e^{-\alpha_2(-\ln x_2)^{\beta_2}}\left[1+\theta\left(1-e^{-\alpha_1(-\ln x_1)^{\beta_1}}\right)\left(1-e^{-\alpha_2(-\ln x_2)^{\beta_2}}\right)\right]}$$
$$\times \left(1+\theta\left(1-2e^{-\alpha_1(-\ln x_1)^{\beta_1}}\right)\left(1-2e^{-\alpha_2(-\ln x_2)^{\beta_2}}\right)\right).$$

3. Statistical Inference for the BUW Distribution

In this section, two distinct estimation techniques are presented that are employed to estimate the parameters of the BVUW distribution, namely, maximum likelihood estimation (MLE) and estimation by inference functions for margins (IFM). These methods were also considered by Almetwally et al. [25].

3.1. Maximum Likelihood Estimation (MLE)

Under this parametric method, the MLE is found to estimate the parameters of the model jointly and in a single step. The log-likelihood function is given by

$$\ln L = \sum_{i=1}^{n}[\ln(f_1(x_{1i})f_2(x_{2i})c(F_1(x_{1i}), F_2(x_{2i}); \theta))]$$

To obtain estimates for each parameter, we use the log-likelihood function. We define

$$\begin{aligned} a(x_j; \alpha_j, \beta_j) &= 1 - 2F(x_j; \alpha_j, \beta_j) \\ &= 1 - 2e^{-\alpha_j(-\ln x_j)^{\beta_j}}, \quad j = 1, 2. \end{aligned}$$

A bivariate unit-Weibull distribution has a likelihood function defined by

$$L = (\alpha_1\alpha_2\beta_1\beta_2)^n \prod_{i=1}^{n}\left(\frac{1}{x_{1i}}\frac{1}{x_{2i}}(-\ln x_{1i})^{\beta_1-1}(-\ln x_{2i})^{\beta_2-1}\right)e^{-\sum_{i=1}^{n}\alpha_1(-\ln x_1)^{\beta_1}-\sum_{i=1}^{n}\alpha_2(-\ln x_2)^{\beta_2}}$$
$$\times \prod_{i=1}^{n}(1+\theta(a(x_{1i}; \alpha_1, \beta_1))(a(x_{2i}; \alpha_2, \beta_2))).$$

The corresponding log-likelihood function of a BVUW distribution is subsequently provided by

$$\begin{aligned} \ln L = {} & n(\ln\alpha_1 + \ln\beta_1 + \ln\alpha_2 + \ln\beta_2) + \sum_{i=1}^{n}\ln\left(\frac{1}{x_{1i}}\right) + \sum_{i=1}^{n}\ln\left(\frac{1}{x_{2i}}\right) \\ & + (\beta_1 - 1)\sum_{i=1}^{n}\ln(-\ln x_{1i}) + (\beta_2 - 1)\sum_{i=1}^{n}\ln(-\ln x_{2i}) \\ & - \sum_{i=1}^{n}\alpha_1(-\ln x_1)^{\beta_1} - \sum_{i=1}^{n}\alpha_2(-\ln x_2)^{\beta_2} \\ & + \sum_{i=1}^{n}\ln\big(1 + \theta(a(y_{1i}; \alpha_1, \beta_1))(a(y_{2i}; \alpha_2, \beta_2))\big) \end{aligned}$$

To obtain the estimations, we equate the derivatives of the log-likelihood function with respect to each parameter to zero. This is performed by determining the equation score in the following manner:

$$\frac{\partial \ln L}{\partial \alpha_1} = \frac{n}{\alpha_1} - \sum_{i=1}^{n} (-\ln x_{1i})^{\beta_1} + \sum_{i=1}^{n} \frac{2\theta(a(x_{2i};\alpha_2,\beta_2))(-\ln x_{1i})^{\beta_1} e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))}$$

$$\frac{\partial \ln L}{\partial \alpha_2} = \frac{n}{\alpha_2} - \sum_{i=1}^{n} (-\ln x_{2i})^{\beta_2} + \sum_{i=1}^{n} \frac{2\theta(a(x_{1i};\alpha_1,\beta_1))(-\ln x_{2i})^{\beta_2} e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \beta_1} = {} & \frac{n}{\beta_1} + \sum_{i=1}^{n} \ln(-\ln x_{1i}) - \sum_{i=1}^{n} \alpha_1(-\ln x_{1i})^{\beta_1} \ln(-\ln x_{1i}) \\ & - \sum_{i=1}^{n} \frac{2\alpha_1\theta(a(x_{2i};\alpha_2,\beta_2)) \ln(-\ln x_{1i})(-\ln x_{1i})^{\beta_1} e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))}\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \beta_2} = {} & \frac{n}{\beta_2} + \sum_{i=1}^{n} \ln(-\ln x_{2i}) - \sum_{i=1}^{n} \alpha_1(-\ln x_{2i})^{\beta_2} \ln(-\ln x_{2i}) \\ & - \sum_{i=1}^{n} \frac{2\alpha_2\theta(a(x_{1i};\alpha_1,\beta_1)) \ln(-\ln x_{2i})(-\ln x_{2i})^{\beta_2} e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))}\end{aligned}$$

and

$$\frac{\partial \ln L}{\partial \theta} = \sum_{i=1}^{n} \frac{(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2))}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))}$$

The MLE $\widehat{\boldsymbol{\varphi}} = (\widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\alpha}_2, \widehat{\beta}_2, \widehat{\theta})$ can be obtained by simultaneously solving the score equations.

It is possible to obtain the MLE $\widehat{\boldsymbol{\varphi}} = (\widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\alpha}_2, \widehat{\beta}_2, \widehat{\theta})$ by solving the score equation,

$$\frac{\partial \ln L}{\partial \theta} = 0, \quad \frac{\partial \ln L}{\partial \beta_j} = 0, \quad \frac{\partial \ln L}{\partial \alpha_j} = 0, \quad j = 1, 2$$

The estimates are obtained by performing iterative numerical methods such as Newton–Raphson or quasi-Newton methods.

3.2. Information Matrix

In this section, we shall explicate the components of the information matrix for the BUW distribution that have been observed and expected. According to the definition, the elements of the observed information matrix are found using the expression

$$j_{\varphi_p,\varphi_q} = -\frac{\partial^2 \ln L(\boldsymbol{\varphi})}{\partial \varphi_p \partial \varphi_q}, \quad \varphi_p, \varphi_q \in (\alpha_1, \alpha_2, \beta_1, \beta_2, \theta) \tag{7}$$

where $L(\boldsymbol{\varphi})$ is the log-likelihood function associated with the parameter vector $\boldsymbol{\varphi} = (\alpha_1, \alpha_2, \beta_1, \beta_2, \theta)$. These elements are presented in detail in Appendix A. The expected information matrix is determined by taking the expected value of the observed information matrix and using the expression.

$$i_{\varphi_p,\varphi_q} = -\frac{1}{n} E\left(\frac{\partial^2 \ln L(\boldsymbol{\varphi})}{\partial \varphi_p \partial \varphi_q}\right) \tag{8}$$

The above values are obtained numerically by using iterative methods, and therefore the information matrix can also be obtained numerically. Using the theory of large samples, the maximum likelihood estimator $\widehat{\boldsymbol{\varphi}}$ asymptotically follows a normal distribution with

$$\widehat{\boldsymbol{\varphi}} \xrightarrow{d} \mathrm{N}_5(\boldsymbol{\varphi}, \boldsymbol{\Sigma}_{\boldsymbol{\varphi}})$$

where $\boldsymbol{\Sigma}_{\boldsymbol{\varphi}} = \mathrm{Var}(\widehat{\boldsymbol{\varphi}}) = (i_{\varphi_p,\varphi_q})^{-1}$. The approximation to $\mathrm{N}_5\big(\boldsymbol{\varphi}, (i_{\varphi_p,\varphi_q})^{-1}\big)$ can be used to construct confidence intervals for α_1, α_2, β_1, β_2 and θ; these are given by $\varphi_p \mp z_{1-\delta/2}\sqrt{\widehat{\sigma}_{pp}}$ for $\varphi_p \in (\alpha_1, \alpha_2, \beta_1, \beta_2, \theta)$, where $\widehat{\sigma}_{pp}$ is on the diagonal of the matrix $\boldsymbol{\Sigma}_{\boldsymbol{\varphi}}$ for each parameter and is the quantile $100(\delta/2)\%$ of the standard normal distribution.

3.3. Estimation by Inference Functions for Margins (IFM)

This two-step estimation parametric method was proposed by Joe [30]. Each marginal distribution parameter is evaluated separately in the initial stage.

$$\ln L_1 = \sum_{j=1}^{n} \ln f_1(y_{1j}, \delta_1); \qquad \ln L_2 = \sum_{j=1}^{n} \ln f_2(y_{2j}, \delta_2)$$

The subsequent step entails maximizing the log-likelihood function of the copula density by utilizing the maximum likelihood estimates of the marginals $\widehat{F}_1(x_{1j}, \delta)$ and $\widehat{F}_2(x_{2j}, \delta)$ to estimate the copula parameter. For a UW marginal distribution, the log-likelihood function is defined as follows.

$$\ln L_j = \sum_{i=1}^{n} \ln\left(\frac{1}{x_{ji}}\right) + n\big(\ln(\alpha_j) + \ln(\beta_j)\big) + (\beta_j - 1)\sum_{i=1}^{n} \ln(-\ln x_{ji}) - \alpha_j \sum_{i=1}^{n} (-\ln x_{ji})^{\beta_j}$$

The maximum likelihood estimators $(\widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\alpha}_2, \widehat{\beta}_2)$ can be obtained by simultaneously solving the likelihood equations,

$$\frac{\partial \ln L}{\partial \beta_j} = 0, \quad \frac{\partial \ln L}{\partial \alpha_j} = 0, \quad j = 1, 2$$

That is,

$$\frac{\partial \ln L_j}{\partial \beta_j} = \frac{n}{\beta_j} + \sum_{i=1}^{n} (-\ln x_{ji}) - \alpha_j \sum_{i=1}^{n} (-\ln x_{ji})^{\beta_j} \ln(-\ln x_{ji}) = 0$$

$$\frac{\partial \ln L_j}{\partial \alpha_j} = \frac{n}{\alpha_j} - \sum_{i=1}^{n} (-\ln x_{ji})^{\beta_j} = 0$$

then,

$$\widehat{F}_j(x_j) = e^{-\alpha_j(-\ln x_j)^{\beta_j}}.$$

As per the preceding procedure, the IFM estimation of a bivariate unit-Weibull distribution is defined as follows,

$$\ln L_{IFM} = \sum_{i=1}^{n} \ln\Big(1 + \theta(1 - 2\widehat{F}_1(x_{1i}))(1 - 2\widehat{F}_2(x_{2i}))\Big).$$

Based on this, the differentiation of the log-likelihood function with respect to θ thus is

$$\frac{\partial \ln L_{IFM}}{\partial \theta} = \sum_{i=1}^{n} \frac{(a(x_{1i}, \widehat{\alpha}_1, \widehat{\beta}_1))(a(x_{2i}, \widehat{\alpha}_2, \widehat{\beta}_2))}{(1 + \theta(a(x_{1i}, \widehat{\alpha}_1, \widehat{\beta}_1))(a(x_{2i}, \widehat{\alpha}_2, \widehat{\beta}_2)))}.$$

The estimate of the θ parameter is computed numerically by letting

$$\frac{\partial \ln L_{IFM}}{\partial \theta} = 0$$

The closed form expression for the maximum likelihood estimator $\widehat{\theta}$ is not available, and its computation necessitates numerical execution using a nonlinear optimization algorithm such as optimize or uniroot in R Development Core Team [28].

For the IFM method, the asymptotic variance matrix does not have a closed form. Joe [30] defines the inference functions as,

$$\sum_{i=1}^{n} \boldsymbol{g}(\boldsymbol{Y};\boldsymbol{\eta})$$

where $\boldsymbol{g}^\top = (g_1^\top, \ldots, g_m^\top, g_d^\top)$, $g_j = \frac{\partial l_j}{\partial \alpha_j}$ for $j = 1, \ldots, m$, corresponding to the equation,

$$g_d = \frac{\partial l}{\partial \delta}$$

Thus, the asymptotic variance matrix is given by,

$$\boldsymbol{V} = (-\boldsymbol{D}_g^{-1})\boldsymbol{M}_g(-\boldsymbol{D}_g^{-1}), \quad (9)$$

where $\boldsymbol{M}_g = \mathrm{Cov}(\boldsymbol{g}(\boldsymbol{Y};\boldsymbol{\eta})) = E(\boldsymbol{g}\boldsymbol{g}^\top)$ and $\boldsymbol{D}_g = E\left(\frac{\partial \boldsymbol{g}(\boldsymbol{Y};\boldsymbol{\eta})}{\partial \boldsymbol{\eta}^\top}\right)$.

Other approaches can be used to perform the inference on the model in practical situations, for example, Bayesian methods. Nevertheless, in this work, the classical estimation of the parameters is considered using the methods described above. Based on the research conducted by Kundu and Gupta [31], Zhou et al. [32] and Peralta et al. [33], Bayesian methods can be proposed.

4. Monte Carlo Simulation Study

In this particular section, a Monte Carlo simulation study was conducted with the objective of comparing two copula-based estimation techniques, namely maximum likelihood and estimation by inference of marginal functions, in order to determine the BVUW distribution parameters.

For this simulation, the maxLik function of the statistical software [28] was used and data from the BVUW distribution were generated considering the following values of the parameters α_1, β_1, α_2, β_2 and θ:

- Scenario 1: $\alpha_1 = 1.5, \beta_1 = 3.5, \alpha_2 = 0.5, \beta_2 = 2.0, \theta = 0.25$.
- Scenario 2: $\alpha_1 = 1.5, \beta_1 = 3.5, \alpha_2 = 0.5, \beta_2 = 2.0, \theta = 0.75$.
- Scenario 3: $\alpha_1 = 1.5, \beta_1 = 3.5, \alpha_2 = 0.5, \beta_2 = 2.0, \theta = -0.25$.
- Scenario 4: $\alpha_1 = 1.5, \beta_1 = 3.5, \alpha_2 = 0.5, \beta_2 = 2.0, \theta = -0.75$.
- Scenario 5: $\alpha_1 = 2.5, \beta_1 = 1.5, \alpha_2 = 1.5, \beta_2 = 1, \theta = 0.25$.
- Scenario 6: $\alpha_1 = 2.5, \beta_1 = 1.5, \alpha_2 = 1.5, \beta_2 = 1, \theta = 0.75$.

For each scenario, 5000 random samples were generated using the algorithm presented in Section 2.1 for the sample sizes $n = 30, 50, 75, 100, 200$ and 500. As quality measures to evaluate the benefits of the MLE and IFM methods, the bias, mean square error (MSE), confidence interval length (CIL) and the interval coverage rate (ICR) were used. These measurements were calculated as

$$Bias(\hat{\delta}^{(j)}) = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{\delta}_i^{(j)} - \delta^{(j)}), \quad MSE(\hat{\delta}^{(j)}) = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{\delta}_i^{(j)} - \delta)^2$$

and

$$CIL(\hat{\delta}^{(j)}) = \frac{1}{5000}\sum_{i=1}^{5000} 2z_{(0.975)} \times ee(\hat{\delta}^{(j)})$$

where $\hat{\delta}_i^{(j)}$ is the estimate of $\delta^{(j)}$ for the ith sample, and $ee(\hat{\delta}^{(j)})$ is the standard error of $\hat{\delta}_i^{(j)}$. The ICR is calculated as the proportion of the confidence interval that contain the true value of the parameter below the nominal 95% level. The outcomes of the simulation study are shown in Tables A1–A3 in Appendix B.

Based on the data presented in the tables, it can be observed that, in general, as the sample size increases, the bias, MSE and CIL of all parameter estimators tend to decrease with the implementation of MLE and IFM methods. In regard to the ICR, the values obtained are very similar to the reference value of 95%. On the other hand, for large n sample sizes, the MLE and IFM methods produce very similar results in terms of bias, MSE and CIL values. Furthermore, both estimation methods have better results in terms of estimations when the copula parameter is not high, that is, for values of the parameter θ in the interval $(-0.6, 0.6)$.

The IFM method produces better results when estimating the copula parameter theta. This phenomenon can be attributed in part to the fact that the IFM methodology comprises two estimation steps: the first step involves the estimation of the marginal distribution parameters, followed by the estimation of the copula parameter by incorporating the estimates of the previous marginal distribution parameters.

5. Illustration

This application demonstrates the relevance of the WUB model in a practical context. To accomplish this goal, a collection of facts is considered, focusing on the legislation governing drunk driving and fatalities from traffic collisions in states of the United States of America during the period from 1980 to 2004, as revealed by a thorough investigation conducted by Freeman [23].

The database is accessible within the woolridge library of R Development Core Team [28] software, where it is designated as driving. It contains information related to current legislation, accident records and demographic characteristics. For this particular application, the variables unemployment rate (x_1) and the percent population aged 14 to 24 (x_2) were employed, and the BUW distribution was subsequently adjusted. Likewise, to compare with our proposal, the bivariate Johnson SB (BVSJB) distribution of Lemonte and Moreno-Arenas [18], the bivariate normal (BVN) distribution and the conditional bivariate skew normal (BVSN) distribution of Arnold et al. [34] were studied.

For the purpose of comparison, the Akaike information criterion (AIC) [35] and the Bayesian information criterion (BIC) [36] were employed. These measures are defined in the following:

$$\text{AIC} = -2\ell(\widehat{\theta}) + 2p \quad \text{and} \quad \text{BIC} = -2\ell(\widehat{\theta}) + n\log(n)$$

The parameter p represents the number of parameters and the log-likelihood function $\widehat{\ell}(\cdot)$ was evaluated at the MLEs of the parameters. It is best to choose the model with the smallest AIC or BIC. We used the maxLik function of the statistical package [28] to fit the bivariate model.

The parameters of these models were estimated using the maximum likelihood method. The standard errors are shown in parentheses in Table 2. According to the AIC and BIC criteria, the BVUW distribution provides the best fit.

The graphs in Figure 2 show the contours of the BVN, BVSN and BVUW distributions. We carried out a bivariate Kolmogorov–Smirnov test. For the special case of the BVUW model,

$$d_n(\text{WUB}) = \max\{3.1061671 \times 10^{-21}, 5.1055311 \times 10^{-20}\} = 5.105531 \times 10^{-20};$$

therefore, it is possible to conclude that the BVUW model is well suited to the present data set.

Table 2. MLE (SE) for the fitted models.

Parameters	BVSJB	BVN	BVSN	BVUW
α_1	0.1554(0.0010)	0.1532(0.0005)	0.1546(0.0005)	18.0497(0.3906)
α_2	0.0624(0.0018)	0.0595(0.0005)	0.0602(0.0005)	9.3641(0.1949)
β_1	5.0188(0.9010)			1.9351(0.0032)
β_2	2.7763(0.1020)			3.0170(0.0096)
σ_1	1.7204(0.6972)	0.0003(0.00001)	0.0188(0.0003)	
σ_2	8.0839(0.2439)	0.0004(0.00001)	0.0207(0.0004)	
λ	3.2156(0.5746)	0.0001(0.00001)	0.6973(0.0594)	0.9990(0.0038)
AIC	−10,980.07	−12,250.64	−12,245.92	−12,447.81
BIC	−10,944.44	−12,225.19	−12,220.47	−12,422.36

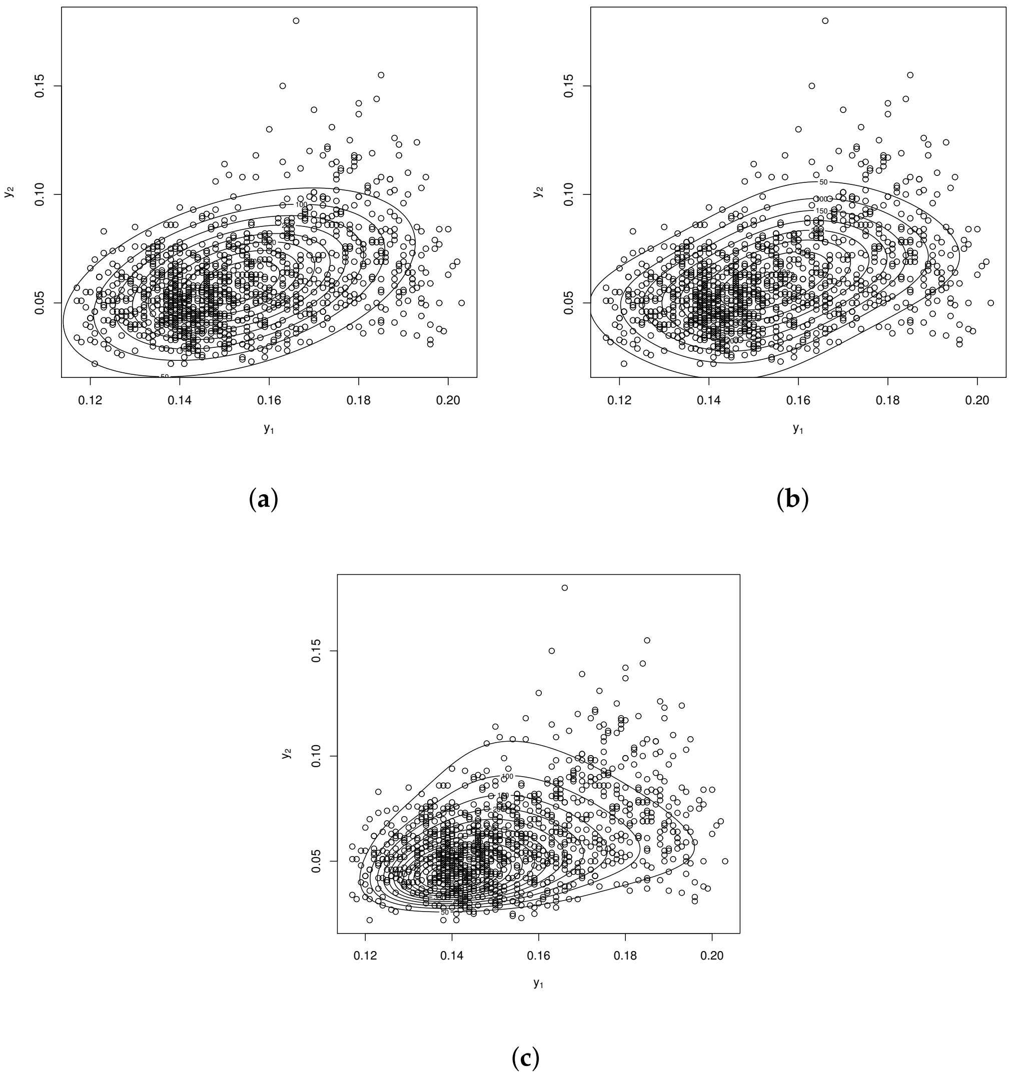

Figure 2. Contour plots for the fitted models. (**a**) BVN, (**b**) BVSN and (**c**) BVUW.

6. Concluding Remarks

Although there are several proposals for fitting proportion, rate or index data, very few focus on the problem of fitting multivariate responses, for example, in the region $(0, 1) \times (0, 1)$.

In this work, a new bivariate distribution was proposed to model data whose response falls within the region $(0,1)\times(0,1)$. It is possible to obtain the new distribution by extending the univariate unit-Weibull distribution introduced by Mazucheli et al. [12]. For the proposed distribution, the main properties such as the joint density function and the joint distribution function were studied. The moments were also studied and two methods of estimation of the parameters involved in the model were considered. The results of a Monte Carlo-type simulation study showed good asymptotic properties of the parameter estimators and that the two estimation methods are equally efficient. Finally, application to a real data set showed that the proposed model is a viable alternative to analyze proportion, rate or index data in a multivariate context.

Future work could use the results of this proposal to consider how to extend the case of regression models and make inferences about the models from a Bayesian perspective.

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Appendix A. Elements of the Observed Information Matrix

This section presents the elements of the observed information matrix presented in Section 3.

The following elements are obtained using the expression in (7) and after some algebraic simplifications:

$$j_{\alpha_1,\alpha_1} = \frac{n}{\alpha_1^2} + \sum_{i=1}^{n} \frac{2\theta(a(x_2;\alpha_2,\beta_2))(-\ln x_{1i})^{2\beta_1} e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}((1+\theta(a(x_1;\alpha_1,\beta_1))(a(x_2;\alpha_2,\beta_2)))}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2}$$
$$+ \sum_{i=1}^{n} \frac{2\theta(a(x_2;\alpha_2,\beta_2))(-\ln x_{1i})^{2\beta_1} e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}\left(2\theta(a(x_2;\alpha_2,\beta_2))e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}\right)}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2}$$

$$j_{\alpha_1,\alpha_2} = \sum_{i=1}^{n} \frac{-4\theta(-\ln x_{1i})^{\beta_1} e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}(-\ln x_{2i})^{\beta_2} e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2}$$

$$j_{\alpha_1,\beta_1} = \frac{-2\theta(a(x_{2i};\alpha_2,\beta_2))(-\ln x_{1i})^{\beta_1}\ln(-\ln x_{1i})e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))} \times \frac{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))\left(1-\alpha_1(-\ln x_{1i})^{\beta_1}\right)}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))} + \frac{2\theta(a(x_{2i};\alpha_2,\beta_2))(-\ln x_{1i})^{\beta_1}\ln(-\ln x_{1i})e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))} \times \frac{\left(2\theta\alpha_1(a(x_{2i};\alpha_2,\beta_2))(-\ln x_{1i})^{\beta_1}e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}\right)}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))} + \sum_{i=1}^{n}(-\ln x_{1i})^{\beta_1}\ln(-\ln x_{1i})$$

$$j_{\alpha_1,\beta_2} = \sum_{i=1}^{n}\frac{-4\theta\alpha_2(-\ln x_{1i})^{\beta_1}e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}\ln(-\ln x_{2i})(-\ln x_{2i})^{\beta_2}e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2}$$

$$j_{\alpha_1,\theta} = \sum_{i=1}^{n}\frac{-2(a(x_{2i};\alpha_2,\beta_2))(-\ln x_{1i})^{\beta_1}e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2}$$

$$j_{\alpha_2,\alpha_2} = \frac{n}{\alpha_2^2} + \sum_{i=1}^{n}\frac{2\theta(a(x_1;\alpha_1,\beta_1))(-\ln x_{2i})^{2\beta_2}e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}((1+\theta(a(x_1;\alpha_1,\beta_1))(a(x_2;\alpha_2,\beta_2)))}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2} + \sum_{i=1}^{n}\frac{2\theta(a(x_1;\alpha_1,\beta_1))(-\ln x_{2i})^{2\beta_2}e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}\left(2\theta(a(x_2;\alpha_2,\beta_2))e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}\right)}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2}$$

$$j_{\alpha_2,\beta_1} = \sum_{i=1}^{n}\frac{-4\theta\alpha_1(-\ln x_{2i})^{\beta_2}e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}\ln(-\ln x_{1i})(-\ln x_{1i})^{\beta_1}e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2}$$

$$j_{\alpha_2,\beta_2} = \sum_{i=1}^{n}\frac{-2\theta(a(x_{1i};\alpha_1,\beta_1))(-\ln x_{2i})^{\beta_2}\ln(-\ln x_{2i})e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))} \times \frac{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))\left(1-\alpha_2(-\ln x_{2i})^{\beta_2}\right)}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))} + \sum_{i=1}^{n}\frac{2\theta(a(x_{1i};\alpha_1,\beta_1))(-\ln x_{2i})^{\beta_2}\ln(-\ln x_{2i})e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))} \times \frac{\left(2\theta\alpha_2(a(x_{1i};\alpha_1,\beta_1))(-\ln x_{2i})^{\beta_2}e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}\right)}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))} + \sum_{i=1}^{n}(-\ln x_{2i})^{\beta_2}\ln(-\ln x_{2i})$$

$$j_{\alpha_2,\theta} = \sum_{i=1}^{n}\frac{-2(a(x_{1i};\alpha_1,\beta_1))(-\ln x_{2i})^{\beta_2}e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2}$$

$$\begin{aligned}
j_{\beta_1,\beta_1} = {} & \sum_{i=1}^{n} \frac{-2\alpha_1\theta(a(x_{2i};\alpha_2,\beta_2))\ln(-\ln x_{1i})^2(-\ln x_{1i})^{\beta_1}e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))} \\
& \times \frac{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))\left(1-\alpha_1(-\ln x_{1i})^{\beta_1}\right)}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))} \\
& + \sum_{i=1}^{n} \frac{4\alpha_1^2\theta^2(a(x_{2i};\alpha_2,\beta_2)^2)\ln(-\ln x_{1i})^2(-\ln x_{1i})^{2\beta_1}e^{-2\alpha_1(-\ln x_{1i})^{\beta_1}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2} \\
& + \sum_{i=1}^{n} \alpha_1 \ln(-\ln x_{1i})^2(-\ln x_{1i})^{\beta_1} + \frac{n}{\beta_1^2}
\end{aligned}$$

$$j_{\beta_1,\beta_2} = \sum_{i=1}^{n} \frac{-2\alpha_1\theta\ln(-\ln x_{1i})(-\ln x_{1i})^{\beta_1}e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2}$$

$$j_{\beta_1,\theta} = \sum_{i=1}^{n} \frac{-2\alpha_1(a(x_{2i};\alpha_2,\beta_2))\ln(-\ln x_{1i})(-\ln x_{1i})^{\beta_1}e^{-\alpha_1(-\ln x_{1i})^{\beta_1}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2}$$

$$\begin{aligned}
j_{\beta_2,\beta_2} = {} & \sum_{i=1}^{n} \frac{-2\alpha_2\theta(a(x_{1i};\alpha_1,\beta_1))\ln(-\ln x_{2i})^2(-\ln x_{2i})^{\beta_2}e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))} \\
& \times \frac{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))\left(1-\alpha_2(-\ln x_{2i})^{\beta_2}\right)}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))} \\
& + \sum_{i=1}^{n} \frac{4\alpha_2^2\theta^2(a(x_{1i};\alpha_1,\beta_1)^2)\ln(-\ln x_{2i})^2(-\ln x_{2i})^{2\beta_2}e^{-2\alpha_2(-\ln x_{2i})^{\beta_2}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2} \\
& + \sum_{i=1}^{n} \alpha_2 \ln(-\ln x_{2i})^2(-\ln x_{2i})^{\beta_2} + \frac{n}{\beta_2^2}
\end{aligned}$$

$$j_{\beta_2,\theta} = \sum_{i=1}^{n} \frac{-2\alpha_2(a(x_{1i};\alpha_1,\beta_1))\ln(-\ln x_{2i})(-\ln x_{2i})^{\beta_2}e^{-\alpha_2(-\ln x_{2i})^{\beta_2}}}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2}$$

$$j_{\theta,\theta} = \sum_{i=1}^{n} \frac{(a(x_{1i};\alpha_1,\beta_1)^2)(a(x_{2i};\alpha_2,\beta_2)^2)}{(1+\theta(a(x_{1i};\alpha_1,\beta_1))(a(x_{2i};\alpha_2,\beta_2)))^2}$$

Appendix B. Simulation Tables

Table A1. Quality measure (QM) of the estimates obtained under the MLE and IMF methods: Scenarios 1 and 2.

Scenario 1		**MLE Method**					**IFM Method**				
n	**QM**	$\widehat{\alpha}_1$	$\widehat{\beta}_1$	$\widehat{\alpha}_2$	$\widehat{\beta}_2$	$\widehat{\theta}$	$\widehat{\alpha}_1$	$\widehat{\beta}_1$	$\widehat{\alpha}_2$	$\widehat{\beta}_2$	$\hat{\theta}$
30	Bias	0.0405	0.1680	0.0048	0.0845	−0.0647	0.0408	0.1676	0.0044	0.0854	−0.0701
	MSE	0.0907	0.3265	0.0168	0.1211	0.1956	0.0910	0.3229	0.0166	0.1195	0.2005
	CIL	1.1173	2.0657	0.4760	1.1742	2.0777	1.1157	2.0645	0.4749	1.1736	2.0586
	ICR	0.9549	0.9542	0.9246	0.9386	0.9689	0.9573	0.9542	0.9245	0.9390	0.9741
50	Bias	0.0235	0.0906	0.0029	0.0543	−0.0320	0.0248	0.0926	0.0029	0.0543	−0.0342
	MSE	0.0482	0.1672	0.0097	0.0617	0.1535	0.0489	0.1682	0.0097	0.0614	0.1558
	CIL	0.8501	1.5614	0.3678	0.8916	1.6133	0.8508	1.5624	0.3678	0.8917	1.6055
	ICR	0.9550	0.9543	0.9331	0.9338	0.9451	0.9533	0.9533	0.9331	0.9393	0.9428
75	Bias	0.0189	0.0551	0.0048	0.0296	−0.0181	0.0197	0.0552	0.0048	0.0296	−0.0206
	MSE	0.0323	0.1062	0.0063	0.0375	0.1121	0.0326	0.1067	0.0062	0.0373	0.1113
	CIL	0.6904	1.2591	0.3011	0.7176	1.3245	0.6909	1.2596	0.3013	0.7177	1.3186
	ICR	0.9448	0.9550	0.9373	0.9366	0.9441	0.9455	0.9544	0.9408	0.9374	0.9415
100	Bias	0.0119	0.0405	0.0026	0.0222	−0.0154	0.0123	0.0400	0.0029	0.0219	−0.0166
	MSE	0.0239	0.0808	0.0047	0.0260	0.0869	0.0239	0.0808	0.0047	0.0259	0.0859
	CIL	0.5944	1.0851	0.2601	0.6189	1.1532	0.5948	1.0853	0.2604	0.6190	1.1485
	ICR	0.9477	0.9510	0.9403	0.9544	0.9477	0.9450	0.9490	0.9389	0.9537	0.9477
200	Bias	0.0079	0.0258	0.0008	0.0083	−0.0007	0.0080	0.0258	0.0009	0.0081	−0.0013
	MSE	0.0119	0.0383	0.0022	0.0125	0.0439	0.0119	0.0383	0.0022	0.0125	0.0437
	CIL	0.4184	0.7630	0.1836	0.4342	0.8182	0.4187	0.7633	0.1838	0.4343	0.8160
	ICR	0.9467	0.9533	0.9453	0.9540	0.9393	0.9473	0.9527	0.9467	0.9527	0.9387
500	Bias	0.0025	0.0063	−0.0008	0.0062	−0.0010	0.0025	0.0062	−0.0008	0.0061	−0.0012
	MSE	0.0044	0.0146	0.0009	0.0052	0.0172	0.0044	0.0146	0.0009	0.0052	0.0172
	CIL	0.2634	0.4794	0.1160	0.2743	0.5192	0.2636	0.4796	0.1160	0.2744	0.5183
	ICR	0.9507	0.9547	0.9467	0.9427	0.9547	0.9513	0.9527	0.9480	0.9420	0.9540
Scenario 2											
30	Bias	0.0593	0.1801	0.0029	0.0908	−0.2892	0.0625	0.1920	0.0034	0.0916	−0.2993
	MSE	0.0997	0.3186	0.0159	0.1133	0.2248	0.1049	0.3375	0.0165	0.1185	0.2341
	CIL	1.1301	2.0703	0.4740	1.1771	1.9724	1.1329	2.0781	0.4744	1.1778	2.0189
	ICR	0.9568	0.9568	0.9290	0.9424	0.9302	0.9557	0.9547	0.9241	0.9389	0.9505
50	Bias	0.0291	0.1007	0.0012	0.0637	−0.1823	0.0263	0.1021	0.0006	0.0637	−0.1826
	MSE	0.0495	0.1678	0.0093	0.0644	0.1246	0.0495	0.1703	0.0094	0.0651	0.1251
	CIL	0.8510	1.5651	0.3658	0.8948	1.5255	0.8517	1.5682	0.3669	0.8962	1.5299
	ICR	0.9545	0.9516	0.9370	0.9302	0.9486	0.9536	0.9536	0.9366	0.9290	0.9612
75	Bias	0.0229	0.0679	0.0026	0.0417	−0.1263	0.0238	0.0673	0.0030	0.0404	−0.1264
	MSE	0.0321	0.1070	0.0060	0.0378	0.0777	0.0326	0.1077	0.0061	0.0381	0.0783
	CIL	0.6900	1.2622	0.2990	0.7203	1.2294	0.6927	1.2644	0.3006	0.7215	1.2356
	ICR	0.9486	0.9593	0.9398	0.9486	0.9575	0.9477	0.9582	0.9425	0.9442	0.9747
100	Bias	0.0172	0.0522	0.0015	0.0302	−0.1007	0.0172	0.0523	0.0015	0.0297	−0.1016
	MSE	0.0240	0.0827	0.0047	0.0266	0.0576	0.0243	0.0830	0.0047	0.0266	0.0575
	CIL	0.5946	1.0875	0.2586	0.6200	1.0706	0.5967	1.0897	0.2599	0.6211	1.0661
	TCR	0.9452	0.9486	0.9351	0.9503	0.9646	0.9432	0.9491	0.9307	0.9524	0.9766
200	Bias	0.0095	0.0325	0.0004	0.0101	−0.0368	0.0102	0.0313	0.0005	0.0099	−0.0363
	MSE	0.0120	0.0391	0.0022	0.0126	0.0277	0.0122	0.0390	0.0022	0.0126	0.0278
	CIL	0.4169	0.7626	0.1823	0.4334	0.7398	0.4193	0.7647	0.1837	0.4347	0.7382
	ICR	0.9463	0.9553	0.9433	0.9530	0.9605	0.9453	0.9542	0.9424	0.9549	0.9749
500	Bias	0.0023	0.0078	−0.0009	0.0063	−0.0068	0.0026	0.0070	−0.0008	0.0061	−0.0052
	MSE	0.0044	0.0145	0.0009	0.0052	0.0125	0.0044	0.0147	0.0009	0.0052	0.0129
	CIL	0.2619	0.4782	0.1151	0.2735	0.4667	0.2636	0.4797	0.1160	0.2744	0.4621
	TCR	0.9563	0.9549	0.9494	0.9460	0.9706	0.9506	0.9526	0.9438	0.9445	0.9675

Table A2. Quality measure of the estimates obtained under the MLE and IMF methods: Scenario 3 and 4.

Scenario 3		**MLE Method**					**IFM Method**				
n	**QM**	$\widehat{\alpha}_1$	$\widehat{\beta}_1$	$\widehat{\alpha}_2$	$\widehat{\beta}_2$	$\widehat{\theta}$	$\widehat{\alpha}_1$	$\widehat{\beta}_1$	$\widehat{\alpha}_2$	$\widehat{\beta}_2$	$\widehat{\theta}$
30	Bias	0.0374	0.1699	0.0068	0.0826	0.0878	0.0429	0.1724	0.0074	0.0821	0.0927
	MSE	0.0868	0.3288	0.0167	0.1187	0.2061	0.0940	0.3299	0.0168	0.1182	0.2156
	CIL	1.1148	2.0670	0.4777	1.1732	2.0815	1.1176	2.0667	0.4767	1.1715	2.0661
	ICR	0.9527	0.9566	0.9310	0.9380	0.9744	0.9533	0.9563	0.9337	0.9352	0.9721
50	Bias	0.0225	0.0900	0.0043	0.0517	0.0400	0.0214	0.0903	0.0047	0.0511	0.0450
	MSE	0.0487	0.1688	0.0099	0.0602	0.1509	0.0486	0.1682	0.0098	0.0598	0.1514
	CIL	0.8496	1.5605	0.3685	0.8906	1.6170	0.8489	1.5606	0.3686	0.8904	1.6097
	ICR	0.9532	0.9546	0.9362	0.9376	0.9489	0.9519	0.9534	0.9385	0.9350	0.9519
75	Bias	0.0175	0.0519	0.0045	0.0300	0.0158	0.0176	0.0518	0.0044	0.0303	0.0138
	MSE	0.0328	0.1063	0.0063	0.0370	0.1133	0.0326	0.1062	0.0062	0.0370	0.1136
	CIL	0.6896	1.2578	0.3009	0.7181	1.3217	0.6899	1.2582	0.3011	0.7184	1.3177
	ICR	0.9420	0.9556	0.9413	0.9365	0.9427	0.9450	0.9552	0.9409	0.9382	0.9402
100	Bias	0.0114	0.0393	0.0031	0.0214	−0.0051	0.0113	0.0392	0.0031	0.0212	−0.0044
	MSE	0.0239	0.0807	0.0047	0.0257	0.0850	0.0238	0.0809	0.0047	0.0258	0.0844
	CIL	0.5940	1.0847	0.2603	0.6189	1.1487	0.5943	1.0851	0.2605	0.6191	1.1441
	ICR	0.9489	0.9482	0.9408	0.9509	0.9509	0.9456	0.9489	0.9415	0.9489	0.9469
200	Bias	0.0079	0.0258	0.0010	0.0082	−0.0005	0.0080	0.0258	0.0009	0.0083	0.0001
	MSE	0.0119	0.0383	0.0022	0.0124	0.0446	0.0119	0.0383	0.0022	0.0124	0.0444
	CIL	0.4184	0.7630	0.1837	0.4342	0.8182	0.4187	0.7633	0.1838	0.4344	0.8158
	ICR	0.9493	0.9513	0.9507	0.9507	0.9407	0.9473	0.9527	0.9540	0.9527	0.9400
500	Bias	0.0025	0.0062	−0.0008	0.0060	−0.0013	0.0025	0.0062	−0.0008	0.0061	−0.0011
	MSE	0.0044	0.0146	0.0009	0.0051	0.0173	0.0044	0.0146	0.0009	0.0051	0.0172
	CIL	0.2634	0.4794	0.1160	0.2743	0.5193	0.2636	0.4796	0.1160	0.2744	0.5184
	ICR	0.9513	0.9540	0.9507	0.9447	0.9573	0.9513	0.9527	0.9513	0.9447	0.9573
Scenario 4											
30	Bias	0.0402	0.1742	0.0083	0.0899	0.2911	0.0463	0.1864	0.0093	0.0926	0.2995
	MSE	0.0844	0.3241	0.0166	0.1144	0.2160	0.0988	0.3481	0.0172	0.1169	0.2258
	CIL	1.1148	2.0707	0.4777	1.1762	1.9978	1.1207	2.0754	0.4778	1.1774	2.0209
	ICR	0.9588	0.9631	0.9262	0.9360	0.9393	0.9508	0.9559	0.9231	0.9364	0.9579
50	Bias	0.0235	0.1047	0.0044	0.0544	0.1941	0.0271	0.1017	0.0049	0.0550	0.1966
	MSE	0.0496	0.1712	0.0099	0.0591	0.1292	0.0515	0.1708	0.0103	0.0603	0.1295
	CIL	0.8488	1.5659	0.3677	0.8918	1.5209	0.8523	1.5661	0.3687	0.8926	1.5376
	ICR	0.9553	0.9553	0.9329	0.9377	0.9436	0.9478	0.9573	0.9307	0.9402	0.9658
75	Bias	0.0167	0.0658	0.0027	0.0365	0.1315	0.0198	0.0598	0.0021	0.0373	0.1324
	MSE	0.0325	0.1136	0.0060	0.0379	0.0813	0.0329	0.1130	0.0061	0.0381	0.0809
	CIL	0.6872	1.2606	0.2992	0.7198	1.2148	0.6910	1.2610	0.3003	0.7215	1.2337
	ICR	0.9400	0.9472	0.9463	0.9418	0.9346	0.9435	0.9496	0.9426	0.9417	0.9664
100	Bias	0.0114	0.0495	0.0025	0.0245	0.0905	0.0104	0.0495	0.0016	0.0261	0.0895
	MSE	0.0238	0.0842	0.0045	0.0250	0.0544	0.0238	0.0850	0.0045	0.0253	0.0543
	CIL	0.5918	1.0856	0.2588	0.6192	1.0481	0.5940	1.0885	0.2601	0.6213	1.0537
	ICR	0.9503	0.9435	0.9409	0.9572	0.9563	0.9452	0.9427	0.9435	0.9536	0.9755
200	Bias	0.0084	0.0297	0.0013	0.0100	0.0387	0.0089	0.0287	0.0011	0.0100	0.0399
	MSE	0.0120	0.0386	0.0022	0.0125	0.0285	0.0120	0.0383	0.0022	0.0125	0.0285
	CIL	0.4166	0.7617	0.1825	0.4337	0.7375	0.4189	0.7638	0.1839	0.4350	0.7376
	ICR	0.9482	0.9512	0.9497	0.9474	0.9684	0.9461	0.9528	0.9506	0.9513	0.9813
500	Bias	0.0024	0.0071	−0.0008	0.0063	0.0040	0.0023	0.0069	−0.0007	0.0061	0.0036
	MSE	0.0044	0.0147	0.0009	0.0051	0.0127	0.0044	0.0148	0.0009	0.0051	0.0129
	CIL	0.2619	0.4781	0.1151	0.2735	0.4650	0.2635	0.4797	0.1161	0.2744	0.4621
	ICR	0.9526	0.9560	0.9464	0.9444	0.9547	0.9517	0.9523	0.9510	0.9462	0.9598

Table A3. Quality measure of the estimates obtained under the MLE and IMF methods: Scenario 5 and 6.

Scenario 5		**MLE Method**					**IFM Method**				
n	**QM**	$\widehat{\alpha}_1$	$\widehat{\beta}_1$	$\widehat{\alpha}_2$	$\widehat{\beta}_2$	$\widehat{\theta}$	$\widehat{\alpha}_1$	$\widehat{\beta}_1$	$\widehat{\alpha}_2$	$\widehat{\beta}_2$	$\widehat{\theta}$
30	Bias	0.1408	0.0687	0.0665	0.0435	−0.0683	0.1419	0.0716	0.0653	0.0424	−0.0720
	MSE	0.3577	0.0572	0.1152	0.0302	0.1945	0.3623	0.0592	0.1125	0.0299	0.1997
	CIL	2.0970	0.8835	1.1390	0.5880	2.0834	2.0962	0.8848	1.1362	0.5866	2.0627
	ICR	0.9664	0.9578	0.9445	0.9391	0.9766	0.9633	0.9541	0.9442	0.9388	0.9755
50	Bias	0.0780	0.0391	0.0420	0.0274	−0.0310	0.0804	0.0397	0.0429	0.0271	−0.0353
	MSE	0.1716	0.0309	0.0613	0.0155	0.1540	0.1740	0.0309	0.0611	0.0154	0.1553
	CIL	1.5612	0.6693	0.8616	0.4460	1.6160	1.5625	0.6696	0.8620	0.4458	1.6061
	ICR	0.9592	0.9529	0.9424	0.9339	0.9487	0.9588	0.9532	0.9455	0.9392	0.9441
75	Bias	0.0552	0.0239	0.0313	0.0149	−0.0176	0.0559	0.0237	0.0319	0.0148	−0.0206
	MSE	0.1094	0.0197	0.0370	0.0094	0.1123	0.1101	0.0196	0.0370	0.0093	0.1113
	CIL	1.2555	0.5397	0.6963	0.3588	1.3253	1.2558	0.5398	0.6967	0.3589	1.3186
	ICR	0.9632	0.9537	0.9469	0.9367	0.9442	0.9592	0.9544	0.9462	0.9374	0.9415
100	Bias	0.0367	0.0174	0.0206	0.0110	−0.0149	0.0374	0.0172	0.0212	0.0109	−0.0166
	MSE	0.0800	0.0148	0.0268	0.0065	0.0872	0.0803	0.0148	0.0267	0.0065	0.0859
	CIL	1.0753	0.4651	0.5980	0.3094	1.1532	1.0757	0.4651	0.5984	0.3095	1.1484
	ICR	0.9631	0.9510	0.9370	0.9544	0.9484	0.9624	0.9490	0.9362	0.9537	0.9477
200	Bias	0.0237	0.0111	0.0065	0.0041	−0.0012	0.0238	0.0110	0.0065	0.0041	−0.0013
	MSE	0.0393	0.0070	0.0118	0.0031	0.0436	0.0393	0.0070	0.0118	0.0031	0.0437
	CIL	0.7532	0.3270	0.4180	0.2171	0.8183	0.7535	0.3271	0.4183	0.2172	0.8160
	ICR	0.9453	0.9533	0.9460	0.9540	0.9400	0.9480	0.9527	0.9487	0.9527	0.9387
500	Bias	0.0069	0.0027	0.0015	0.0031	−0.0010	0.0069	0.0027	0.0015	0.0031	−0.0012
	MSE	0.0147	0.0027	0.0046	0.0013	0.0172	0.0147	0.0027	0.0046	0.0013	0.0172
	CIL	0.4716	0.2055	0.2633	0.1372	0.5192	0.4717	0.2055	0.2634	0.1372	0.5183
	ICR	0.9533	0.9547	0.9507	0.9427	0.9547	0.9533	0.9527	0.9493	0.9420	0.9533
Scenario 6											
30	Bias	0.1775	0.0747	0.0646	0.0428	−0.2893	0.1901	0.0834	0.0664	0.0470	−0.2994
	MSE	0.3854	0.0579	0.1121	0.0281	0.2221	0.4210	0.0615	0.1114	0.0298	0.2341
	CIL	2.1327	0.8858	1.1364	0.5872	2.0001	2.1476	0.8912	1.1369	0.5895	2.0130
	ICR	0.9701	0.9590	0.9369	0.9435	0.9391	0.9684	0.9568	0.9399	0.9389	0.9515
50	Bias	0.0854	0.0433	0.0442	0.0308	−0.1791	0.0858	0.0435	0.0437	0.0320	−0.1838
	MSE	0.1687	0.0306	0.0552	0.0157	0.1200	0.1726	0.0312	0.0559	0.0162	0.1253
	CIL	1.5643	0.6708	0.8600	0.4471	1.5351	1.5669	0.6721	0.8619	0.4482	1.5338
	ICR	0.9639	0.9513	0.9561	0.9318	0.9581	0.9630	0.9526	0.9554	0.9298	0.9649
75	Bias	0.0669	0.0297	0.0356	0.0200	−0.1269	0.0672	0.0288	0.0359	0.0202	−0.1264
	MSE	0.1082	0.0197	0.0365	0.0095	0.0776	0.1091	0.0198	0.0366	0.0095	0.0783
	CIL	1.2607	0.5412	0.6959	0.3600	1.2439	1.2628	0.5419	0.6984	0.3607	1.2354
	ICR	0.9654	0.9592	0.9504	0.9459	0.9690	0.9616	0.9582	0.9538	0.9442	0.9738
100	Bias	0.0486	0.0223	0.0244	0.0150	−0.0983	0.0500	0.0224	0.0234	0.0148	−0.1014
	MSE	0.0807	0.0153	0.0266	0.0067	0.0576	0.0823	0.0152	0.0268	0.0066	0.0575
	CIL	1.0801	0.4660	0.5972	0.3100	1.0676	1.0827	0.4670	0.5992	0.3106	1.0655
	ICR	0.9582	0.9465	0.9406	0.9498	0.9632	0.9608	0.9491	0.9391	0.9525	0.9750
200	Bias	0.0307	0.0139	0.0068	0.0049	−0.0361	0.0294	0.0134	0.0071	0.0048	−0.0359
	MSE	0.0402	0.0072	0.0118	0.0032	0.0277	0.0402	0.0072	0.0119	0.0031	0.0278
	CIL	0.7541	0.3268	0.4160	0.2167	0.7413	0.7557	0.3277	0.4184	0.2173	0.7375
	ICR	0.9480	0.9554	0.9546	0.9524	0.9673	0.9454	0.9543	0.9550	0.9550	0.9742
500	Bias	0.0070	0.0033	0.0011	0.0030	−0.0059	0.0073	0.0029	0.0014	0.0030	−0.0051
	MSE	0.0147	0.0027	0.0045	0.0013	0.0126	0.0147	0.0027	0.0046	0.0013	0.0129
	CIL	0.4702	0.2049	0.2617	0.1367	0.4666	0.4718	0.2056	0.2634	0.1372	0.4620
	ICR	0.9544	0.9558	0.9483	0.9455	0.9741	0.9527	0.9520	0.9466	0.9446	0.9655

References

1. Ferrari, S.; Cribari-Neto, F. Beta regression for modelling rates and proportions. *Appl. Stat.* **2004**, *31*, 799–815. [CrossRef]
2. Branscum, A.J.; Johnson, W.O.; Thurmond, M.C. Bayesian beta regression: Applications to household expenditure data and genetic distance between foot-and-mouth diseases viruses. *Aust. N. Z. J. Stat.* **2007**, *49*, 287–301. [CrossRef]
3. Ospina, R.; Ferrari, S.L.P. Inflated beta distribution. *Stat. Pap.* **2010**, *51*, 111–126. [CrossRef]

4. Ospina, R.; Ferrari, S.L.P. A general class of zero-or-one inflated beta regression models. *Comput. Stat. Data Anal.* **2012**, *56*, 1609–1623. [CrossRef]
5. Paolino, P. Maximum likelihood estimation of models with beta-distributed dependent variables. *Political Anal.* **2001**, *9*, 325–346. [CrossRef]
6. Cribari-Neto, F.; Vasconcellos, K.L.P. Nearly unbiased maximum likelhood estimation for the beta distribution. *J. Stat. Comput. Simul.* **2002**, *72*, 107–118. [CrossRef]
7. Kieschnick, R.; Mccullough, B.D. Regression analysis of variates observed on $(0, 1)$. *Stat. Model.* **2003**, *3*, 193–213. [CrossRef]
8. Vasconcellos, K.L.P.; Cribari-Neto, F. Improved maximum likelihood estimation in a new class of beta regression models. *Braz. J. Probab. Stat.* **2005**, *19*, 13–31.
9. Martínez-Flórez, G.; Bolfarine, H.; Gómez, H.W. Power-models for proportions with zero/one excess. *Appl. Math. Inf. Sci.* **2018**, *12*, 293–303. [CrossRef]
10. Grassia, A. On a family of distributions with argument between 0 and 1 obtained by transformation of the Gamma distribution and derived compound distributions. *Aust. J. Stat.* **1977**, *19*, 108–114. [CrossRef]
11. Gómez-Déniz, E.; Sordo, M.A.; Calderín-Ojeda, E. The Log-Lindley distribution as an alternative to the Beta regression model with applications in insurance. *Insur. Math. Econ.* **2013**, *54*, 49–57. [CrossRef]
12. Mazucheli, J.; Menezes, A.F.B.; Ghitany, M.E. The unit-Weibull distribution and associated inference. *J. Appl. Probab. Stat.* **2018**, *13*, 1–22.
13. Mazucheli, J.; Menezes, A.; Dey, S. The unit-Birnbaum-Saunders distribution with applications. *Chil. J. Stat.* **2018**, *9*, 47–57.
14. Mazucheli, J.; Menezes, A.F.B.; Chakraborty, S. On the one parameter unit-Lindley distribution and its associated regression model for proportion data. *J. Appl. Stat.* **2019**, *49*, 700–714. [CrossRef]
15. Martínez-Flórez, G.; Azevedo-Farias, R.B.; Tovar-Falón, R. New Class of Unit-Power-Skew-Normal Distribution and Its Associated Regression Model for Bounded Responses. *Mathematics* **2022**, *10*, 3035. [CrossRef]
16. Martínez-Flórez, G.; Tovar-Falón, R. New Regression Models Based on the Unit-Sinh-Normal Distribution: Properties, Inference, and Applications. *Mathematics* **2021**, *9*, 1231. [CrossRef]
17. Martínez-Flórez, G.; Gómez, H.W.; Tovar-Falón, R. Modeling Proportion Data with Inflation by Using a Power-Skew-Normal/Logit Mixture Model. *Mathematics* **2021**, *9*, 1989. [CrossRef]
18. Lemonte, A.J.; Moreno-Arenas, G. On a multivariate regression model for rates and proportions. *J. Appl. Stat.* **2019**, *46*, 1084–1106. [CrossRef]
19. Martínez-Flórez, G.; Lemonte, A.J.; Moreno-Arenas, G.; Tovar-Falón, R. The Bivariate Unit-Sinh-Normal Distribution and Its Related Regression Model. *Mathematics* **2022**, *10*, 3125. [CrossRef]
20. Gumbel, E.G. Bivariate exponential distributions. *J. Am. Stat. Assoc.* **1960**, *292*, 698–707. [CrossRef]
21. Gokhale, S.; Khare, M. Vehicle wake factor for heterogeneous traffic in urban environments. *Int. J. Environ. Pollut.* **2007**, *30*, 97–105. [CrossRef]
22. Martínez-Flórez, G.; Azevedo-Farias, R.B.; Tovar-Falón, R. An Exponentiated Multivariate Extension for the Birnbaum-Saunders Log-Linear Model. *Mathematics* **2022**, *10*, 1299. [CrossRef]
23. Freeman, D.G. Drunk Driving Legislation and Traffic Fatalities: New Evidence on BAC 08 Laws. *Contemp. Econ. Policy* **2007**, *25*, 293–308. [CrossRef]
24. Martínez-Flórez, G.; Vergara-Cardozo, S.; Tovar-Falón, R.; Rodriguez-Quevedo, L. The Multivariate Skewed Log-Birnbaum–Saunders Distribution and Its Associated Regression Model. *Mathematics* **2023**, *11*, 1095. [CrossRef]
25. Almetwally, E.M.; Muhammed, H.Z.; El-Sherpieny, E.S.A. Bivariate Weibull Distribution: Properties and Different Methods of Estimation. *Ann. Data. Sci.* **2020**, *7*, 163–193. [CrossRef]
26. Sklar, A. Random variables, joint distributions, and copulas. *Kybernetica* **1973**, *9*, 449–460.
27. Nelsen, R.B. *An Introduction to Copulas*, 2nd ed.; Springer: New York, NY, USA, 2010.
28. R Development Core Team. *R: A Language and Environment for Statistical Computing*; R Foundation for Statistical Computing: Vienna, Austria, 2021. Available online: http://www.R-project.org (accessed on 31 July 2021).
29. Basu, A. Bivariate failure rate. *J. Am. Stat. Assoc.* **1971**, *66*, 103–104. [CrossRef]
30. Joe, H. Asymptotic efficiency of the two-stage estimation method for copula-based models. *J. Multivar. Anal.* **2005**, *94*, 401–419. [CrossRef]
31. Kundu, D.; Gupta, A.K. Bayes estimation for the Marshall-Olkin bivariate Weibull distribution. *Comput. Stat. Data Anal.* **2013**, *57*, 271–281. [CrossRef]
32. Zhou, S.; Xu, A.; Tang, Y.; Shen, L. Fast Bayesian Inference of Reparameterized Gamma Process With Random Effects. *IEEE Trans. Reliab.* **2023**, 1–14. [CrossRef]
33. Peralta, D.; DeOliveira, R.P.; Achcar, J.A. A hierarchical Bayesian analysis for bivariate Weibull distribution under left-censoring scheme. *J. Appl. Stat.* **2023**, 1–20. [CrossRef]
34. Arnold, B.C.; Castillo, E.; Sarabia, J.M. Conditionally specified multivariate skewed distributions. *Sankhya Indian J. Stat. Ser. A* **2002**, *64*, 206–226.

35. Akaike, H. A new look at statistical model identification. *IEEE Trans. Autom. Control* **1974**, *19*, 716–722. [CrossRef]
36. Schwarz, G. Estimating the dimension of a model. *Ann. Stat.* **1978**, *6*, 461–464. [CrossRef]