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Two-Dimensional Moran Model: Final Altitude and Number of Resets

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Abstract: In this paper, we consider a two-dimension symmetric random walk with reset. We give, in the first part, some results about the distribution of every component. In the second part, we give some results about the final altitude Z_n. Finally, we analyse the statistical properties of N_n^X, the number of resets (the number of returns to state 1 after n steps) of the first component of the random walk. As a principal tool in these studies, we use the probability generating function.

Keywords: random structure; random walk; probability generating function; height

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1. Introduction

The two-dimensional Moran model is a simple discrete process used in many fields to describe the evolution of two discrete random walks in each unit of time.

Moran's random walk model can be applied in the field of renewable energy. Many renewable energy situations can be modelled as Moran's random walk. This modelling has the advantage of minimizing expenses to guarantee the proper functioning of such a system by avoiding surprise breakdowns. In certain ecosystems, and more particularly in certain tropical forests, different species with the same ecological requirements coexist in the same environment. For example, some forests have more than a hundred different tree species on one hectare. To explain this astonishing diversity, scientists have constructed models in which community composition is solely based on the stochastic dispersal of individuals. The mathematical model studied in our paper is in line with this. It was suggested by M. Kalyuzhni [1] in an article where he justifies its relevance. It is known as Moran's model in a random environment. It is therefore a question of studying a process of birth and death which takes into account environmental hazards (climates, diseases, etc.) that randomly favour or disadvantage certain species.

In this work, our goal is to study the statistical properties of some discrete statistics such as the limiting distributions, the mean and the variance of two discrete walks in our model, X_n and Y_n, and the maximum of two walks Z_n (called final altitude) using very elegant tools called probability generating functions. Also, we analyse the return time, N_n^X, of the random walk X_n and find its mean and variance.

In the literature, these properties of discrete random walks are studied in one dimension and in higher dimensions via the kernel method and singularity analysis (see [2,3]). For example, for one dimension, if one focuses on articles which play a role in our analysis, Banderier and Flajolet have proven in [2] that the limiting distribution of the final altitude of a random meander of length n converges to a Rayleigh distribution (drift $\delta = 0$) and normal distribution ($\delta > 0$). Furthermore, the height of discrete bridges/meanders/excursions for bounded discrete walks has been analysed by Banderier and Nicodème [4]. Also, Aguech, Althagafi , and Banderier in [5] have studied the height of walks with resets and the Moran Model. Similar extremal parameters were studied for trees in [6,7], and by Gafni [8] for the asymptotic distribution of the length of the longest run of consecutive equal parts.

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Finally, Banderier and Wallner treated the number of catastrophes of a random excursion of size n, which converges to a Gaussian, Rayleigh, or discrete distribution depending on the drift (see Theorem 4.12 in [9]).

For higher dimensions, still in connection with our model, we can mention the two-dimensional Moran model, investigated by Abdelkader and Althagafi in [10], where they showed that the age of each component converges to a shifted geometric distribution in law. Furthermore, the limiting distribution for the lifetime of an individual converges to a (shifted) geometric distribution in law, proven by Itoh and Mahmoud [11]. Itoh, Mahmoud, and Takahashi in [12] proved that the wavelength converges, in distribution, to a convolution of geometric random variables. Other papers are related to the Moran process (in biology and population genetics); see, e.g., [13–15]. The models in the papers [16,17] can be modelled as a Moran process.

This paper is organized as follows. In Section 2, we present our model in detail and define some statistics. In Section 3, in order to obtain the probability generating functions of the random walks, X_n and Y_n, we give some recursive equations for the sequence of multivariate polynomials in our model. We show that the two Moran random walks X_n and Y_n converge to shifted geometric distributions in law asymptotically. Also, we calculate their means and variances using the probability generating function of the random walks X_n and Y_n. In Section 4, we study the statistical properties of the maximum age, Z_n, between two random walks X_n and Y_n. In Section 5, we analyse the number of returns up to time n, N_n^X. We start with a simulation of the random walk N_n^X with different lengths: 100, 1000, 10,000, and 100,000 according to the initial probability q using R software. Also, we obtain the distribution and the probability generating function. In Section 6, we determine the general probability generating function of the two-dimensional random walk, which can be useful to extract the distribution of the height H_n. In Section 7, we present some conclusions concerning our results and some perspectives. In Appendix A, we give some technical lemmas useful for studying the final altitude.

2. Definitions and Presentation of the Model

In this section, we introduce our model: the two-dimensional symmetric Moran model. We define some statistics such as the final altitude, the height, and the return time. We present an elegant tool called the probability generating function, which plays an important role in finding the statistical properties of discrete random walks.

2.1. Presentation of the Model

Our model is presented as follows: At time 0, the random walk starts from the origin. After one unit of time, (a) the first random walk shifts by one positive unit, but the second random walk returns to 1 with probability $1-q$; (b) the second walk shifts by one positive unit, but the first random walk returns to 1 with same probability $1-q$; (c) the two random walks shift by one positive unit with probability $2q-1$, where $q \in (1/2, 1)$. Mathematically, our model is given by the following system: for all $n \in \mathbb{N}$

$$(X_{n+1}, Y_{n+1}) = \begin{cases} (1+X_n, 1), & \text{with probability } 1-q, \\ (1, 1+Y_n), & \text{with probability } 1-q, \\ (1+X_n, 1+Y_n), & \text{with probability } 2q-1, \end{cases} \quad (1)$$

where $q \in (1/2, 1)$. The process (X_n, Y_n) is considered a stochastic process with dimension two defined on the state space $S^2 = \{0, \ldots, n\}^2$, and started from the initial state $(X_0, Y_0) = (0, 0)$.

2.2. Definitions

In this subsection, we present some definitions concerning some discrete random walks.

1. The final altitude, Z_n, of the two-dimensional Moran random walk is defined by

$$\forall n \in \mathbb{N}, \quad Z_n = \max(X_n, Y_n),$$

2. The height, H_n, of the two-dimensional Moran random walk is defined by

$$\forall n \in \mathbb{N}, \quad H_n = \max(Z_0, Z_1, Z_2, \ldots, Z_n),$$

3. The return time, N_n^X, of the Moran random walk $X_.$ equals the number when $X_.$ returns to 1.

where $Z_0 = 0$, $H_0 = 0$ and $N_0^X = N_1^X = 0$.

Our goal is to study the statistical properties of the following discrete random walks: X_n, Y_n, Z_n, H_n, N_n^X. Precisely, we want to find their limiting distributions, their means, and their variances. As mentioned before, as a tool, we use the probability generating function.

Definition 1. *Let U be a discrete random variable with distribution $\mathbb{P}(U = r) = p_r$, $r \in \mathbb{N}$. The probability generating function, denoted by G, of the variable U is defined by*

$$G_U(u) = \mathbb{E}(u^U) = \sum_{r=0}^{\infty} u^r \, p_r,$$

for all $u \in \mathbb{R}$ such that $|u| \leq 1$.

Due to their numerous uses, probability generating functions constitute an elegant tool to study the characteristic of a distribution. Mainly, the probability density functions associated with discrete stochastic processes and their moments can be obtained from the derivatives of the probability generating function. In fact, the mean and the variance of the process (the first and second centred moments of the distribution of U) are related to the derivatives of the probability generating function at $u = 1$. More precisely, the next folklore lemma explains this link.

Lemma 1 ([3]). *Let G_U be the probability generating function of a the discrete random process U. For all $k \in \mathbb{N}$, the k^{th} factorial moment of U is given by*

$$\frac{\partial^k G_U(u)}{\partial^k u}\Big|_{u=1} = \mathbb{E}\Big[U(U-1)(U-2)\ldots(U-k+1)\Big].$$

In addition, if the limits of $\frac{\partial G_U(u)}{\partial u}$ and $\frac{\partial^2 G_U(u)}{\partial^2 u}$ exist at $u = 1$, then we have the following two important equations, which are related to the mean and variance of U and $G_U(u)$:

$$\mathbb{E}\Big(U\Big) = \frac{\partial G_U(u)}{\partial u}\Big|_{u=1} \quad \text{and} \quad \mathbb{V}ar\Big(U\Big) = \frac{\partial^2 G_U(u)}{\partial^2 u}\Big|_{u=1} + \mathbb{E}\Big(U\Big) - \mathbb{E}\Big(U\Big)^2. \tag{2}$$

3. Distributions of X_n and Y_n

In this section, firstly, we derive a conditional probability of the position of the process defined in (1) at time $(n+1)$ given that we know its position at time n. Secondly, we determine the sequence of multivariate polynomials, denoted by $f_n(x, y) = \mathbb{E}\Big(x^{X_n} y^{Y_n}\Big)$, and find the recursive equation related to this sequence between two consecutive times n and $n+1$. Finally, we show that the two symmetric random Moran walks X_n and Y_n converge to the shifted geometric distribution, and we compute their means and variances. Using the definitions in Section 2, we define the joint probability mass function of (X_n, Y_n). Denote, for all $r, s \in \{0, \ldots, n+1\}$,

$$\mathbb{P}_{n+1}(r, s) = \mathbb{P}(X_{n+1} = r, Y_{n+1} = s), \tag{3}$$

this is the probability that the process is in the position $(r,\, s)$ at time $n+1$.

We start this section with a technical lemma. It involves a recursive equation between the probability of our model for two consecutive times, n and $n+1$, to be used in Theorem 1. It is based on the following conditional probability:

Lemma 2. *For all $n \geq 2$, we have*

$$\mathbb{P}_{n+1}(r,s) = \begin{cases} (2q-1)\,\mathbb{P}_n(r-1,s-1), & \text{if } r \geq 2,\ s \geq 2, \\ (1-q)\displaystyle\sum_{l=1}^{n} \mathbb{P}_n(l,s-1), & \text{if } r = 1,\ s \geq 2, \\ (1-q)\displaystyle\sum_{k=1}^{n} \mathbb{P}_n(r-1,k), & \text{if } s = 1,\ r \geq 2. \end{cases}$$

Proof. This proof is based on the utility of the conditional probability that the Moran walks X and Y are aged r and s at time $n+1$, given that they are aged l and k at time n, and then

1. For $r \geq 2$ and $s \geq 2$, we have

$$\begin{aligned} \mathbb{P}_{n+1}(r,s) =& \mathbb{P}(X_{n+1} = r, Y_{n+1} = s) \\ =& \mathbb{P}(X_{n+1} = r, Y_{n+1} = s, X_n = r-1, Y_n = s-1) \\ =& \mathbb{P}(X_{n+1} = r, Y_{n+1} = s | X_n = r-1, Y_n = s-1) \\ & \times \mathbb{P}(X_n = r-1, Y_n = s-1) = (2q-1)\,\mathbb{P}_n(r-1,s-1), \end{aligned}$$

2. For $r = 1$ and $s \geq 2$, we have:

$$\begin{aligned} \mathbb{P}_{n+1}(1,s) =& \sum_{l=1}^{n} \mathbb{P}(X_{n+1} = 1, Y_{n+1} = s, X_n = l, Y_n = s-1) \\ =& \sum_{l=1}^{n} \mathbb{P}(X_{n+1} = 1, Y_{n+1} = s | X_n = l, Y_n = s-1) \\ & \times \mathbb{P}(X_n = l, Y_n = s-1) \\ =& (1-q)\sum_{l=1}^{n} \mathbb{P}_n(l,s-1), \end{aligned}$$

3. For $r \geq 2$ and $s = 1$, we have by symmetry:

$$\mathbb{P}_{n+1}(r,1) = (1-q)\sum_{k=1}^{n} \mathbb{P}_n(r-1,k).$$

□

Remark 1. *Consider two consecutive times n and $(n+1)$, r and s days, starting from 1 to $(n+1)$, and the ages of two components X and Y are equal at time $n+1$, respectively. We give some comments on the different cases of the age of two components X and Y at time $n+1$:*

1. *If $r \geq 2$ and $s \geq 2$, then the probability that X and Y are aged r and s at time $(n+1)$ is equal to the probability that (X,Y) is aged $(r-1,s-1)$ days at the preceding time n multiplied by $(2q-1)$.*
2. *If $r = 1$ and $s \geq 2$, the probability that (X,Y) is aged $(1,s)$ days equals $(1-q)$ multiplied by the sum of all probabilities of X and Y that are aged l and $s-1$ days at time n, where l starts from 1 to n, respectively.*
3. *If $r \geq 2$ and $s = 1$, the probability that (X,Y) is aged $(r,1)$ days equals $(1-q)$ multiplied by the sum of all probabilities of X and Y that are aged $r-1$ and k days at time n, where k starts from 1 to n, respectively.*

Next, we define the sequence of multivariate polynomials $f_n(x, y)$ (for $n \in \mathbb{N}$) associated with the two-dimensional process (X_n, Y_n), by

$$f_n(x,y) = \mathbb{E}\left(x^{X_n} y^{Y_n}\right) = \sum_{r=0}^{n} \sum_{s=0}^{n} x^r\, y^s \mathbb{P}_n(r,s). \tag{4}$$

The coefficient of $x^r\, y^s$ in $f_n(x,y)$ represents the probability that the position of the two-dimensional process (X, Y) is at level (r,s) at time n.

When $x = y = 1$, we have the special case

$$f_n(1,1) = \sum_{r=0}^{n} \sum_{s=0}^{n} \mathbb{P}_n(r,s) = 1. \tag{5}$$

By Equation (4) and Lemma 2, we deduce a recursive equation related to $f_{n+1}(x,y)$, $f_n(x,1)$, $f_n(1,y)$, and $f_n(x,y)$. It is presented in the next proposition.

Proposition 1. *For all $(x,y) \in \mathbb{R}^2$, the explicit expression of the sequence of multivariate polynomials $f_n(x,y)$ holds the following recurrence:*

$$\begin{cases} f_{n+1}(x,y) = (1-q)\, x\, y\, f_n(x,1) + (1-q)\, x\, y\, f_n(1,y) + (2q-1)\, x\, y\, f_n(x,y), \\ f_0(x,y) = P_0(0,0) = 1. \end{cases} \tag{6}$$

Proof. Using Equation (4) and for all $n \geq 1$, the function $f_{n+1}(x,y)$ can be developed as

$$f_{n+1}(x,y) = \underbrace{\sum_{r=2}^{n+1} \sum_{s=2}^{n+1} x^r\, y^s\, \mathbb{P}_{n+1}(r,s)}_{A} + y \underbrace{\sum_{r=1}^{n+1} x^r\, \mathbb{P}_{n+1}(r,1)}_{B} + x \underbrace{\sum_{s=1}^{n+1} y^s\, \mathbb{P}_{n+1}(1,s)}_{C}. \tag{7}$$

Due to Lemma (2), we can compute A, B, and C as follows:

$$\begin{aligned} A =& (2q-1) \sum_{r=2}^{n+1} \sum_{s=2}^{n+1} x^r\, y^s\, \mathbb{P}_n(r-1,s-1) = (2q-1) \sum_{r=1}^{n} \sum_{s=1}^{n} x^{r+1}\, y^{s+1}\, \mathbb{P}_n(r,s) \\ =& (2q-1)\, x\, y \sum_{r=0}^{n} \sum_{s=0}^{n} x^r\, y^s\, \mathbb{P}_n(r,s) = (2q-1)\; x\, y\, f_n(x,y), \end{aligned} \tag{8}$$

$$C = (1-q) \sum_{s=1}^{n+1} \sum_{l=1}^{n} y^s \mathbb{P}_n(l,s-1) = y(1-q) \sum_{s=0}^{n} \sum_{l=0}^{n} y^s \mathbb{P}_n(r,s) = (1-q) y f_n(1,y), \tag{9}$$

finally, via symmetry, we deduce

$$B = (1-q) x f_n(x,1). \tag{10}$$

We obtain Equation (6) by combining Equations (7) and (10). □

In this part, we study some statistical characteristics such that the probability generating function, the asymptotic distribution, the mean, and the variance of the final altitude of each component X_n and Y_n at time n can be obtained. Precisely, we start by finding the probability generating function of each component $f_n^X(x) = \mathbb{E}\left[x^{X_n}\right] = f_n(x,1)$ and $f_n^Y(y) = \mathbb{E}\left[y^{Y_n}\right] = f_n(1,y)$. Next, we show that the final altitude of the two random walks X_n and Y_n converge to a shifted geometric distribution asymptotically. Finally, we finish this section by computing the mean and the variance of the two random walks. The following theorem introduces the probability generating function and the asymptotic limit distributions of X_n and Y_n.

Theorem 1. *X_n and Y_n converge to a shifted geometric distribution with parameter $(1-q)$ in law asymptotically, with the same probability generating function given by the following: for all $n \geq 0$*

$$f_n(x) = \mathbb{E}\left(x^{X_n}\right) = (q\,x)^n + (1-q)\frac{x - q^n x^{n+1}}{1 - q\,x}, \tag{11}$$

for all $x \in \mathbb{R}$, such that $|1 - q\,x| < 1$.

Proof. Using Equations (5) and (6) with $y = 1$, we obtain

$$\begin{aligned} f_n(x,1) &= (1-q)\,x\,f_{n-1}(x,1) + (1-q)\,x\;f_{n-1}(1,1) + (2q-1)\,x\,f_{n-1}(x,1) \\ &= (1-q)\,x\,f_{n-1}(1,1) + q\,x\,f_{n-1}(x,1) \\ &= (1-q)\,x + q\,x\,f_{n-1}(x,1). \end{aligned}$$

We iterate $f_n(x,1)$ n times and we obtain

$$\mathbb{E}\left(x^{X_n}\right) = f_n(x,1) = (1-q)\,x\sum_{s=0}^{n-1}(q\,x)^s + (q\,x)^n = (q\,x)^n + (1-q)\frac{x - q^n\,x^{n+1}}{1 - q\,x}.$$

Hence, passing to the limit of $f_n(x,1)$, we the have

$$\lim_{n\to\infty} f_n(x,1) = \lim_{n\to\infty}\left[(q\,x)^n + (1-q)\frac{x - (1-q)^n\,x^{n+1}}{1 - q\,x}\right] = \frac{(1-q)x}{1-qx}, \tag{12}$$

it is exactly the generating function of a shifted geometric distribution with parameter $1-q$.

By symmetry, we have

$$\lim_{n\to\infty} f_n(1,y) = \lim_{n\to\infty}\left[(q\,y)^n + (1-q)\frac{y - (1-q)^n\,y^{n+1}}{1 - q\,y}\right] = \frac{(1-q)y}{1-qy}.$$

□

Theorem 1 leads us to find the explicit expressions of the means and variances of X_n and Y_n, which depend on the first and the second derivatives of the probability generating function $f_n(u)$, $u = x,\,y$.

Corollary 1. *The means and the variances of X_n and Y_n are given by*

$$\mathbb{E}\left(X_n\right) = \mathbb{E}\left(Y_n\right) = \frac{1-q^n}{1-q}, \tag{13}$$

and

$$\mathbb{V}ar\left(X_n\right) = \mathbb{V}ar\left(Y_n\right) = \frac{1}{(1-q)^2}\left(q - q^n\left\{q^n + (2\,n\,q - 1)\,(1-q)\right\}\right). \tag{14}$$

Proof. Calculating the first derivative of $f_n(u,1)$ defined in Equation (11) with respect to u,

$$\begin{aligned} \frac{\partial f_n(u)}{\partial u} =& \frac{\partial}{\partial u}\left\{q^n\,u^n + (1-q)\,\frac{u - q^n\,u^{n+1}}{1 - q\,u}\right\} \\ =& n\,q^n\,u^{n-1} + (1-q)\,\frac{1 - (n+1)\,q^n\,u^n}{1 - q\,u} + (1-q)\,q\,\frac{u - q^n\,u^{n+1}}{(1-q\,u)^2}, \end{aligned} \tag{15}$$

evaluating with $u = 1$,

$$\begin{aligned}\frac{\partial f_n(u)}{\partial u}\Big|_{u=1} =& n\,q^n + (1-q)\,\frac{1-(n+1)\,q^n}{1-q} + (1-q)\,q\,\frac{1-q^n}{(1-q)^2} \\ =& n\,q^n + 1-(n+1)\,q^n + \frac{q\,(1-q^n)}{1-q} = \frac{1-q^n}{1-q}.\end{aligned}$$

Using Equation (2), we obtain

$$\mathbb{E}\Big(X_n\Big) = \mathbb{E}\Big(Y_n\Big) = \frac{\partial f_n(u)}{\partial u}\Big|_{u=1} = \frac{1-q^n}{1-q}.$$

To derive the variance of X_n and Y_n, we need to define the following sequences of functions:

$$\begin{aligned}K_n(u) =& \frac{1-q^n\,u^n}{1-q\,u}, \\ L_n(u) =& \frac{u^n}{1-q\,u}, \\ M_n(u) =& \frac{u-q^n\,u^{n+1}}{(1-q\,u)^2}.\end{aligned}$$

Observe that

$$\frac{\partial f_n(u)}{\partial u} = n\,q^n\,u^{n-1} + (1-q)\,K_n(u) - n\,(1-q)\,q^n\,L_n(u) + (1-q)\,q\,M_n(u), \quad (16)$$

using Equation (16) and computing the second derivative of $f_n(u)$ with respect u, one has

$$\begin{aligned}\frac{\partial^2 f_n(u)}{\partial u^2} =& n\,q^n\,\frac{\partial(u^{n-1})}{\partial u} + (1-q)\,\frac{\partial K_n(u)}{\partial u} \\ & - n\,(1-q)\,q^n\,\frac{\partial L_n(u)}{\partial u} + (1-q)\,q\,\frac{\partial M_n(u)}{\partial u}.\end{aligned} \quad (17)$$

The first derivatives of the functions $K_n(u)$, $L_n(u)$, and $M_n(u)$ are given by

$$\begin{aligned}\frac{\partial K_n(u)}{\partial u} =& \frac{-nq^n\,u^{n-1}}{1-q\,u} + q\frac{1-q^n\,u^n}{(1-q\,u)^2}, \\ \frac{\partial L_n(u)}{\partial u} =& \frac{n\,u^{n-1}}{1-q\,u} + q\frac{u^n}{(1-q\,u)^2}, \\ \frac{\partial M_n(u)}{\partial u} =& \frac{1-(n+1)q^n\,u^n}{(1-q\,u)^2} + 2q\,\frac{u-q^n\,u^{n+1}}{(1-q\,u)^3}.\end{aligned}$$

Let $u=1$ and multiply by $(1-q)$, $n(1-q)q^n$, and $(1-q)\,q$, respectively, we can obtain

$$\begin{aligned}(1-q)\frac{\partial K_n(u)}{\partial u}\Big|_{u=1} =& (1-q)\Big(\frac{-nq^n}{1-q} + q\frac{1-q^n}{(1-q)^2}\Big) = -nq^n + \frac{q\,(1-q^n)}{1-q}, \\ n(1-q)q^n\,\frac{\partial L_n(u)}{\partial u}\Big|_{u=1} =& n(1-q)q^n\left(\frac{n}{1-q} + \frac{q}{(1-q)^2}\right) = n^2q^n + \frac{n\,q^{n+1}}{1-q}, \\ (1-q)\,q\frac{\partial M_n(u)}{\partial u}\Big|_{u=1} =& q\,(1-q)\Big(\frac{1-(n+1)q^n}{(1-q)^2} + 2q\,\frac{1-q^n}{(1-q)^3}\Big) \\ =& \frac{q\Big(1-(n+1)q^n\Big)}{(1-q)} + \frac{2\,q^2\,(1-q^n)}{(1-q)^2}.\end{aligned}$$

Replacing, in Equation (17), the first derivatives of $K_n(u)$, $L_n(u)$, and $M_n(u)$, with respect the variable u, with 1, we obtain

$$\begin{aligned}\frac{\partial^2 f_n(u)}{\partial u^2}\Big|_{u=1} =& n(n-1)q^n - nq^n + \frac{q\,(1-q^n)}{1-q} - n^2 q^n - \frac{n\,q^{n+1}}{1-q} \\ &+ \frac{q-(n+1)q^{n+1}}{(1-q)} + \frac{2\,q^2\,(1-q^n)}{(1-q)^2} \\ =& -2\,n\,q^n + \frac{q\,(1-q^n)}{1-q} - \frac{2n\,q^{n+1}}{1-q} + \frac{q}{1-q} - \frac{q^{n+1}}{1-q} + \frac{2\,q^2\,(1-q^n)}{(1-q)^2}.\end{aligned} \tag{18}$$

Using the following equalities,

$$2\,n\,q^n + \frac{2n\,q^{n+1}}{1-q} = \frac{2\,n\,q^n}{1-q}, \tag{19}$$

$$\frac{q\,(1-q^n)}{1-q} + \frac{q}{1-q} - \frac{q^{n+1}}{1-q} + \frac{2\,q^2\,(1-q^n)}{(1-q)^2} = \frac{2\,q\,(1-q^n)}{(1-q)^2}, \tag{20}$$

and combining Equations (18)–(20), the second derivative of $f_n(u)$ evaluated at $u = 1$ is given by

$$\frac{\partial^2 f_n(u)}{\partial u^2}\Big|_{u=1} = -\frac{2\,n\,q^n}{1-q} + \frac{2\,q\,(1-q^n)}{(1-q)^2}. \tag{21}$$

Applying Equation (2), and using Equations (13) and (18), we obtain

$$\begin{aligned}\mathbb{V}ar(X_n) = \mathbb{V}ar(Y_n) =& -\frac{2\,n\,q^n}{1-q} + \frac{2\,q\,(1-q^n)}{(1-q)^2} + \frac{1-q^n}{1-q} - \left(\frac{1-q^n}{1-q}\right)^2 \\ =& \frac{1}{(1-q)^2}\Big(-2\,n\,q^n + 2\,n\,q^{n+1} + 2\,q - 2\,q^{n+1}\Big) \\ &+ \frac{1}{(1-q)^2}\Big(1 - q - q^n + q^{n+1} - 1 + 2\,q^n - q^{2n}\Big) \\ =& \frac{1}{(1-q)^2}\Big(q - q^n\,\Big\{q^n + (2\,n-1)\,(1-q)\Big\}\Big).\end{aligned}$$

□

4. Statistical Properties of the Maximum Age Z_n

In this section, we analyse the final altitude of the maximum age, $Z_n = \max(X_n, Y_n)$, between two components at time n. Precisely, we determine the explicit form of $\Phi_n(v)$, the probability generating function of Z_n. It is defined as follows: for all $v \in \mathbb{R}$ such that $|v| \leq 1$

$$\Phi_n(v) = \mathbb{E}\Big[v^{Z_n}\Big] = \sum_{s=0}^{n} v^s\,\mathbb{P}\Big(Z_n = s\Big).$$

4.1. Moment Generating Function of Z_n

In the following theorem, we give the explicit expression of $\Phi_n(v)$:

Theorem 2. *The probability generating function, $\Phi_n(v)$, of the final altitude Z_n between the two components X_n and Y_n is given by the following expression:*

For all $v \in [-1, 1]$ and for all $n \in \mathbb{N}^*$,

$$\Phi_n(v) = \Big(2q^n - (2q-1)^{n-1}\Big) v^n + 2\,(1-q)\,\frac{q\,v^2 - q^n\,v^{n+1}}{1 - q\,v} - \frac{2\,(1-q)}{1-(2q-1)v}\Big((2q-1)\,v^2 - (2q-1)^{n-1}\,v^n\Big),$$

with $\Phi_0(v) = 1$.

Proof. The probability generating function, $\Phi_n(v)$, of the final altitude $Z_n = \max(X_n, Y_n)$, can be developed as follows: $\forall n \in \mathbb{N}^*, v \in \mathbb{R}$

$$\Phi_n(v) = \sum_{r=1}^{n} v^r\,\mathbb{P}(Z_n = r) = \sum_{r=1}^{n} v^r\,\Big(\mathbb{P}(X_n = r, Y_n \le r) + \mathbb{P}(X_n < r, Y_n = r)\Big)$$
$$= 2\sum_{r=1}^{n}\sum_{s=1}^{r} v^r\,\mathbb{P}(X_n = r, Y_n = s) - \sum_{s=1}^{n} v^r \mathbb{P}(X_n = Y_n = r),$$

and

$$\mathbb{P}(X_n = n, Y_n = n) = (2q-1)^{n-1}, \quad \text{where } n \ge 1. \tag{22}$$

Using the last equation and Equation (A8), in Appendix A, we deduce that for all $n \ge 1$

$$\Phi_n(v) = \Big(2q^n - (2q-1)^{n-1}\Big) v^n + \frac{2(1-q)}{1-q\,v}\Big(q\,v^2 - q^n\,v^{n+1}\Big) - \frac{2\,(1-q)}{1-(2q-1)v}\Big((2q-1)\,v^2 - (2q-1)^{n-1}\,v^n\Big).$$

□

Remark 2. *The probability generating function $\Phi_n(v)$ of the random walk Z_n, satisfies*

$$\Phi_n(v) = 2\,S_n(v) - (2q-1)^{n-1}\,v^n, \tag{23}$$

where $S_n(v)$ is given in Appendix A. The coefficient $(2q-1)^{n-1}$ represents the probability that both random walks X_n and Y_n are always increasing from $(1,1)$ at time 1 to (n,n) at time n with probability $(2q-1)$. Also, this probability equals 1 when $n = 1$, (i.e., $P_1(1,1) = P_0(0,0) = 1$). The coefficient 2 in Equation (23) reflects the symmetry between X_n and Y_n.

4.2. Moments of Z_n

Due to the explicit form of $\Phi_n(v)$ and using the first and the second derivatives in $v = 1$, we are able to compute the mean and the variance of the final altitude Z_n.

Corollary 2. *The explicit expressions of the mean and the variance of the final altitude Z_n are given by*

$$\mathbb{E}(Z_n) = \frac{1}{2(1-q)}\Big(3 - 4\,q^n + (2q-1)^n\Big), \tag{24}$$

$$\mathbb{V}ar(Z_n) = \frac{1}{2(1-q)^2}\Big(1 + 6q + 8q^n\Big[(n-1)q - n\Big] + (2q-1)^n\Big[2n(1-q) + (2q-1)\Big]\Big) + \frac{1}{2(1-q)}\Big(3 - 4\,q^n + (2q-1)^n\Big) - \Big[\frac{1}{2(1-q)}\Big(3 - 4\,q^n + (2q-1)^n\Big)\Big]^2.$$

Proof. In order to obtain the mean and the variance of Z_n, we need to compute the first and the second derivatives of the probability generating function, $\Phi_n(v)$, given in Theorem 2. Next, we evaluate two derivatives of $\Phi_n(v)$ at $v = 1$.

Define the following sequences of functions: $\forall n \geq 1$, $\forall v \in \mathbb{R}$ such that $qv < 1$ and $(2q-1)v < 1$

$$
\begin{aligned}
H_n(v) =& \frac{1}{1-q\,v}\Big(q\,v^2 - q^n\,v^{n+1}\Big),\\
L_n(v) =& \frac{1}{1-(2q-1)v}\,\Big((2q-1)\,v^2 - (2q-1)^{n-1}\,v^n\Big).
\end{aligned}
$$

The first derivative of $\Phi_n(v)$ defined in Theorem 2 with respect v is given by

$$
\begin{aligned}
\frac{\partial \Phi_n(v)}{\partial v} =& \frac{\partial}{\partial v}\Big\{\Big(2\,q^n - (2q-1)^{n-1}\Big)\,v^n + \frac{2\,(1-q)}{1-q\,v}\Big(q\,v^2 - q^n\,v^{n+1}\Big)\\
&- \frac{2\,(1-q)}{1-(2q-1)v}\,\Big((2q-1)\,v^2 - (2q-1)^{n-1}\,v^n\Big)\Big\}\\
=& n\,\Big(2\,q^n - (2q-1)^{n-1}\Big)\,v^{n-1} + 2(1-q)\,\frac{\partial H_n(v)}{\partial v} - 2\,(1-q)\,\frac{\partial L_n(v)}{\partial v},
\end{aligned} \tag{25}
$$

where

$$
\begin{aligned}
\frac{\partial H_n(v)}{\partial v} =& \frac{1}{1-q\,v}\Big(2q\,v - (n+1)\,q^n\,v^n\Big) + \frac{1}{(1-q\,v)^2}\Big(q^2\,v^2 - q^{n+1}\,v^{n+1}\Big),\\
\frac{\partial L_n(v)}{\partial v} =& \frac{1}{1-(2q-1)\,v}\Big(2(2q-1)\,v - n\,(2q-1)^{n-1}\,v^{n-1}\Big)\\
&+ \frac{1}{(1-(2q-1)\,v)^2}\Big((2q-1)^2\,v^2 - (2q-1)^n\,v^n\Big),
\end{aligned}
$$

and evaluating at $v = 1$, we obtain

$$
\begin{aligned}
2(1-q)\,\frac{\partial H_n(v)}{\partial v}\Big|_{v=1} =& 2(1-q)\,\Big(\frac{2q-(n+1)\,q^n}{1-q} + \frac{q^2-q^{n+1}}{(1-q)^2}\Big)\\
=& 4q - 2(n+1)\,q^n + \frac{2(q^2-q^{n+1})}{1-q},
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
2(1-q)\,\frac{\partial L_n(v)}{\partial v}\Big|_{v=1} =& 2(1-q)\,\Big(\frac{2(2q-1) - n\,(2q-1)^{n-1}}{1-(2q-1)} + \frac{(2q-1)^2-(2q-1)^n}{(1-(2q-1))^2}\Big)\\
=& 2(2q-1) - n\,(2q-1)^{n-1} + \frac{(2q-1)^2-(2q-1)^n}{(1-(2q-1))}.
\end{aligned} \tag{27}
$$

By combining Equations (25)–(27), we obtain

$$
\begin{aligned}
\frac{\partial \Phi_n(v)}{\partial v}\Big|_{u=1} =& 2\,n\,q^n - n\,(2q-1)^{n-1} + 4q - 2(n+1)\,q^n + \frac{2(q^2-q^{n+1})}{1-q}\\
&- 2(2q-1) + n\,(2q-1)^{n-1} - \frac{(2q-1)^2-(2q-1)^n}{2(1-q)}\\
=& \frac{1}{2(1-q)}\Big(3 - 4\,q^n + (2q-1)^n\Big).
\end{aligned} \tag{28}
$$

Using Equations (2) and (28), we obtain

$$
\mathbb{E}\Big(Z_n\Big) = \frac{\partial \Phi_n(v)}{\partial v}\Big|_{v=1} = \frac{1}{2(1-q)}\Big(3 - 4\,q^n + (2q-1)^n\Big).
$$

Observe that

$$\frac{\partial^2\Phi_n(v)}{\partial^2 v} = n\,(n-1)\Big(2\,q^n - (2q-1)^{n-1}\Big)\,v^{n-2} + 2(1-q)\,\frac{\partial^2 H_n(v)}{\partial^2 v} - 2\,(1-q)\,\frac{\partial^2 L_n(v)}{\partial^2 v}. \tag{29}$$

The second derivative of the function $H_n(v)$ can be computed as

$$\begin{aligned}\frac{\partial^2 H_n(v)}{\partial^2 v} =& \frac{\partial}{\partial v}\Big\{\frac{1}{1-q\,v}\Big(2q\,v - (n+1)\,q^n\,v^n\Big) + \frac{1}{(1-q\,v)^2}\Big(q^2\,v^2 - q^{n+1}\,v^{n+1}\Big)\Big\}\\ =& \frac{1}{1-q\,v}\Big(2q - n\,(n+1)\,q^n\,v^{n-1}\Big) + \frac{1}{(1-q\,v)^2}\Big(2q^2\,v - (n+1)\,q^{n+1}\,v^n\Big)\\ &+ \frac{1}{(1-q\,v)^2}\Big(2q^2\,v - (n+1)\,q^{n+1}\,v^n\Big) + \frac{1}{(1-q\,v)^3}\Big(2q^3\,v^2 - 2q^{n+2}\,v^{n+1}\Big).\end{aligned} \tag{30}$$

We evaluate $\frac{\partial^2 H_n(v)}{\partial^2 v}$ at $v=1$, and we multiply it by $2(1-q)$ to obtain

$$\begin{aligned}2(1-q)\,\frac{\partial^2 H_n(v)}{\partial^2 v}\Big|_{v=1} =& 2(1-q)\,\Big(\frac{2q - n\,(n+1)\,q^n}{1-q} + \frac{2q^2 - (n+1)\,q^{n+1}}{(1-q)^2}\Big)\\ &+ 2(1-q)\,\Big(\frac{2q^2 - (n+1)\,q^{n+1}}{(1-q)^2} + \frac{2q^3 - 2q^{n+2}}{(1-q)^3}\Big)\\ =& 4q - 2n\,(n+1)\,q^n + \Big(\frac{8q^2 - 4(n+1)\,q^{n+1}}{(1-q)}\Big) + \Big(\frac{4q^3 - 4q^{n+2}}{(1-q)^2}\Big).\end{aligned} \tag{31}$$

The second derivative of the function $L_n(v)$ is given by

$$\begin{aligned}\frac{\partial^2 L_n(v)}{\partial^2 v} =& \frac{\partial^2}{\partial^2 v}\Big(\frac{2(2q-1)\,v - n\,(2q-1)^{n-1}\,v^{n-1}}{1-(2q-1)\,v} + \frac{(2q-1)^2\,v^2 - (2q-1)^n\,v^n}{(1-(2q-1)\,v)^2}\Big)\\ =& \frac{1}{1-(2q-1)\,v}\Big(2(2q-1) - n(n-1)\,(2q-1)^{n-1}\,v^{n-2}\Big)\\ &+ \frac{2q-1}{(1-(2q-1)\,v)^2}\Big(2(2q-1)\,v - n\,(2q-1)^{n-1}\,v^{n-1}\Big)\\ &+ \frac{1}{(1-(2q-1)\,v)^2}\Big(2(2q-1)^2\,v - n\,(2q-1)^n\,v^{n-1}\Big)\\ &+ \frac{2(2q-1)}{(1-(2q-1)\,v)^3}\Big((2q-1)^2\,v^2 - (2q-1)^n\,v^n\Big).\end{aligned}$$

The last second derivative evaluated at $v=1$ gives

$$\begin{aligned}\frac{\partial^2 L_n(v)}{\partial^2 v}\Big|_{v=1} =& \frac{1}{2(1-q)}\Big(2(2q-1) - n(n-1)\,(2q-1)^{n-1}\Big)\\ &+ \frac{2q-1}{4(1-q)^2}\Big(2(2q-1) - n\,(2q-1)^{n-1}\Big)\\ &+ \frac{1}{4(1-q)^2}\Big(2(2q-1)^2 - n\,(2q-1)^n\Big)\\ &+ \frac{(2q-1)}{4(1-q)^3}\Big((2q-1)^2 - (2q-1)^n\Big).\end{aligned}$$

We simplify $\frac{\partial^2 L_n(v)}{\partial^2 v}\Big|_{v=1}$, and we multiply it by $2(1-q)$ to obtain

$$\begin{aligned}2(1-q)\,\frac{\partial^2 L_n(v)}{\partial^2 v}\Big|_{v=1} =& \Big(2(2q-1) - n(n-1)\,(2q-1)^{n-1}\Big)\\ &+ \frac{1}{(1-q)}\Big(2(2q-1)^2 - n\,(2q-1)^n\Big)\\ &+ \frac{1}{2(1-q)^2}\Big((2q-1)^3 - (2q-1)^{n+1}\Big).\end{aligned} \tag{32}$$

Since

$$n\,(n-1)\Big(2\,q^n-(2q-1)^{n-1}\Big)-2n\,(n+1)\,q^n+n\,(n-1)(2q-1)^{n-1}=-4n\,q^n, \tag{33}$$

by (31)–(33), we deduce

$$\begin{aligned}\frac{\partial^2\Phi_n(v)}{\partial^2 v}\Big|_{v=1} &= -\,4n\,q^n-2(2q-1)+\Big(\frac{8q^2-4(n+1)\,q^{n+1}}{(1-q)}\Big)+\Big(\frac{4q^3-4q^{n+2}}{(1-q)^2}\Big)\\ &\quad-\frac{1}{(1-q)}\Big(2(2q-1)^2-n\,(2q-1)^n\Big)\\ &\quad-\frac{1}{2(1-q)^2}\Big((2q-1)^3-(2q-1)^{n+1}\Big).\end{aligned} \tag{34}$$

Define A_q by

$$A_q=\frac{1}{1-q}\Big(8q^2-2(2q-1)^2\Big)+\frac{1}{2(1-q)^2}\Big(8q^3-(2q-1)^3\Big)-2(2q-1)+4q.$$

Via a simple calculation, we obtain

$$A_q=\frac{1}{2(1-q)^2}(1+6q). \tag{35}$$

Since

$$-4nq^n-\frac{1}{1-q}\Big(4(n+1)q^{n+1}\Big)-\frac{4q^{n+1}}{(1-q)^2}=\frac{4q^n}{(1-q)^2}\Big((n-1)q-n\Big), \tag{36}$$

$$\frac{1}{1-q}n(2q-1)^n+\frac{1}{2(1-q)^2}(2q-1)^{n+1}=\frac{(2q-1)^n}{2(1-q)^2}\Big(2n(1-q)+(2q-1)\Big), \tag{37}$$

and combining Equations (34)–(37), we obtain

$$\begin{aligned}\frac{\partial^2\Phi_n(v)}{\partial^2 v}\Big|_{v=1} &=\frac{1}{2(1-q)^2}\Big(1+6q+8q^n\Big[(n-1)q-n\Big]\\ &\quad+(2q-1)^n\Big[2n(1-q)+(2q-1)\Big]\Big).\end{aligned} \tag{38}$$

The variance of Z_n is finally obtained using Equations (2), (24), and (38). □

5. Return Time N_n^X of the Random Walk X_n

In this section, we analyse the number of return times N_n^X at time n of the process $(X_\cdot)$ to position 1 at time n. Precisely, we start with a simulation of the process N_n^X and determine the explicit form of $G_n(x)$, i.e., the probability generating function of N_n^X.

5.1. Simulations of N_n^X

In this subsection, we give some simulations with R-program using N_n^X with different lengths: 100, 1000, 10,000, and 100,000 for different values of the given probabilities $q = 0.6$, 0.75, 0.9.

Figure 1 shows that the return time, N_X^n, of the random walk, X, with length 100 is increasing from 0 to 40 from time 0 to time 100, when the random walk X alternates between 0 and 9 with initial probability q equal to 0.6. Also, we observe that the return time of random walk N_n^X with lengths 1000, 10,000, and 100,000 is increasing from 0 to 400, 4000, and 40,000, when the evolution of the random walk, X, is about 0 (very small variation in X), respectively.

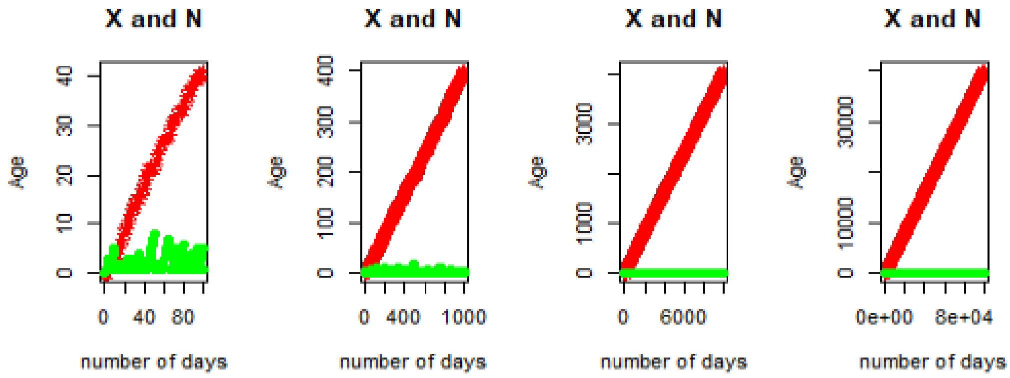

Figure 1. Return time of X (in red) and the random walk X (in green) of lengths 1000 and 500 and for $q = 0.6$.

Figure 2 shows that the return time, N_X^n, of the random walk, X, with length 100 is increasing from 0 to 40 from time 0 to time 100, when the evolution of the random walk X alternates between 0 and at most 12 with initial probability q equal to 0.75. Also, we observe that the return time of random walk N_n^X with length 1000 is increasing from 0 to 300, when the evolution of the random walk, X, alternates between 0 and 25. Furthermore, Figure 2 shows that the return time of random walk N_n^X with lengths 10,000 and 100,000 is increasing from 0 to 2500 and 25,000, when the evolution of the random walk X is about 0, respectively.

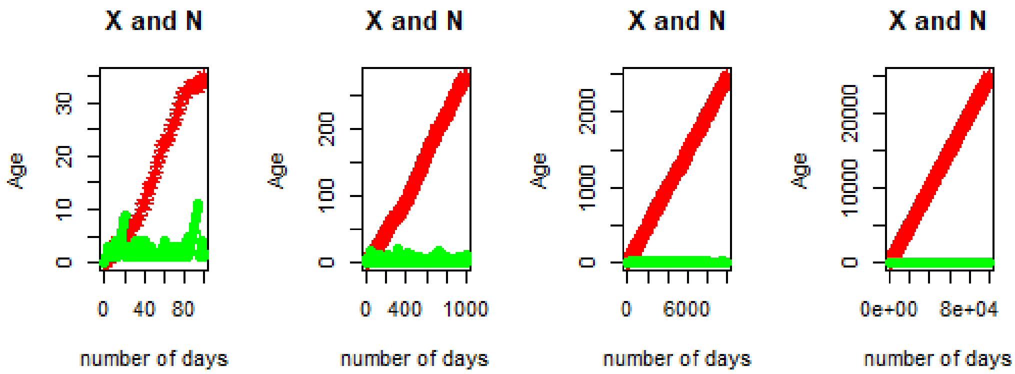

Figure 2. Return time of X (in red) and the random walk X (in green) of lengths 1000 and 500 and for $q = 0.75$.

Figure 3 shows that the return time, N_X^n, of the random walk, X, with length 100 is increasing slowly from 0 to 10 from time 0 to time 100, when the evolution of the random walk X alternates between 0 and 17 with initial probability q equal to 0.9. Also, it shows that the return time of random walk N_n^X with length 1000 is increasing from 0 to 100, when the evolution of the random walk X alternates between 0 and 50. Furthermore, we observe that the return time of random walk N_n^X with lengths 10,000 and 100,000 is increasing from 0 to 1000 and from 0 to 10,000, when the evolution of the random walk X alternates between 0 and 80, and about 0, respectively.

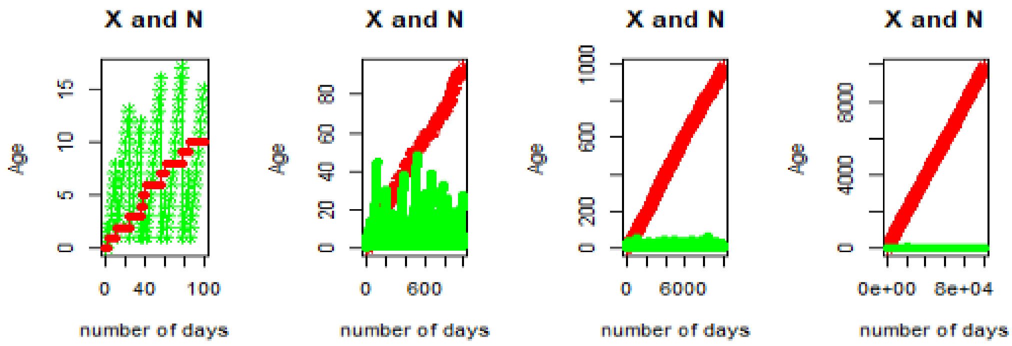

Figure 3. Return time of X (in red) and the random walk X (in green) of lengths 1000 and 500 and for $q = 0.9$.

5.2. Probability Distribution of N_n^X

In this section, we give the probability distribution of N_n^X.

Theorem 3. *The exact distribution of N_n^X is given by*

$$\mathbb{P}(N_n^X = k) = q^n \left(\frac{1-q}{q}\right)^{k+1} \sum_{s=1}^{n-k-1} \binom{n-1-s}{k} q^s.$$

Remark 3. *Through an easy computation, we prove that this probability can be given by*

$$\mathbb{P}(N_n^X = k) = \frac{q^{n-k-1}}{k!} \left(\frac{x^{n-1} - x^k}{x-1}\right)^{(k)} |_{x=q^{-1}},$$

where for a k differentiable function g, the notation $g^{(k)}$ denotes the k^{th} derivative of g.

Proof. For the proof of Theorem 3, we start by computing the joint distribution of the discrete return time N_n^X and the discrete random walk X. To this end, for all $k \in \{0, \ldots, n-2\}$ and for all $s \in \{1, \ldots, n-1\}$, we compute, as a first step, the probability of intersection between the return time equal to k, and the random walk equal to s. As a second step, we deduce the marginal distribution of N_n^X.

Consider N_n^X the number of visits of the process $X_.$ to the state 1 up to time n.

$$(N_n^X = k | X_n = s) = \begin{cases} X_. & \text{increases from time } n-s+1 \text{ to time } n, \\ X_. & \text{has } k \text{ resets from time 2 to time } n-s. \end{cases} \tag{39}$$

We start by giving the joint distribution of (N_n^X, X_n).

Lemma 3. *The joint distribution of (N_n^X, X_n) satisfies the following relation:*

$$\mathbb{P}(N_n^x = k, X_n = s) = (1-q)q^s\, \mathbb{P}\Big(Bin(n-1-s, 1-q) = k\Big), \tag{40}$$

and is given as

$$\mathbb{P}(N_n^x = k, X_n = s) = \begin{cases} \binom{n-1-s}{k}(1-q)^{k+1} q^{n+s-k-1} & \text{if } k \in \{1, \ldots, n-s-1\}, \\ 0 & \text{other wise,} \end{cases} \tag{41}$$

where $Bin(n-1-s, 1-q)$ is a binomial distribution with parameters $n-1-s$ and $1-q$.

Proof. Using Equation (39), we have

$$\mathbb{P}(N_n^X = k|X_n = s) = \begin{cases} \binom{n-s-1}{k}(1-q)^k q^{n-1-k} & \text{if } k \in \{1, \dots n-1-s\}, \\ 0 & \text{other wise.} \end{cases} \tag{42}$$

By Theorem 1, X_n follows a shifted geometric distribution with parameter $1-q$

$$\mathbb{P}(X_n = s) = q^s(1-q). \tag{43}$$

We conclude the proof by using the fact that

$$\mathbb{P}(N_n^X = k, X_n = s) = \mathbb{P}(N_n^x = k|X_n = s)\mathbb{P}(X_n = s). \tag{44}$$

□

By summing with respect to s and using the known distribution of X_n, we deduce the result of Theorem 3. □

Remark 4. *In the particular cases,*

- *If* $k = n-2$, *we obtain*
$$\mathbb{P}\left(N_n^X = n-2\right) = q^2(1-q)^{n-1},$$
- *If* $k = 1$, *we obtain*
$$\mathbb{P}\left(N_n^X = 1\right) = q^{n-2}(1-q)\frac{(n-2)(n-1)}{2},$$
- *If* $k = 0$, *we obtain*
$$\mathbb{P}\left(N_n^X = 0\right) = \frac{q^{2n} + q^n - q^{n+1} - q^{2n-1}}{1-q}.$$

- *If the return time of X equals 0 ($k = 0$), then the probability that the random walk X is strictly increasing from time 0 to time n equals q^{n-1} for any age of the random walk Y at time n.*
- *If the return time of X equals $n-2$ ($k = n-2$), given that the age of X increases from 0 at time 0 to $n-2$ at time $n-2$, the probability that the age of X equals 1 at time $n-1$ comes from the probability of X at time $n-2$ multiplied by $(1-q)$.*

Remark 5. *The probability generating function $G_n(x)$ of N_n^X can be expressed, and we prove that it is given by*

$$G_n(x) = \frac{1}{nx(q+x)}\Big[q^{n-1}(q-1)(q+x)x\Big(\sum_{k=0}^{n-2}(1-q)^k\left(xq^{-1}\right)^k(n-k)\binom{n}{k}g(n,k,q)\Big) - n\left(q^{2n+1}\left(-xq^{-1}\right)^n + xq^{2n}\right)\Big],$$

where the function $g(n,k,q)$ is the well-known hyper-geometric function given by

$$g(n,\, k,\, q) = hypergeom\left(1, n;\, [n-k];\, q^{-1}\right).$$

From $G_n(x)$, we can compute the mean and the variance of N_n^X, but the expressions are very complicated.

6. The Probability Generating Function of the Random Walks X_n and Y_n

Next, consider the generating function associated to the above Moran process defined in (1), where the time is encoded by the exponent of t:

$$F(t,x,y) := \sum_{n\geq 0} \mathbb{E}\left[x^{X_n}\, y^{Y_n}\right] t^n = \sum_{n\geq 0} f_n(x,y)\, t^n, \quad \forall (x,y) \in \mathbb{R}^2. \tag{45}$$

Starting from the functional equation defined in (6) and using the kernel method (for more details view [2,18]), we obtain the PGF. It is very important to obtain this the probability generating function because it contains all information about the past of the random walk; in particular, it will be very useful for studying the height H_n. This point will be one of our objectives in a future work.

Theorem 4. *The probability generating function of the final altitude of the two-dimensional Moran walk is given as follows: for all $(x,y) \in \mathbb{R}^2$ such that $|t| < 1$, $|t\,x\,y| < 1/(2q-1)$, $|t\,x| < 1/q$ and $|t\,y| < 1/q$*

$$F(t,x,y) = \frac{1 + Q(x,y)\,\dfrac{t}{1-t}}{1-(2q-1)\,t\,x\,y}, \tag{46}$$

where

$$Q(x,y) = \frac{(1-q)xy\Big[1-t(1-(1-q)x)\Big]}{1-q\,t\,x} + \frac{(1-q)xy\Big[1-t(1-(1-q)y)\Big]}{1-q\,t\,y}.$$

Proof. By Equation (4), and for all $n \geq 1$, we have

$$f_n(x,y) = (1-q)\,x\,y\,f_{n-1}(x,1) + (1-q)\,x\,y\,f_{n-1}(1,y) + (2q-1)\,x\,y\,f_{n-1}(x,y).$$

Using the following identities

$$\begin{aligned}
\sum_{n\geq 1} f_{n-1}(1,x_2)\,t^n &= t \sum_{n\geq 0} f_n(1,x_2)\,t^n = t\,F(t,1,x_2),\\
\sum_{n\geq 1} f_{n-1}(x_1,1)\,t^n &= t \sum_{n\geq 0} f_n(x_1,1)\,t^n = t\,F(t,x_1,1),\\
\sum_{n\geq 1} f_{n-1}(x_1,x_2)\,t^n &= t \sum_{n\geq 0} f_n(x_1,x_2)\,t^n = t\,F(t,x_1,x_2),
\end{aligned}$$

the probability generating function $F(t,x,y)$ satisfies the recursive equation

$$F(t,x,y) = 1 + (1-q)\,x\,y\,t\,F(t,1,y) + (1-q)\,x\,y\,t\,F(t,x,1) + (2q-1)\,x\,y\,t\,F(t,x,y). \tag{47}$$

From Equation (47), $F(t,x,1)$ and $F(t,1,y)$ are given by

$$F(t,x,1) = \frac{1}{1-qtx} + \frac{(1-q)t\,x}{1-qtx}F(t,1,1), \tag{48}$$

$$F(t,1,y) = \frac{1}{1-qty} + \frac{(1-q)t\,y}{1-qty}F(t,1,1). \tag{49}$$

From Equations (47)–(49), we obtain

$$\begin{aligned}
\Big(1-(2q-1)\,x\,y\,t\Big)\,F(t,x,y) =& 1 + t\left(\frac{(1-q)^2txy^2}{1-qty} + \frac{(1-q)^2tx^2y}{1-qtx}\right)F(t,1,1)\\
&+ t\left(\frac{(1-q)xy}{1-qty} + \frac{(1-q)xy}{1-qtx}\right).
\end{aligned}$$

Since $F(t,1,1) = 1/(1-t)$ for $|t| < 1$, we deduce

$$\begin{aligned}\Big(1-(2q-1)\,x\,y\,t\Big)\,F(t,x,y) &= 1 + \Big(\frac{(1-q)^2txy^2}{1-qty} + \frac{(1-q)^2tx^2y}{1-qtx}\Big)\frac{t}{1-t} \\ &\quad + (1-t)\Big(\frac{(1-q)xy}{1-qty} + \frac{(1-q)xy}{1-qtx}\Big)\frac{t}{1-t} \\ &= 1 + \Big(\frac{(1-q)xy\Big[1-t(1-qx)\Big]}{1-qtx}\Big)\frac{t}{1-t} \\ &\quad + \Big(\frac{(1-q)xy\Big[1-t(1-qy)\Big]}{1-qty}\Big)\frac{t}{1-t}.\end{aligned} \tag{50}$$

Divide Equation (50) by $((1-(2q-1)\,x\,y\,t)$; this concludes the proof. □

Remark 6. *The term $1 - t\,p\,x_1\,x_2$ in Equation (46) is called the kernel factor.*

Remark 7. *We can factorize the probability generating function given by Equation (46) as*

$$F(t,x,y) = \frac{1}{1 - Q(x,y)\,t\,\dfrac{1}{1-t\Big(1-Q(x,y)\Big)}}\;\frac{1}{\Big(1-(2q-1)\Big)t\,x\,y}, \tag{51}$$

with $Q(1,1) = 2(1-q)$.

From the previous theorem, we can find the probability generating function of two random Moran walks X_n and Y_n.

Corollary 3. *The probability generating functions, denoted by $F_X(t,x) := F(t,x,1)$ and $F_Y(t,y)F(t,1,y) :=$, of X_n and Y_n are given by*

$$F_X(t,x) = \frac{1}{1-(2q-1)t\,x}\Big(1 + (1-q)\,x + \frac{(1-q)x\Big[1-t(1-(1-q)x)\Big]}{1-qtx}\Big), \tag{52}$$

and

$$F_Y(t,y) = F(t,y) = \frac{1}{1-(2q-1)t\,y}\Big(1 + (1-q)\,y + \frac{(1-q)y\Big[1-t(1-(1-q)y)\Big]}{1-qty}\Big), \tag{53}$$

$\forall (x,y) \in \mathbb{R}^2$ such that $|t\,x| < 1/(2q-1)$, $|t\,y| < 1/(2q-1)$, where $|t| < 1$.

Proof. The proof is a direct consequence of the previous theorem: if we take $x = 1$ in (46), then we obtain the expression of the probability generating function of X_n, and similarly for Y_n. □

The previous result is very important. It allows us to know the probability generating function of the two-dimensional walk $(X_n,\,Y_n)$. Combining Equations (45), (52), and (53), and evaluating at $t = 1$, then we have the expressions of $f_n(x,1)$ and $f_n(1,y)$.

7. Conclusions and Perspectives

In this current paper, we use very useful tools called probability generating functions to find the statistical properties, i.e., the mean, the variance, and the limiting distribution, of the random walks X_n, Y_n, Z_n, and N_n^X. Firstly, we prove that both symmetric random Moran walks X_n and Y_n converge to a shifted geometric distribution with parameter $(1-q)$ using the probability generating functions asymptotically. Also, the means and the variances of

X_n and Y_n are calculable explicitly using the same tools. Secondly, we use the symmetry of two random walks X_n and Y_n to find the statistical properties of the maximum age Z_n between two components, such as the mean and the variance, derived from the probability generating function. Finally, we analyse the return time, N_n^X, of the random Moran walk X_n. From the simulation of N_n^X, we observe that the return time is affected according to the initial probability q and the length of the random walk. Precisely, we distinguish two cases:

1. When the initial probability q approaches 1 (q = 0.9), the return time with a small length $(n = 100)$ is increasing slowly and remains lower than the final altitude of X_n at time n (see Figure 3). In this case, the Moran random walk increases often and returns to 1 few times. That means the number of increases in X_n is greater than the number of times that N_n^X returns to 1.
2. When the length of the random walk X_n is very large, $(n = 1000)$ or 10,000 or 100,000, the return time N_n^X is not affected by the initial probability q and increases quickly (see Figures 1–3). In this case, the Moran random walk often returns to 1 but X_n alternates between 1 and at most 50. That means the number of times that X_n returns to 1 is greater than the increase in X_n.

Here, the initial probability q represents the probability that the random walk X_n increases. This increase in X_n happens intwo ways: in the first way, X_n increases but Y_n stops at 1 with probability $(1 - q)$; in the second, both walks X_n and Y_n increase in the same time with probability $(2q - 1)$ (see Equation (1)).

In the next work, we will use the probability generating function to study the statistical properties of the height statistics, H_n. Precisely, we will find the distribution of H_n and compute its mean and variance based on the return time N_n^X. Firstly, we will start with the following conditional probability:

$$\mathbb{P}(H_n \le k | N_n = r) = \sum_{I_{n,r,k}} \mathbb{P}\Big(G_1 = n_1,\, G_2 = n_2 - n_1, \dots, G_r = n_r - n_{r-1},\, X_{n_r+1} \cdots X_n \neq 0\Big),$$

where the random walk H_n is bounded by an integer $k > 0$ given that the random walk N_n^X equals $r \geq 1$. Secondly, we will try to obtain the joint distribution of (H_n, N_n^X). Finally, we can extract the distribution of the bounded random walk H_n and determine its statistical properties.

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Appendix A. Technical Lemmas

In this subsection, we study the following sequences of polynomials, $R_n(v)$, $T_n(v)$, $N_n(v)$, and $S_n(v)$, defined by

$$R_n(v) := \sum_{r=1}^{n} v^r\, \mathbb{P}(X_n = r, Y_n = 1) = \sum_{r=1}^{n} v^r\, \mathbb{P}_n(r, 1),$$

$$T_n(v) := \sum_{r=2}^{n} \sum_{s=2}^{r} v^r \, \mathbb{P}(X_n = r, Y_n = s) = \sum_{r=2}^{n} \sum_{s=2}^{r} v^r \mathbb{P}_n(r,s),$$
$$N_n(v) := \sum_{r=1}^{n} \sum_{s=r+1}^{n} v^r \, \mathbb{P}(X_n = r, Y_n = s) = \sum_{r=1}^{n} \sum_{s=r+1}^{n} v^r \, \mathbb{P}_n(r,s),$$
$$S_n(v) := \sum_{r=1}^{n} \sum_{s=1}^{r} v^r \, \mathbb{P}(X_n = r, Y_n = s) = \sum_{r=1}^{n} \sum_{s=1}^{r} v^r \mathbb{P}_n(r,s),$$

where $R_0(v) = 0$, $T_0(v) = T_1(v) = 0$, $N_0(v) = N_1(v) = 0$, and $S_0(v) = 0$.

The sequence $S_n(v)$ is essential for the proof of Theorem 2. The sequences $R_n(v)$, $T_n(v)$, $N_n(v)$ are needed to study $S_n(v)$.

The expressions of these sequences are given in the following lemmas.

Lemma A1. *The sequence of polynomials $N_n(v)$ holds the following recursive equation: for all $n \geq 1$,*

$$N_n(v) = (1-q)\, v + (2q-1)\, v \, N_{n-1}(v),$$

and, then, the explicit form of $N_n(v)$ is given by

$$N_n(v) = (1-q)\, \frac{v - (2q-1)^{n-1}\, v^n}{1-(2q-1)\, v}, \tag{A1}$$

for all $v \in \mathbb{R}$ such that $|(1-p)\, v| < 1$.

Proof. Developing the sequence of polynomials $N_n(v)$

$$N_n(v) := \sum_{r=1}^{n} \sum_{s=r+1}^{n} v^r \, \mathbb{P}_n(r,s) = \sum_{s=2}^{n} v \, \mathbb{P}_n(1,s) + \sum_{r=2}^{n} \sum_{s=r+1}^{n} v^r \, \mathbb{P}_n(r,s), \tag{A2}$$

applying Lemma 2

$$\sum_{s=2}^{n} v \, \mathbb{P}_n(1,s) = v\,(1-q) \sum_{r=1}^{n-1} \sum_{s=2}^{n} \mathbb{P}_{n-1}(r, s-1) = v\,(1-q) \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} \mathbb{P}_{n-1}(r,s),$$

and using

$$\sum_{r=0}^{n-1} \sum_{s=0}^{n-1} \mathbb{P}_{n-1}(r,s) = f_{n-1}(1,1) = 1,$$

we obtain

$$\sum_{s=2}^{n} v \, \mathbb{P}_n(1,s) \quad = \quad v\,(1-q) \sum_{r=0}^{n-1} \sum_{s=0}^{n-1} \mathbb{P}_{n-1}(r,s) = v\,(1-q), \tag{A3}$$

and

$$\sum_{r=2}^{n} \sum_{s=r+1}^{n} v^r \, \mathbb{P}_n(r,s) = (2q-1)\, v \sum_{r=1}^{n-1} \sum_{s=r+1}^{n-1} v^r \, \mathbb{P}_{n-1}(r,s) := (2q-1)\, v \, N_{n-1}(v). \tag{A4}$$

Combining Equations (A2)–(A4), we obtain

$$N_n(v) = (1-q)\, v + (2q-1)\, v \, N_{n-1}(v).$$

The last equation is iterated n times, and the equation defined in (A1) is proved. □

The next lemma is used to compute the sequence of polynomials $T_n(v)$ when the final ages of two components, X_n and Y_n, are started at time 2 from $(2,2)$ and are finished at two different times, n and r, where $r < n$, respectively. It is introduced as follows:

Lemma A2. *The sequence of polynomials $T_n(v)$ verifies the following recursive equation: for all $n \geq 2$*

$$T_n(v) = (2q-1)\,v\Big(f_{n-1}(v,1) - N_{n-1}(v)\Big).$$

Its explicit expression is given by

$$\begin{aligned} T_n(v) &= (2q-1)\,q^{n-1}\,v^n + (2q-1)\,(1-q)\,\frac{v^2 - q^{n-1}\,v^{n+1}}{1-q\,v} \\ &\quad - (2q-1)\,(1-q)\,\frac{v^2 - (2q-1)^{n-2}\,v^n}{1-(2q-1)\,v}, \end{aligned} \tag{A5}$$

for all $v \in \mathbb{R}$ such that $|(2q-1)\,v| < 1$ and $|q\,v| < 1$.

Proof. Applying Lemma 2,

$$T_n(v) \;:=\; \sum_{r=2}^{n}\sum_{s=2}^{r} v^r\,\mathbb{P}_n(r,s) = (2q-1)\,v\sum_{r=1}^{n-1}\sum_{s=1}^{r} v^r\,\mathbb{P}_{n-1}(r,s),$$

and using Equation defined in (4), with $x = v$ and $y = 1$,

$$\begin{aligned} T_n(v) &= (2q-1)\,v\Big(\sum_{r=1}^{n-1}\sum_{s=1}^{n-1} v^r\,\mathbb{P}_{n-1}(r,s) - \sum_{r=1}^{n-1}\sum_{s=r+1}^{n-1} v^r\,\mathbb{P}_{n-1}(r,s)\Big) \\ &= (2q-1)\,v\Big(f_{n-1}(v,1) - N_{n-1}(v)\Big), \end{aligned}$$

hence, the explicit expression of $T_n(v)$ is obtained directly from Equations (11) and (A1). □

Lemma A3. *The sequence of polynomials $R_n(v)$ satisfies the following: for all $n \geq 1$*

$$R_n(v) = (1-q)\,v\,f_{n-1}(v,1), \quad \textit{where } q = 1-(p/2), \tag{A6}$$

the explicit form of $R_n(v)$ is given by

$$R_n(v) = (1-q)^{n-1}\,v^n + (1-q)^2\,\frac{v^2 - q^{n-1}\,v^{n+1}}{1-q\,v}, \tag{A7}$$

for all $v \in \mathbb{R}$ such that $|q\,v| < 1$.

Proof. Using Lemma 2 and Equation (4), we have

$$R_n(v) = (1-q)\,v\sum_{r=0}^{n-1}\sum_{s=1}^{n-1} v^r\,\mathbb{P}_{n-1}(r,s) = (1-q)\,v\,f_{n-1}(v,1),$$

also, the explicit expression of $R_n(v)$ is obtained directly from Equation (11). □

Lemma A4. *The sequence of polynomials $S_n(v)$ satisfies the following: for all $n \geq 1$*

$$S_n(v) = q\,v\,f_{n-1}(v,1) - (1-p)\,v\,N_{n-1}(v),$$

the form of $S_n(v)$ is given by

$$\begin{aligned} S_n(v) =& (q\,v)^n + \frac{1-q}{1-q\,v}\Big(q\,v^2 - q^n\,v^{n+1}\Big) \\ & - \frac{1-q}{1-(2q-1)\,v}\Big((2q-1)\,v^2 - (2q-1)^{n-1}\,v^n\Big), \end{aligned} \tag{A8}$$

for all $v \in \mathbb{R}$ such that $|q\, v| < 1$ and $|(2q-1)\, v| < 1$.

Proof. Developing the sequence of polynomials $S_n(v)$

$$S_n(v) := \sum_{r=1}^{n}\sum_{s=1}^{r} v^r\, \mathbb{P}_n(r,s) = \sum_{r=1}^{n} v^r\, \mathbb{P}_n(r,1) + \sum_{r=2}^{n}\sum_{s=2}^{r} v^r\, \mathbb{P}_n(r,s) = R_n(v) + T_n(v),$$

using Lemmas A2 and A4, we obtain

$$\begin{aligned} S_n(v) &= (1-q)\, v\, f_{n-1}(v,1) + (2q-1)\, v\, f_{n-1}(v,1) - (2q-1)\, v\, N_{n-1}(v) \\ &= q\, v\, f_{n-1}(v,1) - (2q-1)\, v\, N_{n-1}(v). \end{aligned}$$

The explicit expression of $S_n(v)$ is obtained by combining Equations (11) and (A1). □

Remark A1. *Consider two consecutive times n and $(n-1)$, r and s days, the age of two components X and Y at time n, respectively. We present some remarks concerning the sequences of polynomials $N_n(v)$, $T_n(v)$, $R_n(v)$,and $S_n(v)$.*

(1) *From Equation (A3), if X is aged 1 day at time n and the age of Y is strictly increasing from 2 to n, we deduce*

$$\sum_{s=2}^{n} \mathbb{P}_n(1,s) = (1-q) = \mathbb{P}_2(1,2).$$

By symmetry, we have

$$\sum_{r=2}^{n} \mathbb{P}_n(r,1) = (1-q) = \mathbb{P}_2(2,1).$$

(2) *The sequence $T_n(v)$ is given by the multiplication between $(2q-1)v$ and the difference between the explicit expressions of the probability generating function $f_{n-1}(v,1)$ and the sequence of polynomials $N_{n-1}(v)$.*

(3) *The sequence $R_n(v)$ is expressed by the product between $(1-q)v$ and the explicit expression of the probability generating function $f_{n-1}(v,1)$.*

(4) *The sequence $S_n(v)$ is expressed from the explicit expressions of two sequences $T_n(v)$ and $R_n(v)$. The sequence $S_n(v)$ depends on the probability generating function $f_{n-1}(v,1)$ and the sequence of polynomials $N_{n-1}(v)$.*

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