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Inequalities for Riemann–Liouville-Type Fractional Derivatives of Convex Lyapunov Functions and Applications to Stability Theory

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Abstract: In recent years, various qualitative investigations of the properties of differential equations with different types of generalizations of Riemann–Liouville fractional derivatives were studied and stability properties were investigated, usually using Lyapunov functions. In the application of Lyapunov functions, we need appropriate inequalities for the fractional derivatives of these functions. In this paper, we consider several Riemann–Liouville types of fractional derivatives and prove inequalities for derivatives of convex Lyapunov functions. In particular, we consider the classical Riemann–Liouville fractional derivative, the Riemann–Liouville fractional derivative with respect to a function, the tempered Riemann–Liouville fractional derivative, and the tempered Riemann–Liouville fractional derivative with respect to a function. We discuss their relations and their basic properties, as well as the connection between them. We prove inequalities for Lyapunov functions from a special class, and this special class of functions is similar to the class of convex functions of many variables. Note that, in the literature, the most common Lyapunov functions are the quadratic ones and the absolute value ones, which are included in the studied class. As a result, special cases of our inequalities include Lyapunov functions given by absolute values, quadratic ones, and exponential ones with the above given four types of fractional derivatives. These results are useful in studying types of stability of the solutions of differential equations with the above-mentioned types of fractional derivatives. To illustrate the application of our inequalities, we define Mittag–Leffler stability in time on an interval excluding the initial time point. Several stability criteria are obtained.

Keywords: Riemann–Liouville-type fractional derivative; tempered fractional derivative; fractional derivative with respect to another function; Lyapunov functions; Mittag–Leffler stability in time

MSC: 34A34; 34A08; 34D20

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1. Introduction

In this paper, we consider Riemann–Liouville-type fractional derivatives. These types of derivatives have a singularity at the initial time point and give us a new tool for modeling anomalies in the dynamics of processes. The study of linear systems of fractional differential equations with Riemann–Liouville-type fractional derivatives was considered in [1], nonlinear systems in [2], existence and Ulam stability was studied in [3], and for basic concepts on stability for Riemann–Liouville fractional differential equations, we refer the reader to [4]. A general fractional derivative of arbitrary order in the Riemann–Liouville sense was defined and applied to Cauchy problems for single- and multi-term linear fractional differential equations by Luchko in [5]. In addition, generalized proportional fractional integrals and derivatives were defined, studied, and applied in [6,7]. These derivatives are similar to tempered fractional ones. The exponential tempering has many merits, both in mathematical and practical terms, and we mention the applications in finance [8,9], in geophysics [10,11], and in Brownian motion [12].

One of basic qualitative properties in differential equations is stability. There are different approaches for studying stability properties of the solutions. Stability is applied for theoretical study as well as for the practical investigation of various dynamical models with different types of derivatives. One of these approaches is the application of Lyapunov functions. Lyapunov functions are important in stability theory of dynamical systems and control theory (see, for example, the epidemiological models [13,14], the fractional models in biology [15], and the fractional SIR and SIRS models [16]). When Lyapunov functions are applied to study stability, we need inequalities for these functions with the applied derivatives. In the case when integer order derivatives are applied, these inequalities are known and applied to many different problems. This is not the situation with the application of fractional derivatives. In the case of the Riemann–Liouville type of fractional derivative, the inequalities for the Lyapunov functions are not well studied and so this restricted the study and the application of stability properties for the corresponding differential equations. In connection with this, we first present the basic types of Riemann–Liouville-type fractional derivatives defined and studied in the literature. We consider the classical Riemann–Liouville fractional derivative (RLFD), the tempered Riemann–Liouville fractional derivative (TRLFD), the Riemann–Liouville fractional derivative with respect to a function (RLFDF) (also called the ψ—fractional derivative), and the tempered Riemann–Liouville fractional derivative with respect to a function (TRLFDF). We provide some of their basic properties as well as the connection between them. Then, we prove inequalities for Lyapunov functions from a special class with these types of fractional derivatives. This special class of functions is similar to the class of convex functions of many variables and includes the quadratic ones and the absolute value ones. This allows us to obtain as a special case of our inequalities some results for the well-known Lyapunov functions defined by absolute values, quadratic ones, and exponential ones.

Lyapunov functions are one of the most powerful tools for studying stability properties of fractional physical and biological systems, such as the Duffing oscillator [17], neural networks [18,19], and predator–prey models [20]. Note that, recently, the Lyapunov method has been applied to analyze Mittag–Leffler stability of different fractional systems (see, for example, [21–23]). In [23] the authors studied Mittag–Leffler stability of tempered fractional dynamical systems. Unfortunately, the inequality applied for Lyapunov functions defined by absolute values is not correctly proved (inequality (12) does not follow from (15)). Motivated by this, it is necessary and meaningful to prove inequalities for convex Lyapunov functions with various types of fractional derivatives. In this way, a tool for investigating stability properties is built. In addition, in this paper, to illustrate an application of our inequalities for Lyapunov functions, we define generalized Mittag–Leffler stability in time for the above mentioned Riemann-Liouvile-type fractional derivatives and apply our inequalities to obtain sufficient conditions for this type of stability. The Mittag–Leffler stability is studied by Lyapunov functions and applied to some models, such as some epidemiological models with Caputo fractional derivative in [24], and tempered fractional neural networks in [25].

The main contributions in the paper are summarized by the following:

- An overview of the literature on Riemann–Liouville-type fractional derivatives and integrals, which are generalizations of the classical one as well as the relation between them are given;
- Some inequalities for convex Lyapunov-type functions with the defined Riemann–Liouville-type fractional derivatives are proven.
- Inequalities for Lyapunov functions defined by absolute values and quadratic Lyapunov functions with the defined fractional derivatives are obtained.
- The generalized Mittag–Leffler stability in time for differential equations with the given fractional derivatives is defined.

- Sufficient conditions for the generalized Mittag–Leffler stability in time are obtained in the cases of all types of the considered fractional derivatives.

The basic definitions are given in Section 2, and some additional results necessary to the proofs of the main results are provided in Section 2.2. The main inequalities for convex Lyapunov functions are proven in Section 3. Special case results for Lyapynov function defined by absolute values and quadratic Lyapunov function are obtained. These inequalities are applied in Section 4 to study the defined generalized Mittag–Leffler stability in time. Some concluding comments are stated in the last section.

2. Basic Definitions for Riemann–Liouville-Type Fractional Integrals and Derivatives

We will provide an overview of the literature on basic generalizations of the classical Riemann–Liouville fractional integrals and derivatives. We will assume that all functions in the defined below integrals and derivatives are "smooth" enough such that all integrals exit. Later, we will define the appropriate set of functions.

Let $t_0 \geq 0$ be a given number, $t_0 < b \leq \infty$ (in the case $b = \infty$ we consider the open interval (t_0, ∞) instead of $(t_0, b]$).

2.1. Riemann–Liouville Fractional Integrals and Derivatives

Definition 1 ([26,27])**.** *The Riemann–Liouville fractional integral (RLFI) of order $\gamma > 0$ of a given function $v : [t_0, b] \to \mathbb{R}$ is defined by*

$$ {}_{t_0}I^{\gamma} v(t) = \frac{1}{\Gamma(\gamma)} \int_{t_0}^{t} (t-s)^{\gamma-1} v(s)\, ds, \quad t \in (t_0, b], $$

and the Riemann–Liouville fractional derivative (RLFD) of order $\gamma \in (0,1)$ of a given function $v : [t_0, b] \to \mathbb{R}$ is defined by

$$ {}_{t_0}^{RL}D_t^{\gamma} v(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-\gamma} v(s)\, ds, \quad t \in (t_0, b]. $$

Consider the set

$$ C_{\gamma}^{RL}([t_0, b]) = \{ \nu : [t_0, b] \to \mathbb{R} : \ \forall t \in (t_0, b] \ \exists \ {}_{t_0}^{RL}D_t^{\gamma} \nu(t) \}. $$

2.2. Tempered Riemann–Liouville Fractional Integrals and Derivatives

The tempered fractional integral and derivative of Riemann–Liouville type are defined as follows:

Definition 2 ([28,29])**.** *Let $\lambda > 0$. The tempered Riemann–Liouville fractional integral (TRLFI) of order $\gamma > 0$ of a given function $v : [t_0, b] \to \mathbb{R}$ is defined by*

$$ {}_{t_0}^{\mathcal{T}}\mathcal{I}_t^{\gamma,\lambda} v(t) = \frac{1}{\Gamma(\gamma)} \int_{t_0}^{t} (t-s)^{\gamma-1} e^{-\lambda(t-s)} v(s)\, ds, \quad t \in (t_0, b]. $$

and the tempered Riemann–Liouville fractional derivative (TRLFD) of order $\gamma \in (0,1)$ of a given function $v : [t_0, b] \to \mathbb{R}$ is defined by

$$ \begin{aligned} {}_{t_0}^{\mathcal{TRL}}\mathcal{D}_t^{\gamma,\lambda} v(t) &= \Big(\lambda + \frac{d}{dt}\Big) {}_{t_0}^{\mathcal{T}}\mathcal{I}_t^{1-\gamma,\lambda} v(t) \\ &= \frac{\lambda}{\Gamma(1-\gamma)} \int_{t_0}^{t} (t-s)^{-\gamma} e^{-\lambda(t-s)} v(s)\, ds \\ &\quad + \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-\gamma} e^{-\lambda(t-s)} v(s)\, ds, \quad t \in (t_0, b]. \end{aligned} $$

Consider the set

$$C^{\mathcal{TRL}}_{\gamma,\lambda}([t_0,b]) = \{\nu : [t_0,b] \to \mathbb{R} : \ \forall t \in (t_0,b] \ \exists \, {}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{t}\nu(t)\}.$$

2.3. Riemann–Liouville Fractional Integral and Derivative with Respect to a Function

Definition 3 ([26,27]). *Let the function $\psi \in C^1([t_0,b],\mathbb{R})$, $b \le \infty$ be a given strictly increasing function with $\psi(s) > 0$, $s \in [t_0,b]$. The Riemann–Liouville fractional integral with respect to $\psi(t)$ (RLFIF) of order $\gamma > 0$ of a given function $v : [t_0,b] \to \mathbb{R}$ is defined by*

$${}_{t_0}\mathcal{I}^{\gamma}_{\psi(t)}v(t) = \frac{1}{\Gamma(\gamma)}\int_{t_0}^{t}(\psi(t)-\psi(s))^{\gamma-1}\psi'(s)v(s)\,ds, \quad t \in (t_0,b].$$

and the Riemann–Liouville fractional derivative with respect to $\psi(t)$ (RLFDF) of order $\gamma \in (0,1)$ of a given function $v : [t_0,b] \to \mathbb{R}$ is defined by

$$\begin{aligned}{}^{RL}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)}v(t) &= \Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big){}_{t_0}\mathcal{I}^{1-\gamma}_{\psi(t)}v(t)\\ &= \frac{1}{\psi'(t)\Gamma(1-\gamma)}\frac{d}{dt}\int_{t_0}^{t}(\psi(t)-\psi(s))^{-\gamma}\psi'(s)v(s)\,ds, \quad t \in (t_0,b].\end{aligned}$$

Consider the set

$$C^{RL}_{\gamma,\psi}([t_0,b]) = \{\nu : [t_0,b] \to \mathbb{R} : \ \forall t \in (t_0,b] \ \exists \, {}^{RL}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)}\nu(t)\}.$$

Remark 1. *Note in Definition 3, RLFIF and RLFDF are also called the ψ-fractional integral and the ψ-Riemann–Liouville fractional derivative, respectively.*

2.4. Tempered Riemann–Liouville Fractional Integral and Derivative with Respect to a Function

Definition 4 ([30]). *Let the function $\psi \in C^1([t_0,b],\mathbb{R})$, $b \le \infty$ be a given strictly increasing function with $\psi(s) > 0$, $s \in [t_0,b]$ and $\lambda > 0$. The tempered Riemann–Liouville fractional integral with respect to $\psi(t)$ (TRLFIF) of order $\gamma > 0$ of a given function $v : [t_0,b] \to \mathbb{R}$ is defined by*

$${}^{\mathcal{TF}}_{t_0}\mathcal{I}^{\gamma,\lambda}_{\psi(t)}v(t) = \frac{1}{\Gamma(\gamma)}\int_{t_0}^{t}(\psi(t)-\psi(s))^{\gamma-1}\psi'(s)e^{-\lambda(\psi(t)-\psi(s))}v(s)\,ds, \quad t \in (t_0,b].$$

and the tempered Riemann–Liouville fractional derivative with respect to $\psi(t)$ (TRLFDF) of order $\gamma \in (0,1)$ of a given function $v : [t_0,b] \to \mathbb{R}$ is defined by

$$\begin{aligned}&{}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)}v(t) = \Big(\lambda + \frac{1}{\psi'(t)}\frac{d}{dt}\Big){}_{t_0}\mathcal{I}^{1-\gamma}_{\psi(t)}v(t)\\ &= \frac{1}{\Gamma(1-\gamma)}\bigg(\lambda\int_{t_0}^{t}(\psi(t)-\psi(s))^{-\gamma}\psi'(s)e^{-\lambda(\psi(t)-\psi(s))}v(s)\,ds\\ &\quad + \frac{1}{\psi'(t)}\frac{d}{dt}\int_{t_0}^{t}(\psi(t)-\psi(s))^{-\gamma}\psi'(s)e^{-\lambda(\psi(t)-\psi(s))}v(s)\,ds\bigg), \quad t \in (t_0,b].\end{aligned}$$

Consider the set

$$C^{\mathcal{TRLF}}_{\gamma,\lambda,\psi}([t_0,b]) = \{\nu : [t_0,b] \to \mathbb{R} : \ \forall t \in (t_0,b] \ \exists \, {}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)}\nu(t)\}.$$

Remark 2. *Note that all above definitions for fractional integrals and derivative could be generalized to vector function pointwise.*

2.5. Relations between the Riemann–Liouville-Type Fractional Integrals and Derivatives

Note the following relations

$$\begin{aligned}&\text{TRLFIF (Definition 4)} \xrightarrow{\lambda=1} \text{RLFIF (Definition 3)} \xrightarrow{\psi\equiv 1} \text{RLFI (Definition 1)};\\&\text{TRLFIF (Definition 4)} \xrightarrow{\psi\equiv 1} \text{TRLFI (Definition 2)} \xrightarrow{\lambda=1} \text{RLFI (Definition 1)},\end{aligned} \quad (1)$$

and

$$\begin{aligned}&\text{TRLFDF (Definition 4)} \xrightarrow{\lambda=1} \text{RLFDF (Definition 3)} \xrightarrow{\psi\equiv 1} \text{RLFD (Definition 1)};\\&\text{TRLFDF (Definition 4)} \xrightarrow{\psi\equiv 1} \text{TRLFD (Definition 2)} \xrightarrow{\lambda=1} \text{RLFD (Definition 1)}.\end{aligned} \quad (2)$$

are valid.

Remark 3. *From (1) and (2), it follows that the TRLFDF and TRLFIF are the most general. In connection with the above, we will prove an inequality for convex Lyapunov-type functions with TRLFDF.*

2.6. Some Preliminary Results for Fractional Derivatives

Lemma 1. *(Conjugation relations) (Equations (2.3)–(2.5) [31])*

$$\begin{aligned}&{}^{\mathcal{T}}_{t_0}\mathcal{I}^{\gamma,\lambda}_t y(t) = e^{-\lambda t}\ {}_{t_0}I^{\gamma}(e^{\lambda t}y(t)),\\&{}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t y(t) = e^{-\lambda t}\ {}^{RL}_{t_0}D^{\gamma}_t(e^{\lambda t)}y(t)),\\&{}^{\mathcal{TF}}_{t_0}\mathcal{I}^{\gamma,\lambda}_{\psi} y(t) = e^{-\lambda\psi(t)}\ {}_{t_0}\mathcal{I}^{\gamma}_{\psi}(e^{\lambda\psi(t)}y(t)),\\&{}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi} y(t) = e^{-\lambda\psi(t)}\ {}^{RLF}_{t_0}\mathcal{D}^{\gamma}_{\psi}(e^{\lambda\psi(t)}y(t)).\end{aligned} \quad (3)$$

Lemma 2 (Theorem 4.1 [32])**.** *Let $\gamma \in (0,1), C, y_0 \in \mathbb{R}$ and $f \in C[0,b] \to \mathbb{R}$, where $b \le \infty$. Then the solution of the fractional differential equation with RLFDF*

$$^{\mathcal{RLF}}_{0}\mathcal{D}^{\gamma}_{\psi(t)}y(t) - Cy(t) = f(t),\ \ t \in (0,b] \quad (4)$$

with initial condition

$$^{\mathcal{RLF}}_{0}\mathcal{I}^{1-\gamma,\lambda}_{\psi(t)}\Big(y(t)\Big)\Big|_{t=0} = y_0 \quad (5)$$

is given by

$$\begin{aligned}y(t) &= y_0(\psi(t))^{\gamma-1}E_{\gamma,\gamma}(C(\psi(t))^{\gamma})\\&+\int_0^t (\psi(t)-\psi(s)^{\gamma-1}E_{\gamma,\gamma}(C(\psi(t)-\psi(s))^{\gamma})f(s)\psi'(s)ds, \quad t\in(0,b].\end{aligned} \quad (6)$$

Corollary 1. *Let $\gamma \in (0,1), C, y_0, \lambda \in \mathbb{R}$ and $f \in C([0,b])$. Then the solution of the fractional differential equation with TRLFDF*

$$^{\mathcal{TRLF}}_{0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)}y(t) - Cy(t) = f(t),\ \ t \in (0,b] \quad (7)$$

with the initial condition

$$^{\mathcal{TF}}_{0}\mathcal{I}^{1-\gamma,\lambda}_{\psi(t)}y(t)\Big|_{t=0} = y_0 \quad (8)$$

is given by

$$\begin{aligned}&y(t) = y_0e^{-\lambda\psi(t)}(\psi(t))^{\gamma-1}E_{\gamma,\gamma}(C(\psi(t))^{\gamma})\\&+\int_0^t e^{-\lambda(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\gamma-1}E_{\gamma,\gamma}(C(\psi(t)-\psi(s))^{\gamma})f(s)\psi'(s)ds, \quad t\in(t_0,b].\end{aligned} \quad (9)$$

Proof. Let $u(t) = e^{\lambda\psi(t)}y(t)$, $t \in [0,b]$. According to Lemma 1 and the conjugation relations (3), we obtain

$$ {}_0^{\mathcal{TRLF}}\mathcal{D}_{\psi}^{\gamma,\lambda}y(t) = e^{-\lambda\psi(t)}\,{}_0^{RLF}\mathcal{D}_{\psi}^{\gamma}u(t) $$

and

$$ {}_0^{\mathcal{TF}}\mathcal{I}_{\psi}^{1-\gamma,\lambda}y(t) = e^{-\lambda\psi(t)}\,{}_0\mathcal{I}_{\psi}^{1-\gamma}(e^{\lambda\psi(t)}y(t)). $$

Then the initial value problem (7), (8) can be written in the form

$$ \begin{aligned} &{}_0^{\mathcal{RLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}u(t) - Cu(t) = e^{\lambda\psi(t)}f(t), \;\; t \in (0,b] \\ &{}_0\mathcal{I}_{\psi(t)}^{1-\gamma,\lambda}u(t)\Big|_{t=0} = e^{\lambda\psi(0)}y_0. \end{aligned} \tag{10} $$

According to Lemma 2 with $y(t) \equiv u(t)$, the solution of (10) is given by

$$ \begin{aligned} u(t) &= e^{\lambda\psi(0)}y_0(\psi(t))^{\gamma-1}E_{\gamma,\gamma}(C(\psi(t))^{\gamma}) \\ &\quad + \int_0^t(\psi(t) - \psi(s)^{\gamma-1}E_{\gamma,\gamma}(C(\psi(t) - \psi(s))^{\gamma})e^{\lambda\psi(s)}f(s)\psi'(s)ds. \end{aligned} \tag{11} $$

□

Lemma 3. *Let* $\gamma \in (0,1), \lambda \in \mathbb{R}$ *and* $\xi > -1$. *Then*

$$ {}_{t_0}^{\mathcal{TRLF}}\mathcal{I}_{\psi(t)}^{\gamma,\lambda}\Big(e^{-\lambda\psi(t)}(\psi(t) - \psi(t_0))^{\gamma-1}\Big) = \frac{\Gamma(\gamma)}{\Gamma(2\gamma)}e^{-\lambda\psi(t)}(\psi(t) - \psi(t_0))^{2\gamma-1}, \tag{12} $$

$$ {}_{t_0}^{\mathcal{TRLF}}\mathcal{I}_{\psi(t)}^{\gamma,\lambda}\Big(e^{-\lambda\psi(t)}(\psi(t) - \psi(t_0))^{-\gamma}\Big) = \Gamma(1-\gamma)e^{-\lambda\psi(t)} \tag{13} $$

$$ {}_{t_0}^{\mathcal{TRLF}}\mathcal{I}_{\psi(t)}^{1-\gamma,\lambda}\Big(e^{-\lambda\psi(t)}(\psi(t) - \psi(t_0))^{\gamma-1}\Big) = \Gamma(\gamma)e^{-\lambda\psi(t)} \tag{14} $$

and

$$ {}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}\Big(e^{-\lambda\psi(t)}(\psi(t) - \psi(t_0))^{\gamma-1}\Big) = \Gamma(\gamma)e^{-\lambda\psi(t)}(\psi(t) - \psi(t_0))^{-1} \tag{15} $$

$$ {}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}\Big(e^{-\lambda\psi(t)}(\psi(t) - \psi(t_0))^{\gamma}\Big) = \Gamma(\gamma+1)e^{-\lambda\psi(t)} \tag{16} $$

Lemma 4 (Proposition 3.13 [30]). *If* $\gamma > 0, \lambda > 0, \xi > -1$ *then*

$$ {}_{t_0}^{\mathcal{TRLF}}\mathcal{I}_{\psi(t)}^{\gamma,\lambda}\Big(e^{-\lambda\psi(t)}(\psi(t) - \psi(t_0))^{\xi}\Big) = \frac{\Gamma(\xi+1)}{\Gamma(\xi+\gamma+1)}e^{-\lambda\psi(t)}(\psi(t) - \psi(t_0))^{\xi+\gamma}, $$

$$ {}_{t_0}^{\mathcal{TRLF}}\mathcal{I}_{\psi(t)}^{\gamma,\lambda}(1) = e^{-\lambda(\psi(t)-\psi(t_0))}(\psi(t) - \psi(t_0))^{\gamma}E_{1,1+\gamma}\Big(\lambda(\psi(t) - \psi(t_0)\Big). $$

Corollary 2. *For* $\gamma > 0, \lambda > 0$ *we have*

$$ {}_{t_0}^{\mathcal{TRLF}}\mathcal{I}_{\psi(t)}^{1-\gamma,\lambda}\Big(e^{-\lambda\psi(t)}(\psi(t) - \psi(t_0))^{\gamma-1}\Big) = \Gamma(\gamma)e^{-\lambda\psi(t)}. $$

Lemma 5. *For any* $T > t_0$ *the equality*

$$ \Gamma(1-\gamma){}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}(1)|_{t=T} = \lambda\Psi(T,t_0) + (\psi(T) - \psi(t_0))^{-\gamma}e^{-\lambda(\psi(T)-\psi(t_0))} \tag{17} $$

holds, here

$$ {}_1F_1(a,b,z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b)}{\Gamma(b+k)\Gamma(a)} \frac{z^k}{k!} $$

and (the Krummer function)

$$ \Psi(T,\sigma) = \frac{\Gamma(1-\gamma)}{\Gamma(2-\gamma)}(\psi(T)-\psi(\sigma))^{1-\gamma}\,{}_1F_1(1-\gamma,2-\gamma,-\lambda(\psi(T)-\psi(\sigma))),\ \ \sigma \in [t_0,T]. \tag{18} $$

Proof. From Definition 4 we have

$$ \begin{aligned} {}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} 1 = \frac{1}{\Gamma(1-\gamma)} \Bigg(\lambda \int_{t_0}^{t} (\psi(t)-\psi(s)^{-\gamma}\psi'(s)e^{-\lambda(\psi(t)-\psi(s))}\, ds \\ + \frac{1}{\psi'(t)} \frac{d}{dt} \int_{t_0}^{t} (\psi(t)-\psi(s)^{-\gamma}\psi'(s)e^{-\lambda(\psi(t)-\psi(s))}\, ds \Bigg). \end{aligned} \tag{19} $$

For any $T > t_0$ and $\sigma \in [t_0, T]$, we have

$$ \begin{aligned} &\int_{\sigma}^{T} (\psi(T)-\psi(s))^{-\gamma}\psi'(s)e^{-\lambda(\psi(T)-\psi(s))}ds \\ &= \sum_{k=0}^{\infty} \frac{(-\lambda(\psi(T)-\psi(s))^k}{k!} \int_{\sigma}^{T} (\psi(T)-\psi(s))^{-\gamma}\psi'(s)ds \\ &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \frac{1}{-1-k+\gamma} \int_{\sigma}^{T} \frac{d}{ds}(\psi(T)-\psi(s))^{1+k-\gamma}ds \\ &= (\psi(T)-\psi(\sigma))^{1-\gamma} \sum_{k=0}^{\infty} \frac{(-\lambda(\psi(T)-\psi(\sigma)))^k}{k!} \frac{1}{1+k-\gamma} \\ &= \frac{\Gamma(1-\gamma)}{\Gamma(2-\gamma)}(\psi(T)-\psi(\sigma))^{1-\gamma} \sum_{k=0}^{\infty} \frac{(-\lambda(\psi(T)-\psi(\sigma))^k}{k!} \frac{\Gamma(1+k-\gamma)\Gamma(2-\gamma)}{\Gamma(1-\gamma)\Gamma(2+k-\gamma)} \\ &= \frac{\Gamma(1-\gamma)}{\Gamma(2-\gamma)}(\psi(T)-\psi(\sigma))^{1-\gamma}\,{}_1F_1(1-\gamma,2-\gamma,-\lambda(\psi(T)-\psi(\sigma))) \\ &= \Psi(T,\sigma), \quad \sigma \in [t_0,T], \end{aligned} \tag{20} $$

note ${}_1F_1(1-\gamma, 2-\gamma, -\lambda(\psi(T)-\psi(\sigma))) \geq 0$ for $\gamma \in (0,1)$ and $\sigma \in [t_0, T]$.

For any $T > t_0$ and $\sigma \in [t_0, T]$, we get

$$ \begin{aligned} &\frac{d}{dT} \int_{\sigma}^{T} (\psi(T)-\psi(s))^{-\gamma}\psi'(s)e^{-\lambda(\psi(T)-\psi(s))}ds \\ &= \frac{d}{dT} \sum_{k=0}^{\infty} \frac{(-\lambda(\psi(T)-\psi(s))^k}{k!} \int_{\sigma}^{T} (\psi(T)-\psi(s))^{-\gamma}\psi'(s)ds \\ &= \frac{d}{dT} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \frac{1}{-1-k+\gamma} \int_{\sigma}^{T} \frac{d}{ds}(\psi(T)-\psi(s))^{1+k-\gamma}ds \\ &= \frac{d}{dT} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \frac{1}{1+k-\gamma}(\psi(T)-\psi(\sigma))^{1+k-\gamma} \\ &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!}(\psi(T)-\psi(\sigma))^{k-\gamma}\psi'(T) \\ &= (\psi(T)-\psi(\sigma))^{-\gamma}\psi'(T) \sum_{k=0}^{\infty} \frac{(-\lambda(\psi(T)-\psi(\sigma))^k}{k!} \\ &= (\psi(T)-\psi(\sigma))^{-\gamma}\psi'(T)e^{-\lambda(\psi(T)-\psi(\sigma))}. \end{aligned} \tag{21} $$

From equalities (20) and (21) with $T = t$ and $\sigma = t_0$ and Equation (19), we obtain (17). □

3. Inequalities for Riemann–Liouville-Type Fractional Derivatives of Convex Functions

We will use the following class of functions:

$$\Omega = \{V \in C^2(\mathbb{R}^n, \mathbb{R}) : \ V(0) = 0, \\ V(\lambda x + (1-\lambda)y) \le \lambda V(x) + (1-\lambda)V(y) \text{ for } \lambda \in [0,1], \ x, y \in \mathbb{R}^n\}.$$

Remark 4. *The function* $V \in \Omega$ *iff* $V \in C^2(\mathbb{R}^n, \mathbb{R})$ *and* $V(y) \ge V(x) + \sum_{i=1}^n \frac{\partial V(x)}{\partial x_i}(y_i - x_i)$ *for all* $x, y \in \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n), \ y = (y_1, y_2, \ldots, y_n)$.

Remark 5. *We will study Lyapunov functions from the class* Ω*, which is similar to the set of convex functions. The proofs of all inequalities for Lyapunov functions is deeply connected with the restriction that these functions are from the set* Ω*. At the same time, most of the practically applied Lyapunov functions in the literature, such as the quadratic Lyapunov functions, the Lyapunov functions defined by absolute values are in the set* Ω*. The Lyapunov functions suggested in [13] for the study stability of epidemiological models are also from the set* Ω*.*

For functions of the set Ω, we will prove some inequalities, applying different types of fractional derivatives.

3.1. Tempered Riemann–Liouville Fractional Derivative with Respect to a Function

Theorem 1 (TRLFDF). *Suppose the functions* $V \in \Omega : \quad V(.) \ge 0, \ x_k \in C_{q,\xi}([t_0, b]), \ k = 1, 2 \ldots, n, \ x = (x_1, x_2, \ldots, x_n)$*, and* $V(x(.)) \in C_{q,\xi}([t_0, b])$*. Then, the inequality*

$$\left({}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} V(x(t)) \right) \le \sum_{k=1}^n \left({}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} x_k(t) \right) \frac{\partial V(x(t))}{\partial x_k}, \quad t \in (t_0, b] \tag{22}$$

holds.

Proof. Fix an arbitrary point $T \in (t_0, b]$. The inequality (22) is equivalent to

$$ {}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} V(x(t)) \Big|_{t=T} - \sum_{k=1}^n \left({}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} x_k(t) \right) \Big|_{t=T} \frac{\partial V(x(T))}{\partial x_k} \le 0. \tag{23}$$

From Definition 4, we obtain

$$\begin{aligned}
&\Gamma(1-\gamma)\Big({}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} V(x(t)) \Big|_{t=T} - \sum_{k=1}^n \left({}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} x_k(t) \right) \Big|_{t=T} \frac{\partial V(x(T))}{\partial x_k} \Big) \\
&= \lambda \int_{t_0}^T \psi(T) - \psi(s)^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} V(x(s))\, ds \\
&\quad + \frac{1}{\psi'(T)} \frac{d}{dT} \int_{t_0}^T \psi(T) - \psi(s)^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} V(x(s)\, ds \\
&- \sum_{k=1}^n \frac{\partial V(x(T))}{\partial x_k} \Big[\lambda \int_{t_0}^T \psi(T) - \psi(s)^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} x_k(s)\, ds \\
&\quad + \frac{1}{\psi'(T)} \frac{d}{dT} \int_{t_0}^T \psi(T) - \psi(s)^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} x_k(s)\, ds \Big] \\
&= \lambda \int_{t_0}^T \psi(T) - \psi(s)^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} \Big(V(x(s)) - \sum_{k=1}^n \frac{\partial V(x(T))}{\partial x_k}\, x_k(s) \Big) ds \\
&\quad + \frac{1}{\psi'(T)} \frac{d}{dT} \int_{t_0}^T \psi(T) - \psi(s)^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} V(x(s))\, ds \\
&- \sum_{k=1}^n \frac{\partial V(x(T))}{\partial x_k} \frac{1}{\psi'(T)} \frac{d}{dT} \int_{t_0}^T \psi(T) - \psi(s)^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} x_k(s)\, ds.
\end{aligned} \tag{24}$$

We have

$$x_k(s) = x_k(t_0) + \int_{t_0}^{s} \frac{d}{d\sigma} x_k(\sigma) d\sigma, \;\; k = 1, 2, \ldots, n, \;\; s \in [t_0, T], \tag{25}$$

and

$$V(x(s)) = V(x(t_0)) + \sum_{i=1}^{n} \int_{t_0}^{s} \frac{\partial V(x(\sigma))}{\partial x_i} x_i'(\sigma) d\sigma, \;\; s \in [t_0, T]. \tag{26}$$

Apply (25) and (26) to (24) and obtain

$$\begin{aligned}
&\Gamma(1-\gamma)\Big({}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)} V(x(t)) \Big|_{t=T} - \sum_{k=1}^{n} \Big({}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)} x_k(t) \Big) \Big|_{t=T} \frac{\partial V(x(T))}{\partial x_k} \Big) \\
&= \lambda \int_{t_0}^{T} (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} \Big\{ \Big(V(x(t_0)) - \sum_{k=1}^{n} \frac{\partial V(x(T)}{\partial x_k} x_k(t_0) \Big) \\
&\quad + \int_{t_0}^{s} \Big(\sum_{k=1}^{n} \frac{\partial V(x(\sigma))}{\partial x_k} x_k'(\sigma) - \sum_{k=1}^{n} \frac{\partial V(x(T))}{\partial x_k} x_k'(\sigma) \Big) d\sigma \Big\} ds \\
&\quad + \frac{1}{\psi'(T)} \frac{d}{dT} \int_{t_0}^{T} (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} \Big(V(x(t_0)) + \int_{t_0}^{s} \sum_{k=1}^{n} \frac{\partial V(x(\sigma))}{\partial x_k} x_k'(\sigma) d\sigma \Big) ds \\
&\quad - \frac{1}{\psi'(T)} \sum_{k=1}^{n} \frac{\partial V(x(T))}{\partial x_k} \frac{d}{dT} \int_{t_0}^{T} (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} \Big(x_k(t_0) + \int_{t_0}^{s} x_k'(\sigma) d\sigma \Big) ds \\
&= \Big(V(x(t_0)) - \sum_{k=1}^{n} \frac{\partial V(x(T))}{\partial x_k} x_k(t_0) \Big) \Big\{ \lambda \int_{t_0}^{T} (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} ds \\
&\quad + \frac{1}{\psi'(T)} \frac{d}{dT} \int_{t_0}^{T} (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} ds \Big\} \\
&+ \lambda \int_{t_0}^{T} (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} \int_{t_0}^{s} \Big[\sum_{k=1}^{n} \Big(\frac{\partial V(x(\sigma))}{\partial x_k} - \frac{\partial V(x(T))}{\partial x_k} \Big) x_k'(\sigma) \Big] d\sigma \, ds \\
&\quad + \frac{1}{\psi'(T)} \frac{d}{dT} \int_{t_0}^{T} (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} \int_{t_0}^{s} \sum_{k=1}^{n} \frac{\partial V(x(\sigma))}{\partial x_k} x_k'(\sigma) d\sigma \, ds \\
&\quad - \frac{1}{\psi'(T)} \sum_{k=1}^{n} \frac{\partial V(x(T))}{\partial x_k} \frac{d}{dT} \int_{t_0}^{T} (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} \int_{t_0}^{s} x_k'(\sigma) d\sigma ds.
\end{aligned} \tag{27}$$

Use the equality

$$\int_{t_0}^{T} \int_{t_0}^{s} f(s,\sigma) d\sigma ds = \int_{t_0}^{T} \int_{\sigma}^{T} f(s,\sigma) ds d\sigma$$

for $f(s,\sigma) = (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} \frac{\partial V(x(\sigma))}{\partial x_k} x_k'(\sigma)$

or $f(s,\sigma) = (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} x_k'(\sigma)$, Lemma 5 with $t = T$ and Definition 4 and we obtain, from (27), the equality

$$\begin{aligned}
&\Gamma(1-\gamma)\Big({}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)} V(x(t)) \Big|_{t=T} - \sum_{k=1}^{n} \Big({}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)} x_k(t) \Big) \Big|_{t=T} \frac{\partial V(x(T))}{\partial x_k} \Big) \\
&= \Big(V(x(t_0)) - \sum_{k=1}^{n} \frac{\partial V(x(T))}{\partial x_k} x_k(t_0) \Big) \Xi(T) \\
&+ \lambda \int_{t_0}^{T} \Big[\sum_{k=1}^{n} \Big(\frac{\partial V(x(\sigma))}{\partial x_k} - \frac{\partial V(x(T))}{\partial x_k} \Big) x_k'(\sigma) \Big] \int_{\sigma}^{T} (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} ds \, d\sigma \\
&\quad + \frac{1}{\psi'(T)} \frac{d}{dT} \int_{t_0}^{T} \sum_{k=1}^{n} \frac{\partial V(x(\sigma))}{\partial x_k} x_k'(\sigma) \int_{\sigma}^{T} (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} ds \, d\sigma \\
&\quad - \frac{1}{\psi'(T)} \sum_{k=1}^{n} \frac{\partial V(x(T))}{\partial x_k} \frac{d}{dT} \int_{t_0}^{T} x_k'(\sigma) \int_{\sigma}^{T} (\psi(T) - \psi(s))^{-\gamma} \psi'(s) e^{-\lambda(\psi(T)-\psi(s))} ds d\sigma,
\end{aligned} \tag{28}$$

where $\Xi(T) = \lambda\Psi(T, t_0) + (\psi(T) - \psi(t_0))^{-\gamma} e^{-\lambda(\psi(T) - \psi(t_0))}$ and $\Psi(t, \sigma)$ is defined by (18).

Note we have the equalities

$$\begin{aligned} &\frac{d}{dT}\int_{t_0}^{T} \frac{\partial V(x(\sigma))}{\partial x_k} x_k'(\sigma) \int_{\sigma}^{T} (\psi(T) - \psi(s))^{-\gamma}\psi'(s) e^{-\lambda(\psi(T) - \psi(s))} ds\, d\sigma \\ &= \int_{t_0}^{T} \frac{\partial V(x(\sigma))}{\partial x_k} x_k'(\sigma) \frac{d}{dT}\int_{\sigma}^{T} (\psi(T) - \psi(s))^{-\gamma}\psi'(s) e^{-\lambda(\psi(T) - \psi(s))} ds\, d\sigma \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\frac{d}{dT}\int_{t_0}^{T} x_k'(\sigma) \int_{\sigma}^{T} (\psi(T) - \psi(s))^{-\gamma}\psi'(s) e^{-\lambda(\psi(T) - \psi(s))} ds\, d\sigma \\ &= \int_{t_0}^{T} x_k'(\sigma) \frac{d}{dT}\int_{\sigma}^{T} (\psi(T) - \psi(s))^{-\gamma}\psi'(s) e^{-\lambda(\psi(T) - \psi(s))} ds\, d\sigma. \end{aligned} \tag{30}$$

Apply equalities (20) with $t = T$, (29) and (30) to (28) and we get

$$\begin{aligned} &\Gamma(1-\gamma)\Big({}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} V(x(t))\Big|_{t=T} - \sum_{k=1}^{n}\Big({}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} x_k(t)\Big)\Big|_{t=T} \frac{\partial V(x(T))}{\partial x_k}\Big) \\ &= \Big(V(x(t_0)) - \sum_{k=1}^{n} \frac{\partial V(x(T))}{\partial x_k} x_k(t_0)\Big)\Xi(T) \\ &+ \lambda \int_{t_0}^{T} \Big[\sum_{k=1}^{n}\Big(\frac{\partial V(x(\sigma))}{\partial x_k} - \frac{\partial V(x(T))}{\partial x_k}\Big) x_k'(\sigma)\Big]\Psi(T,\sigma) d\sigma \\ &\quad + \frac{1}{\psi'(T)}\int_{t_0}^{T} \Big[\sum_{k=1}^{n}\Big(\frac{\partial V(x(\sigma))}{\partial x_k} - \frac{\partial V(x(T))}{\partial x_k}\Big) x_k'(\sigma)\Big] \\ &\quad \times \frac{d}{dT}\int_{\sigma}^{T} (\psi(T) - \psi(s))^{-\gamma}\psi'(s) e^{-\lambda(\psi(T) - \psi(s))} ds\, d\sigma. \end{aligned} \tag{31}$$

Apply the equality (21) to (31) and we get

$$\begin{aligned} &\Gamma(1-\gamma)\Big({}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} V(x(t))\Big|_{t=T} - \sum_{k=1}^{n}\Big({}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} x_k(t)\Big)\Big|_{t=T} \frac{\partial V(x(T))}{\partial x_k}\Big) \\ &= \Big(V(x(t_0)) - \sum_{k=1}^{n} \frac{\partial V(x(T))}{\partial x_k} x_k(t_0)\Big)\Xi(T) \\ &+ \lambda \int_{t_0}^{T} \Big[\sum_{k=1}^{n}\Big(\frac{\partial V(x(\sigma))}{\partial x_k} - \frac{\partial V(x(T))}{\partial x_k}\Big) x_k'(\sigma)\Big]\Psi(T,\sigma) d\sigma \\ &\quad + \int_{t_0}^{T} \Big[\sum_{k=1}^{n}\Big(\frac{\partial V(x(\sigma))}{\partial x_k} - \frac{\partial V(x(T))}{\partial x_k}\Big) x_k'(\sigma)\Big] (\psi(T) - \psi(\sigma))^{-\gamma} e^{-\lambda(\psi(T) - \psi(\sigma))} d\sigma. \end{aligned} \tag{32}$$

Define the function $P(s) = V(x(s)) - V(x(T)) - \sum_{k=1}^{n} \frac{\partial V(x(T))}{\partial x_k})[x_k(s) - x_k(T)]$ for $s \in [t_0, T]$. From $V \in \Omega$, it follows that $P(s) \geq 0$ for all $s \in [t_0, T]$, $P(T) = 0$ and $\frac{dP(s)}{ds} = \sum_{k=1}^{n}\Big(\frac{\partial V(x(s))}{\partial x_k} - \frac{\partial V(x(T))}{\partial x_k}\Big) x_k'(s)$.

Integrate by parts, use

$$\lim_{s \to T-} \frac{P(s)}{(\psi(T) - \psi(s))^{\gamma}} = \lim_{s \to T-} \frac{P''(s)}{-\gamma(1-\gamma)(\psi'(s))^2 - \gamma(\psi(T) - \psi(s))\psi''(s)} (\psi(T) - \psi(s))^{2-\gamma} = 0$$

and

$$\begin{aligned}
&\frac{d}{ds}\Big((\psi(T)-\psi(s))^{-\gamma} e^{-\lambda(\psi(T)-\psi(s))} \Big) \\
&= e^{-\lambda(\psi(T)-\psi(s))}\psi'(s)(\psi(T)-\psi(s))^{-\gamma}\Big(\gamma(\psi(T)-\psi(s))^{-1}+\lambda\Big) \\
&\geq e^{-\lambda(\psi(T)-\psi(s))}\psi'(s)(\psi(T)-\psi(s))^{-\gamma}\lambda.
\end{aligned} \tag{33}$$

From equality (18) we have

$$\begin{aligned}
&\frac{d}{d\sigma}\Psi(T,\sigma) = \frac{\Gamma(1-\gamma)}{\Gamma(2-\gamma)}\sum_{k=0}^{\infty}\frac{(-\lambda)^k \frac{d}{d\sigma}(\psi(T)-\psi(\sigma))^{1-\gamma+k}}{k!}\frac{\Gamma(1+k-\gamma)\Gamma(2-\gamma)}{\Gamma(1-\gamma)\Gamma(2+k-\gamma)} \\
&= -\psi'(\sigma)(\psi(T)-\psi(\sigma))^{-\gamma}\sum_{k=0}^{\infty}\frac{(-\lambda)^k(\psi(T)-\psi(\sigma))^k}{k!} \\
&= -\psi'(\sigma)(\psi(T)-\psi(\sigma))^{-\gamma}e^{-\lambda(\psi(T)-\psi(\sigma))}, \quad \sigma\in[t_0,T].
\end{aligned} \tag{34}$$

From inequality (33) and equality (34) we obtain

$$\begin{aligned}
&\int_{t_0}^{T}\Big[\sum_{k=1}^{n}\Big(\frac{\partial V(x(s))}{\partial x_k}-\frac{\partial V(x(T))}{\partial x_k}\Big)x_k'(s)\Big](\psi(T)-\psi(s))^{-\gamma}e^{-\lambda(\psi(T)-\psi(s))}\,ds \\
&= \int_{t_0}^{T}(\psi(T)-\psi(s))^{-\gamma}e^{-\lambda(\psi(T)-\psi(s))}P'(s)ds \\
&= e^{-\lambda(\psi(T)-\psi(s))}\frac{P(s)}{(\psi(T)-\psi(s))^{\gamma}}\Big|_{t_0}^{T} \\
&\quad -\int_{t_0}^{T}\Big(\frac{d}{ds}(\psi(T)-\psi(s))^{-\gamma}e^{-\lambda(\psi(T)-\psi(s))}\Big)P(s)ds \\
&\leq -e^{-\lambda(\psi(T)-\psi(t_0))}\frac{P(t_0)}{(\psi(T)-\psi(t_0))^{\gamma}} \\
&\quad -\lambda\int_{t_0}^{T}e^{-\lambda(\psi(T)-\psi(s))}\psi'(s)(\psi(T)-\psi(s))^{-\gamma}P(s)ds,
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
&\int_{t_0}^{T}\Big[\sum_{k=1}^{n}\Big(\frac{\partial V(x(\sigma))}{\partial x_k}-\frac{\partial V(x(T))}{\partial x_k}\Big)x_k'(\sigma)\Big]\Psi(T,\sigma)\,d\sigma \\
&= \int_{t_0}^{T}P'(\sigma)\Psi(T,\sigma)d\sigma \\
&= P(\sigma)\Psi(T,\sigma)\Big|_{\sigma=t_0}^{\sigma=T}-\int_{t_0}^{T}P(\sigma)\frac{d}{d\sigma}\Psi(T,\sigma)d\sigma \\
&= -P(t_0)\Psi(T,t_0)+\int_{t_0}^{T}P(\sigma)(\psi(T)-\psi(\sigma))^{-\gamma}\psi'(\sigma)e^{-\lambda(\psi(T)-\psi(\sigma))}d\sigma.
\end{aligned} \tag{36}$$

From $V \in \Omega$ and Remark 4 with $y = 0$, it follows that $-V(x(T)) + \sum_{k=1}^{n} \frac{\partial V(x(T))}{\partial x_k} x_k(T) \geq 0$ and thus from (32), (35), and (36) we get

$$\begin{aligned}
&\Gamma(1-\gamma)\Big({}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} V(x(t))\Big|_{t=T} - \sum_{k=1}^{n}\Big({}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} x_k(t)\Big)\Big|_{t=T} \frac{\partial V(x(T))}{\partial x_k}\Big) \\
&\leq P(t_0)\Xi(T) + \Big(V(x(T)) - \sum_{k=1}^{n} \frac{\partial V(x(T))}{\partial x_k}) x_k(T)\Big)\Xi(T) \\
&\quad - \lambda P(t_0)\Psi(T,t_0) + \lambda \int_{t_0}^{T} P(\sigma)(\psi(T)-\psi(\sigma))^{-\gamma}\psi'(\sigma)e^{-\lambda(\psi(T)-\psi(\sigma))}d\sigma \\
&\quad - e^{-\lambda(\psi(T)-\psi(t_0))}\frac{P(t_0)}{(\psi(T)-\psi(t_0))^{\gamma}} \\
&\quad - \lambda \int_{t_0}^{T} e^{-\lambda(\psi(T)-\psi(s))}\psi'(s)(\psi(T)-\psi(s))^{-\gamma}P(s)ds \\
&\leq P(t_0)(\Xi(T) - \lambda\Psi(T,t_0) - e^{-\lambda(\psi(T)-\psi(t_0))}(\psi(T)-\psi(t_0))^{-\gamma}) = 0.
\end{aligned} \tag{37}$$

Inequality (37) proves the claim. □

As special cases of the result in Theorem 1, we obtain some results about particular Lyapunov functions.

Consider the quadratic Lyapunov function given by $V(x) = x^T x$, $x \in \mathbb{R}^n$. The function $V \in \Omega$.

Lemma 6. *Suppose the function* $v = (v_1, v_2, \dots, v_n) : v_k \in C_{\gamma,\lambda}^{\mathcal{TRLF}}([t_0, b])$, $k = 1, 2, \dots, n$, *and* $v_k^2 \in C_{\gamma,\lambda}^{\mathcal{TRLF}}([t_0, b])$, $k = 1, 2, \dots, n$,*. Then the inequality*

$$\sum_{k=1}^{n} {}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} v_k^2(t) \leq 2\sum_{k=1}^{n} v_k(t)\, {}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} v_k(t), \quad t \in (t_0, b]$$

holds.

Similarly, for the Lyapunov function $V(x) = x^T P x$, $x \in \mathbb{R}^n$, with $P \in \mathbb{R}^{n\times n}$, a positive semidefinite, symmetric, square, and constant matrix, the following result is true:

Lemma 7. *Suppose the function* $v = (v_1, v_2, \dots, v_n) : v_k \in C_{\gamma,\lambda}^{\mathcal{TRLF}}([t_0, b])$, $k = 1, 2, \dots, n$, *and* $v_k^2 \in C_{\gamma,\lambda}^{\mathcal{TRLF}}([t_0, b])$, $k = 1, 2, \dots, n$, *and* $P \in \mathbb{R}^{n\times n}$ *is a positive semidefinite, symmetric, square, and constant matrix. Then the inequality*

$${}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}\Big(v^T(t) P\, v(t)\Big) \leq 2\Big(v^T(t) P\, {}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} v(t)\Big), \quad t \in (t_0, b]$$

holds.

Consider the quartic Lyapunov function given by $V(x) = \sum_{i=1}^{n} x_i^4$, $x \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$.

Lemma 8. *Suppose the function* $v = (v_1, v_2, \dots, v_n) : v_k \in C_{\gamma,\lambda}^{\mathcal{TRLF}}([t_0, b])$, $k = 1, 2, \dots, n$, *and* $v_k^4 \in C_{\gamma,\lambda}^{\mathcal{TRLF}}([t_0, b])$, $k = 1, 2, \dots, n$,*. Then the inequality*

$$\sum_{i=1}^{n} {}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} v_i^4(t) \leq 4\sum_{i=1}^{n} v_i^3(t)\, {}_{t_0}^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} v_i(t), \quad t \in (t_0, b]$$

holds.

The result is true for the exponential Lyapunov function $V(x) = e^{\sum_{i=1}^{n} x_i}$, with $x = (x_1, x_2, \ldots, x_n)$.

Lemma 9. *Suppose the function* $v = (v_1, v_2, \ldots, v_n) : \ v_k \in C^{\mathcal{TRLF}}_{\gamma,\lambda}([t_0, b]), \ k = 1, 2, \ldots, n.$ *Then the inequality*

$$ {}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)} e^{\sum_{i=1}^{n} v_i(t)} \leq \sum_{i=1}^{n} e^{v_i(t)} \, {}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)} v_i(t), \quad t \in (t_0, b] $$

holds.

Consider the Lyapunov function defined by absolute values, i.e., $V(x) = \sum_{i=1}^{n} |x_i|$, $x \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$. This function is not differentiable at 0, so Lemma 1 cannot be applied directly.

Lemma 10. *Let* $v \in C^{\mathcal{TRLF}}_{\gamma,\lambda}([t_0, b])$*. Then, for any* $t \in (t_0, b] : \ v(t) \neq 0$ *the inequality*

$$ {}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)} |v(t)| \leq sign(v(t)) \, {}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)} v(t) \tag{38} $$

holds.

Proof. The proof is similar to the one of Theorem 1 with the function $V(y) = |y|$ and for any fixed point $T \in [t_0, b]$ applying the function $P(s) = |x(s)| - |x(T)| - sign(x(T))[x(s) - x(T)] = |x(s)| - sign(x(T))x(s) \geq 0$ for all $s \in [t_0, T]$. □

Corollary 3. *Let* $v = (v_1, v_2, \ldots, v_n) : \ v_k \in C^{\mathcal{TRLF}}_{\gamma,\lambda}([t_0, b]), \ k = 1, 2, \ldots, n.$ *Then, for any point* $t \in (t_0, b]$ *such that* $v_i(t) \neq 0, \ i = 1, 2, \ldots, n$*, the inequality*

$$ \sum_{i=1}^{n} {}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)} |v_i(t)| \leq \sum_{i=1}^{n} sign(v_i(t)) \, {}^{\mathcal{TRLF}}_{t_0}\mathcal{D}^{\gamma,\lambda}_{\psi(t)} v_i(t) \tag{39} $$

holds.

3.2. Riemann–Liouville Fractional Derivative with Respect to a Function

Theorem 2 (RLFDF). *Suppose the functions* $V \in \Omega : \quad V(.) \geq 0, \ x_k \in C^{RL}_{\gamma,\psi}([t_0, b]), \ k = 1, 2 \ldots, n, \ x = (x_1, x_2, \ldots, x_n)$*, and* $V(x(.)) \in C^{RL}_{\gamma,\psi}([t_0, b])$*. Then the inequality*

$$ \left({}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} V(x(t)) \right) \leq \sum_{k=1}^{n} \left({}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} x_k(t) \right) \frac{\partial V(x(t))}{\partial x_k}, \quad t \in (t_0, b] \tag{40} $$

holds.

The proof of Theorem 2 follows from Theorem 1, applying the results of Section 2.1 and Definition 3 instead of Definition 4.

As special cases of the result in Theorem 2, we obtain some results about particular Lyapunov functions.

Consider the quadratic Lyapunov function given by $V(x) = x^T x, \ x \in \mathbb{R}^n$. The function $V \in \Omega$.

Lemma 11. *Suppose the function* $v = (v_1, v_2, \ldots, v_n) : \ v_k \in C^{RL}_{\gamma,\psi}([t_0, b]), \ k = 1, 2, \ldots, n,$ *and* $v_k^2 \in C^{RL}_{\gamma,\psi}([t_0, b]), \ k = 1, 2, \ldots, n,$*. Then the inequality*

$$ \sum_{k=1}^{n} {}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} v_k^2(t) \leq 2 \sum_{k=1}^{n} v_k(t) {}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} v_k(t), \quad t \in (t_0, b] $$

holds.

Similarly, for the Lyapunov function $V(x) = x^T P x,\ x \in \mathbb{R}^n$, with $P \in \mathbb{R}^{n\times n}$, a positive semidefinite, symmetric, square, and constant matrix, the following result is true:

Lemma 12. *Suppose the function* $\nu = (\nu_1, \nu_2, \dots, \nu_n) :\ \nu_k \in C^{RL}_{\gamma,\psi}([t_0, b]),\ k = 1, 2, \dots, n$, *and* $\nu_k^2 \in C^{RL}_{\gamma,\psi}([t_0, b]),\ k = 1, 2, \dots, n$, *and* $P \in \mathbb{R}^{n\times n}$ *is a positive semidefinite, symmetric, square, and constant matrix. Then the inequality*

$$ {}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)}\Big(\nu^T(t) P\, \nu(t)\Big) \le 2\Big(\nu^T(t) P\, {}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} \nu(t)\Big), \quad t \in (t_0, b] $$

holds.

Consider the quartic Lyapunov function given by $V(x) = \sum_{i=1}^n x_i^4,\ x \in \mathbb{R}^n,\ x = (x_1, x_2, \dots, x_n)$.

Lemma 13. *Suppose the function* $\nu = (\nu_1, \nu_2, \dots, \nu_n) :\ \nu_k \in C^{RL}_{\gamma,\psi}([t_0, b]),\ k = 1, 2, \dots, n$, *and* $\nu_k^4 \in C^{RL}_{\gamma,\psi}([t_0, b]),\ k = 1, 2, \dots, n,$ *. Then the inequality*

$$ \sum_{i=1}^n {}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} \nu_i^4(t) \le 4 \sum_{i=1}^n \nu_i^3(t) {}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} \nu_i(t), \quad t \in (t_0, b] $$

holds.

The result is true for the exponential Lyapunov function $V(x) = e^{\sum_{i=1}^n x_i}$, with $x = (x_1, x_2, \dots, x_n)$.

Lemma 14. *Suppose the function* $\nu = (\nu_1, \nu_2, \dots, \nu_n) :\ \nu_k \in C^{RL}_{\gamma,\psi}([t_0, b]),\ k = 1, 2, \dots, n$. *Then the inequality*

$$ {}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} e^{\sum_{i=1}^n \nu_i(t)} \le \sum_{i=1}^n e^{\nu_i(t)}\ {}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} \nu_i(t), \quad t \in (t_0, b] $$

holds.

Consider the Lyapunov function defined by absolute values, i.e., $V(x) = \sum_{i=1}^n |x_i|$, $x \in \mathbb{R}^n,\ x = (x_1, x_2, \dots, x_n)$. This function is not differentiable at 0, so Lemma 1 cannot be applied directly.

Lemma 15. *Let* $\nu \in C^{RL}_{\gamma,\psi}([t_0, b])$. *Then, for any* $t \in (t_0, b] :\ \nu(t) \ne 0$, *the inequality*

$$ {}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} |\nu(t)| \le sign(\nu(t))\ {}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} \nu(t) \tag{41} $$

holds.

Corollary 4. *Let* $\nu = (\nu_1, \nu_2, \dots, \nu_n) :\ \nu_k \in C^{RL}_{\gamma,\psi}([t_0, b]),\ k = 1, 2, \dots, n$. *Then, for any point* $t \in (t_0, b]$ *such that* $\nu_i(t) \ne 0,\ i = 1, 2, \dots, n$, *the inequality*

$$ \sum_{i=1}^n {}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} |\nu_i(t)| \le \sum_{i=1}^n sign(\nu_i(t))\ {}^{\mathcal{RL}}_{t_0}\mathcal{D}^{\gamma}_{\psi(t)} \nu_i(t) \tag{42} $$

holds.

3.3. Tempered Riemann–Liouville Fractional Derivative

Theorem 3 (TRLFD). *Suppose the functions $V \in \Omega: \quad V(.) \geq 0$, $x_k \in C^{\mathcal{TRL}}_{\gamma,\lambda}([t_0, b])$, $k = 1, 2 \ldots, n$, $x = (x_1, x_2, \ldots, x_n)$, and $V(x(.)) \in C^{\mathcal{TRL}}_{\gamma,\lambda}([t_0, b])$. Then the inequality*

$$\left({}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t V(x(t))\right) \leq \sum_{k=1}^{n} \left({}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t x_k(t)\right) \frac{\partial V(x(t))}{\partial x_k}, \quad t \in (t_0, b] \tag{43}$$

holds.

The proof follows from Theorem 1 and the results of Section 2.1, with the application of Definition 2 instead of Definition 4.

As special cases of the result in Theorem 3, we obtain some results about particular Lyapunov functions.

Consider the quadratic Lyapunov function given by $V(x) = x^T x$, $x \in \mathbb{R}^n$. The function $V \in \Omega$.

Lemma 16. *Suppose the function $v = (v_1, v_2, \ldots, v_n): \ v_k \in C^{\mathcal{TRL}}_{\gamma,\lambda}([t_0, b])$, $k = 1, 2, \ldots, n$, and $v_k^2 \in C^{\mathcal{TRL}}_{\gamma,\lambda}([t_0, b])$, $k = 1, 2, \ldots, n,$. Then the inequality*

$$\sum_{k=1}^{n} {}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t v_k^2(t) \leq 2 \sum_{k=1}^{n} v_k(t) {}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t v_k(t), \quad t \in (t_0, b]$$

holds.

Similarly, for the Lyapunov function $V(x) = x^T P x$, $x \in \mathbb{R}^n$, with $P \in \mathbb{R}^{n \times n}$ a positive semidefinite, symmetric, square, and constant matrix, the following result is true:

Lemma 17. *Suppose the function $v = (v_1, v_2, \ldots, v_n): \ v_k \in C^{\mathcal{TRL}}_{\gamma,\lambda}([t_0, b])$, $k = 1, 2, \ldots, n$, and $v_k^2 \in C^{\mathcal{TRL}}_{\gamma,\lambda}([t_0, b])$, $k = 1, 2, \ldots, n$, and $P \in \mathbb{R}^{n \times n}$ is a positive semidefinite, symmetric, square, and constant matrix. Then the inequality*

$${}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t \left(v^T(t) P \, v(t)\right) \leq 2\left(v^T(t) P \, {}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t v(t)\right), \quad t \in (t_0, b]$$

holds.

Consider the quartic Lyapunov function given by $V(x) = \sum_{i=1}^{n} x_i^4$, $x \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$.

Lemma 18. *Suppose the function $v = (v_1, v_2, \ldots, v_n): \ v_k \in C^{\mathcal{TRL}}_{\gamma,\lambda}([t_0, b])$, $k = 1, 2, \ldots, n$, and $v_k^4 \in C^{\mathcal{TRL}}_{\gamma,\lambda}([t_0, b])$, $k = 1, 2, \ldots, n, .$ Then the inequality*

$$\sum_{i=1}^{n} {}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t v_i^4(t) \leq 4 \sum_{i=1}^{n} v_i^3(t) {}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t v_i(t), \quad t \in (t_0, b]$$

holds.

The result is true for the exponential Lyapunov function $V(x) = e^{\sum_{i=1}^{n} x_i}$, with $x = (x_1, x_2, \ldots, x_n)$.

Lemma 19. *Suppose the function $v = (v_1, v_2, \ldots, v_n): \ v_k \in C^{\mathcal{TRL}}_{\gamma,\lambda}([t_0, b])$, $k = 1, 2, \ldots, n$. Then the inequality*

$${}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t e^{\sum_{i=1}^{n} v_i(t)} \leq \sum_{i=1}^{n} e^{v_i(t)} \, {}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t v_i(t), \quad t \in (t_0, b]$$

holds.

Consider the Lyapunov function defined by absolute values, i.e., $V(x) = \sum_{i=1}^{n} |x_i|$, $x \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$. This function is not differentiable at 0, so Theorem 3 cannot be applied directly.

Lemma 20. *Let $\nu \in C^{\mathcal{TRL}}_{\gamma,\lambda}([t_0, b])$. Then, for any $t \in (t_0, b] : \ \nu(t) \neq 0$ the inequality*

$$ {}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t |\nu(t)| \leq sign(\nu(t))\, {}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t \nu(t) \tag{44} $$

holds.

Corollary 5. *Let $\nu = (\nu_1, \nu_2, \dots, \nu_n) : \ \nu_k \in C^{\mathcal{TRL}}_{\gamma,\lambda}([t_0, b])$, $k = 1, 2, \dots, n$. Then, for any point $t \in (t_0, b]$ such that $\nu_i(t) \neq 0$, $i = 1, 2, \dots, n$, the inequality*

$$ \sum_{i=1}^{n} {}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t |\nu_i(t)| \leq \sum_{i=1}^{n} sign(\nu_i(t))\, {}^{\mathcal{TRL}}_{t_0}\mathcal{D}^{\gamma,\lambda}_t \nu_i(t) \tag{45} $$

holds.

3.4. Riemann–Liouville Fractional Derivative

Theorem 4 (RLFD). *Suppose the functions $V \in \Omega : \quad V(.) \geq 0$, $x_k \in C^{RL}_{\gamma}([t_0, b])$, $k = 1, 2 \dots, n$, $x = (x_1, x_2, \dots, x_n)$, and $V(x(.)) \in C^{RL}_{\gamma}([t_0, b])$. Then the inequality*

$$ \left({}^{RL}_{t_0}D^{\gamma}_t V(x(t)) \right) \leq \sum_{k=1}^{n} \left({}^{RL}_{t_0}D^{\gamma}_t x_k(t) \right) \frac{\partial V(x(t))}{\partial x_k}, \quad t \in (t_0, b] \tag{46} $$

holds.

The proof follows from Theorem 1 and the results of Section 2.1, with the application of Definition 1 instead of Definition 4.

As special cases of the result in Theorem 4,we obtain some results about particular Lyapunov functions.

Consider the quadratic Lyapunov function given by $V(x) = x^T x$, $x \in \mathbb{R}^n$. The function $V \in \Omega$.

Lemma 21. *Suppose the function $\nu = (\nu_1, \nu_2, \dots, \nu_n) : \ \nu_k \in C^{RL}_{\gamma}([t_0, b])$, $k = 1, 2, \dots, n$, and $\nu_k^2 \in C^{RL}_{\gamma}([t_0, b])$, $k = 1, 2, \dots, n$,. Then the inequality*

$$ \sum_{k=1}^{n} {}^{RL}_{t_0}D^{\gamma}_t \nu_k^2(t) \leq 2 \sum_{k=1}^{n} \nu_k(t) {}^{RL}_{t_0}D^{\gamma}_t \nu_k(t), \quad t \in (t_0, b] $$

holds.

Similarly, for the Lyapunov function $V(x) = x^T P x$, $x \in \mathbb{R}^n$, with $P \in \mathbb{R}^{n \times n}$ a positive semidefinite, symmetric, square, and constant matrix, the following result is true:

Lemma 22. *Suppose the function $\nu = (\nu_1, \nu_2, \dots, \nu_n) : \ \nu_k \in C^{RL}_{\gamma}([t_0, b])$, $k = 1, 2, \dots, n$, and $\nu_k^2 \in C^{RL}_{\gamma}([t_0, b])$, $k = 1, 2, \dots, n$, and $P \in \mathbb{R}^{n \times n}$ is a positive semidefinite, symmetric, square, and constant matrix. Then the inequality*

$$ {}^{RL}_{t_0}D^{\gamma}_t \left(\nu^T(t) P\, \nu(t) \right) \leq 2 \left(\nu^T(t) P\, {}^{RL}_{t_0}D^{\gamma}_t \nu(t) \right), \quad t \in (t_0, b] \tag{47} $$

holds.

Remark 6. *Note that the Caputo version of the inequality (47) cannot be applied to study Riemann–Liouville fractional differential equations (as is done, for example, in Lemma 2.2 [33], in Lemma 3 [34], in Lemma 2.2 [35], and in Lemma 2 [36,37]).*

Consider the quartic Lyapunov function given by $V(x) = \sum_{i=1}^{n} x_i^4$, $x \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$.

Lemma 23. *Suppose the function $\nu = (\nu_1, \nu_2, \ldots, \nu_n) : \nu_k \in C_\gamma^{RL}([t_0, b])$, $k = 1, 2, \ldots, n$, and $\nu_k^4 \in C_\gamma^{RL}([t_0, b])$, $k = 1, 2, \ldots, n$, . Then the inequality*

$$\sum_{i=1}^{n} {}_{t_0}^{RL}D_t^\gamma \nu_i^4(t) \le 4 \sum_{i=1}^{n} \nu_i^3(t) {}_{t_0}^{RL}D_t^\gamma \nu_i(t), \quad t \in (t_0, b]$$

holds.

The result is true for the exponential Lyapunov function $V(x) = e^{\sum_{i=1}^{n} x_i}$, with $x = (x_1, x_2, \ldots, x_n)$.

Lemma 24. *Suppose the function $\nu = (\nu_1, \nu_2, \ldots, \nu_n) : \nu_k \in C_\gamma^{RL}([t_0, b])$, $k = 1, 2, \ldots, n$. Then the inequality*

$${}_{t_0}^{RL}D_t^\gamma e^{\sum_{i=1}^{n} \nu_i(t)} \le \sum_{i=1}^{n} e^{\nu_i(t)} \; {}_{t_0}^{RL}D_t^\gamma \nu_i(t), \quad t \in (t_0, b]$$

holds.

Consider the Lyapunov function defined by absolute values, i.e., $V(x) = \sum_{i=1}^{n} |x_i|$, $x \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$. This function is not differentiable at 0, so Theorem 4 cannot be applied directly.

Lemma 25. *Let $\nu \in C_\gamma^{RL}([t_0, b])$. Then, for any $t \in (t_0, b] : \nu(t) \neq 0$ the inequality*

$${}_{t_0}^{RL}D_t^\gamma |\nu(t)| \le sign(\nu(t)) \; {}_{t_0}^{RL}D_t^\gamma \nu(t) \tag{48}$$

holds.

Corollary 6. *Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n) : \nu_k \in C_\gamma^{RL}([t_0, b])$, $k = 1, 2, \ldots, n$. Then, for any point $t \in (t_0, b]$ such that $\nu_i(t) \neq 0$, $i = 1, 2, \ldots, n$, the inequality*

$$\sum_{i=1}^{n} {}_{t_0}^{RL}D_t^\gamma |\nu_i(t)| \le \sum_{i=1}^{n} sign(\nu_i(t)) \; {}_{t_0}^{RL}D_t^\gamma \nu_i(t) \tag{49}$$

holds.

Remark 7. *The inequalities for Lyapunov functions with various types of fractional derivatives such as RLFD, TRLFD, RLFDF, and TRLFDF look similar, but their proofs are quite different and they deeply depend on the definitions of the corresponding applied fractional derivative.*

4. Mittag–Leffler Stability in Time for Differential Equations with Riemann–Liouville-Type Fractional Derivatives

Consider the following system of nonlinear differential equations with the Riemann–Liouville-type fractional derivative discussed above

$${}_0D\, y(t) = F(t, y(t)) \;\text{ for } t > 0, \tag{50}$$

with initial conditions

$$ {}_0I\, y(t)|_{t=0} = y_0, \tag{51} $$

where $\gamma \in (0,1)$, $\lambda \in \mathbb{R}$, and $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $F = (F_1, F_2, \ldots, F_n)$, and both the fractional integral ${}_0I$ and the fractional derivative ${}_0D$ are

- both are replaced by RLFI ${}_0I_t^{1-\gamma}$ and RLFD ${}_0^{RL}D_t^{\gamma}$, respectively;
- both are replaced by TRLFI ${}_0^{\mathcal{T}}\mathcal{I}_t^{1-\gamma,\lambda}$ and TRLFD ${}_0^{\mathcal{TRL}}\mathcal{D}_t^{\gamma,\lambda}$, respectively;
- both are replaced by RLFIF ${}_0\mathcal{I}_{\psi(t)}^{1-\gamma}$ and RLFDF ${}_0^{RL}\mathcal{D}_{\psi(t)}^{\gamma}$, respectively;
- both are replaced by TRLFIF ${}_0^{\mathcal{TF}}\mathcal{I}_{\psi(t)}^{1-\gamma,\lambda}$ and TRLFDF ${}_0^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}$, respectively.

We will assume that for any initial value y_0, the problems (50) and (51) has a solution $y(t; y_0) :\ y = (y_1, y_2, \ldots, y_n)$ defined for $t > 0$.

Note that any type of Riemann–Liouville fractional derivative defined above has a singularity at the initial time point. For example, consider the classical Riemann–Liouville fractional derivative and the simple scalar equation ${}_0^{RL}D_t^{\gamma} y(t) = -y(t)$, with $\gamma \in (0,1)$. The initial condition cannot be of the type $y(y) = y_0$ as in the case of the ordinary derivative or the Caputo derivative. The initial condition is ${}_0I^{1-\gamma}y(t)|_{t=0} = y_0$ and the solution is $y(t) = y_0 t^{\gamma-1}E_{\gamma,\gamma}(-t^{\gamma})$, with $E_{\gamma,\gamma}(.)$ being the Mittag–Leffler function. It is obvious that, for any positive number ε, the inequality $|y(t)| < \varepsilon$ cannot be satisfied for values of $t > 0$ and close to 0. This requires, for the stability to be defined, the initial time point 0 to be excluded, i.e., the special type of stability is deeply connected with the applied type of Riemann–Liouville fractional derivative (see, for example, refs. [2,4,38]). In some papers, for example, in [39], the authors do not exclude the initial time point and this leads to misunderstandings (see, for example the main condition of Theorem 1, $t^{\gamma-1}E_{\gamma,\gamma}(At^{\gamma}) \le Me^{-\gamma t}$, which is not satisfied on the whole interval $[0, \infty)$).

Definition 5. *We will say that the initial value problems* (50) *and* (51) *are generalized Mittag–Leffler stable in time if there exists positive constants T_γ, C and nondecreasing functions $P(.), Q(.) : [0.\infty) \to [0, \infty)$, $P(0) = 0, Q(0) = 0$, such that for any solutions $y(t; y_0)$ and $x(t; x_0)$ of* (50) *and* (51)

- *if both RLFI ${}_0I_t^{1-\gamma}$ and RLFD ${}_0^{RL}D_t^{\gamma}$ are applied, then the inequality*

$$ ||y(t) - x(t)|| \le P\Big(Q(||y_0 - x_0||E_{\gamma,\gamma}(-C(t)^{\gamma})\Big), \ \ t \ge T_\gamma $$

  holds;
- *if both TRLFI ${}_0^{\mathcal{T}}\mathcal{I}_t^{1-\gamma,\lambda}$ and TRLFD ${}_0^{\mathcal{TRL}}\mathcal{D}_t^{\gamma,\lambda}$ are applied, then the inequality*

$$ ||y(t) - x(t)|| \le P\Big(Q(||y_0 - x_0||e^{-\lambda t}E_{\gamma,\gamma}(-C\, t^{\gamma})\Big), \ \ t \ge T_\gamma $$

  holds;
- *if both RLFIF ${}_0\mathcal{I}_{\psi(t)}^{1-\gamma}$ and RLFDF ${}_0^{RL}\mathcal{D}_{\psi(t)}^{\gamma}$ are applied, then the inequality*

$$ ||y(t) - x(t)|| \le P\Big(Q(||y_0 - x_0||E_{\gamma,\gamma}(-C(\psi(t))^{\gamma})\Big), \ \ t \ge T_\gamma $$

  holds;
- *if both TRLFIF ${}_0^{\mathcal{TF}}\mathcal{I}_{\psi(t)}^{1-\gamma,\lambda}$ and TRLFDF ${}_0^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}$ are applied, then the inequality*

$$ ||y(t) - x(t)|| \le P\Big(Q(||y_0 - x_0||e^{-\lambda\psi(t)}E_{\gamma,\gamma}(-C(\psi(t))^{\gamma})\Big), \ \ t \ge T_\gamma $$

  holds.

Theorem 5. *Suppose there exists a function $V \in \Omega$ such that* *(i)* *there exist functions* $a, b \in \mathcal{K}$ *such that* $a(||x||) \le V(x) \le b(||x||)$ *for* $x \in \mathbb{R}^n$;

(ii) *For any solution* $y(t; y_0) : \ y = (y_1, y_2, \ldots, y_n)$ *such that*

- *if both RLFI* ${}_0I_t^{1-\gamma}$ *and RLFD* ${}_0^{RL}D_t^{\gamma}$ *are applied, then* $y_k \in C_\gamma^{RL}([0,\infty)), k = 1, 2, \ldots, n$ *and the fractional inequalities*

$$\begin{aligned} &{}_0I_t^{1-\gamma}V(y(t))|_{t=0} \le V(y_0), \\ &{}_0^{RL}D_t^{\gamma}V(y(t)) \le -CV(y(t)), \ \ t > 0 \end{aligned} \tag{52}$$

  hold where $C > 0$;

- *if both TRLFI* ${}_0^{\mathcal{T}}\mathcal{I}_t^{1-\gamma,\lambda}$ *and TRLFD* ${}_0^{\mathcal{TRL}}\mathcal{D}_t^{\gamma,\lambda}$ *are applied, then* $y_k \in C_{\gamma,\lambda}^{\mathcal{TRL}}([0,\infty))$, $k = 1, 2, \ldots, n$ *and the inequalities*

$$\begin{aligned} &{}_0^{\mathcal{T}}\mathcal{I}_t^{1-\gamma,\lambda}V(y(t))|_{t=0} \le V(y_0), \\ &{}_0^{\mathcal{TRL}}\mathcal{D}_t^{\gamma,\lambda}V(y(t)) \le -CV(y(t)), \ \ t > 0 \end{aligned} \tag{53}$$

  hold where $C > 0$;

- *if both RLFIF* ${}_0\mathcal{I}_{\psi(t)}^{1-\gamma}$ *and RLFDF* ${}_0^{RL}\mathcal{D}_{\psi(t)}^{\gamma}$ *are applied, then* $y_k \in C_{\gamma,\psi}^{RL}([0,\infty))$, $k = 1, 2, \ldots, n$ *and the inequalities*

$$\begin{aligned} &{}_0\mathcal{I}_{\psi(t)}^{1-\gamma}V(y(t))|_{t=0} \le V(y_0), \\ &{}_0^{RL}\mathcal{D}_{\psi(t)}^{\gamma}V(y(t)) \le -CV(y(t)), \ \ t > 0 \end{aligned} \tag{54}$$

  hold where $C > 0$;

- *if both TRLFIF* ${}_0^{\mathcal{TF}}\mathcal{I}_{\psi(t)}^{1-\gamma,\lambda}$ *and TRLFDF* ${}_0^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}$ *are applied, then the functions* $y_k \in C_{\gamma,\lambda,\psi}^{\mathcal{TRLF}}([0,\infty))$, $k = 1, 2, \ldots, n$ *and the inequalities*

$$\begin{aligned} &{}_0^{\mathcal{TF}}\mathcal{I}_{\psi(t)}^{1-\gamma,\lambda}V(y(t))|_{t=0} \le V(y_0), \\ &{}_0^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}V(y(t)) \le -CV(y(t)), \ \ t > 0 \end{aligned} \tag{55}$$

  hold where $C > 0$.

Then, the initial value problem (50) and (51) is generalized Mittag–Leffler stable in time.

Proof. We will prove the case when both TRLFIF ${}_0^{\mathcal{TF}}\mathcal{I}_{\psi(t)}^{1-\gamma,\lambda}$ and TRLFDF ${}_0^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}$ are applied. The proofs of the other fractional diffintegrals are similar.

Consider two solutions $y(t) = y(t; y_0)$ and $x(t) = x(t; x_0)$ of (50) and (51) with both TRLFIF ${}_0^{\mathcal{TF}}\mathcal{I}_{\psi(t)}^{1-\gamma,\lambda}$ and TRLFDF ${}_0^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}$, respectively, being applied and with the initial values $y_0, x_0 \in \mathbb{R}^n$. Define the function $u(t) = y(t) - x(t), \ t \ge 0$. Note the function $u(t)$ is a solution of the fractional Equation (50) with initial condition

$${}_0^{\mathcal{TF}}\mathcal{I}_{\psi(t)}^{1-\gamma,\lambda}u(t)|_{t=0} = y_0 - x_0. \tag{56}$$

Define the function $v(t) = V(y(t) - x(t)), \ t \ge 0$. From inequalities (55) it follows there exists a function $m \in C[0,\infty) : \ m(t) \le 0$ and a constant $K \ge 0$, such that

$$\begin{aligned} &{}_0^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}v(t) + Cv(t) = m(t), \quad t > 0, \\ &{}_0^{\mathcal{TF}}\mathcal{I}_{\psi(t)}^{1-\gamma,\lambda}v(t)|_{t=0} = v_0, \end{aligned} \tag{57}$$

where $v_0 = V(y_0 - x_0) - K$.

According to Corollary 1, the solution of the initial value problem (57) is given by

$$\begin{aligned} &V(u(t)) = v_0 e^{-\lambda\psi(t)}(\psi(t))^{\gamma-1}E_{\gamma,\gamma}(-C(\psi(t))^{\gamma}) \\ &+ \int_0^t e^{-\lambda(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\gamma-1}E_{\gamma,\gamma}(-C(\psi(t)-\psi(s))^{\gamma})m(s)\psi'(s)ds \\ &\leq V(y_0 - x_0)e^{-\lambda\psi(t)}(\psi(t))^{\gamma-1}E_{\gamma,\gamma}(-C(\psi(t))^{\gamma}). \end{aligned} \tag{58}$$

The function ψ is a nondecreasing continuous function. Therefore, there exists a number $T_\gamma > 0$, such that $(\psi(t))^{\gamma-1} \leq 1$ for $t \geq T_\gamma$. Then, from condition (i) and inequality (58), it follows that

$$||y(t) - x(t)|| \leq a^{-1}\Big(b(||y_0 - x_0||)e^{-\lambda\psi(t)}E_{\gamma,\gamma}(-C(\psi(t))^{\gamma})\Big), \ t \geq T_\gamma.$$

□

In the case of the Lyapunov function being a quadratic one, we obtain the following result (we will consider only the case when both TRLFIF ${}_0^{\mathcal{TF}}\mathcal{I}_{\psi(t)}^{1-\gamma,\lambda}$ and TRLFDF ${}_0^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda}$ are applied. The cases with the other Riemann–Liouville type of fractional derivatives are similar).

Corollary 7. *Suppose for any solution $y(t;y_0) : \ y = (y_1, y_2, \dots, y_n), \ y_0 = (y_{1,0}, y_{2,0}, \dots, y_{n,0})$, of (50) and (51) is such that $y_k^2 \in C_{\gamma,\lambda,\psi}^{\mathcal{TRLF}}([0,\infty)), \ k = 1,2,\dots,n$, and*

$$\begin{aligned} &{}_0^{\mathcal{TF}}\mathcal{I}_{\psi}^{1-\gamma,\lambda}(y_k^2(t))|_{t=0} \leq y_{k,0}^2, \\ &{}_0^{\mathcal{TRLF}}\mathcal{D}_{\psi(t)}^{\gamma,\lambda} y_k^2(t) \leq -C_k y_k^2(t), \ \ t > 0, \ \ k = 1,2,\dots,n, \end{aligned} \tag{59}$$

hold where $C_k > 0, \ k = 1,2,\dots,n$. Then, the initial value problem (50) and (51) is generalized Mittag–Leffler stable in time, i.e., there exists a constant $T_\gamma > 0$ such that for any two solutions $y(t;y_0), \ x(t;x_0)$ of (50) and (51) the inequality

$$\sum_{k=1}^n (y_k(t;y_0) - x_k(t;x_0))^2 \leq \sqrt{(y_{k,0} - x_{k,0})^2} e^{-\lambda\psi(t)}E_{\gamma,\gamma}(-C(\psi)^{\gamma}), \ \ t \geq T_\gamma$$

holds.

The proof follows from Theorem 5 by taking the Lyapunov function $V(x) = \sum_{i=1}^n x_i^2$ and using the norm $|| \cdot ||_2$.

Remark 8. *Note that the stability criteria in Corollary 7 is connected mainly with the assumptions about the solutions of the studied system, i.e., they have to be squared fractional integrable.*

5. Conclusions

The basic definitions of Riemann–Liouville types of fractional derivatives are given, such as the classical Riemann-Liuoville fractional derivative, the generalizations of Riemann–Liouville fractional derivative with respect to a function, as well as the tempered Riemann Liouville fractional derivative and the one with respect to a function. The main aim of the paper is to prove some inequalities for Lyapunov-type convex functions and the above mentioned fractional derivatives. As special cases, several inequalities for the widely applied Lyapunov functions defined by absolute values and the quadratic Lyapunov functions are obtained. In the partial case, when the classical fractional derivatives are applied, these inequalities provide results in the literature. The generalized Mittag-Lefler stability in time is defined and studied with the help of some of our inequalities.

Our inequalities are a practical tool for the study of various types of stability properties of Riemann–Liouville fractional differential equations by Lyapunov functions, such as stability, asymptotic stability, practical stability, exponential stability, and Mittag–Leffler stability. Note in the case of the Riemann–Liouville type of fractional derivatives, the initial time has to be excluded and modified types of stability have to be defined and studied. In addition, these inequalities are very useful in investigating the stability of equilibrium of real models such as the Hopsfield model of neural networks, the Cohen–Grossberg model of neural networks, and many others in which the dynamics of the units is modeled by a Riemann–Liouville-type fractional derivative. The inequalities give us the opportunity to study the stability of fractional models, and their applications will increase the interdisciplinary collaborations between the fields of mathematics, physics, engineering, epidemiology, and other relevant disciplines. These inequalities are the basis of the application of the Lyapunov method for fractional differential equations, and they could be used for teaching advanced concepts on stability analysis of fractional differential equations. Note, similar inequalities for the Lyapunov type of functions could be proven for alternative fractional derivatives and applied to other types of fractional models.

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Abbreviations

In this paper, the following abbreviations are used:

- RLFI and RLFD—Riemann–Liouville fractional integral and Riemann–Liouville fractional derivative, respectively;
- TRLFI and TRLFD—tempered Riemann–Liouville fractional integral and tempered Riemann–Liouville fractional derivative, respectively;
- RLFIF and RLFDF—Riemann–Liouville fractional integral with respect to a function and Riemann–Liouville fractional derivative with respect to a function, respectively;
- TRLFIF and TRLFDF—tempered Riemann–Liouville fractional integral with respect to a function and tempered Riemann–Liouville fractional derivative with respect to a function, respectively.

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