mathematics

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Prabhakar Functions of Le Roy Type: Inequalities and Asymptotic Formulae

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Abstract: In this paper, the four-index generalization of the classical Le Roy function is considered on a wider set of parameters and its order and type are given. Letting one of the parameters take non-negative integer values, a family of functions with such a type of index is constructed. The behaviour of these functions is studied in the complex plane $\mathbb{C}$ and in different domains thereof. First, several inequalities are obtained in $\mathbb{C}$, and then they are modified on its compact subsets as well. Moreover, an asymptotic formula is proved for 'large' values of the indices of these functions. Additionally, the multi-index analogue of the abovementioned four-index Le Roy type function is considered and its basic properties are obtained. Finally, several special cases of the two functions under consideration are discussed.

Keywords: special functions; Le Roy function; Mittag-Leffler function; entire functions; inequalities; asymptotic formula

MSC: 30D20; 33E20; 33E12; 30A10; 30E15; 30D15

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Citation: Paneva-Konovska, J. Prabhakar Functions of Le Roy Type: Inequalities and Asymptotic Formulae. *Mathematics* **2023**, *11*, 3768. https://doi.org/10.3390/math11173768

Academic Editor: Marius Radulescu

Received: 27 July 2023
Revised: 29 August 2023
Accepted: 30 August 2023
Published: 1 September 2023

1. Introduction

The special function $F^{(\gamma)}$, defined in the whole complex plane $\mathbb{C}$ by the power series

$$F^{(\gamma)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(k+1)]^{\gamma}}, \quad z \in \mathbb{C}, \tag{1}$$

is known as the Le Roy function. It was named after the French mathematician Édouard Louis Emmanuel Julien Le Roy (1870–1954) who introduced it in [1]. He himself used it in studying the asymptotics of the analytic continuation of the sum of power series. In his paper [2], Kolokoltsov used this function (with $\gamma = 1/2$), namely,

$$R(z) = F^{(1/2)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}}, \tag{2}$$

in evaluating the solution of initial stochastic DE. As he himself commented therein, 'the function $R(z)$ plays the same role for stochastic equations as the exponential function and Mittag-Leffler functions for deterministic equations'.

We have to admit that, probably because of its purely theoretical origin, the Le Roy function remains relatively little known and used. However, due to various problems of analysis and probability theory, interest in the Le Roy function has recently been revived. These are problems related to integral-differential operators involving fractional Hadamard derivatives or hyper-Bessel operators, finding of solutions of some integral-differential equations involving such operators by using operational methods, probability density functions of probability distributions, solutions of initial stochastic differential equations, and so on. Because of growing interest and possible further applications, many new generalizations of it have appeared.

Recently, Gerhold [3] and Garra and Polito [4], independently of each other, have introduced the new special function $F_{\alpha,\beta}^{(\gamma)}$, generalizing in such a manner the Mittag-Leffler function $E_{\alpha,\beta}$:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \tag{3}$$

with two indices (parameters) by adding an index γ, namely,

$$F_{\alpha,\beta}^{(\gamma)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(\alpha k + \beta)]^{\gamma}}, \quad z \in \mathbb{C}, \tag{4}$$

for complex values of the variable z and positive values of the parameters α, β, γ. In fact, Gerhold [3] found an asymptotic formula that holds true in different sectors of the complex plane and thus extended the work of Le Roy [1]. Garra and Polito [4] have dealt with some operators involving Hadamard derivatives.

At a later stage, imposing more general conditions on the parameters, its definition was extended by Garrappa, Rogosin and Mainardi [5]. However, to ensure the existence of the coefficients $[\Gamma(\alpha k + \beta)]^{-\gamma}$ in the expansion (4), the values of the parameters should be constrained. In this direction, the restriction $\alpha,\ \beta \in \mathbb{C}, \gamma > 0$ would be quite natural. It turns out that (4) is an entire function of the complex variable z for all parameter values such that [5]

$$\alpha,\ \beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \gamma > 0. \tag{5}$$

It should be noted that if the condition $\gamma > 0$ is changed with $\Re(\gamma) > 0$ and additionally $\Re(\alpha\gamma) > 0$, (4) remains an entire function, i.e., the condition (5) can be replaced from

$$\alpha,\ \beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\gamma) > 0,\ \Re(\alpha\gamma) > 0. \tag{6}$$

Here, it is appropriate to note that the conditions $\Re(\alpha) > 0$ and $\Re(\gamma) > 0$ are imposed in connection with the expression $[\Gamma(\alpha k + \beta)]^{-\gamma}$. Furthermore, $\Re(\gamma) > 0$ ensures the existence of $[\Gamma(\alpha k + \beta)]^{-\gamma}$ without any additional conditions for β. However, if $\Re(\alpha k + \beta) > 0$, then the constraint $\Re(\gamma) > 0$, concerning γ, is no longer needed. The condition $\Re(\alpha\gamma) > 0$ provides the convergence of the series (4) in the whole complex plane.

The function $F_{\alpha,\beta}^{(\gamma)}$ is said to be a Le Roy-type function [5] (it is also known as a Mittag-Leffler function of Le Roy type). It is clear that (4) is a natural generalization of the Le Roy function (1).

In their paper [6], Tomovski and Mehrez, in studying Mathieu series and their generalizations and associated probability distributions, introduced the following generalization of the function (4)

$$F_{\alpha,\beta;\tau}^{(\gamma)}(z) = \sum_{k=0}^{\infty} \frac{(\tau)_k}{[\Gamma(\alpha k + \beta)]^{\gamma}} \frac{z^k}{k!}, \quad z \in \mathbb{C},\ \tau \in \mathbb{C}, \tag{7}$$

adding one more index. Here, $(\tau)_k$ denotes the Pochhammer symbol ([7] [2.1.1])

$$(\tau)_0 = 1, \quad (\tau)_k = \tau(\tau+1)\ldots(\tau+k-1).$$

Obviously, if $\tau = 1$, then the function (7) coincides with (4). However, if $\gamma = 1$, then (7) is the so-called Mittag-Leffler function $E_{\alpha,\beta}^{\tau}$ with three parameters [8]:

$$E_{\alpha,\beta}^{\tau}(z) = \sum_{k=0}^{\infty} \frac{(\tau)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad z \in \mathbb{C},\ \tau \in \mathbb{C}, \tag{8}$$

(also known as the Prabhakar function). Being close to the Mittag-Leffler function, this function is widely used in the modelling of various processes (for its applications in diffusion and random search processes, random search and stochastic resetting processes, many examples of anomalous diffusion and other fields, see, e.g., the recently published books [9,10]).

It is worth noting that if $\gamma = m$ is a positive integer, then (4) becomes a case of the so-called multi-index (2m-index) Mittag-Leffler function $E_{(\alpha_i),(\beta_i)}$, and (7) becomes a case of the 3m-index Mittag-Leffler function $E^{(\tau_i),m}_{(\alpha_i),(\beta_i)}$ (for more details for domains of parameters, properties and applications of these functions, see, e.g., the books [10,11] and also the papers [12,13]. Their definitions are given below, namely,

$$E_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1)\dots\Gamma(\alpha_m k + \beta_m)}, \quad z, \alpha_i, \beta_i \in \mathbb{C}, \Re(\alpha_i) > 0, \tag{9}$$

respectively,

$$E^{(\tau_i),m}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{(\tau_1)_k \dots (\tau_m)_k}{\Gamma(\alpha_1 k + \beta_1)\dots\Gamma(\alpha_m k + \beta_m)} \frac{z^k}{(k!)^m}, \quad z, \tau_i, \alpha_i, \beta_i \in \mathbb{C}, \Re(\alpha_i) > 0. \tag{10}$$

More specifically,

$$F^{(m)}_{\alpha,\beta}(z) = E_{(\alpha_i),(\beta_i)}(z), \quad F^{(m)}_{\alpha,\beta;\tau}(z) = E^{(\tau_i),m}_{(\alpha_i),(\beta_i)}(z), \tag{11}$$

with parameters

$$\alpha_i = \alpha, \ \beta_i = \beta, \text{ and } \tau_i = \tau \quad (i = 1, \dots, m). \tag{12}$$

Let us also note that, similarly to the Mittag-Leffler functions, more general analogues of (4) and (7) with multi-indices were defined and studied by Rogosin and Dubatovskaya [14] and Kiryakova and Paneva-Konovska [15] almost simultaneously.

The function (7) is called a Le Roy-type function (with four indices), or generalized Le Roy-type function, or Prabhakar function of Le Roy type. It was studied in detail by Paneva-Konovska [16], and its basic properties were obtained. It was proved that if $\alpha, \ \beta, \ \tau \in \mathbb{C}, \ \Re(\alpha) > 0, \ \gamma > 0$, then this function is an entire function and its order and type and different integral representations of Mellin–Barnes type were obtained. It is also established that the nth integer derivatives of the function (4) are Le Roy-type functions with four indices. Analogical relations for the integrals and derivatives of fractional orders have been obtained as well. Further, the resulting derivatives of the nth order are used for representing the three-index function (4) in a Taylor series at an arbitrary point $z_0 \in \mathbb{C}$.

The main objective of this paper is to study the function (7) and its multi-index analogue (44) under more extended parameter domains and to establish their basic properties, such as order and type and related asymptotic inequalities, and also different representations of them depending on the parameters. We also intend to provide upper estimations in different domains of the complex plane and asymptotic formulae for 'large values' of the parameters. Our motivation to consider this topic is provoked by the increasing interest in the Le Roy function and its generalizations and their possible applications.

The paper is organized in the following way. The definitions and historical overview are given in Section 1. The Prabhakar function of Le Roy type is considered in Section 2 under a more extended domain of the parameters. Its order and type are established. In Section 3, families of such a type of function are considered and their representations are given, depending on the parameters. They are used in Section 4 for obtaining inequalities and asymptotic formulae in the complex plane. Section 5 is devoted to the multi-index Prabhakar function of Le Roy type which is a 4m-index analogue of (7). In Section 6, several specific cases are considered. In the concluding Section 7, the validity of the results obtained in the previous sections is discussed for them.

2. Order and Type of the Four-Parametric Le Roy-Type Function

A careful follow-up of the proof in [16] shows that if the condition $\gamma > 0$ is replaced by conditions $\Re(\gamma) > 0$ and $\Re(\alpha\gamma) > 0$, then all the coefficients in (7) exist and it remains an entire function. In what follows, we consider the function (7) under the weakened condition

$$\alpha,\ \beta,\ \gamma,\ \tau \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\gamma) > 0,\ \Re(\alpha\gamma) > 0, \tag{13}$$

thus making the results stronger than the ones obtained in [16]. The order and type of the function (7) are given below under the condition (13), imposed on the parameters.

Theorem 1. *Let α, β, τ, and γ be the parameters satisfying the condition (13) and let τ be neither a negative integer nor zero. Then, (7) is an entire function.*

Proof. According to Cauchy–Hadamard's formula, the radius of convergence of the series (7) is $R \geq 0$, and

$$R = \frac{1}{\limsup_{k\to\infty} |c_k|^{\frac{1}{k}}}, \quad \text{where} \quad c_k = \frac{(\tau)_k}{[\Gamma(\alpha k+\beta)]^{\gamma}}\,\frac{1}{k!}. \tag{14}$$

After equivalent analytical manipulations and applying both, Stirling's asymptotic formula for the Γ-function and Γ-functions quotient property (see e.g., [11] [Rem. 6.5, (iii)] for them), namely:

$$\Gamma(z+\alpha) \sim \sqrt{(2\pi)}z^{z+\alpha-\frac{1}{2}}\exp(-z), \quad |\arg(z+\alpha)| < \pi, \tag{15}$$

for large values of z, respectively,

$$\frac{\Gamma(z)}{\Gamma(z+\alpha)} = O\left(\frac{1}{z^{\alpha}}\right), \quad |\arg(z)| < \pi,\ |\arg(z+\alpha)| < \pi, \tag{16}$$

and, in view of (14) along with the relation $(\tau)_k = \frac{\Gamma(\tau+k)}{\Gamma(\tau)}$, we obtain

$$c_k = \frac{1}{\Gamma(\tau)}\,\frac{\Gamma(\tau+k)}{\Gamma(k+1)}\,\frac{1}{[\Gamma(\alpha k+\beta)]^{\gamma}}. \tag{17}$$

From here, it follows that

$$(|c_k|)^{\frac{1}{k}} \sim \frac{1}{|\sqrt{2\pi}^{\,\gamma/k}|}\,\frac{|k^{(\tau-1)/k}|}{|\Gamma^{1/k}(\tau)|}\,\frac{\exp(\Re(\alpha\gamma))}{|\,(\alpha k)^{\alpha\gamma+\beta\gamma/k-\gamma/(2k)}\,|} \to 0, \text{ when } k\to\infty. \tag{18}$$

Due to (14) and (18), the radius of convergence of the series in the Formula (7) is $R = \infty$, which means that the function (7) is an entire function. □

Let us recall ([17] [Chapter 7, §1]) that an important characteristic of a given entire function f is the maximum of its modulus $M(r) = \max\limits_{|z|=r} |f(z)|$. More precisely, if there exists a positive number μ such that $M(r) < \exp(r^{\mu})$, for all r sufficiently large, then f is said to be a function of a finite order $\rho = \inf \mu \geq 0$. Further, if f has a finite order ρ and there exists a positive number κ such that $M(r) < \exp(\kappa r^{\rho})$, then f is said to be a function of a finite type. The infimum of this κ for which the above inequality is valid for r sufficiently large, is denoted by σ and is called type of f, namely, $\sigma = \inf \kappa \geq 0$. In view of the definitions recalled, the following asymptotic inequality holds true

$$|\ M(r)\ | < \exp((\sigma+\varepsilon)r^{\rho}), \quad \forall\, r > r_0(\varepsilon) > 0, \tag{19}$$

for each $\varepsilon > 0$ and r_0 sufficiently large.

The order and type of the function (7) are given by the following theorem.

Theorem 2. *Let α, β, τ, and γ be the parameters satisfying the condition (13) and let τ be neither a negative integer nor zero. Then, the order ρ and type σ of the entire function (7) are connected by the relations*

$$\frac{1}{\rho} = \Re(\alpha\,\gamma), \tag{20}$$

respectively,

$$\frac{1}{\sigma} = |(\alpha\rho)^{\,\alpha\gamma\rho}|, \tag{21}$$

i.e.,

$$\frac{1}{\sigma} = \frac{|\alpha^{\alpha\gamma}|^{\,1/\Re(\alpha\gamma)}}{\Re(\alpha\gamma)}. \tag{22}$$

Proof. In order to calculate the order ρ of (7), we use Stirling's formula in the logarithmic form:

$$\ln\Gamma(z) = \left(z - \frac{1}{2}\right)\ln z + \frac{1}{2}\ln(2\pi) + O\left(\frac{1}{z}\right), \tag{23}$$

the relation $\ln|z| = \Re(\ln z)$, and the well-known formula, expressing the order ρ of the entire function $\sum_{k=0}^{\infty} c_k z^k$, namely:

$$\rho = \limsup_{k\to\infty} \frac{k\ln k}{\ln(1/|c_k|)}. \tag{24}$$

Further, in view of (17), the denominator in (24) becomes:

$$\ln\frac{1}{|c_k|} = \ln|\Gamma(\tau)| + \ln\Gamma(k+1) - \Re(\ln\Gamma(\tau+k)) + \Re(\gamma\,\ln\Gamma(\alpha k+\beta)), \tag{25}$$

and additionally writing the functions $\ln\Gamma(k+1)$, $\ln(\Gamma(\tau+k))$, and $\ln(\Gamma(\alpha k+\beta))$ in the form (23), we get to the relation:

$$\ln\frac{1}{|c_k|} = \ln|\Gamma(\tau)| + \left(k+\frac{1}{2}\right)\ln\Gamma(k+1) - \Re\left[\left(\tau+k-\frac{1}{2}\right)\ln\Gamma(\tau+k)\right] \tag{26}$$
$$+\Re\left[\gamma\left(\alpha k+\beta-\frac{1}{2}\right)\ln(\alpha k+\beta) + \frac{\gamma}{2}\ln(2\pi)\right] + O\left(\frac{1}{k}\right).$$

Hence,

$$\frac{1}{\rho} = \lim_{k\to\infty}\frac{\ln(1/|c_k|)}{k\ln k} = 1 + \Re(\alpha\gamma) - 1 = \Re(\alpha\gamma),$$

which proves the formula (20).

Further, the type of the entire function $\sum_{k=0}^{\infty} c_k z^k$ of order ρ, is expressed by the formula

$$(\sigma e\rho)^{1/\rho} = \limsup_{k\to\infty}\left(k^{1/\rho}|c_k|^{1/k}\right). \tag{27}$$

The above equality, due to (20) and (18), produces the following result:

$$k^{1/\rho}|c_k|^{1/k} \sim \frac{k^{\,\Re(\alpha\gamma)}}{|\sqrt{2\pi}^{\,\gamma/k}|}\,\frac{|k^{(\tau-1)/k}|}{|\Gamma^{1/k}(\tau)|}\,\frac{\exp(\Re(\alpha\gamma))}{|\,(\alpha k)^{\alpha\gamma+\beta\gamma/k-\gamma/(2k)}\,|}, \text{ when } k\to\infty.$$

Now, by taking the limit in the above formula, the relation below follows

$$\lim_{k\to\infty} k^{1/\rho}|c_k|^{1/k} = \lim_{k\to\infty}\left(k^{\,\Re(\alpha\gamma)}\,\frac{\exp(\Re(\alpha\gamma))}{|(\alpha k)^{\alpha\gamma}|}\right),$$

and the last limit becomes

$$\lim_{k\to\infty} k^{1/\rho}|c_k|^{1/k} = \frac{\exp(\Re(\alpha\gamma))}{|\alpha^{\alpha\gamma}|}.$$

Then, the above equality along with (27) implies:

$$(\sigma e \rho)^{\Re(\alpha\gamma)} = \frac{\exp(\Re(\alpha\gamma))}{|\alpha^{\alpha\gamma}|}, \quad \text{i.e.,} \quad (\sigma\rho)^{\Re(\alpha\gamma)} = \frac{1}{|\alpha^{\alpha\gamma}|}.$$

From here, taking into account (20), the relation (21) immediately follows. The relation (22) follows after the calculations written below, namely,

$$\frac{1}{\sigma} = |\alpha^{\alpha\gamma\rho}||\rho^{\alpha\gamma\rho}| = |\alpha^{\alpha\gamma\rho}|\, \rho^{\rho\Re(\alpha\gamma)} = \frac{|\alpha^{\alpha\gamma}|^{1/\Re(\alpha\gamma)}}{\Re(\alpha\gamma)},$$

which ends the proof of the theorem. □

By the general theory of entire functions, in particular according to the formula (19), an upper asymptotic estimate is valid for the entire function (7). Namely, the following corollary can be formulated.

Corollary 1. *Let the parameters α, β, τ, and γ satisfy the condition* (13), *ε be an arbitrary positive number, and let τ be neither a negative integer nor zero. Then, there exists a positive number $r_0(\varepsilon) > 0$, depending only on ε, such that the following asymptotic estimation*

$$|\, F^{(\gamma)}_{\alpha,\beta;\tau}(z)\, | < \exp((\sigma+\varepsilon)|z|^{\rho}), \quad \forall\, |z| > r_0(\varepsilon) > 0, \tag{28}$$

holds true, with ρ and σ like in (20) *and* (21).

3. Auxiliary Statements

In the recent papers [18–20], Paneva-Konovska considered series in systems of the three-parametric Le Roy-type functions and some of their special cases, as representatives of the Special Functions of Fractional Calculus ([21]). Different representations, inequalities, and asymptotic formulae, concerning these systems, were obtained and discussed there. They were further used in order to study the convergence of such series in the complex plane $\mathbb{C}$ in proving Cauchy–Hadamard, Abel, and Tauberian-type theorems. Such a type of problem was also considered for other type of functions. Among them are the Bessel and Mittag-Lefler-type functions (for details, see e.g., [11]).

To be able to prove similar convergence theorems for series in the four-parametric Le Roy-type functions (7), we need first some inequalities in the complex plane, as well as on its compact subsets, and asymptotic formulae for 'large' values of indices of these functions.

Remark 1. *In what follows, we will use the notations $\mathbb{Z}^-$ (resp. $\mathbb{N}$) for the set of negative (resp. positive) integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$.*

Consider now the Prabhakar function of Le Roy type (7), satisfying the condition (13), for indices of the kind $\beta = n$; $n = 0, 1, 2, \ldots$, namely, the family of functions:

$$F^{(\gamma)}_{\alpha,n;\tau}(z) = \sum_{k=0}^{\infty} \frac{(\tau)_k}{[\Gamma(\alpha k+n)]^{\gamma}} \frac{z^k}{k!},\ \alpha,\gamma,\tau \in \mathbb{C},\ \Re(\alpha) > 0, \Re(\gamma) > 0, \Re(\alpha\gamma) > 0,\ n \in \mathbb{N}_0. \tag{29}$$

Remark 2. *For a given number τ, it is possible that some coefficients in (29) become equal to zero, that is, there exists a number $p \in \mathbb{N}_0$, such that the representation (29) can be written as follows:*

$$F^{(\gamma)}_{\alpha,n;\tau}(z) = z^p \sum_{k=p}^{\infty} \frac{(\tau)_k}{[\Gamma(\alpha k+n)]^{\gamma}} \frac{z^{k-p}}{k!}. \tag{30}$$

Further, let us set

$$a_k = \frac{1}{[\Gamma(\alpha k+n)]^{\gamma}}, \quad b_k = (\tau)_k, \quad c_k = a_k b_k / k!, \quad k = 0, 1, 2, \dots. \tag{31}$$

Depending on τ, we consider three main cases separately. The first of them is $\tau \notin \mathbb{Z}_0^-$.

Lemma 1. *Let z, α, and γ satisfy the conditions in (13), $\tau \in \mathbb{C}$, but $\tau \notin \mathbb{Z}_0^-$. Then the Formula (30) holds true with:*

1. $p = 0$, *for* $n \in \mathbb{N}$,
2. $p = 1$, *for* $n = 0$.

Proof. Obviously, in the first case, $b_k \neq 0$ and $\alpha k + n$ are neither negative integers nor zero. Because of that, $a_k \neq 0$ and therefore $c_k \neq 0$ for all the values of k. In the second case, $n = 0$ and therefore $\alpha k + n = \alpha k$, which means that $c_0 = 0$ but $c_k \neq 0$ for all the natural values of k, since only a_0 equals zero. □

Remark 3. *Actually, in the case $\tau \in \mathbb{Z}_0^-$, the functions (29) reduce to polynomials of the kind (30) of power $m = -\tau$, and then their representation can be rewritten in the alternative forms:*

$$F^{(\gamma)}_{\alpha,n;-m}(z) = z^p \sum_{k=p}^{m} \frac{(-m)_k}{[\Gamma(\alpha k+n)]^{\gamma}} \frac{z^{k-p}}{k!} = z^p \sum_{k=p}^{m} (-1)^k \binom{m}{k} \frac{z^{k-p}}{[\Gamma(\alpha k+n)]^{\gamma}}. \tag{32}$$

The second case is that τ is a negative integer.

Lemma 2. *Let z, α, and γ satisfy the conditions in (13), $\tau \in \mathbb{Z}^-$, $m = -\tau$. Then, (29) can be expressed by the formula (32) with the following values of p:*

1. $p = 0$, *for* $n \in \mathbb{N}$,
2. $p = 1$, *for* $n = 0$.

Proof. The numbers a_k are given by (31) and their values are the same as in the proof of Lemma 1. Moreover,

$$b_k = (-m)_k = -m(-m+1)\dots(-m+k-1) = (-1)^k m \dots (m-k+1) = (-1)^k \binom{m}{k}.$$

Then $b_k \neq 0$ only for $k \leq m$ and hence $c_k = 0$ for all $k > m$, and therefore the values of p are the same as required. □

Lemma 3. *Let z, α, and γ satisfy the conditions in (13), and $\tau = 0$. Then:*

1. $F^{(\gamma)}_{\alpha,n;0}(z) = \dfrac{1}{[\Gamma(n)]^{\gamma}}$, *for* $n \in \mathbb{N}$,
2. $E^{(\gamma)}_{\alpha,0;0}(z) = 0$.

Proof. It automatically follows, taking in view that $b_k = 0$ for all $k \in \mathbb{N}$. □

Remark 4. *Let us mention that if τ is a non-positive integer, as is seen above, the functions (7) reduce to polynomials, but when $\tau \notin \mathbb{Z}_0^-$, they are entire functions of z.*

The above lemmas show that the functions $F^{(\gamma)}_{\alpha,n,\tau}(z)$ can be written in the following form

$$F^{(\gamma)}_{\alpha,n;\tau}(z) = \frac{(\tau)_k}{[\Gamma(\alpha k+n)]^\gamma}\, z^p \left(1+\theta^\gamma_{\alpha,n}(z)\right), \tag{33}$$

with

$$\theta^\gamma_{\alpha,n;\tau}(z) = \sum_{k=p+1}^{\infty} \frac{(\tau)_k}{(\tau)_p}\, \frac{[\Gamma(\alpha p+n)]^\gamma}{[\Gamma(\alpha k+n)]^\gamma}\, \frac{z^{k-p}}{k!} \quad \text{for} \quad \tau \in \mathbb{C}\setminus\mathbb{Z}_0^-, \tag{34}$$

and, respectively:

$$\theta^\gamma_{\alpha,n;-m}(z) = \sum_{k=p+1}^{m} \frac{(-m)_k}{(-m)_p}\, \frac{[\Gamma(\alpha p+n)]^\gamma}{[\Gamma(\alpha k+n)]^\gamma}\, \frac{z^{k-p}}{k!} \tag{35}$$

$$= \sum_{k=p+1}^{m} \frac{(-1)^{k-p}\binom{m}{k}}{\binom{m}{p}}\, \frac{[\Gamma(\alpha p+n)]^\gamma}{[\Gamma(\alpha k+n)]^\gamma}\, z^{k-p}, \quad \text{for} \quad \tau=-m,\ m\in\mathbb{N}.$$

Remark 5. *The parameter τ in the representations (33)–(35) is nonzero, and the parameter p is determined by Lemmas 1 and 2. More precisely, $p=0$ for all the natural values of n and $p=1$ for $n=0$. If $\tau=0$, then $\theta^{(\gamma)}_{\alpha,n;0} = \frac{1}{[\Gamma(n)]^\gamma}$ for $n\in\mathbb{N}$ and $\theta^{(\gamma)}_{\alpha,0;0}=0$, according to Lemma 3.*

4. Inequalities and Asymptotic Formulae

Our goal is to find upper estimations of the moduli of the entire functions $\theta^\gamma_{\alpha,n;\tau}(z)$. To this end, we transform the expressions in the equalities (34) and (35), to the following forms:

$$\theta^\gamma_{\alpha,n;\tau}(z) = \frac{[\Gamma(\alpha p+n)]^\gamma}{[\Gamma(\alpha(p+1)+n)]^\gamma} \sum_{k=p+1}^{\infty} \widetilde{g}_{n,k}\, \frac{(\tau)_k}{(\tau)_p}\, \frac{z^{k-p}}{k!} \tag{36}$$

and, respectively,

$$\theta^\gamma_{\alpha,n;-m}(z) = \frac{[\Gamma(\alpha p+n)]^\gamma}{[\Gamma(\alpha(p+1)+n)]^\gamma} \sum_{k=p+1}^{m} (-1)^{k-p}\, \widetilde{g}_{n,k}\, \frac{\binom{m}{k}}{\binom{m}{p}}\, z^{k-p}, \tag{37}$$

with

$$\widetilde{g}_{n,k} = (g_{n,k})^\gamma, \quad g_{n,k} = \frac{\Gamma(\alpha(p+1)+n)}{\Gamma(\alpha k+n)} \qquad (n\in\mathbb{N}_0). \tag{38}$$

Theorem 3. *Let $\alpha,\ \gamma$ satisfy the conditions in (13) and $\tau\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then, there exists an entire function φ such that*

$$\left|\theta^\gamma_{\alpha,n;\tau}(z)\right| \;\le\; \frac{|[\Gamma(\alpha p+n)]^\gamma|}{|[\Gamma(\alpha(p+1)+n)]^\gamma|}\; \varphi(|z|;\alpha,\gamma,\tau), \tag{39}$$

for all the values of $z\in\mathbb{C}$.

Proof. In order to find such a function φ and to prove the inequality (39), we estimate the function (36) beginning with the values of (38). Since

$$g_{0,\,k} = \frac{\Gamma(\alpha(p+1))}{\Gamma(\alpha k)},$$

$$g_{n,k} = \frac{\Gamma(\alpha(p+1))}{\Gamma(\alpha k)}\, \frac{(\alpha(p+1))}{(\alpha k)} \cdots \frac{(\alpha(p+1)+n-1)}{(\alpha k)+n-1} \quad \text{for } n\in\mathbb{N},$$

and due to the following inequality

$$\frac{|\alpha(p+1)|}{|\alpha k|} \cdots \frac{|\alpha(p+1)+n-1|}{|\alpha k+n-1|} \leq 1,$$

we obtain that

$$|g_{n,k}| \leq \frac{|\Gamma(\alpha(p+1))|}{|\Gamma(\alpha k)|}, \quad \text{for all the possible values of } n \text{ and } k.$$

Finally, by taking

$$\varphi(z;\alpha,\gamma,\tau) = \sum_{k=p+1}^{\infty} \frac{|[\Gamma(\alpha(p+1))]^{\gamma}|}{|[\Gamma(\alpha k)]^{\gamma}|} \frac{|(\tau)_k|}{|(\tau)_p|} \frac{z^{k-p}}{k!},$$

the proof of the theorem ends. □

Theorem 4. *Let $\alpha,\ \gamma$ satisfy the conditions in (13), $\tau \in \mathbb{Z}^-$, and $m = -\tau$. Then, there exists a polynomial $\widetilde{\varphi}$ such that for all $z \in \mathbb{C}$:*

$$\left|\theta^{\gamma}_{\alpha,\,n;\,-m}(z)\right| \ \leq\ \frac{|[\Gamma(\alpha p+n)]^{\gamma}|}{|[\Gamma(\alpha(p+1)+n)]^{\gamma}|}\ \widetilde{\varphi}(|z|;\alpha,\gamma,m). \tag{40}$$

Proof. Denoting

$$\widetilde{\varphi}(z;\alpha,\gamma,m) = \sum_{k=p+1}^{m} \frac{|[\Gamma(\alpha(p+1))]^{\gamma}|}{|[\Gamma(\alpha k)]^{\gamma}|} \frac{\binom{m}{k}}{\binom{m}{p}} z^{k-p},$$

and following the idea of the proof of Theorem 3, we complete the proof. The details are omitted. □

The inequalities from Theorems 2 and 3 are valid in the whole complex plane. Moreover, in the case that the analytic functions φ and $\tilde{\varphi}$ are considered only on a given compact subset of $\mathbb{C}$, then they are modulo bounded. In this case, the inequalities (39) and (40) can be combined. Namely, the following remark can be written.

Remark 6. *If $\alpha,\ \gamma,\ \tau$ satisfy the conditions in (13) and K is a compact subset of the complex plane $\mathbb{C}$, then there exists a constant $C = C(K)$ such that*

$$\left|\theta^{\gamma}_{\alpha,\,n;\,\tau}(z)\right| \ \leq\ C\, \frac{|[\Gamma(\alpha p+n)]^{\gamma}|}{|[\Gamma(\alpha(p+1)+n)]^{\gamma}|}\,, \quad \forall z \in K,\ n = 0,1,2,\ldots, \tag{41}$$

for $\tau \neq 0$ and p defined by Lemmas in Section 3.

Further, an asymptotic formula for 'large' values of the indices n is proved.

Theorem 5. *Let $\alpha,\ \gamma,\ \tau$ satisfy the conditions in (13), $n \in \mathbb{N}_0,\ \tau \neq 0$, and $\theta^{\gamma}_{\alpha,n,\tau}$ be given by the formulae (36)–(38). Then, the Prabhakar functions of Le Roy type (29) satisfy the following asymptotic formulae*

$$F^{(\gamma)}_{\alpha,n;\tau}(z) = \frac{(\tau)_p}{[\Gamma(\alpha p+n)]^{\gamma}}\, z^p \left(1+\theta^{\gamma}_{\alpha,\,n;\tau}(z)\right) \text{ and } \theta^{\gamma}_{\alpha,\,n;\tau}(z) \to 0 \text{ as } n \to \infty \quad (z \in \mathbb{C}), \tag{42}$$

with the corresponding p, depending on τ. Moreover, the convergence of $\theta^{\gamma}_{\alpha,\,n;\tau}$ is uniform on the compact subsets of the complex plane $\mathbb{C}$, and

$$\theta^{\gamma}_{\alpha,\,n;\tau}(z) \;=\; O\left(\frac{1}{n^{\Re(\alpha\gamma)}}\right) \quad (n\;\in\mathbb{N}). \tag{43}$$

Proof. Using the formulae (33), (41), and the Γ-functions quotients formula (16), the proof is evident. The details are omitted. □

Remark 7. *According to the asymptotic formula (42), it follows there exists a natural number M such that the functions $F^{(\gamma)}_{\alpha,\,n;\tau}$ have no zeros at all for $n > M$, possibly except for the zero.*

5. Multi-Index Analogue of the Prabhakar Function of Le Roy Type

In the recent paper [15], a multi-index version of the Prabhakar function of Le Roy type (7), which is also a generalization of (10), has been introduced and studied by Kiryakova and Paneva-Konovska. This function (multi-index Mittag-Leffler-Prabhakar functions of Le Roy type and abbrev. as multi-MLPR) has been defined analogously to (10), taking $4m$ parameters (α_i, β_i, τ_i and γ_i for $i = 1, \ldots, m$) instead of four: α, β, τ and γ, namely:

$$\begin{aligned}\mathbb{F}_m(z) &:= \mathbb{F}^{\gamma_i;m}_{\alpha_i,\beta_i;\tau_i}(z)\\ &= \sum_{k=0}^{\infty}\frac{(\tau_1)_k\ldots(\tau_m)_k}{[\Gamma(\alpha_1 k+\beta_1)]^{\gamma_1}\ldots[\Gamma(\alpha_m k+\beta_m)]^{\gamma_m}}\cdot\frac{z^k}{(k!)^m}\\ &= \sum_{k=0}^{\infty} c_k\, z^k,\;\text{ with } c_k = \prod_{i=1}^{m}\left\{\frac{\Gamma(k+\tau_i)}{\Gamma(k+1)}\cdot\frac{1}{\Gamma(\tau_i)}\cdot\frac{1}{[\Gamma(\alpha_i k+\beta_i)]^{\gamma_i}}\right\},\end{aligned} \tag{44}$$

with $4m$ parameters $\alpha_i > 0, \beta_i > 0, \gamma_i > 0, \tau_i > 0,\ \forall i = 1, ..., m$. It has been established that (44) is an entire function and its order and type have been determined.

As already mentioned in the Introduction, almost in parallel, Rogosin and Dubatovskaya [14] have studied a multi-index analog of (4) with $3m$ parameters, denoted by them with $F^{(\gamma)_m}_{(\alpha,\beta)_m}$. It can be considered as the above functions (44) when $\tau_i = 1, \forall i = 1, ..., m$, i.e.,

$$F^{(\gamma)_m}_{(\alpha,\beta)_m} = \mathbb{F}^{\gamma_i;m}_{\alpha_i,\beta_i;1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{[\Gamma(\alpha_1 k+\beta_1)]^{\gamma_1}\ldots[\Gamma(\alpha_m k+\beta_m)]^{\gamma_m}}. \tag{45}$$

Extending the domains of the parameters, we consider the functions (44) under the weakened conditions:

$$\alpha_i,\ \beta_i,\ \gamma_i,\ \tau_i\in\mathbb{C},\ \Re(\alpha_i)>0,\ \Re(\gamma_i)>0,\ \sum_{i=1}^{m}\Re(\alpha_i\gamma_i)>0. \tag{46}$$

It turns out, that under the new conditions, the function (44) is also an entire function Moreover, if at least one of τ_i is either a negative integer or zero, then (44) is reduced to a finite sum (polynomial); say if

$$\exists i_0:\ -\tau_{i_0} = M\in\mathbb{N}_0 \quad (1\le i_0\le m),$$

then, similarly to the statements in Section 3, the function (44) has the form:

$$\mathbb{F}(z) = \mathbb{F}^{\gamma_i;m}_{\alpha_i,\beta_i;\tau_i}(z) = \sum_{k=0}^{M}\frac{(\tau_1)_k\ldots(\tau_m)_k}{[\Gamma(\alpha_1 k+\beta_1)]^{\gamma_1}\ldots[\Gamma(\alpha_m k+\beta_m)]^{\gamma_m}}\cdot\frac{z^k}{(k!)^m}. \tag{47}$$

Oppositely, if none of the parameters τ_i is a non-positive integer, the proof is very similar to this one for the positive parameters, given in [15]. Its order and type are given below.

Theorem 6. *The multi-index MLPR-function* (44)*, with the condition* (46) *on the parameters and* $\tau_i \notin \mathbb{Z}_0^-$*, is an entire function of the complex variable z of order* ρ *and type* σ*, expressed as follows:*

$$\frac{1}{\rho} = \Re(\alpha_1\gamma_1) + \cdots + \Re(\alpha_m\gamma_m), \tag{48}$$

and

$$\frac{1}{\sigma} = \prod_{i=1}^{m} |(\rho\alpha_i)^{\rho\alpha_i\gamma_i}|, \tag{49}$$

that is,

$$\frac{1}{\sigma} = \frac{\left|\alpha_1^{\alpha_1\gamma_1} \cdots \alpha_m^{\alpha_m\gamma_m}\right|^{1/(\Re(\alpha_1\gamma_1)+\cdots+\Re(\alpha_m\gamma_m))}}{\Re(\alpha_1\gamma_1) + \cdots + \Re(\alpha_m\gamma_m)}. \tag{50}$$

Proof. In the part concerning (48) and (49), it goes in the same way as in [15] for the positive parameters. The details are omitted here. The relation (50) follows replacing ρ from (48) into (49) and taking in view that

$$|(\rho\alpha_i)^{\rho\alpha_i\gamma_i}| = |\alpha_i^{\rho\alpha_i\gamma_i}||\rho^{\rho\alpha_i\gamma_i}| = |\alpha_i^{\rho\alpha_i\gamma_i}|\,\rho^{\rho\Re(\alpha_i\gamma_i)},$$

and also $\rho\,(\Re(\alpha_1\gamma_1) + \cdots + \Re(\alpha_m\gamma_m)) = 1.$ □

Remark 8. *Note that the order and type of the multi-MLPR-function* (44)*, given by* (48)–(50)*, do not depend on the parameters* β_i *and* τ_i*. Moreover, if all the* $\tau_i = 1$*, they coincide with the results concerning the order and type of the function* $F^{(\gamma)_m}_{(\alpha,\beta)_m}$*, obtained in ([14] [Theorem 1]); if* $m = 1$*, they produce the order and type of the Prabhakar functions of Le Roy type* (7) *with four parameters. If the parameters are positive, the formulae* (48)–(50) *lead to the results obtained in ([15] [Theorem 2]).*

According to the general theory of entire functions, again according to the Formula (19), an upper asymptotic estimate holds true for the entire function (44). Namely, the following corollary can be formulated.

Corollary 2. *Let the parameters* α_i*,* β_i*,* τ_i*, and* γ_i *satisfy the condition* (13) *and let* τ_i *be neither negative integers nor zero. Then, for any* $\varepsilon > 0$ *there exists a positive number* $r_0(\varepsilon) > 0$*, depending only on* ε*, such that the asymptotic estimate*

$$|\mathbb{F}_m(z)| < \exp((\sigma+\varepsilon)|z|^{\rho}), \quad \forall\,|z| \geq r_0(\varepsilon) > 0, \tag{51}$$

holds, with ρ *and* σ *like in* (48) *and* (49)*, and* $r_0(\varepsilon)$ *being sufficiently large.*

6. Special Cases of the Multi-MLPR-Function

Let us summarize that the multi-MLPR-function (44) (considered as above under the conditions (46) on the parameters), being more general than multi-index function (45), Prabhakar functions of Le Roy Type (7), and multi-index Mittag-Leffler functions (9) and (10), leads to them for special choices of the parameters. For example, if $\gamma_i = 1$ or $\tau_i = 1$, $\forall i = 1, ..., m$, then (44) becomes (10), respectively (45); if $\gamma_i = \tau_i = 1$, the function (44) is the $2m$-index Mittag-Lefler function (9):

$$\mathbb{F}^{1;m}_{\alpha_i,\beta_i;\tau}(z) = E^{(\tau_i),m}_{(\alpha_i),(\beta_i)}(z), \quad \mathbb{F}^{\gamma_i;m}_{\alpha_i,\beta_i;1}(z) = F^{(\gamma)_m}_{(\alpha,\beta)_m}, \quad \mathbb{F}^{1;m}_{\alpha_i,\beta_i;1}(z) = E^{m}_{(\alpha_i),(\beta_i)}(z). \tag{52}$$

An interesting special case is mentioned by Pogány as an example only, in the paper [22], devoted to the search for an integral form of the Le Roy-type function (4). It is a special function of the form (his denotations are kept here):

$$F^{\alpha,\beta}_{(p,q;r,s)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(pk+q)]^{\alpha}\,[\Gamma(rk+s)]^{\beta}}. \tag{53}$$

This function illustrates the case $m = 2$ of the special functions (44), which we consider here; namely, it is:

$$\mathbb{F}^{(\alpha,\beta);2}_{(p,r),(q,s);(1,1)} = \sum_{k=0}^{\infty} \frac{(1)_k(1)_k}{[\Gamma(pk+q)]^{\alpha}\,[\Gamma(rk+s)]^{\beta}} \cdot \frac{z^k}{(k!)^2}.$$

Then, in this case, we obtain that it is an entire function with:

$$\rho = \frac{1}{\Re(p\alpha + r\beta)}, \quad \sigma = \frac{\Re(p\alpha + r\beta)}{\left|p^{p\alpha} \cdot r^{r\beta}\right|^{1/\Re(p\alpha+r\beta)}}. \tag{54}$$

The Bessel function J_ν of the first kind and its numerous generalizations can also be represented by the cases of multi-index Mittag-Leffler functions (see e.g., [11] for such type of representations) and thus, they can also be considered as special cases of the multi-MLPR-function (44). Further, if $m = 1$, then (44) is the four-parametric Prabhakar function of Le Roy type, given by (7). On the other hand, the function (7) produces different special functions as particular cases. Some of them are written below. For example, if γ is a positive integer, say $\gamma = m \in \mathbb{N}$, the function (7) is a multi-index Mittag-Leffler function of the kind (10), or (in particular, if additionally $\tau = 1$), it is of the kind (9). Both examples are expressed trough the multi-index Mittag-Leffler functions by the Formula (11) and their parameters are given in (12). Further, for $\tau = 1$ the function (7) is a Le Roy-type function given by (4), for $\gamma = 1$, it is Prabhakar's $E^{\tau}_{\alpha,\beta}$ function (8), namely:

$$F^{(\gamma)}_{\alpha,\beta;1}(z) = F^{(\gamma)}_{\alpha,\beta}(z), \quad F^{(1)}_{\alpha,\beta;\tau}(z) = E^{\tau}_{\alpha,\beta}(z). \tag{55}$$

Finally, the Mittag-Leffler functions $E_{\alpha,\beta}$, E_α, the classical Le Roy function (1) (and therefore Kolokoltsov' s function R), are also obtained as particular cases, i.e.,:

$$F^{(1)}_{\alpha,\beta;1}(z) = E_{\alpha,\beta}(z),\ F^{(1)}_{\alpha,1;1}(z) = E_{\alpha}(z),\ F^{(\gamma)}_{1,1;1}(z) = F^{(\gamma)}(z),\ R(z) = F^{(1/2)}(z). \tag{56}$$

7. Conclusions

In this paper, the Prabhakar function (7) of Le Roy type as well as its $4m$-index analogue (44) are considered. Their basic properties, such as orders and types are established, and also the corresponding asymptotic inequalities (28) and (51), resulting from the general theory of entire functions. Moreover, by choosing non-negative integer values of the beta parameter of (7), we construct the family (29). For its functions, different representations and inequalities are found, depending on the parameters, and an asymptotic formula for large values of the integer parameters as well. Some of the inequalities proved are valid in the whole complex plane, and others in compact subsets of it. Additionally, numerous special cases of the four-parametric Le Roy type function and its multi-index analogue are listed.

In conclusion, let us emphasize that the results established in Sections 2–4 can be automatically applied to the special cases (55) and (56). The results in Section 5 concerning the order and type of (44), as well as the inequality (51), are applicable to all the special cases discussed above in Section 6.

Funding: This research received no financial funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

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