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Inverses and Determinants of $n \times n$ Block Matrices

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Abstract: Block matrices play an important role in all branches of pure and applied mathematics. In this paper, we study the two fundamental concepts: inverses and determinants of general $n \times n$ block matrices. In the first part, the inverses of 2×2 block matrices are given, where one of the blocks is a non-singular matrix, a result which can be generalised to a block matrix of any size, by splitting it into four blocks. The second part focuses on the determinants, which is covered in two different methods. In the first approach, we revise a formula for the determinant of a block matrix A, with blocks elements of R; a commutative subring of $M_{n\times n}(F)$. The determinants of tensor products of two matrices are also given in this part. In the second method for computing the determinant, we give the general formula, which would work for any block matrix, regardless of the ring or the field under consideration. The individual formulas for determinants of 2×2 and 3×3 block matrices are also produced here.

Keywords: block matrices; inverses; determinants; tensor products

MSC: 15A09; 15A29; 15A15

1. Introduction

Block matrices arise in many fields of pure and applied mathematics and also applied sciences. Since these matrices are widely used in so many different fields, it is important to know about their algebraic properties. This paper aims to bring together the two fundamental concepts: inverses and determinants of $n \times n$ block matrices. Exercises on some inverse and determinant properties of block matrices can be found in standard linear algebra textbooks. However, in this study, we discuss these two concepts with a complete theory, hoping that it will serve as a primary reference for those interested in the subject or would like to use these matrices in their research. There are many published works in the literature concerning these two algebraic concepts. For example, the inverses of 2×2 block matrices have been studied by Lu and Shiou in [1], the determinants of these matrices have been examined by Powell and Silvester in [2] and [3], respectively, and the generalized inverses and ranks for 2×2 block matrices are given in [4]. Obtaining formula for the inverse and the determinant of a 2×2 block matrix is crucial, as these results can always be expanded to the block matrices of larger sizes by either splitting it into 2×2 blocks for the inverse case or by expressing the determinants in terms of the determinants of the 2×2 block matrices using the cofactor expansion in the determinant case. Therefore, in this paper, the two theories are first established for the 2×2 block matrices.

Block matrices are used in the proofs of many critical theorems in linear algebra. For example, the determinants of block upper triangular matrices are used to prove that the dimension of the eigenspace corresponding to an eigenvalue λ is always less than or equal to the algebraic multiplicity of the given λ. For a linear operator T on a finite dimensional vector space V, and for a T-invariant subspace W of V, the proof of the theorem stating that the characteristic polynomial of T_W (T restricted to W) divides the characteristic polynomial of T also uses the determinants of block matrices. This is an important result

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used in the proof of the famous Cayley–Hamilton theorem [5]. Last but not least, the Jordan canonical form theory is also based on block diagonal matrices. Remember that a linear operator T on n dimensional vector space V is invertible if and only if its standard matrix is invertible, and in fact, the matrix for the inverse operator is just the inverse of the standard matrix of T [5]. Therefore, the invertibility of a block matrix will give information on the invertibility of the so-called 'block operators'; operators whose standard matrices are the block matrices (standard matrices are obtained by finding the images of the bases vectors of the given vector space V under the linear operator). In functional analysis, these linear operators can help solve integral equations (see, for example, [6], where the block operator diagrams are used, or [7], where the strict positivity (invertibility and the positivity) of the operators on Hilbert spaces are identified with the strict positivity of the 2×2 block matrices). Other crucial problems under pure mathematics can also be solved using these linear operators. Hence, our paper can help to obtain information on them via their matrices.

Many algebraic operations can be simplified by using blocks in a matrix as elements instead of using matrices of much larger sizes. We give some references for the applications of block matrices in applied mathematics and general sciences. However, let us remind the reader that this is an algebraic paper, and the scope of it is not in the application direction. Generalised inverses of block matrices are used in [8] to establish conditions for the values of two matrix functions to be equal, intuitionistic fuzzy block matrices (generalisations of subsets fuzzy block matrices) and their properties are studied in [9], and the solution of linear systems using block tridiagonal coefficient matrices via the block cyclic reduction method is used to study the roots of the characteristic polynomials of matrices in [10]. To give more examples on applications in the numerical analysis field, incomplete block-matrix factorisations are used to precondition the iterative methods in [11], where applications for the solution of the Dirichlet problem of Laplace's equation on a rectangle using the finite difference method with classical rectangular grids are given. Moreover, in [12], an incomplete block factorisation for symmetric positive definite block tridiagonal matrices is given; this factorization is used to precondition the conjugate gradient method and is applied to solve the Dirichlet boundary value problem of the heat equation on a rectangle by developing a new difference method on a hexagonal grid.

The details on the inverse and determinant theories of block matrices are discussed in this paper. All proofs are independently produced by the authors unless explicitly stated and cited in this paper. For examples on the covered topics, one may refer to the Eastern Mediterranean University MSc Thesis of the second named author [13]. The organization of this paper is as follows: in the next section, we start by giving the preliminaries. In Section 3, we first give formulas for the inverses of 2×2 block diagonal and block triangular matrices; the techniques of proofs here can be generalised to block diagonal and block triangular matrices of higher dimensions. In the same section, we give the inverse formulas for the 2×2 block matrices in case one of the blocks is invertible. Proofs are provided via block Gaussian elimination and also block elementary matrices. LDU decompositions of the given block matrix are also provided in this section. Once the inverse formula for 2×2 block matrices is obtained, this can be generalised to $n \times n$ block matrices by diving it into four blocks (by producing a 2×2 block matrix). Section 4 deals with the determinant concept in two different approaches. First, a special commutative case is revised, where the determinants of matrices with blocks belonging to a commutative subring of a field or a commutative ring are studied. The determinants of 2×2 block diagonal and block triangular matrices, together with the determinants of a general 2×2 block matrix, are provided. In this section, we also give a determinant formula for tensor products of two matrices. Next, the general formula existing in the literature is presented, which works for matrices in any ring or field. This formula can get very complex if the block matrix has a large size. Therefore, in specific useful cases, the determinants for 2×2, and 3×3 block matrices are also provided. To demonstrate the $p = 3$ case, we also give a numerical example and compute the determinant of a 12×12 matrix. With this example, one can easily see the efficiency of the method compared to computing the determinant via a cofactor expansion or row reduction. Finally, we conclude this paper in Section 5.

2. Preliminaries

Basic concepts are defined below; one may also refer to [14] for more details.

Definition 1. *A block matrix (partitioned matrix) is a matrix having split sections referred to as blocks or submatrices. These blocks are separated with horizontal or vertical lines. The blocks (submatrices) of a block matrix must fit together to form a rectangle or a square shape.*

Example 1. *Let* $C = \begin{bmatrix} 8 & 6 & 2 & -5 & -9 \\ 2 & 5 & 11 & 8 & 15 \\ 13 & 16 & 4 & 12 & 29 \\ 10 & 7 & 32 & 6 & 5 \end{bmatrix}$

We can partition this matrix in different ways. We can create a maximum of 20 different submatrices. The maximum number of submatrices is equal to the total number of entries in this matrix, i.e., the size of the matrix. Below, we give an example of a partition for block matrix C.

$$c_1 = \begin{bmatrix} 8 & 6 & 2 \\ 2 & 5 & 11 \end{bmatrix}, c_2 = \begin{bmatrix} -5 & -9 \\ 8 & 15 \end{bmatrix}, c_3 = \begin{bmatrix} 13 & 16 & 4 \\ 10 & 7 & 32 \end{bmatrix}, c_4 = \begin{bmatrix} 12 & 29 \\ 6 & 5 \end{bmatrix}$$

$$C = \left(\begin{array}{c|c} c_1 & c_2 \\ \hline c_3 & c_4 \end{array} \right) = \left(\begin{array}{ccc|cc} 8 & 6 & 2 & -5 & -9 \\ 2 & 5 & 11 & 8 & 15 \\ \hline 13 & 16 & 4 & 12 & 29 \\ 10 & 7 & 32 & 6 & 5 \end{array} \right)$$

Definition 2. *The block diagonal matrix is a matrix with square blocks in the main diagonal position and zero matrices elsewhere. A block matrix is called block upper triangular/block lower triangular if all the submatrices below/above the main diagonal are zero matrices. A matrix is called block triangular if it is either block upper triangular or block lower triangular.*

Definition 3. *A block matrix is called a block identity matrix if all the off-diagonal matrices are zero matrices.*

Definition 4. *A block elementary matrix is the matrix obtained from the block identity matrix by applying only one elementary row operation.*

Definition 5. *Let* $U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in M_{2\times 2}(F)$ *and let* $V \in M_{n\times n}(F)$*. Then, the tensor product* $U \otimes V$ *is defined as the* $2n \times 2n$ *matrix* $\begin{pmatrix} u_{11}V & u_{12}V \\ u_{21}V & u_{22}V \end{pmatrix}$*. Similarly, if* $U \in M_{m\times m}(F)$*, and* $V \in M_{n\times n}(F)$*, then the tensor product* $U \otimes V$ *is the* $mn \times mn$ *matrix*

$$U \otimes V = \begin{pmatrix} u_{11}V & u_{12}V & \cdots & u_{1m}V \\ u_{21}V & u_{22}V & \cdots & u_{2m}V \\ \vdots & \ddots & \ddots & \vdots \\ u_{m1}V & u_{m2}V & \cdots & u_{mm}V \end{pmatrix}.$$

3. Inverses of Block Matrices

3.1. Inverses of Block Diagonal and Block Triangular Matrices

In this first section, we start by giving the inverses of 2×2 block diagonal and block triangular matrices. The same techniques of the proofs can be applied to the block diagonal and block triangular matrices of larger sizes.

Proposition 1. *If* $D = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix}$ *is a block diagonal matrix with square and invertible blocks* D_{ii}*, for* $i \in \{1, 2\}$*, then* D *is invertible, and the inverse of* D *is given via*

$$D^{-1} = \begin{pmatrix} D_{11}^{-1} & 0 \\ 0 & D_{22}^{-1} \end{pmatrix}.$$

Proof. One can easily observe that the multiplication of the two matrices will produce the block identity matrix. The Gauss elimination method is another way to see the result. □

Proposition 2. *If* $U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$ *is a block upper triangular matrix with square and invertible main blocks* U_{ii}*, for* $i \in \{1, 2\}$*, then* U *is invertible and the inverse is given by*

$$U^{-1} = \begin{pmatrix} U_{11}^{-1} & -U_{11}^{-1}U_{12}U_{22}^{-1} \\ 0 & U_{22}^{-1} \end{pmatrix}.$$

Proof. Using the block Gauss Elimination, we obtain

$$\left(\begin{array}{cc|cc} U_{11} & U_{12} & I & 0 \\ 0 & U_{22} & 0 & I \end{array}\right) R_1 \to U_{11}^{-1} R_1 \left(\begin{array}{cc|cc} I & U_{11}^{-1}U_{12} & U_{11}^{-1} & 0 \\ 0 & U_{22} & 0 & I \end{array}\right) R_2 \to U_{22}^{-1} R_2$$

$$\left(\begin{array}{cc|cc} I & U_{11}^{-1}U_{12} & U_{11}^{-1} & 0 \\ 0 & I & 0 & U_{22}^{-1} \end{array}\right) R_1 \to -U_{11}^{-1}U_{12}R_2 + R_1 \left(\begin{array}{cc|cc} I & 0 & U_{11}^{-1} & -U_{11}^{-1}U_{12}U_{22}^{-1} \\ 0 & I & 0 & U_{22}^{-1} \end{array}\right)$$

$$UU^{-1} = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} \begin{pmatrix} U_{11}^{-1} & -U_{11}^{-1}U_{12}U_{22}^{-1} \\ 0 & U_{22}^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \quad \square$$

Similarly, we provide the inverse of the 2×2 lower triangular matrix below; the idea of the proof works in a very similar manner to the proof of Proposition 2.

Proposition 3. *If* $L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}$ *is a block lower triangular matrix with square and invertible main blocks* L_{ii}*, for* $i \in \{1, 2\}$*, then* L *is invertible and the inverse is given via*

$$L^{-1} = \begin{pmatrix} L_{11}^{-1} & 0 \\ -L_{22}^{-1}L_{21}L_{11}^{-1} & L_{22}^{-1} \end{pmatrix}.$$

3.2. Inverses of 2×2 *Block Matrices*

Assume that A is a 2×2 non-singular square block matrix $A = \begin{pmatrix} T & E \\ M & N \end{pmatrix}$, and its inverse is $A^{-1} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$. Let T, E, M, and N be the partitioned matrices in A, with sizes $k \times m$, $k \times n$, $l \times m$, $l \times n$ (with $k + l = m + n$), and P, Q, R, and S be the submatrices in A^{-1} with sizes $m \times k$, $m \times l$, $n \times k$, and $n \times l$, respectively, for the multiplications to be compatible. We can verify A^{-1}; here, we consider the following two cases. The case where all the blocks are square matrices (i.e., if $k = m = n = l$) is discussed in Remark 1.

- Square matrices in diagonal positions in A and A^{-1}, implying $k = m$ and $l = n$.
- Square matrices in anti-diagonal positions of A and A^{-1} implying $k = n$ and $m = l$.

The following two theorems deal with the first case where matrices in the diagonal positions are all square.

Theorem 1. *Let* T *be non-singular. Then,* A^{-1} *exists if and only if the matrix* $N - MT^{-1}E$ *is invertible and*

$$A^{-1} = \begin{pmatrix} T^{-1} + T^{-1}E(N - MT^{-1}E)^{-1}MT^{-1} & -T^{-1}E(N - MT^{-1}E)^{-1} \\ -(N - MT^{-1}E)^{-1}MT^{-1} & (N - MT^{-1}E)^{-1} \end{pmatrix}.$$

Proof. First, we use the block Gauss elimination on block matrix A. The Gauss elimination method on the block matrices is not as straightforward as the one on standard matrices without blocks. At each stage of performing an elementary row operation, one must check that the sizes of the matrices are compatible.

$$\left(\begin{array}{cc|cc} T & E & I & 0 \\ M & N & 0 & I \end{array}\right) R_1 \to T^{-1}R_1 \left(\begin{array}{cc|cc} I & T^{-1}E & T^{-1} & 0 \\ M & N & 0 & I \end{array}\right) R_2 \to -MR_1 + R_2$$

$$\left(\begin{array}{cc|cc} I & T^{-1}E & T^{-1} & 0 \\ 0 & N - MT^{-1}E & -MT^{-1} & I \end{array}\right) R_2 \to (N - MT^{-1}E)^{-1}R_2$$

$$\left(\begin{array}{cc|cc} I & T^{-1}E & T^{-1} & 0 \\ 0 & I & -(N - MT^{-1}E)^{-1}MT^{-1} & (N - MT^{-1}E)^{-1} \end{array}\right) R_1 \to (-T^{-1}E)R_2 + R_1$$

$$\left(\begin{array}{cc|cc} I & 0 & T^{-1} + T^{-1}E(N - MT^{-1}E)^{-1}MT^{-1} & -T^{-1}E(N - MT^{-1}E)^{-1} \\ 0 & I & -(N - MT^{-1}E)^{-1}MT^{-1} & (N - MT^{-1}E)^{-1} \end{array}\right).$$

Another way of proving this inverse is via realising that A can be written as a product of four block elementary matrices. Note that

$$A = \begin{pmatrix} T & E \\ M & N \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ M & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & N - MT^{-1}E \end{pmatrix}\begin{pmatrix} I & T^{-1}E \\ 0 & I \end{pmatrix}.$$

In Section 3.1, the inverses of these block elementaries can easily be computed, which would give a second way of proving the theorem. Also, note that $\begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ M & I \end{pmatrix} = \begin{pmatrix} T & 0 \\ M & I \end{pmatrix}$, which would give the LDU decomposition of the matrix A above. □

Theorem 2. *Let N be non-singular. Then, A^{-1} exists if and only if the matrix $T - EN^{-1}M$ is invertible, and*

$$A^{-1} = \begin{pmatrix} (T - EN^{-1}M)^{-1} & -(T - EN^{-1}M)^{-1}EN^{-1} \\ -N^{-1}M(T - EN^{-1}M)^{-1} & N^{-1} + N^{-1}M(T - EN^{-1}M)^{-1}EN^{-1} \end{pmatrix}.$$

Proof. Again, by using block Gauss elimination method, we obtain

$$\left(\begin{array}{cc|cc} T & E & I & 0 \\ M & N & 0 & I \end{array}\right) R_2 \to N^{-1}R_2 \left(\begin{array}{cc|cc} T & E & I & 0 \\ N^{-1}M & I & 0 & N^{-1} \end{array}\right) R_1 \to -ER_2 + R_1$$

$$\left(\begin{array}{cc|cc} T - EN^{-1}M & 0 & I & -EN^{-1} \\ N^{-1}M & I & 0 & N^{-1} \end{array}\right) R_1 \to (T - EN^{-1}M)^{-1}R_1$$

$$\left(\begin{array}{cc|cc} I & 0 & (T - EN^{-1}M)^{-1} & -(T - EN^{-1}M)^{-1}EN^{-1} \\ N^{-1}M & I & 0 & N^{-1} \end{array}\right)$$

$$R_2 \to -(N^{-1}M)R_1 + R_2$$

$$\left(\begin{array}{cc|cc} I & 0 & (T - EN^{-1}M)^{-1} & -(T - EN^{-1}M)^{-1}EN^{-1} \\ 0 & I & -N^{-1}M(T - EN^{-1}M)^{-1} & N^{-1} + N^{-1}M(T - EN^{-1}M)^{-1}EN^{-1} \end{array}\right).$$

As we illustrated above, we next express A as a product of four block elementary matrices. These block elementary matrices can be observed by applying the inverses of the elementary row operations to the block identity matrix. Note that

$$A = \begin{pmatrix} T & E \\ M & N \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} I & E \\ 0 & I \end{pmatrix} \begin{pmatrix} T - EN^{-1}M & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ N^{-1}M & I \end{pmatrix}.$$

From this equation, it is also very straightforward to observe A^{-1}. As we did above in the proof of Theorem 1, the two elementary matrices on the left can be multiplied to give a lower triangular matrix, which produces the LDU decomposition of the given matrix A. □

If the square blocks are now in the anti-diagonal positions of A and A^{-1}, a small trick can be applied to move these square blocks to the diagonal positions. Assume that J is a matrix with 1's in the reverse diagonal position and 0's elsewhere. Then, AJ reverses the order of columns of A, and JA reverses the order of rows of A.

$$AJ = \begin{pmatrix} T & E \\ M & N \end{pmatrix} \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} = \begin{pmatrix} EJ & TJ \\ NJ & MJ \end{pmatrix}.$$

Note that $J(AJ)^{-1} = A^{-1}$. Therefore, Theorems 1 and 2 can be used to obtain the inverse formulas for the 2×2 block matrices when the square matrices are in the anti-diagonal positions.

Remark 1. *Note that inverse formulas in Theorems 1 and 2 are equivalent, if T and N are both non-singular. Also note that if we have square matrices in all positions of A and A^{-1}, that is to say, if $k = m = n = l$, then Theorems 1 and 2 and also the remaining two theorems coming from square matrices in anti-diagonal positions of A, must coincide and produce the same inverse formula, depending, of course, on the invertibility of the square blocks. This technique will always work by splitting a block matrix of any size into four blocks and using theorems above by considering the positions of the square blocks.*

Remark 2. *The inverses of the diagonal and triangular matrices in Section 3.1 can also be computed using Theorems 1 and 2.*

4. Determinants of Block Matrices

4.1. The Commutative Case

The set of all $n \times n$ matrices with entries from a field forms a ring with unity but not a field, as multiplicative inverses of elements may not exist. Therefore, 2×2 block matrix A, with entries as the matrices of sizes $n \times n$ can be thought of as having a size of 2×2 in the ring of matrices but of a size of $2n \times 2n$ in the considered field. For a 2×2 matrix, $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = ad - bc$. For the same formula to make sense for the 2×2 block matrix $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, (each block an element of $M_{n\times n}(F)$, where F will denote the field under consideration), we cannot just use any subring R of $M_{n\times n}(F)$, but we would need a commutative one for the expressions $AD - BC, AD - CB, DA - BC, DA - CB$ to be all equivalent. As it is in [3], we will use the notation $\det_R T$ to take the determinant in this commutative subring, where elements are considered as matrices, and the notation $\det_F T$, where the determinant is taken in the usual way with elements in a field, like the field of real numbers or complex numbers. Therefore, note that $\det_R T$, here, will not be a scalar, but it will be a matrix over F. We state the main result of this section below; for proof, please refer to [3].

Theorem 3. *For a given field F, let R be a commutative subring of $M_{n\times n}(F)$, and let $T \in M_{p\times p}(R)$. Then,*

$$\det_F T = \det_F(\det_R T).$$

Below, we start with the determinants of 2×2 block matrices, where each block is an element of $M_{n\times n}(F)$. First, we consider the block diagonal block upper/lower triangular matrices.

Proposition 4. *Given* $T = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$*, a block diagonal matrix,* $\det_F T = \det_F(MN) = \det_F M \det_F N$.

Proof. This can be proved via induction on the size of the matrix M in the top left corner. Here, $n = 1$ case can be observed immediately. Next, the cofactor expansion is applied in the first row or column, and we assume that the results holds for the $n-1$ case to prove the general $n \times n$ case. Actually, the same proof will apply when the block matrices on the main diagonal position are of different sizes. For details of the proofs, please refer to [13]. □

The similar inductive proofs can be used to prove the following, where one needs to use cofactor expansion along rows for lower triangular block matrices and along columns for upper triangular block matrices.

Proposition 5. *If* $T = \begin{pmatrix} M & 0 \\ R & N \end{pmatrix}$*, or* $T = \begin{pmatrix} M & P \\ 0 & N \end{pmatrix}$*, then* $\det_F T = \det_F(MN) = \det_F M \det_F N$.

Proposition 6. *If* $T = \begin{pmatrix} M & P \\ R & N \end{pmatrix}$*, and if at least one of the blocks* M, P, R, N *is a zero matrix, then* $\det_F T = \det_F(MN - PR)$.

Proof. Propositions 4 and 5 show the cases where the off-diagonal matrices are the zero matrices. If one or both of the main diagonal matrices are zero, then we use the following approach:

$$\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}\begin{pmatrix} M & P \\ R & N \end{pmatrix} = \begin{pmatrix} -R & -N \\ M & P \end{pmatrix},$$

giving $\det_F \begin{pmatrix} M & P \\ R & N \end{pmatrix} = \det_F \begin{pmatrix} -R & -N \\ M & P \end{pmatrix}$. Hence,

$$\det_F \begin{pmatrix} M & P \\ R & 0 \end{pmatrix} = \det_F \begin{pmatrix} -R & 0 \\ M & P \end{pmatrix} = \det_F(-R)\det_F(P) = \det_F \begin{pmatrix} -R & -N \\ 0 & P \end{pmatrix} \det_F \begin{pmatrix} 0 & P \\ R & N \end{pmatrix}.$$

Morover,

$$\begin{pmatrix} M & 0 \\ -I & N \end{pmatrix}\begin{pmatrix} I & N \\ 0 & I \end{pmatrix} = \begin{pmatrix} M & MN \\ -I & 0 \end{pmatrix}$$

proving also the multiplicative property $\det_F(MN) = \det_F M \det_F N$ for matrices. □

We give the following a more general result, where the proof can be found in [3] and involves working not in a particular field (or ring) F but instead in the polynomial ring $F[x]$.

Theorem 4. *Given* $T = \begin{pmatrix} M & P \\ R & N \end{pmatrix}$*, where* M, P, R, N *are* $n \times n$ *matrices, with coefficients in a field, and* $RN = NR$*,*

$$\det_F T = \det_F(MN - PR).$$

Remark 3. *Given* $T = \begin{pmatrix} M & P \\ R & N \end{pmatrix}$, *where* M, P, R, N *are* $n \times n$ *matrices, with coefficients in a field, and* N *is invertible, note that* $\begin{vmatrix} M & P \\ R & N \end{vmatrix} = \det\big(MN - PN^{-1}RN\big)$. *The result can be seen from the following matrix equation.*

$$\begin{pmatrix} M & P \\ R & N \end{pmatrix}\begin{pmatrix} I & 0 \\ -N^{-1}R & I \end{pmatrix} = \begin{pmatrix} M - PN^{-1}R & P \\ 0 & N \end{pmatrix}.$$

Remark 4. *For a matrix* $U \in M_{2\times 2}(F)$ *and* $V \in M_{n\times n}(F)$,

$$\begin{aligned} \det_F(U \otimes V) &= \det_F((u_{11}V)(u_{22}V) - (u_{12}V)(u_{21}V)) \\ &= \det_F\Big((u_{11}u_{22} - u_{12}u_{21})V^2\Big) \\ &= \det_F\Big((\det_F U)V^2\Big) \\ &= (\det_F U)^n(\det_F V)^2 \end{aligned}$$

Following from the remark above, we generalise the results for the determinant of the tensor product of two matrices as follows:

Proposition 7. *Let* $U \in M_{m\times m}(F)$ *and* $V \in M_{n\times n}(F)$. *Then,*

$$\det(U \otimes V) = (\det_F U)^n(\det_F V)^m.$$

Proof. We can prove the result via induction on m; the size of the matrix U. If $m = 1$, then U is 1×1, and $U \otimes V = [u_{11}V]$ with $\det_F(u_{11}V) = (u_{11})^n \det_F(V)$. Next, we assume that result holds for the case $m - 1$ and try to prove it for case m.

$$\begin{aligned} \det_R(U \otimes V) &= u_{11}V\Big[\det_R(U^{11}_{m-1} \otimes V)\Big] - u_{12}V\Big[\det_R(U^{12}_{m-1} \otimes V)\Big] + \cdots + \\ &\quad (-1)^{m+1}u_{1m}V\Big[\det_R(U^{1m}_{m-1} \otimes V)\Big] \\ &= u_{11}V\Big((\det_F U^{11}_{m-1})V^{m-1}\Big) - u_{12}V\Big((\det_F U^{12}_{m-1})V^{m-1}\Big) + \cdots + \\ &\quad (-1)^{m+1}u_{1m}V\Big((\det_F U^{1m}_{m-1})V^{m-1}\Big) \\ &= (\det_F U)V^m \end{aligned}$$

where U^{ij}_{m-1} denotes the square matrix of size $m-1$, when the ith row and jth column are deleted from U. Next, using Theorem 3, we obtain

$$\begin{aligned} \det_F(U \otimes V) &= \det_F(\det_R(U \otimes V)) = \det_F((\det_F U)V^m) \\ &= (\det_F U)^n(\det_F V^m) = (\det_F U)^n(\det_F V)^m. \end{aligned}$$

□

4.2. The General Case

In this section, we give a general formula for the determinant of an $m \times m$ block matrix, a result due to [2].

Theorem 5. *Given a* $(np) \times (np)$ *block* M, *divided into* p^2 *blocks each of them of size* $n \times n$,

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1p} \\ M_{21} & M_{22} & \cdots & M_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ M_{p1} & M_{p2} & \cdots & M_{pp} \end{bmatrix}$$

the determinant of M is given via

$$\det(M) = \prod_{k=1}^{p} \det\left(\alpha_{kk}^{p-k}\right),$$

where the α matrices are defined as

$$\begin{aligned} \alpha_{ij}^{(0)} &= M_{ij} \\ \alpha_{ij}^{(k)} &= M_{ij} - \sigma_{i,p-k+1}^{T} \widetilde{M}_k^{-1} m_{p-k+1,j} \qquad k \geq 1 \end{aligned}$$

where the vectors σ_{ij}^T, m_{ij} are $\sigma_{ij}^T = (M_{ij}, M_{i,j+1}, \cdots, M_{ip})$ and $m_{ij} = (M_{ij}, M_{i+1,j}, \cdots, M_{pj})^T$. Here, $\widetilde{M}_k$ matrix is given via

$$\widetilde{M}_k = \begin{bmatrix} M_{p-k+1,p-k+1} & M_{p-k+1,p-k+2} & \cdots & M_{p-k+1,p} \\ M_{p-k+2,p-k+1} & M_{p-k+2,p-k+2} & \cdots & M_{p-k+2,p} \\ \vdots & \vdots & \cdots & \vdots \\ M_{p,p-k+1} & M_{p,p-k+2} & \cdots & M_{p,p} \end{bmatrix}.$$

Next, we will give the determinant formulas for the cases $p = 2$ and $p = 3$. Consequently, we will show that the formula for $p = 2$ case coincides with the formula in Remark 3 of the previous section.

1. The case where block matrix M has four blocks (i.e., $p = 2$).
   In this case, $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ with $\det(M) = \prod_{k=1}^{2} \det\left(\alpha_{kk}^{2-k}\right) = \det\left(\alpha_{11}^{1}\right) \det\left(\alpha_{22}^{0}\right)$.
   Here, $\alpha_{22}^0 = M_{22}$ and $\alpha_{11}^1 = M_{11} - \sigma_{12}^T M_{22}^{-1} m_{21} = M_{11} - M_{12} M_{22}^{-1} M_{21}$, giving

$$\begin{aligned} \det(M) = \det\left(\alpha_{11}^1\right) \det\left(\alpha_{22}^0\right) &= \det\left(M_{11} - M_{12} M_{22}^{-1} M_{21}\right) \det(M_{22}) \\ &= \det\left(M_{11} M_{22} - M_{12} M_{22}^{-1} M_{21} M_{22}\right) \end{aligned}$$

   which is the formula stated in the previous section in Remark 3.

2. The case where block matrix M has nine blocks (i.e., $p = 3$).
   For this case, $M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$, with $\det(M) = \prod_{k=1}^{3} \det\left(\alpha_{kk}^{3-k}\right)$
   $= \det\left(\alpha_{11}^2\right) \det\left(\alpha_{22}^1\right) \det\left(\alpha_{33}^0\right)$.

   Here, $\alpha_{33}^0 = M_{33}$, $\alpha_{22}^1 = M_{22} - \sigma_{23}^T \widetilde{M}_1^{-1} m_{32}$, and $\alpha_{11}^2 = M_{11} - \sigma_{12}^T \widetilde{M}_2^{-1} m_{21}$.

$$\begin{aligned} \det(M) &= \det\left(\alpha_{11}^2\right) \det\left(\alpha_{22}^1\right) \det\left(\alpha_{33}^0\right) \\ &= \det\left[M_{11} - \begin{pmatrix} M_{12} & M_{13} \end{pmatrix} \begin{pmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{pmatrix}^{-1} \begin{pmatrix} M_{21} \\ M_{31} \end{pmatrix}\right] \\ &\quad \times \det\left(M_{22} - M_{23} M_{33}^{-1} M_{32}\right) \det(M_{33}). \end{aligned} \tag{1}$$

Using Theorem 2 for the inverse of the 2×2 block matrix and after some simplification, we obtain the following formulae.

$$\begin{aligned}\det(M) = \Big[& \det\left(M_{11} - M_{13}M_{33}^{-1}M_{31}\right) \\ & - \left(M_{12} - M_{13}M_{33}^{-1}M_{32}\right)\left(M_{22} - M_{23}M_{33}^{-1}M_{32}\right)^{-1}\left(M_{21} - M_{23}M_{33}^{-1}M_{31}\right)\Big] \\ & \times \det\left(M_{22} - M_{23}M_{33}^{-1}M_{32}\right)\det(M_{33}).\end{aligned}$$

Example 2. *Given the following matrix Q, we compute $|Q|$ using the formula (1) for the determinant of the 3×3 block matrix given above.*

$$Q = \left(\begin{array}{cccc|cccc|cccc} 17 & -8 & 5 & 7 & 13 & -6 & -112 & 99 & -5 & 2 & 11 & 9 \\ 5 & 6 & 11 & 14 & 17 & 21 & 20 & 18 & 13 & 12 & 11 & 10 \\ -15 & 18 & 13 & 16 & 19 & -16 & -18 & 17 & 13 & 15 & 10 & 17 \\ -2 & 6 & 2 & 13 & 8 & 5 & 17 & 17 & 6 & 2 & 6 & 3 \\ \hline 18 & 13 & 12 & 16 & 48 & 59 & 21 & 24 & -9 & -7 & -8 & -5 \\ 13 & 14 & 15 & 17 & 21 & 26 & 42 & 38 & 66 & 75 & 21 & 23 \\ 66 & 8 & 14 & 63 & 21 & 12 & 13 & 15 & 2 & 3 & 14 & -18 \\ 11 & 12 & 15 & 21 & 29 & 49 & 12 & 7 & 8 & 4 & 17 & 12 \\ \hline 9 & 13 & -3 & -8 & 17 & 0 & 6 & 60 & 25 & -47 & 18 & 88 \\ 11 & -11 & -15 & -17 & -19 & 21 & 25 & 46 & 68 & 111 & 200 & 300 \\ 22 & 99 & 13 & -2 & -4 & 5 & 6 & -13 & 48 & 54 & -22 & 19 \\ 68 & 57 & 43 & 21 & -30 & 18 & -17 & 5 & 4 & 8 & 16 & -77 \end{array}\right)$$

$$\begin{aligned}\det(Q) = \det\Bigg(& \begin{bmatrix} 17 & -8 & 5 & 7 \\ 5 & 6 & 11 & 14 \\ -15 & 18 & 13 & 16 \\ -2 & 6 & 2 & 13 \end{bmatrix} - \begin{bmatrix} 13 & -6 & -112 & 99 & -5 & 2 & 11 & 9 \\ 17 & 21 & 20 & 18 & 13 & 12 & 11 & 10 \\ 19 & -16 & -18 & 17 & 13 & 15 & 10 & 17 \\ 8 & 5 & 17 & 17 & 6 & 2 & 6 & 3 \end{bmatrix} \\ & \times \begin{bmatrix} 48 & 59 & 21 & 24 & -9 & -7 & -8 & -5 \\ 21 & 26 & 42 & 38 & 66 & 75 & 21 & 23 \\ 21 & 12 & 13 & 15 & 2 & 3 & 14 & -18 \\ 29 & 49 & 12 & 7 & 8 & 4 & 17 & 12 \\ 17 & 0 & 6 & 60 & 25 & -47 & 18 & 88 \\ -19 & 21 & 25 & 46 & 68 & 111 & 200 & 300 \\ -4 & 5 & 6 & -13 & 48 & 54 & -22 & 19 \\ -30 & 18 & -17 & 5 & 4 & 8 & 16 & -77 \end{bmatrix}^{-1} \begin{bmatrix} 18 & 13 & 12 & 16 \\ 13 & 14 & 15 & 17 \\ 66 & 8 & 14 & 63 \\ 11 & 12 & 15 & 21 \\ 9 & 13 & -3 & -8 \\ 11 & -11 & -15 & -17 \\ 22 & 99 & 13 & -2 \\ 68 & 57 & 43 & 21 \end{bmatrix}\Bigg) \\ & \times \det\Bigg(\begin{bmatrix} 48 & 59 & 21 & 24 \\ 21 & 26 & 42 & 38 \\ 21 & 12 & 13 & 15 \\ 29 & 49 & 12 & 7 \end{bmatrix} - \begin{bmatrix} -9 & -7 & -8 & -5 \\ 66 & 75 & 21 & 23 \\ 2 & 3 & 14 & -18 \\ 8 & 4 & 17 & 12 \end{bmatrix} \begin{bmatrix} 25 & -47 & 18 & 88 \\ 68 & 111 & 200 & 300 \\ 48 & 54 & -22 & 19 \\ 4 & 8 & 16 & -77 \end{bmatrix}^{-1} \\ & \begin{bmatrix} 17 & 0 & 6 & 60 \\ -19 & 21 & 25 & 46 \\ -4 & 5 & 6 & -13 \\ -30 & 18 & -17 & 5 \end{bmatrix}\Bigg) \det\begin{bmatrix} 25 & -47 & 18 & 88 \\ 68 & 111 & 200 & 300 \\ 48 & 54 & -22 & 19 \\ 4 & 8 & 16 & -77 \end{bmatrix}.\end{aligned}$$

$$= \det\begin{bmatrix} -\frac{8520955680187479000}{4929589651966029} & -\frac{11264575846381700000}{743587934033933} & -\frac{2736628770512193000}{862678189094055} & -\frac{35301254789755064000}{4140855307651465} \\ \frac{3699705604538041 0}{646058923814921} & \frac{4006106830415344000}{6503176130099903} & \frac{22861318028117330}{1415657388865285} & \frac{202741742474135070}{5935162528718627} \\ -\frac{3672798715419834400}{7127435775330321} & -\frac{5594926198751356000}{1461681164862663} & -\frac{10225134648446195 0}{1484882453193817} & -\frac{1289029481926729 40}{477283645669441} \\ \frac{187543771792049950}{2147154680763521} & \frac{27134787302041260 0}{2683943350954401} & \frac{9304801959579688}{505212866062005} & \frac{313280250781215300}{5591548647821669} \end{bmatrix}$$

$$\times \det \begin{bmatrix} \frac{14491339}{334067} & \frac{21039726}{334067} & \frac{6712187}{334067} & \frac{9636843}{334067} \\ \frac{2055547786}{43762777} & \frac{457642245}{87525554} & \frac{3744329823}{87525554} & \frac{1546420078}{43762777} \\ \frac{1413707764}{43762777} & \frac{209967912}{43762777} & \frac{818862735}{43762777} & \frac{422234741}{43762777} \\ \frac{1508549149}{43762777} & \frac{3855574845}{87525554} & \frac{1157667791}{87525554} & -\frac{126678038}{43762777} \end{bmatrix} \det \begin{bmatrix} 25 & -47 & 18 & 88 \\ 68 & 111 & 200 & 300 \\ 48 & 54 & -22 & 19 \\ 4 & 8 & 16 & -77 \end{bmatrix}.$$

$$= (4.430505045192262 \times 10^{21/445000051556509})(-2.5689046090548023 \times 10^{21/4677548208506851})(87525554)$$
$$= -3.906057144336905 \times 10^{35/816170407800069}.$$

Here, we can use Theorem 2 to find the inverse of the 8×8 *matrix (or the* 2×2 *block matrix, with blocks of size* 4×4*) in the formula above. Note that one would need to calculate determinants of* 11 *matrices (because of the single* 0 *in matrix* Q*), of size* 11×11*, to be able to compute this determinant via cofactor expansion.*

5. Conclusions

This paper aims to serve as a primary reference for block matrices, where the two important concepts, determinants and inverses, are discussed in detail. The inverse formulas for the 2×2 block matrices are given, where one of the four blocks is an invertible matrix. This method could be generalised to block matrices of larger sizes by splitting it into four blocks. On the other hand, the determinant properties of block matrices are also studied; here, we use two main approaches from the literature. First of all, the determinant formula for a block matrix with entries (blocks) belonging to a commutative subring of $M_{n \times n}(F)$ is given. However, as matrix multiplication is not always commutative, this formula unfortunately does not work in many cases. Therefore, in the last part of this paper, we revise the general formulas from the literature for the determinant of any given $n \times n$ block matrix. These formulas also use (as the size of the block matrices gets bigger) inverse formulas for the block matrices. The determinant formula for the tensor product of two matrices is also provided. As we have stated in the introduction, the applications of these matrices arise in many fields; we plan to look into that in our future work.

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