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Quasi-Exponentiated Normal Distributions: Mixture Representations and Asymmetrization

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Abstract: In the paper, quasi-exponentiated normal distributions are introduced for any real power (exponent) no less than two. With natural exponents, the quasi-exponentiated normal distributions coincide with the distributions of the corresponding powers of normal random variables with zero mean. Their representability as scale mixtures of normal and exponential distributions is proved. The mixing distributions are written out in the closed form. Two approaches to the construction of asymmetric quasi-exponentiated normal distributions are described. A limit theorem is proved for sums of a random number of independent random variables in which the asymmetric quasi-exponentiated normal distribution is the limit law.

Keywords: quasi-exponentiated normal distribution; exponential power distribution; generalized gamma distribution; scale mixture; limit theorem; random sum

MSC: 60F05; 60G50; 60G55; 62E20; 62G30

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1. Introduction—History of the Problem and Motivation

This paper was motivated by a suggestion by Yu. V. Prokhorov that he made in the mid-1980s. He suggested that the distributions of all odd powers of a normal random variable with zero mean belong to the class of normal scale mixtures. The validity of this suggestion was announced in the short paper [1] by E. Bagirov, PhD student of Yu. V. Prokhorov. This suggestion was proved in Bagirov's PhD thesis. However, the corresponding mixing distributions were not specified. Some recent results concerning the properties of normal mixtures and related topics (see the references in the lemmas below) have made it possible to give a more or less complete description of the mixture properties of the distributions of real powers of normal random variables with zero mean (to be formally correct, here, we have to speak of 'quasi-powers', since the normal random variables take values of both signs). It is this description that the present paper deals with.

Here, we give an extended solution of Prokhorov's suggestion and introduce *quasi-exponentiated normal distributions* for any real power (exponent) no less than two. With odd natural exponents, the quasi-exponentiated normal distributions coincide with the distributions of the corresponding powers of the normal random variables with zero mean. It is shown that all these distributions belong to the class of normal scale mixtures. The distributions of mixing (scaling) random variables are written out in a closed form, so that it is possible to describe their properties. For example, it is proved that all the mixing distributions are mixed exponential, so actually, all quasi-exponentiated normal distributions are scale mixtures of the Laplace (double-exponential) distributions and are infinitely divisible.

Two approaches are described to the construction of asymmetric generalizations of quasi-exponentiated normal distributions. The first of them, called R-asymmetrization (randomized asymmetrization), is based on the representation of the corresponding asymmetric quasi-exponentiated normal distribution as the discrete Bernoullian mixture of the distribution of the absolute value of the quasi-power of the normal random variable and that of the reflection of such a random variable with different parameters to the negative axis. These R-asymmetric models are rather artificial, and can hardly be seriously substantiated when they are applied to describe real statistical regularities.

The second approach, called M-asymmetrization, is based on the representation of the symmetric quasi-exponentiated normal distribution as the normal scale mixture, and consists of extending the mixing to the normal variance-mean mixture with the same mixing distribution as is used in the symmetric case. The notion of a normal variance-mean mixture was proposed by O. E. Barnorff-Nielsen and his colleagues in [2]. These mixtures proved to provide excellent fit with many statistical regularities observed in various applied problems. In paper [3], this circumstance received a reasonable theoretic explanation. It was proved in that paper that normal variance-mean mixtures appear as limit laws in rather general limit theorems for random sums of independent identically distributed random variables with finite variances. Hence, normal variance-mean mixtures appear to be good asymptotic approximations. This circumstance makes M-asymmetric quasi-exponentiated normal distributions promising candidates for their application as models of some statistical regularities. To illustrate this, we present the corresponding limit theorem, in which the M-asymmetric quasi-exponentiated normal distribution appears as the limit law.

Although the motive to write this paper was rather theoretical—to extend and generalize the Prokhorov–Bagirov suggestion concerning the mixture representability of the exponentiated normal random variable—we dare say that the presented results also have practical importance.

First, as it will be demonstrated in Section 3, the distribution of the exponentiated absolute value of the normal random variable is a special case of the one-sided generalized gamma distribution. The literature on the applications of the generalized gamma models to real data in various fields is almost immense. Moreover, recently, some papers have appeared dealing with two-sided generalizations of the gamma distribution (see, e.g., [4–9]), in which the application of the so-called bilateral gamma distributions to the problems of financial mathematics, including modelling returns and risks distributions and option pricing, is discussed. The bilateral gamma distributions are defined there as the convolutions of (symmetric) two-sided gamma distributions. In paper [10], another approach to the construction of the bilateral generalization of the gamma distributions was proposed, according to which the (in general, asymmetric) bimodal generalized gamma distribution defined on the whole real axis is defined as the Bernoullian mixture of two gamma distributions where the weighting Bernoulli random variable takes values $+1$ and -1. In that paper, some examples were given of fitting this bimodal generalized gamma distribution to real datasets. So, in the present paper in accordance with the first, randomized, approach mentioned above, we actually introduce a similar model with more flexible tail behaviour by the Bernoullian mixing of two generalized gamma distributions.

Second, along with generalizing the generalized gamma model by Bernoullian mixing, as was already said, we propose another approach to the construction of two-sided generalizations based on the normal variance-mean mixture representability of the quasi-exponentiated normal distribution. In applied probability, there is a convention that a model probability distribution can be considered as well justified or adequate if it is an *asymptotic approximation*, that is, if it is possible to suggest a more or less simple limit setting (say, within schemes of maximum or summation of random variables) and a corresponding limit theorem in which the distribution under consideration is a limit law. Apparently, this convention goes far back to the book [11]. For further discussion of this convention, see [12]. The existence of such limit setting can provide a better understanding of real mechanisms that generate observed statistical regularities. The normal variance-mean mixture representability of the asymmetric quasi-exponentiated normal distribution makes it possible to formulate and prove a rather simple limit theorem (see Section 6 below) presenting 'if and only if' conditions for the asymmetric quasi-exponentiated normal distribution to be a limit law for random sums of independent identically distributed random variables. Actually, this theorem describes the limit behaviour of a randomly stopped random walk. Hence, this theorem presents some arguments in favour of the possible utility of the asymmetric quasi-exponentiated normal distribution for modelling data that may be assumed to have an additive structure.

The paper is organized as follows. Section 2 contains auxiliary definitions and notation. Symmetric quasi-exponentiated normal distributions are introduced in Section 3. Some scale mixture representations for this distribution are also proved here. In Section 4, we introduce the R-asymmetric two-sided quasi-exponentiated normal distribution as the Bernoullian scale mixture of two special generalized gamma distributions and present formulas for the moments of this distribution. In Section 5, the M-asymmetric two-sided quasi-exponentiated normal distribution is introduced as a special normal variance-mean mixture, and the formulas for its moments are presented. In Section 6, we formulate and prove a version of the central limit theorem for random sums in which the M-asymmetric two-sided quasi-exponentiated normal distribution is a limit law. Here, we also give an example of the 'doubly stochastic' geometric distribution of the number of summands that provide the validity of the random-sum version of the central limit theorem mentioned above.

2. Auxiliary Definitions and Notation

It is assumed that all the random variables are defined on the same probability space $(\Omega, \mathcal{A}, \mathsf{P})$. In the present paper, we keep to the way of reasoning that can be regarded as arithmetical in the space of random variables. Under this approach, instead of the operation of scale mixing and the corresponding operation of integration of distributions, we consider the operation of multiplication of random variables provided the multipliers are independent. However, speaking of random variables, we actually deal with their distributions. This approach noticeably simplifies the reasoning.

The multiplication of independent random elements is denoted by the symbol $\circ$. The symbols $\stackrel{d}{=}$ and $\Longrightarrow$ denote the coincidence of distributions and convergence in distribution, respectively. The end of the proof is marked be the symbol $\square$. The indicator function of a set A will be denoted $\mathbb{I}_A(z)$: if $z \in A$, then $\mathbb{I}_A(z) = 1$, otherwise $\mathbb{I}_A(z) = 0$.

A random variable with the standard exponential distribution will be denoted as W_1:

$$\mathsf{P}(W_1 < x) = \left[1 - e^{-x}\right]\mathbb{I}_{[0,\infty)}(x).$$

For $x > 0$ and $r > 0$, the (lower) incomplete gamma-function will be denoted as $\Gamma(r; x)$:

$$\Gamma(r; x) = \int_0^x z^{r-1} e^{-z} dz.$$

Let $\Gamma(r) \stackrel{\text{def}}{=} \Gamma(r; \infty)$ be the "usual" Euler's gamma-function.

A random variable having the gamma distribution with shape parameter $r > 0$ and scale parameter $\lambda > 0$ will be denoted as $G_{r,\lambda}$,

$$\mathsf{P}(G_{r,\lambda} < x) = \int_0^x g(z; r, \lambda) dz, \quad \text{with } g(x; r, \lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \mathbb{I}_{[0,\infty)}(x),$$

Obviously, in this notation, $G_{1,1}$ is a random variable with the standard exponential distribution: $G_{1,1} = W_1$.

By generalized gamma distribution, we will mean the absolutely continuous distribution defined by the density

$$gg_{r,\alpha,\mu}(x) = \frac{|\alpha|\mu^r}{\Gamma(r)} x^{\alpha r-1} e^{-\mu x^\alpha} \mathbb{I}_{[0,\infty)}(x) \tag{1}$$

with $\alpha \in \mathbb{R}$, $\mu > 0$, $r > 0$. Generalized gamma distributions were introduced as a separate class in [13]. A random variable with the density $gg_{r,\alpha,\mu}(x)$ will be denoted $\overline{G}_{r,\alpha,\mu}$. It is easy to see that

$$\overline{G}_{r,\alpha,\mu} \stackrel{d}{=} G_{r,\mu}^{1/\alpha} \stackrel{d}{=} \mu^{-1/\alpha} G_{r,1}^{1/\alpha} \stackrel{d}{=} \mu^{-1/\alpha}\overline{G}_{r,\alpha,1}. \tag{2}$$

Let $\gamma > 0$. The distribution of the random variable W_γ:

$$\mathsf{P}(W_\gamma < x) = \left[1 - e^{-x^\gamma}\right]\mathbb{I}_{[0,\infty)}(x),$$

is called the *Weibull distribution* with shape parameter γ. It is easy to see that

$$W_1^{1/\gamma} \stackrel{d}{=} W_\gamma \stackrel{d}{=} \overline{G}_{1,\gamma,1}. \tag{3}$$

The standard normal distribution function and its density will be, respectively, denoted $\Phi(x)$ and $\varphi(x)$,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \Phi(x) = \int_{-\infty}^{x} \varphi(z)dz.$$

A random variable with the standard normal distribution will be denoted by X.

By $\psi_{\alpha,\theta}(x)$, we will denote the probability density of the strictly stable law with characteristic exponent α and parameter θ defined by the characteristic function

$$\mathfrak{g}_{\alpha,\theta}(t) = \exp\left\{ -|t|^\alpha \exp\left\{ -\frac{i\pi\theta\alpha}{2} \mathrm{sign} t \right\}\right\}, \quad t \in \mathbb{R}, \tag{4}$$

with $0 < \alpha \le 2$, $|\theta| \le \theta_\alpha = \min\{1, \frac{2}{\alpha} - 1\}$ (see, e.g., [14]). A random variable with characteristic function (4) will be denoted $S_{\alpha,\theta}$. It is easy to see that $S_{2,0} \stackrel{d}{=} \sqrt{2}\,X$.

If $\theta = 1$ and $0 < \alpha \le 1$, the corresponding strictly stable distribution is concentrated on the nonnegative halfline. If $\alpha = 1$ and $\theta = \pm 1$, then the corresponding stable distributions are degenerate in ± 1, respectively. All the other strictly stable distributions are absolutely continuous. There are no explicit representations for stable distributions in terms of elementary functions with four exceptions: the normal distribution ($\alpha = 2$, $\theta = 0$), the Cauchy distribution ($\alpha = 1$, $\theta = 0$), the Lévy distribution ($\alpha = \frac{1}{2}$, $\theta = 1$), and the distribution symmetric to the Lévy law ($\alpha = \frac{1}{2}$, $\theta = -1$). Expressions for stable densities in terms of generalized Meijer G-functions (Fox functions) can be found in [15,16].

Let $\alpha > 0$. The symmetric exponential power distribution is an absolutely continuous distribution defined by its Lebesgue probability density:

$$p_\alpha(x) = \frac{\alpha}{2\Gamma(\frac{1}{\alpha})} \cdot e^{-|x|^\alpha}, \quad -\infty < x < \infty. \tag{5}$$

To simplify the notation and calculation, here and in what follows, we will use a single parameter α in representation (5), since this parameter is in some sense characteristic and determines the shape of distribution (5). Any random variable with probability density $p_\alpha(x)$ will be denoted Q_α. With $\alpha = 1$, relation (5) defines the classical Laplace distribution

$$p_1(x) = \tfrac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}$$

with zero mean and variance 2. With $\alpha = 2$, relation (5) defines the normal (Gaussian) distribution with zero mean and variance $\frac{1}{2}$:

$$\sqrt{2}\,Q_2 \stackrel{d}{=} X.$$

The class of distributions (5) was introduced and studied in 1923 in paper [17] by M. T. Subbotin. For more detail concerning the properties of exponential power distributions, see [18,19] and the references therein.

It is easy to make sure that

$$|Q_\alpha|^\alpha \stackrel{d}{=} G_{1/\alpha,1}. \tag{6}$$

In paper [20], it was shown that any gamma distribution with a shape parameter no greater than one is mixed exponential. Namely, the following statement holds.

Lemma 1 ([20])**.** *Let $r \in (0,1]$. Then, the density $g(x;r,\mu)$ of a gamma distribution can be represented as*

$$g(x;r,\mu) = \int_0^\infty ze^{-zx}p(z;r,\mu)dz,$$

where

$$p(z;r,\mu) = \frac{\mu^r}{\Gamma(1-r)\Gamma(r)} \cdot \frac{\mathbb{I}_{[\mu,\infty)}(z)}{(z-\mu)^r z}. \tag{7}$$

Moreover, a gamma distribution with shape parameter $r > 1$ cannot be represented as a mixed exponential distribution.

Lemma 2 ([21])**.** *If $r \in (0,1)$, $\mu > 0$ and $G_{r,1}$ and $G_{1-r,1}$ are independent gamma-distributed random variables, then the density $p(z;r,\mu)$ defined by (3) corresponds to the random variable*

$$Z_{r,\mu} = \frac{\mu(G_{r,1}+G_{1-r,1})}{G_{r,1}} \stackrel{d}{=} \mu Z_{r,1} \stackrel{d}{=} \mu\big(1+\tfrac{1-r}{r}R_{1-r,r}\big), \tag{8}$$

where $R_{1-r,r}$ is the r.v. with the Snedecor–Fisher distribution defined by the probability density

$$f(x;1-r,r) = \frac{(1-r)^{1-r}r^r}{\Gamma(1-r)\Gamma(r)} \cdot \frac{\mathbb{I}_{(0,\infty)}(x)}{x^r[r+(1-r)x]}. \tag{9}$$

In other words, if $r \in (0,1)$, then

$$G_{r,\mu} \stackrel{d}{=} W_1 \circ Z_{r,\mu}^{-1}. \tag{10}$$

Lemma 3 ([22])**.** *If $\gamma \in (0,1)$, then the Weibull distribution with parameter γ is mixed exponential:*

$$W_\gamma = W_1 \circ S_\gamma^{-1}. \tag{11}$$

Let $r_1 > 0$, $r_2 > 0$. The Snedecor–Fisher distribution with parameters (r_1, r_2) is defined as the distribution of the random variable

$$P_{r_1,r_2} = G_{r_1,r_1} \circ G_{r_2,r_2}^{-1}.$$

The probability density $p_{r_1,r_2}(x)$ of the Snedecor–Fisher distribution has the form

$$p_{r_1,r_2}(x) = \frac{r_1^{r_1}r_2^{r_2}}{\mathrm{B}(r_1,r_2)} \cdot \frac{x^{r_1-1}}{(r_1x+r_2)^{r_1+r_2}},$$

where $\mathrm{B}(a,b)$ is the beta-function:

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad a > 0,\ b > 0.$$

3. Symmetric Quasi-Exponentiated Normal Distributions

In this section, we introduce quasi-exponentiated normal distributions as the distributions of normal random variables raised to some power, the exponent being an additional parameter.

Since a normal random variable takes values of both signs, it is impossible to formally define the power of this random variable with arbitrary positive exponent no less than two. In paper [1], it was stated that the distributions of all odd powers of the standard normal random variable belong to the class of normal scale mixtures. However, in that paper, the corresponding mixing distributions were not specified. In this subsection, we show that it is possible to give an unambiguous definition of the 'quasi-power' of the normal random variable (and the corresponding 'quasi-exponentiated' normal distribution) for any positive exponent no less than two that coincides with formal powers of the normal random variables with odd exponents and their distributions. Moreover, the corresponding scaling random variables will be defined explicitly. In the next subsection, this notion will be extended to the asymmetric case.

To begin with, consider the random variable $|X|$ (recall that the notation X is used for any random variable with the standard normal distribution). Let $\gamma > 0$. It is obvious that

$$\mathsf{P}(|X|^{\gamma} < x) = 2\Phi(x^{1/\gamma}) - 1, \quad x \geq 0, \tag{12}$$

so the probability density $v_{\gamma}(x)$ of the random variable $|X|^{\gamma}$ has the form

$$v_{\gamma}(x) = \frac{\sqrt{2}}{\gamma\sqrt{\pi}} \cdot x^{1/\gamma - 1} e^{-\frac{1}{2}x^{2/\gamma}}, \quad x \geq 0. \tag{13}$$

It is easily seen that the density $v_{\gamma}(x)$ defined by (13) belongs to the class of generalized gamma distributions. Moreover,

$$v_{\gamma}(x) = gg_{1/2,2/\gamma,1/2}(x), \quad x \in \mathbb{R}.$$

In other words,

$$|X|^{\gamma} \stackrel{d}{=} \overline{G}_{1/2,2/\gamma,1/2} \stackrel{d}{=} G_{1/2,1/2}^{\gamma/2},$$

that is, the distribution of $|X|^{\gamma}$ coincides with that of the power of a chi-square distributed random variable. In what follows, the distribution of the random variable $|X|^{\gamma}$ will be called *exponentiated folded normal distribution*. It should be noted that for the sake of simplicity, by the term 'exponentiated folded normal distribution', we mean the distribution of the exponentiated random variable with the folded normal distribution, although in probability theory and statistics, by 'exponentiated distribution', it is customary to understand the exponentiated distribution function.

As this is so, from Lemma 2, it follows that

$$|X|^{\gamma} \stackrel{d}{=} 2^{\gamma/2} Z_{1/2,1}^{-\gamma/2} \circ W_1^{\gamma/2} \stackrel{d}{=} 2^{\gamma/2} Z_{1/2,1}^{-\gamma/2} \circ W_{2/\gamma}, \tag{14}$$

where $Z_{1/2,1}$ is a random variable with probability density

$$f(x;\,1/2,\,1) = \begin{cases} \dfrac{1}{\pi x\sqrt{x-1}}, & x > 1, \\ 0, & x \leq 1, \end{cases}$$

or, equivalently,

$$Z_{1/2,1} \stackrel{d}{=} 1 + P_{1/2,1/2},$$

where $P_{1/2,1/2}$ is a random variable with the Snedecor–Fisher distribution defined by the density

$$p_{1/2,1/2}(x) = \begin{cases} \dfrac{1}{\pi(1+x)\sqrt{x}}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Now, assume that $\gamma \ge 2$. Continuing (14) under this assumption with the use of Lemma 3, we arrive at the representation

$$|X|^{\gamma} \stackrel{d}{=} 2^{\gamma/2} \big(Z_{1/2,1}^{\gamma/2} \circ S_{2/\gamma,1} \big)^{-1} \circ W_1. \tag{15}$$

This means that the following statement holds.

Proposition 1. *Let $\gamma \ge 2$. Then, the distribution of the random variable $|X|^{\gamma}$ is mixed exponential, that is,*

$$\mathsf{P}(|X|^{\gamma} \ge 2^{\gamma/2} x) = \int_0^{\infty} e^{-zx} h_{\gamma}(z) dz,$$

where the probability density $h_{\gamma}(z)$ has the form

$$h_{\gamma}(z) = \frac{2}{\pi \gamma z} \int_z^{\infty} \frac{y^{1/\gamma} \psi_{2/\gamma,1}(y)\, dy}{\sqrt{z^{2/\gamma} - y^{2/\gamma}}}. \tag{16}$$

In other words, with the account of (12), we can conclude that $h_{\gamma}(z)$ solves the integral equation

$$\int_0^{\infty} e^{-zx} h(z) dz = 2\big(1 - \Phi(\sqrt{2} x^{1/\gamma})\big)$$

with respect to $h(z)$, or, equivalently, the function $2\big(1 - \Phi(\sqrt{2} x^{1/\gamma})\big)$ is the Laplace transform of $h_{\gamma}(z)$.

In [21], it was proved that if the parameters r and α of the generalized gamma distribution (see (1)) satisfy the condition $\alpha r > 1$, then the representation of the generalized gamma distribution as mixed exponential is impossible (see Theorem 3 in that paper). In the case, under consideration condition, $\alpha r > 1$ reduces to $\gamma < 1$. This means that if $\gamma < 1$, then representation of the distribution of $|X|^{\gamma}$ as mixed exponential is impossible.

Corollary 1. *Let $\gamma \ge 2$. Then, the distribution of the random variable $|X|^{\gamma}$ is infinitely divisible.*

Proof. This statement follows from the result of [23], stating that the product of two independent nonnegative random variables is infinitely divisible, if one of the two is exponentially distributed. □

Corollary 2. *Let $\gamma \ge 2$. Then, the distribution of the random variable $|X|^{\gamma}$ is a scale mixture of the folded normal distribution:*

$$|X|^{\gamma} \stackrel{d}{=} \sqrt{Y_{\gamma}} \circ |X|,$$

where

$$Y_{\gamma} \stackrel{d}{=} 2^{\gamma+1} Z_{1/2,1}^{-\gamma} \circ S_{2/\gamma,1}^{-2} \circ W_1. \tag{17}$$

Proof. This statement follows from the easily verified representation $W_1 \stackrel{d}{=} \sqrt{2W_1} \circ |X|$. □

It can be said that the random variable Y_γ solves the 'equation' $|X|^\gamma \stackrel{d}{=} \sqrt{Y} \circ |X|$ with respect to the random variable Y within the class of nonnegative random variables.

As regards the scaling random variable Y_γ, we immediately notice that its distribution is again mixed exponential. Moreover, the following statement holds.

Proposition 2. *Let $\gamma \geq 2$. Then, the distribution of the random variable Y_γ is mixed exponential,*

$$\mathsf{P}(Y_\gamma \geq 2^{\gamma+1} x) = \int_0^\infty e^{-zx} h_\gamma^*(z) dz, \quad x \geq 0,$$

where

$$h_\gamma^*(z) = \frac{1}{\gamma\pi z} \int_{\sqrt{z}}^\infty \frac{y^{1/\gamma} \psi_{2/\gamma,1}(y)\, dy}{\sqrt{z^{1/\gamma} - y^{2/\gamma}}}. \tag{18}$$

Proposition 2 and the result of [23] mentioned above imply the following statement.

Corollary 3. *Let $\gamma \geq 2$. Then, the distribution of the random variable Y_γ is infinitely divisible.*

The random variable Y_γ is nonnegative. Therefore, we can formulate the following definition.

Definition 1. *Let $\gamma \geq 2$. The random variable*

$$X_\gamma \stackrel{\text{def}}{=} \sqrt{Y_\gamma} \circ X \tag{19}$$

will be called the quasi-exponentiated standard normal random variable with exponent γ. Correspondingly, the distribution of X_γ will be called quasi-exponentiated standard normal.

It should be noted that again, for the sake of simplicity, we use one and the same term 'quasi-exponentiated normal' with respect to both the random variable and its distribution.

It can be easily seen that if γ is an odd natural number, $\gamma \geq 3$, then $X_\gamma \stackrel{d}{=} X^\gamma$.

From the formal definition of X_γ it follows that the probability density function $q_\gamma(x)$ of X_γ has the form

$$q_\gamma(x) = \int_0^\infty \frac{1}{\sqrt{z}} \varphi\Big(\frac{x}{\sqrt{z}}\Big) d\mathsf{P}(Y_\gamma < z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{z}} \exp\Big\{-\frac{x^2}{2z}\Big\} \int_0^\infty y e^{-yz} h_\gamma^*(y) dy\, dz,$$

where the function h_γ^* was defined in (18).

However, this expression can be simplified considerably if this problem is approached from another point. Namely, let $I_{1/2}$ be the Bernoulli random variable: $\mathsf{P}(I_{1/2} = 1) = \frac{1}{2} = 1 - \mathsf{P}(I_{1/2} = 0)$. Then, the random variable $2I_{1/2} - 1$ takes two values: 1 and -1, with probability $\frac{1}{2}$ each. It can be easily verified that

$$X_\gamma \stackrel{d}{=} (2I_{1/2} - 1) \circ |X|^\gamma \stackrel{d}{=} (2I_{1/2} - 1) \circ \sqrt{Y_\gamma} \circ |X| \stackrel{d}{=} (2I_{1/2} - 1) \circ \overline{G}_{1/2, 2/\gamma, 1/2}. \tag{20}$$

Hence,

$$q_\gamma(x) = \tfrac{1}{2} v_\gamma(|x|) = \frac{1}{\gamma\sqrt{2\pi}} \cdot |x|^{1/\gamma - 1} e^{-\frac{1}{2}|x|^{2/\gamma}}, \quad x \in \mathbb{R}.$$

Obviously, $q_\gamma(x)$ is symmetric and unimodal with the mode in zero. If $\gamma \geq 2$, then the vertex of $q_\gamma(x)$ is infinite.

Proposition 3. *Let* $\gamma \geq 2$. *Any quasi-exponentiated standard normal distribution with exponent* γ *is a scale mixture of Laplace distributions:*

$$X_\gamma \stackrel{d}{=} \sqrt{Y_\gamma^*} \circ Q_1,$$

where

$$Y_\gamma^* \stackrel{\text{def}}{=} 2^\gamma Z_{1/2,1}^{-\gamma} \circ S_{2/\gamma,1}^{-2}$$

From Definition 1, Corollary 3, and a well-known result that a normal scale mixture is infinitely divisible, if the mixing distribution is infinitely divisible (see, e.g., [24], Chapt. XVII, section 3), we obtain the following statement.

Proposition 4. *All quasi-exponentiated normal distributions with* $\gamma \geq 2$ *are infinitely divisible.*

From (14) and the representation $W_1 \stackrel{d}{=} \sqrt{2W_1} \circ |X|$, it follows that

$$|X|^\gamma \stackrel{d}{=} 2^{\gamma/2} Z_{1/2,1}^{-\gamma/2} \circ W_1^{\gamma/2} \stackrel{d}{=} 2^{3\gamma/4} Z_{1/2,1}^{-\gamma/2} \circ W_1^{\gamma/4} \circ |X|^{\gamma/2}. \tag{21}$$

This means that there holds the following statement, which can be regarded as an analogue of the 'multiplication theorem'.

Proposition 5. *Let* $\gamma \geq 2$. *Then, the exponentiated folded normal distribution with parameter* γ *is a scale mixture of the exponentiated folded normal distribution with half the parameter:*

$$|X|^\gamma \stackrel{d}{=} \sqrt{\bar{Y}_\gamma} \circ |X|^{\gamma/2},$$

where

$$\bar{Y}_\gamma \stackrel{d}{=} 2^{3\gamma/2} Z_{1/2,1}^{-\gamma} \circ W_{2/\gamma}.$$

Consider some properties of the scaling random variable Y_γ in (19).

Proposition 6. *Let* $\gamma \geq 2$. *Then, for any* $\beta > -\frac{1}{\gamma}$

$$\mathsf{E}Y_\gamma^{2\beta} = 2^{\beta(\gamma-1)/2} \frac{\Gamma(\frac{\gamma\beta+1}{2})}{\Gamma(\frac{\beta+1}{2})}.$$

By virtue of identifiability of normal scale mixtures (see [25]), Corollary 6 and Proposition 2 imply the following statement, which can be regarded as an analogue of Proposition 5 for the random variables Y_γ.

Proposition 7. *Let* $\gamma \geq 4$. *Then,*

$$Y_\gamma \stackrel{d}{=} \bar{Y}_\gamma \circ Y_{\gamma/2}.$$

Corollary 4. *Let* $\gamma \geq 2$, $\beta > -\frac{1}{\gamma}$. *Then,*

$$\mathsf{E}\bar{Y}_\gamma^{2\beta} = 2^{\beta\gamma/4} \frac{\Gamma(\frac{\beta\gamma+1}{2})}{\Gamma(\frac{\beta\gamma+2}{4})}.$$

From Proposition 7 and Definition 1, we directly obtain the following 'multiplication theorem' for quasi-exponentiated normal distributions.

Corollary 5. *Let* $\gamma \geq 4$. *Then,*

$$X_\gamma \stackrel{d}{=} \sqrt{\bar{Y}_\gamma} \circ X_{\gamma/2}.$$

4. R-Asymmetric Quasi-Exponentiated Normal Distributions

In order to construct asymmetric generalizations of the quasi-exponentiated normal distribution, at first, we will consider the R-asymmetric (randomized asymmetric) distributions. The starting point is representation (20). The two-parameter asymmetrization of the quasi-exponentiated normal distribution reduces to the replacement of the probability $\frac{1}{2}$ in (20) by an arbitrary probability $p \in [0,1]$. Let I_p be the Bernoulli random variable such that $\mathsf{P}(I_p = 1) = p = 1 - \mathsf{P}(I_p = 0)$. Then, the random variable $2I_p - 1$ takes two values: 1 and -1, with probabilities p and $1-p$. Then, a simple two-parameter R-asymmetrization of the quasi-exponentiated normal distribution is defined as

$$X_{p,\gamma} \stackrel{d}{=} (2I_p - 1) \circ |X|^{\gamma} \stackrel{d}{=} (2I_p - 1) \circ \sqrt{Y_{\gamma}} \circ |X| \stackrel{d}{=} (2I_p - 1) \circ \overline{G}_{1/2, 2/\gamma, 1/2}. \tag{22}$$

Along with (22), $X_{\gamma,p}$ can be represented as

$$X_{p,\gamma} = I_p \circ |X|^{\gamma} - (1 - I_p) \circ |X|^{\gamma} \stackrel{d}{=} I_p \circ \overline{G}_{1/2,2/\gamma,1/2} - (1 - I_p) \circ \overline{G}_{1/2,2/\gamma,1/2}. \tag{23}$$

Let $p \in [0,1]$, $\gamma_1 \geq 2$, $\gamma_2 \geq 2$, $\sigma_1 > 0$, $\sigma_2 > 0$. The general R-asymmetrization $X_{p,\gamma_1,\gamma_2,\sigma_1,\sigma_2}$ of the quasi-exponentiated normal distribution is defined as

$$X_{p,\gamma_1,\gamma_2,\sigma_1,\sigma_2} \stackrel{d}{=} \sigma_1 I_p \circ \overline{G}_{1/2,2/\gamma_1,1/2} - \sigma_2(1 - I_p) \circ \overline{G}_{1/2,2/\gamma_2,1/2}.$$

As this is so,

$$\sqrt{\pi}\mathsf{E}X^{\beta}_{p,\gamma_1,\gamma_2,\sigma_1,\sigma_2} = p2^{\beta\gamma_1/2}\sigma_1^{\beta}\Gamma\left(\tfrac{\beta\gamma_1+1}{2}\right) - (1-p)2^{\beta\gamma_2/2}\sigma_2^{\beta}\Gamma\left(\tfrac{\beta\gamma_2+1}{2}\right),$$

$$\sqrt{\pi}\mathsf{E}|X_{p,\gamma_1,\gamma_2,\sigma_1,\sigma_2}|^{\beta} = p2^{\beta\gamma_1/2}\sigma_1^{\beta}\Gamma\left(\tfrac{\beta\gamma_1+1}{2}\right) + (1-p)2^{\beta\gamma_2/2}\sigma_2^{\beta}\Gamma\left(\tfrac{\beta\gamma_2+1}{2}\right)$$

for β such that $\beta \min\{\gamma_1, \gamma_2\} > -1$.

The R-asymmetrization of the quasi-exponentiated normal representation is formal. This circumstance noticeably hinders the substantiation of R-asymmetric quasi-exponentiated normal models in practical problems of descriptive statistics.

5. M-Asymmetric Quasi-Exponentiated Normal Distributions

Another approach to the de-symmetrization of the quasi-exponentiated normal distribution is based on mixture representations of this distribution. Let $\alpha \in (0,2]$. The starting point here is Definition 1. According to that definition, the quasi-exponentiated normal distribution is a normal scale mixture:

$$X_{\gamma} \stackrel{d}{=} \sqrt{Y_{\gamma}} \circ X.$$

Following the general lines of the construction of normal variance-mean mixtures (see, e.g., [2], two more scalar parameters responsible for scale and asymmetry are introduced: $\sigma > 0$ and $a \in \mathbb{R}$. Define the M-asymmetric (Mixture-asymmetric) quasi-exponentiated normal distribution as

$$F_{\gamma,a,\sigma}(x) \stackrel{\text{def}}{=} \mathsf{P}(X_{\gamma,a,\sigma} < x) = \int_0^{\infty} \Phi\Big(\frac{x - au}{\sigma\sqrt{u}}\Big) d\mathsf{P}(Y_{\gamma} < u), \quad x \in \mathbb{R}. \tag{24}$$

This distribution function corresponds to the random variable

$$X_{\gamma,a,\sigma} = \sigma\sqrt{Y_{\gamma}} \circ X + aY_{\gamma}.$$

Find the moments of the random variable $X_{\gamma,a,\sigma}$. From Proposition 5, it follows that

$$\mathsf{E}X_{\gamma,a,\sigma} = a\mathsf{E}Y_\gamma = a2^{(\gamma-1)/4} \cdot \frac{\Gamma(\frac{\gamma+2}{4})}{\Gamma(\frac{3}{4})}.$$

Further, since Y_γ and X are assumed independent, $\mathsf{E}X = 0$ and $\mathsf{E}X_2 = 1$, from Proposition 5, it follows that

$$\mathsf{E}X^2_{\gamma,a,\sigma} = \mathsf{E}(\sigma\sqrt{Y_\gamma} \circ X + aY_\gamma)^2 = \sigma^2\mathsf{E}Y_\gamma + 2\sigma\mathsf{E}\sqrt{Y_\gamma}\mathsf{E}X + a^2\mathsf{E}Y^2_\gamma$$

$$= \sigma^2\mathsf{E}Y_\gamma + a^2\mathsf{E}Y^2_\gamma = \sigma^2 2^{(\gamma-1)/4} \cdot \frac{\Gamma(\frac{\gamma+2}{4})}{\Gamma(\frac{3}{4})} + a^2 2^{(\gamma-1)/2}\Gamma(\tfrac{\gamma+1}{2}).$$

Hence,

$$\mathsf{D}X_{\gamma,a,\sigma} = \mathsf{E}X^2_{\gamma,a,\sigma} - (\mathsf{E}X_{\gamma,a,\sigma})^2 = 2^{(\gamma-1)/4}\sigma^2 \cdot \frac{\Gamma(\frac{\gamma+2}{4})}{\Gamma(\frac{3}{4})} + 2^{(\gamma-1)/2}a^2 \cdot \left[\Gamma(\tfrac{\gamma+1}{2}) - \left(\frac{\Gamma(\frac{\gamma+2}{4})}{\Gamma(\frac{3}{4})}\right)^2\right].$$

As regards the higher-order moments of the M-asymmetric quasi-exponentiated normal distribution, consider arbitrary $n \in \mathbb{N}$, $n \geq 3$. Using the binomial formula, it is not difficult to show that

$$\mathsf{E}X^n_{\gamma,a,\sigma} = n!\sum_{k=0}^{n} C^k_n \sigma^k a^{n-k}\mathsf{E}X^k \cdot \mathsf{E}Y^{n-k/2}_\gamma$$

$$= 2^{\gamma n/4+1}n!\sum_{k=0}^{[n/2]} \frac{2^{k(1-\gamma/8)}\sigma^{2k}a^{n-2k}}{(2k)!(n-2k)!} \cdot \frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{\gamma(2n-k)}{8}+\frac{1}{2})}{\Gamma(\frac{2n-k}{8})}.$$

6. M-Asymmetric Quasi-Exponentiated Normal Distribution as a Limit Law for Random Sums

Among many possible mixture representations for the quasi-exponentiated normal distribution, the normal mixture model seems most promising for the construction of asymmetric generalizations, since it presents the opportunity to formulate a rather simple limit theorem (more exactly, a transfer theorem) for random sums of independent identically distributed random variables in which the M-asymmetric quasi-exponentiated normal distribution acts as the limit law. Moreover, the conditions imposed on the distribution of the summands are rather loose. For example, the summed random variables may have finite variance and may even be bounded. The corresponding statement is based on the transfer theorem for random sums in the 'if and only if' form (see, e.g., [12]) and the identifiability of normal variance-mean mixtures ([3]).

Let $\{X_{n,j}\}_{j\geq 1}$, $n = 1, 2, \ldots$ be a double array of row-wise (that is, for each fixed n) independent and identically distributed random variables. Let $\{N_n\}_{n\geq 1}$ be a sequence of nonnegative integer-valued random variables such that for each $n \geq 1$, the random variables $N_n, X_{n,1}, X_{n,2}, \ldots$ are independent. Denote $S_{n,k} = X_{n,1} + \ldots + X_{n,k}$. For definiteness, we assume $\sum_{j=1}^{0} = 0$. In the statement below, the convergence is meant as $n \to \infty$.

Lemma 4 ([3])**.** *Assume that there exists a sequence of natural numbers $\{k_n\}_{n\geq 1}$ and numbers $a \in \mathbb{R}$ and $b > 0$ such that*

$$\mathsf{P}\big(S_{n,k_n} < x\big) \Longrightarrow \Phi\Big(\frac{x-a}{b}\Big). \tag{25}$$

Assume that $N_n \to \infty$ in probability (that is, $\mathsf{P}(N_n \geq K) \to 0$ for any $K \in (0, \infty)$). The distributions of random sums S_{N_n} converge to some distribution function $F(x)$:

$$\mathsf{P}\big(S_{n,N_n} < x\big) \Longrightarrow F(x),$$

if and only if there exists a distribution function $H(x)$ *such that* $H(0) = 0$,

$$F(x) = \int_0^\infty \Phi\Big(\frac{x - az}{b\sqrt{z}}\Big)\,dH(z), \tag{26}$$

and

$$\mathsf{P}(N_n < xk_n) \Longrightarrow H(x).$$

It is easily seen that expression (24) defining the M-asymmetric quasi-exponentiated normal distribution coincides with (26), where $H(z) = \mathsf{P}(Y_\gamma < z)$. Hence, Lemma 4 directly yields the following result.

Theorem 1. *Assume that there exists a sequence of natural numbers* $\{k_n\}_{n\geq 1}$ *and numbers* $a \in \mathbb{R}$ *and* $b > 0$ *such that condition* (25) *holds. Assume that* $N_n \to \infty$ *in probability. The distributions of random sums* S_{N_n} *converge to the M-asymmetric quasi-exponentiated normal distribution function* $F_{\gamma,a,\sigma}(x)$ *(see* (24)*),*

$$\mathsf{P}\big(S_{n,N_n} < x\big) \Longrightarrow F_{\gamma,a,\sigma}(x),$$

if and only if

$$\mathsf{P}(N_n < xk_n) \Longrightarrow \mathsf{P}(Y_\gamma < x). \tag{27}$$

Theorem 1 may serve as the theoretic explanation of the possible utility of the M-asymmetric quasi-exponentiated normal distribution as a model of statistical regularities observed in some real phenomena in which the additive structure of the observed process can be assumed. In [26], we have already discussed this question. Although the form of the distribution of Y_γ is rather curious, there are no serious objections against the possibility of application of this distribution for modelling the poorly predictable (or unpredictable) regularities of, say, information flows in financial markets. For more details, see [26].

Consider an example of a random index N_n which satisfies (27). This example is connected with special mixed geometric distributions. First, consider mixed geometric random sums.

Let $p \in (0,1)$ and V_p be a random variable having the geometric distribution with parameter p:

$$\mathsf{P}(V_p = k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots .$$

This means that

$$\mathsf{P}(V_p > m) = \sum_{k=m+1}^{\infty} p(1-p)^{k-1} = (1-p)^m$$

for any $m \in \mathbb{N}$. Let $(\pi_n)_{n\geq 1}$ be a sequence of positive random variables taking values in the interval $(0,1)$, and moreover, for each $n \geq 1$ and all $p \in (0,1)$, the random variables π_n and V_p are independent.

For each $n \in \mathbb{N}$, let $N_n = V_{\pi_n}$. Hence,

$$\mathsf{P}(N_n > m) = \int_0^1 (1-z)^m \, d\mathsf{P}(\pi_n < z) \tag{28}$$

for any $m \in \mathbb{N}$. The distribution of the random variable N_n is called π_n*-mixed geometric* (for more detail, see [27]).

In [27], a generalization of the famous Rényi theorem was proved (see Theorem 1 there), stating that if the random variables π_n in (28) are infinitesimal in the sense that there exist a random variable M such that $\mathsf{P}(0 \leq M < \infty) = 1$ and a sequence of natural numbers $\{k_n\}_{n\in\mathbb{N}}$ such that

$$k_n \pi_n \Longrightarrow M$$

as $n \to \infty$, then

$$\lim_{n\to\infty} \sup_{x\geq 0} \left| \mathsf{P}\Big(\frac{N_n}{k_n} \geq x\Big) - \int_0^\infty e^{-ux} d\mathsf{P}(M < u) \right| = 0. \quad (29)$$

Now it suffices to make use of representation (17) and let $M = 2^{-(\gamma+1)} Z^{\gamma}_{1/2,1} \circ S^2_{2/\gamma,1}$. Then, in accordance with (17) and (29), the mixed geometric random variables N_n defined by (28) will satisfy (27).

7. Conclusions

In the paper, quasi-exponentiated normal distributions were introduced for any power (exponent) no less than two. With natural exponents, the 'quasi-exponentiated' normal distributions coincide with the distributions of the corresponding powers of the normal random variables with zero mean. Their representability as scale mixtures of normal and exponential distributions was proved. The mixing distributions were written out in the closed form. Two approaches to the construction of asymmetric quasi-exponentiated normal distributions were described: one based on randomization, the other based on the representation of the corresponding asymmetric quasi-exponentiated normal distribution as a variance-mean normal mixture. The limit theorem was proved for sums of a random number of independent random variables in which the asymmetric quasi-exponentiated normal distribution is the limit law. An example is given, illustrating that in order to provide the validity of the conditions required for the convergence of the distributions of random sums to the asymmetric quasi-exponentiated normal distribution, the random number of summands may be special mixed geometric. The presented results are valid, if the exponent of the quasi-exponentiated normal distribution is no less than two. It is known that if it is less than one, then it is impossible to represent the asymmetric quasi-exponentiated normal distribution as mixed exponential. The study of mixture properties of quasi-exponentiated normal distributions with the exponent lying in the interval $(1, 2)$ is still an open problem.

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