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Article

Graded Rings Associated with Factorizable Finite Groups

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Abstract: Let R be an associative ring with unity, X be a finite group, H be a subgroup of X, and $\mathfrak{G}$ be a set of left coset representatives for the left action of H on X. In this article, we introduce two different ways to put R into a non-trivial $\mathfrak{G}$-weak graded ring that is a ring graded by the set $\mathfrak{G}$ which is defined with a binary operation $*$ and satisfying an algebraic structure with specific properties. The first one is by choosing a subset S of $\mathfrak{G}$ such that S is a group under the $*$ operation and putting $R_t = 0$ for all $t \in \mathfrak{G}$ and $t \notin S$. The second way, which is the most important, is induced by combining the operation $*$ defined on $\mathfrak{G}$ and the coaction $\triangleleft$ of H on $\mathfrak{G}$. Many examples are provided.

Keywords: weak graded rings; weak graded modules; factorizable finite groups; left coset representatives

MSC: 16W50; 13A02; 16D25

1. Introduction

Recall that, for a group X and a ring R, R is called X-graded if, for each element x in the group X, there is an additive subgroup R_x of R, such that $R = \bigoplus_{x \in X} R_x$ and, for all $x, y \in X$, we have $R_x R_y \subseteq R_{xy}$. If the condition $R_x R_y \subseteq R_{xy}$ is replaced by the stronger one $R_x R_y = R_{xy}$ for all $x, y \in X$, then R is called fully (or strongly) X-graded ring. The theory of group graded ring is a rich area of mathematics with considerable connections to, for example, Clifford's theory, and the research area of operator algebras; see [1–4].

There have been many generalizations for group graded rings via replacing groups by semigroups or monoids for grading; see for example [5–7]. Some mathematicians have used different ways to generalize graded rings and modules as they consider the non-commutative algebraic geometry for quantum algebras to obtain the semi-graded rings and semi-graded modules (see [8]).

There have also been different ways to study their properties, such as categorical methods. For instance, the study of separable functors introduced in [1,9] used this way.Although the emphasis was on the finite case for grading, there have been some works on the infinite case; see for example [10].

In ref. [11], Beggs constructs an algebraic structure consisting of a set $\mathfrak{G}$ of left coset representatives for the left action of a subgroup H on a group X and a binary operation '$*$' on $\mathfrak{G}$. This operation guarantees the left identity and the right division property on $\mathfrak{G}$. '$*$' is not associative in the standard way, though the associativity could be satisfied by applying a "cocycle" $f : \mathfrak{G} \times \mathfrak{G} \longrightarrow H$. Based on this algebraic structure $(\mathfrak{G}, *)$ and the cocycle f together with the action $\triangleright : \mathfrak{G} \times H \to H$ and the coaction $\triangleleft : \mathfrak{G} \times H \to \mathfrak{G}$ defined in [11], many research articles on the non-trivially associated categories and the $\mathfrak{G}$-weak graded rings and modules have been published (see [12–15]). The independence on the choice of representatives was proven in [11].

In refs. [13–15], the concepts of the group graded rings and modules were generalized by using the set $\mathfrak{G}$ of left coset representatives, which was mentioned above with the binary operation '$*$' defined on it. The new generalized concepts were named *$\mathfrak{G}$-weak graded rings* and *$\mathfrak{G}$-weak graded modules*. It was found that many results could been carried out in

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the new setting. Moreover, some properties of these $\mathfrak{G}$-weak graded rings and modules were investigated.

Recently, the researchers have defined analogues of important operator algebras with rings that are equipped with a group grading. In particular, it has been noted that there is a kind of the correspondence between rings graded by a finite group G and rings on which G acts as automorphisms, which has been pointed out by many of the researchers, see [2,16,17]. In fact, the two notions can be identical in specific cases.

In [17], Cohen and Montgomery took advantage of the fact that gradings and group actions are dual concepts to introduce new results about graded rings. It can be noted that an X-grading can be considered as a "coaction" of the finite group X. A certain algebra $A\#k[X]^*$ was formed, where A is a k-algebra graded by X. $A\#k[X]^*$ can be looked at as the graded rings, the skew group algebra $A * G$ can be looked at as the group actions, and a Morita context can be constructed using them. It was found that many graded ring problems can be solved by using the "Duality Theorem".

It was noted that the grading by a set of left coset representatives with the binary operation '$*$' is not always applicable; see [14,15]. In this article, we introduce two different approaches to make it applicable for any ring R with unity. The first one is by choosing a subset S of $\mathfrak{G}$ such that S is a group under the $*$ operation and putting $R_t = 0$ for all $t \in \mathfrak{G}$ and $t \notin S$. The second approach is induced by imposing specific conditions on $\mathfrak{G}$ and the operations defined on it. More specifically, we combine the binary operation with the coaction in the definition of the grading. Many examples are provided throughout the article.

The importance of this work comes from associating the grading with a factorization of a given finite group which may lead to a quantization of the classical results of group graded rings and modules. Moreover, this work may be used to generalize some results in the literature; for example, the work of Cohen and Montgomery [17], which was mentioned above. Throughout, unless otherwise stated, all groups are finite, rings are with unity, and modules are unital.

2. Preliminaries

In this section, we include some definitions that are needed for the present work.

Definition 1 ([11])**.** *Given a finite group X and a subgroup H, call $\mathfrak{G} \subset X$ a set of left coset representatives if for every $x \in X$ there is a unique $s \in \mathfrak{G}$ such that $x = us \in Hs$. Let s, t be elements in $\mathfrak{G}$. Then, $f(s,t)$ in H and $s * t$ in $\mathfrak{G}$ are determined by $st = f(s,t)(s*t)$ in X. Furthermore, the action $\triangleright : \mathfrak{G} \times H \to H$ and the coaction $\triangleleft : \mathfrak{G} \times H \to \mathfrak{G}$ are determined by $su = (s \triangleright u)(s \triangleleft u)$ where $s, s \triangleleft u \in \mathfrak{G}$ and $u, s \triangleright u \in H$. These factorizations are unique.*

The binary operation '$*$' ensures the right division property, i.e., there is a unique solution $p \in \mathfrak{G}$ satisfying the equation $p * s = t$ for all $s, t \in \mathfrak{G}$ and the left identity for each $s \in \mathfrak{G}$ which is denoted by 's^L'. In the case that $e \in \mathfrak{G}$, then $e_{\mathfrak{G}} = e$. Theses are required to prove that the following identities are satisfied for all $s, t, p \in \mathfrak{G}$ and all $u, vs. \in H$ [11]:

$$s \triangleright (t \triangleright u) = f(s,t)\big((s*t) \triangleright u\big) f\big(s \triangleleft (t \triangleright u), t \triangleleft u\big)^{-1}, \quad (s*t) \triangleleft u = \big(s \triangleleft (t \triangleright u)\big) * (t \triangleleft u), \tag{1}$$

$$s \triangleright uv = (s \triangleright u)\big((s \triangleleft u) \triangleright v\big), \quad s \triangleleft uv = (s \triangleleft u) \triangleleft v,$$

$$f(p,s) f(p*s,t) = \big(p \triangleright f(s,t)\big) f\big(p \triangleleft f(s,t), s*t\big),$$

$$\big(p \triangleleft f(s,t)\big) * (s*t) = (p*s)*t,$$

$$e_{\mathfrak{G}} \triangleleft v = e_{\mathfrak{G}}, \qquad e_{\mathfrak{G}} \triangleright v = e_{\mathfrak{G}} v e_{\mathfrak{G}}^{-1}, \qquad t \triangleright e = e, \qquad t \triangleleft e = t, \tag{2}$$

$$f(e_{\mathfrak{G}}, t) = e_{\mathfrak{G}}, \qquad t \triangleright e_{\mathfrak{G}}^{-1} = f\big(t \triangleleft e_{\mathfrak{G}}^{-1}, e_{\mathfrak{G}}\big)^{-1} \quad \text{and} \quad \big(t \triangleleft e_{\mathfrak{G}}^{-1}\big) * e_{\mathfrak{G}} = t.$$

These identities have been used to prove our results and to construct our examples. For more details and properties of the binary operation '$*$', the cocycle f, the action $\triangleright$

and the coaction $\triangleleft$, the reader is refered to [11]. In what follows, whenever $\mathfrak{G}$ and H are mentioned, we mean the set and the subgroup defined above.

Definition 2 ([13])**.** *Let X be a group, H be a subgroup of X, and $\mathfrak{G}$ be a fixed set of left coset representatives associated with a binary operation $*$. Then, a ring R is said to be a $\mathfrak{G}$-weak graded ring if*

$$R = \bigoplus_{s \in \mathfrak{G}} R_s \tag{3}$$

and

$$R_s R_t \subseteq R_{s*t} \qquad \text{for all } s, t \in \mathfrak{G}, \tag{4}$$

where the component R_s is an additive subgroup for each $s \in \mathfrak{G}$. If we have

$$R_s R_t = R_{s*t} \qquad \text{for all } s, t \in \mathfrak{G}, \tag{5}$$

instead of (4), then R is said to be a fully (or strongly) $\mathfrak{G}$-weak graded ring.

Definition 3 ([15])**.** *Let R be a $\mathfrak{G}$-weak graded ring. Then, a $\mathfrak{G}$-weak graded left R-module M is a left R-module, which satisfies:*

$$M = \bigoplus_{s \in \mathfrak{G}} M_s \qquad\qquad (\text{as abelian groups}) \tag{6}$$

and

$$R_s M_t \subseteq M_{s*t} \quad \forall\, s, t \in \mathfrak{G}. \tag{7}$$

If the relation (7) is replaced by

$$R_s M_t = M_{s*t} \quad \forall\, s, t \in \mathfrak{G}, \tag{8}$$

then M is termed a fully (or a strongly) $\mathfrak{G}$-weak graded left R-module.

Theorem 1 ([14])**.** *Let R be a $\mathfrak{G}$-weak graded ring with unity and x be an element in $U(R)$. If $x \in R_s$, for some $s \in \mathfrak{G}$, then $x^{-1} \in R_{s^L}$.*

3. Combining the Operation $*$ and the Coaction $\triangleleft$ to Have $\mathfrak{G}$-Weak Graded Rings

It is known, by definition, that in general $R = \bigoplus_{s \in \mathfrak{G}} R_s$ is not necessarily a non-trivial $\mathfrak{G}$-weak graded ring; see for example [3,13]. However, in this section we define a relation between the operation $*$ and the coaction $\triangleleft$ that makes the ring R into a $\mathfrak{G}$-weak graded ring.

Theorem 2. *Let $X = H\mathfrak{G}$ be a finite group that factorizes into a subgroup H and a set of left coset representatives $\mathfrak{G}$. Then, for any ring $R = \bigoplus_{s \in \mathfrak{G}} R_s$—as additive subgroups—such that $R_s R_t \subseteq R_p$ for some $p \in \mathfrak{G}$, we have $R_s R_t \subseteq R_{(s \triangleleft u)*t}$ for all $s, t \in \mathfrak{G}$ and for some $u \in H$.*

Proof. Let $R_s R_t \subseteq R_p$ for some $p \in \mathfrak{G}$. Hence, to show that $p = (s \triangleleft u) * t$, we consider the following cases:

1. $p = s$
   If $p = s$, then $1_R \in R_t$ which means $t = e_{\mathfrak{G}}$. This leads to the following two sub-cases:
   i - If $s * e_{\mathfrak{G}} = s$, then $(s \triangleleft e) * e_{\mathfrak{G}} = s$.
   ii - If $s * e_{\mathfrak{G}} \neq s$, then as $e_{\mathfrak{G}} \in H \cap \mathfrak{G}$, we have $e_{\mathfrak{G}}^{-1} \in H$. Hence, $(s \triangleleft e_{\mathfrak{G}}^{-1}) * e_{\mathfrak{G}} = s$.
2. $p = t$
   If $p = t$, then we have $1_R \in R_s$, which means $s = e_{\mathfrak{G}}$. Since $e_{\mathfrak{G}} * t = t$, hence $(e_{\mathfrak{G}} \triangleleft e) * t = e_{\mathfrak{G}} * t = t$.
3. $p = e_{\mathfrak{G}}$
   If $p = e_{\mathfrak{G}}$, then we have two sub-cases:
   i - If $s * t = e_{\mathfrak{G}}$, then it is performed as $(s \triangleleft e) * t = e_{\mathfrak{G}}$.

ii - If $s * t \neq e_{\mathfrak{G}}$, then by the right division property there is $t^L \in \mathfrak{G}$ such that $t^L * t = e_{\mathfrak{G}}$. Furthermore, since $(\mathfrak{G}, *)$ is not a group, there is $vs. \in H$ such that $s \triangleleft vs. \neq s$. If $s \triangleleft vs. = t^L$ it is performed. If $s \triangleleft vs. \neq t^L$ suppose that $s \triangleleft vs. = t'$, then again there is $w \in H$ such that $t' \triangleleft w = t''$ with $t'' = t^L$, as $\mathfrak{G}$ is finite. Hence, $(s \triangleleft v) \triangleleft w = s \triangleleft vw$ and since H is a subgroup, we have $vw = u \in H$. Thus, $(s \triangleleft u) * t = e_{\mathfrak{G}}$.

4. $p \in \mathfrak{G}$ and $p \neq s \neq t \neq e_{\mathfrak{G}}$.
We have two sub-cases:

i - If $s * t = p$, then it is performed by choosing $u = e$, i.e., $(s \triangleleft e) * t = p$.

ii - If $s * t \neq p$, then using the same technique as that applied in $3 - (ii)$ yields the result.

Note that if $R_s R_t = 0_R$, then $0_R \in R_p = R_{(s \triangleleft u) * t}$ for all $s, t, p \in \mathfrak{G}$ and for all $u \in H$ since R_p is an additive subgroup of the ring R for all $p \in \mathfrak{G}$ as required. □

Corollary 1. *If the ring R is a $\mathfrak{G}$-weak graded ring, then $R_s R_t \subseteq R_{(s \triangleleft u) * t}$ for all $s, t \in \mathfrak{G}$ and for some $u \in H$.*

Proof. It follows directly by choosing $u = e$ and using the identities in (2). □

Definition 4 ([14])**.** *Let R be a $\mathfrak{G}$-weak graded ring. Then, a non-zero element $r_s \in R$ is said to be a weak graded or $\mathfrak{G}$-homogeneous element of grade s if there exists an s-component R_s of R such that $r_s \in R_s$.*

We recall that a subring K of a $\mathfrak{G}$-weak graded ring R is a $\mathfrak{G}$-weak graded subring if K itself is a $\mathfrak{G}$-weak graded ring [14]. In the next theorem, we discuss when a subring K of a $\mathfrak{G}$-weak graded ring R is a $\mathfrak{G}$-weak graded subring respecting the inclusion property of R that was mentioned in Theorem 2.

Theorem 3. *Let $X = H\mathfrak{G}$ be a finite group that factorizes into a subgroup H and a set of left coset representatives $\mathfrak{G}$ and let R be a $\mathfrak{G}$-weak graded ring such that $R_s R_t \subseteq R_{(s \triangleleft u) * t}$ for all $s, t \in \mathfrak{G}$ and for some $u \in H$. Then, a subring K of R is a $\mathfrak{G}$-weak graded subring respecting the inclusion property above if K contains all the $\mathfrak{G}$-homogeneous components for each $k \in K$.*

Proof. Definition 2 yields every element $r \in R$ has a unique decomposition, written as $r = \sum_{s \in \mathfrak{G}} r_s$ with $r_s \in R_s$ for all $s \in \mathfrak{G}$ and the sum is finite. As K is a subring of R, we can assume that for each $k = \sum_{s \in \mathfrak{G}} k_s$, we have $k_s \in K$ for all $s \in \mathfrak{G}$. So we can write $K = \sum_{s \in \mathfrak{G}} K_s$ as additive subgroups. Again, since K is a subring of R and $R = \bigoplus_{s \in \mathfrak{G}} R_s$, we have $K_s = K \cap R_s$ for all $s \in \mathfrak{G}$. Furthermore, as R is a $\mathfrak{G}$-weak graded ring, the concept of the direct sum condition yields

$$K_s \cap (\sum_{t \in \mathfrak{G}} K_t) = (K \cap R_s) \cap (\sum_{t \in \mathfrak{G}} K \cap R_t) = K \cap (R_s \cap \sum_{t \in \mathfrak{G}} R_t) = K \cap \{0\} = \{0\},$$

for all $s \in \mathfrak{G}$ with $s \neq t$. Consequently, $K = \bigoplus_{s \in \mathfrak{G}} K_s$, as required.

Next, to prove that $K_{(s \triangleleft u)} K_t \subseteq K_{(s \triangleleft u) * t}$. Let $K_{(s \triangleleft u)} = K \cap R_{(s \triangleleft u)}$ and $K_t = K \cap R_t$ for some s, t in $\mathfrak{G}$, and some $u \in H$. Hence,

$$K_{(s \triangleleft u)} K_t = (K \cap R_{(s \triangleleft u)})(K \cap R_t) = K \cap (R_{(s \triangleleft u)} R_t) \subseteq K \cap (R_{(s \triangleleft u) * t}) = K_{(s \triangleleft u) * t},$$

as required. □

Example 1. *Consider the Morita ring $T = \begin{bmatrix} R & M \\ N & S \end{bmatrix}$ mentioned in [13] Example 1. Let the group $X = S_3 = \{e, (12), (13), (23), (123), (132)\}$, the subgroup $H = \{e, (12)\}$, and the set of* *left coset representatives $\mathfrak{G} = \{(12), (13), (23)\}$. Then, the $*$ operation and the cocycle f as well as the action $\triangleright$ and the coaction $\triangleleft$ are given by the following tables (Tables 1 and 2):*

Table 1. The operation $*$ and the cocycle f.

$*$	**(12)**	**(13)**	**(23)**
(12)	(12)	(13)	(23)
(13)	(23)	(12)	(13)
(23)	(13)	(23)	(12)
f	**(12)**	**(13)**	**(23)**
(12)	(12)	(12)	(12)
(13)	(12)	(12)	(12)
(23)	(12)	(12)	(12)

Table 2. The action $s \triangleright u$ and the coaction $s \triangleleft u$.

$s \triangleright u$	e	**(12)**
(12)	e	(12)
(13)	e	(12)
(23)	e	(12)
$s \triangleleft u$	e	**(12)**
(12)	(12)	(12)
(13)	(13)	(23)
(23)	(23)	(13)

Then, $T = T_{(12)} \oplus T_{(13)} \oplus T_{(23)}$, where

$$T_{(12)} = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix} = \left\{ \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} : \quad r \in R, \text{ and } s \in S \right\},$$

$$T_{(13)} = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} : \quad m \in M \right\}$$

and

$$T_{(23)} = \begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} : \quad n \in N \right\}.$$

In this case, regardless of the trivial case, T is not a $\mathfrak{G}$-weak graded ring, as, for example, for all $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \in T_{(13)}$ and $\begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} \in T_{(12)}$, we have $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} = \begin{bmatrix} 0 & ms \\ 0 & 0 \end{bmatrix} \in T_{(13)} \neq T_{(23)} = T_{(13)*(12)}$.

If we replace the relation $s * t$ by the relation $(s \triangleleft u) * t$, then we obtain $T_s T_t \subseteq T_{(s \triangleleft u)*t}$ for all $s, t \in \mathfrak{G}$ and for some $u \in H$. Hence, T is going to be a $\mathfrak{G}$-weak graded ring as follows, where the first table (Table 3) shows our choice of the element u for each time we apply the relation $(s \triangleleft u) * t$ in the second table (Table 4):

Table 3. The choice of the element u.

$s \triangleleft u$	u	u	u
(12)	$(12) \triangleleft e$	$(12) \triangleleft e$	$(12) \triangleleft e$
(13)	$(13) \triangleleft (12)$	$(13) \triangleleft e$	$(13) \triangleleft (12)$
(23)	$(23) \triangleleft (12)$	$(23) \triangleleft (12)$	$(23) \triangleleft e$

Table 4. The relation $(s \triangleleft u) * t$.

$(s \triangleleft u) * t$	(12)	(13)	(23)
$(12) \triangleleft u$	(12)	(13)	(23)
$(13) \triangleleft u$	(13)	(12)	(12)
$(23) \triangleleft u$	(23)	(12)	(12)

It is easy to prove that the inclusion property is satisfied by showing that: $T_{(12)}T_{(12)} \subseteq T_{((12)\triangleleft e)*(12)} = T_{(12)}$, $T_{(12)}T_{(13)} \subseteq T_{((12)\triangleleft e)*(13)} = T_{(13)}$, $T_{(12)}T_{(23)} \subseteq T_{((12)\triangleleft e)*(23)} = T_{(23)}$, $T_{(13)}T_{(12)} \subseteq T_{((13)\triangleleft(12))*(12)} = T_{(13)}$, $T_{(13)}T_{(13)} \subseteq T_{((13)\triangleleft e)*(13)} = T_{(12)}$, $T_{(13)}T_{(23)} \subseteq T_{((13)\triangleleft(12))*(23)} = T_{(12)}$, $T_{(23)}T_{(12)} \subseteq T_{((23)\triangleleft(12))*(12)} = T_{(23)}$, $T_{(23)}T_{(13)} \subseteq T_{((23)\triangleleft(12))*(13)} = T_{(12)}$ *and* $T_{(23)}T_{(23)} \subseteq T_{((23)\triangleleft e)*(23)} = T_{(12)}$.

Thus, T is a $\mathfrak{G}$-weak graded ring but not fully. For instance, $T_{(23)}T_{(23)} \subseteq T_{(12)}$, but the converse is not true. It can be noted that the relation $(s \triangleleft u) * t$ can be satisfied by different choices of u; for example, the following tables also make T into a $\mathfrak{G}$-weak graded ring (Tables 5 and 6):

Table 5. The choice of the element u.

$s \triangleleft u$	u	u	u
(12)	$(12) \triangleleft e$	$(12) \triangleleft e$	$(12) \triangleleft e$
(13)	$(13) \triangleleft (12)$	$(13) \triangleleft (12)$	$(13) \triangleleft (12)$
(23)	$(23) \triangleleft (12)$	$(23) \triangleleft (12)$	$(23) \triangleleft (12)$

Table 6. The relation $(s \triangleleft u) * t$.

$(s \triangleleft u) * t$	(12)	(13)	(23)
$(12) \triangleleft u$	(12)	(13)	(23)
$(13) \triangleleft u$	(13)	(23)	(12)
$(23) \triangleleft u$	(23)	(12)	(13)

Remark 1. *If we replace the relation $R_sR_t \subseteq R_{(s\triangleleft u)*t}$ for all $s,t \in \mathfrak{G}$ and for some $u \in H$ in Theorem 2 by the more general one $R_sR_t \subseteq R_{(s\triangleleft(t\triangleright u))*(t\triangleleft u)}$ for all $s,t \in \mathfrak{G}$ and for some $u \in H$, then the theorem is not going to work. Indeed, if we apply it on Example 1 we obtain $T_{(13)}T_{(23)} \subseteq T_{(12)}$ but $(13) * (23) = (13)$ and there is no $u \in H$ satisfying $(13) \triangleleft u = (12)$. Note that $R_{(s\triangleleft(t\triangleright u))*(t\triangleleft u)} = R_{(s*t)\triangleleft u}$ by the identities in (1).*

Example 2. *Consider the ring of all 2×2 matrices over the ring $\mathbb{Z}$, i.e.,*

$$R = M_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad : \quad a,b,c,d \in \mathbb{Z} \right\}.$$

Let $X = S_3 = \{e,(12),(13),(23),(123),(132)\}$ and the non-normal subgroup $H = \{e,(23)\}$. Choose the set of left coset representatives to be $\mathfrak{G} = \{e,(12),(13)\}$. Then, the $*$ operation and the cocycle f as well as the action $\triangleright$ and the coaction $\triangleleft$ are given by the following tables (Tables 7 and 8):

Table 7. The operation $*$ and the cocycle f.

$*$	e	(12)	(13)
e	e	(12)	(13)
(12)	(12)	e	(12)
(13)	(13)	(13)	e
f	e	(12)	(13)
e	e	e	e
(12)	e	e	(23)
(13)	e	(23)	e

Table 8. The action $s \triangleright u$ and the coaction $s \triangleleft u$.

$s \triangleright u$	e	(23)
e	e	(23)
(12)	e	(23)
(13)	e	(23)
$s \triangleleft u$	e	(23)
e	e	e
(12)	(12)	(13)
(13)	(13)	(12)

Then, R can be written as $R = R_e \oplus R_{(12)} \oplus R_{(13)}$, where

$$R_e = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : \quad a, d \in \mathbb{Z} \right\},\ R_{(12)} = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : \quad c \in \mathbb{Z} \right\} \text{and}$$

$$R_{(13)} = \left\{ \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} : \quad b \in \mathbb{Z} \right\}.$$

However, regardless of the trivial case, R is not a $\mathfrak{G}$-weak graded ring as, for example, for all $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \in R_{(12)}$ and $\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \in R_{(13)}$, we have

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} bc & 0 \\ 0 & 0 \end{bmatrix} \in R_e \neq R_{(12)} = R_{(12)*(13)}.$$

If we replace the relation $s * t$ by the relation $(s \triangleleft u) * t$, then we obtain $R_s R_t \subseteq R_{(s \triangleleft u)*t}$ for all $s, t \in \mathfrak{G}$ and for some $u \in H$. Hence, R is going to be a $\mathfrak{G}$-weak graded ring as follows, where the first table (Table 9) shows our choice of the element u for each time we apply the relation $(s \triangleleft u) * t$ in the second table (Table 10):

Table 9. The choice of the element u.

$s \triangleleft u$	u	u	u
e	$e \triangleleft e$	$e \triangleleft e$	$e \triangleleft e$
(12)	$(12) \triangleleft e$	$(12) \triangleleft (23)$	$(12) \triangleleft (23)$
(13)	$(13) \triangleleft e$	$(13) \triangleleft (23)$	$(13) \triangleleft e$

Table 10. The relation $(s \triangleleft u) * t$.

$(s \triangleleft u) * t$	e	(12)	(13)
$e \triangleleft u$	e	(12)	(13)
$(12) \triangleleft u$	(12)	(13)	e
$(13) \triangleleft u$	(13)	e	e

Now, to show that R is a $\mathfrak{G}$-weak graded ring, we prove that the inclusion property is satisfied, which can be performed easily by showing that: $R_e R_e \subseteq R_{(e \triangleleft e)*e} = R_e$, $R_e R_{(12)} \subseteq R_{(e \triangleleft e)*(12)} = R_{(12)}$, $R_e R_{(13)} \subseteq R_{(e \triangleleft e)*(13)} = R_{(13)}$, $R_{(12)} R_e \subseteq R_{((12) \triangleleft e)*e} = R_{(12)}$, $R_{(12)} R_{(12)} \subseteq R_{\left((12) \triangleleft (23)\right)*(12)} = R_{(13)}$, $R_{(12)} R_{(13)} \subseteq R_{\left((12) \triangleleft (23)\right)*(13)} = R_e$, $R_{(13)} R_e \subseteq R_{((13) \triangleleft e)*e} = R_{(13)}$, $R_{(13)} R_{(12)} \subseteq R_{\left((13) \triangleleft (23)\right)*(12)} = R_e$, and $R_{(13)} R_{(13)} \subseteq R_{((13) \triangleleft e)*(13)} = R_e$.

Thus, R is a $\mathfrak{G}$-weak graded ring but not fully.For instance, $R_{(13)} R_{(13)} \subseteq R_{((13) \triangleleft e)*(13)} = R_e$ but $R_e = R_{((13) \triangleleft e)*(13)} \nsubseteq R_{(13)} R_{(13)}$.

Example 3. *Consider the ring of real quaternions* $(\mathbb{H}, +, .)$. *Let* $X = S_4$ *and let* $H = \{e, (123), (132), (12), (13), (23)\}$ *be a subgroup of* X. *Take* $\mathfrak{G} = \{e, (14)(23), (243), (34)\}$ *to be the set of left coset representative. Then, the* $*$ *operation and the cocycle* f *as well as the action* $\triangleright$ *and the coaction* $\triangleleft$ *are given by the following tables (Tables 11–14):*

Table 11. The $*$ operation.

$*$	e	**(14)(23)**	**(243)**	**(34)**
e	e	(14)(23)	(243)	(34)
(14)(23)	(14)(23)	e	(14)(23)	(14)(23)
(243)	(243)	(34)	(34)	(243)
(34)	(34)	(243)	e	e

Table 12. The cocycle f.

f	e	**(14)(23)**	**(243)**	**(34)**
e	e	e	e	e
(14)(23)	e	e	(123)	(12)
(243)	e	(13)	(23)	(23)
(34)	e	(13)	(23)	e

Table 13. The action $s \triangleright u$.

$s \triangleright u$	e	**(123)**	**(132)**	**(12)**	**(13)**	**(23)**
e	e	(123)	(132)	(12)	(13)	(23)
(14)(23)	e	(13)	(123)	(13)	(123)	(23)
(243)	e	(132)	(12)	(13)	(12)	e
(34)	e	(12)	(13)	(12)	(132)	e

Table 14. The coaction $s \triangleleft u$.

$s \triangleleft u$	e	**(123)**	**(132)**	**(12)**	**(13)**	**(23)**
e	e	e	e	e	e	e
(14)(23)	(14)(23)	(34)	(243)	(243)	(34)	(14)(23)
(243)	(243)	(14)(23)	(34)	(14)(23)	(243)	(34)
(34)	(34)	(243)	(14)(23)	(34)	(14)(23)	(243)

Then, $\mathbb{H} = R_e \oplus R_{(14)(23)} \oplus R_{(243)} \oplus R_{(34)}$, *where*

$$R_e = \mathbb{R}\ ,\ R_{(14)(23)} = \mathbb{R}i\ ,\ R_{(243)} = \mathbb{R}j\ \text{ and }\ R_{(34)} = \mathbb{R}k.$$

But, regardless of the trivial case, the ring $\mathbb{H}$ *is not a* $\mathfrak{G}$*-weak graded ring. For instance,* $R_{(14)(23)}R_{(243)} \nsubseteq R_{(14)(23)*(243)} = R_{(14)(23)}$ *as for all* $ai \in R_{(14)(23)}, bj \in R_{(243)}$, *we have*

$$(ai)(bj) = (ab)k \in R_{(34)} \neq R_{(14)(23)} = R_{(14)(23)*(243)}.$$

If we replace the relation $s * t$ *by the relation* $(s \triangleleft u) * t$, *we obtain* $R_s R_t \subseteq R_{(s \triangleleft u)*t}$ *for all* $s, t \in \mathfrak{G}$ *and for some* $u \in H$. *Hence,* $\mathbb{H}$ *can be a* $\mathfrak{G}$*-weak graded ring as follows, where the first table (Table 15) shows our choice of the element* u *for each time we apply the relation* $(s \triangleleft u) * t$ *in the second table (Table 16):*

Table 15. The choice of the element u.

$s \lhd u$	u	u	u	u
e	$e \lhd e$	$e \lhd e$	$e \lhd e$	$e \lhd e$
$(14)(23)$	$(14)(23) \lhd e$	$(14)(23) \lhd e$	$(14)(23) \lhd (132)$	$(14)(23) \lhd (132)$
(243)	$(243) \lhd e$	$(243) \lhd e$	$(243) \lhd (132)$	$(243) \lhd (123)$
(34)	$(34) \lhd e$	$(34) \lhd e$	$(34) \lhd (132)$	$(34) \lhd e$

Table 16. The relation $(s \lhd u) * t$.

$(s \lhd u) * t$	e	$(14)(23)$	(243)	(34)
$e \lhd u$	e	$(14)(23)$	(243)	(34)
$(14)(23) \lhd u$	$(14)(23)$	e	(34)	(243)
$(243) \lhd u$	(243)	(34)	e	$(14)(23)$
$(34) \lhd u$	(34)	(243)	$(14)(23)$	e

Now, to show that $\mathbb{H}$ is a $\mathfrak{G}$-weak graded ring, we prove that the inclusion property is satisfied, which can be performed easily by showing that:

$R_e R_e \subseteq R_{(e \lhd e)*e} = R_e$, $R_e R_{(14)(23)} \subseteq R_{(e \lhd e)*(14)(23)} = R_{(14)(23)}$, $R_e R_{(243)} \subseteq R_{(e \lhd e)*(243)} = R_{(243)}$, $R_e R_{(34)} \subseteq R_{(e \lhd e)*(34)} = R_{(34)}$, $R_{(14)(23)} R_e \subseteq R_{((14)(23) \lhd e)*e} = R_{(14)(23)}$, $R_{(14)(23)} R_{(14)(23)} \subseteq R_{((14)(23) \lhd e)*(14)(23)} = R_e$, $R_{(14)(23)} R_{(243)} \subseteq R_{((14)(23) \lhd (132))*(243)} = R_{(34)}$, $R_{(14)(23)} R_{(34)} \subseteq R_{((14)(23) \lhd (132))*(34)} = R_{(243)}$, $R_{(243)} R_e \subseteq R_{((243) \lhd e)*e} = R_{(243)}$, $R_{(243)} R_{(14)(23)} \subseteq R_{((243) \lhd e)*(14)(23)} = R_{(34)}$, $R_{(243)} R_{(243)} \subseteq R_{((243) \lhd (132))*(243)} = R_e$, $R_{(243)} R_{(34)} \subseteq R_{((243) \lhd (123))*(34)} = R_{(14)(23)}$, $R_{(34)} R_e \subseteq R_{((34) \lhd e)*e} = R_{(34)}$, $R_{(34)} R_{(14)(23)} \subseteq R_{((34) \lhd e)*(14)(23)} = R_{(243)}$, $R_{(34)} R_{(243)} \subseteq R_{((34) \lhd (132))*(243)} = R_{(14)(23)}$ *and* $R_{(34)} R_{(34)} \subseteq R_{((34) \lhd e)*(34)} = R_e$.

Corollary 2. *Let $X = H\mathfrak{G}$ be a finite group that factorizes into a subgroup H and a set of left coset representatives $\mathfrak{G}$ and let R be a $\mathfrak{G}$-weak graded ring. Then, for any left R-module M such that*

$$M = \bigoplus_{s \in \mathfrak{G}} M_s \qquad (\text{as abelian groups}) \tag{9}$$

and $R_s M_t \subseteq M_p$ for some $p \in \mathfrak{G}$, we have $R_s M_t \subseteq M_{(s \lhd u)*t}$ for all $s, t \in \mathfrak{G}$ and for some $u \in H$.

Proof. It directly follows Definition 3 and Theorem 2. □

Corollary 3. *If M is a $\mathfrak{G}$-weak graded left R-module, then $R_s M_t \subseteq M_{(s \lhd u)*t}$ for all $s, t \in \mathfrak{G}$ and for some $u \in H$.*

Proof. It follows by Theorem 2, Corollary 2, and by choosing $u = e$. □

Example 4. *Consider the ring of all 2×2 matrices over the ring of integers $\mathbb{Z}$, i.e.,*

$$R = M_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \quad a, b, c, d \in \mathbb{Z} \right\}.$$

Consider the group $X = S_3 = \{e, (12), (13), (23), (123), (132)\}$ and its non-normal subgroup $H = \{e, (23)\}$. Choose the set of left coset representatives to be $\mathfrak{G} = \{e, (12), (13)\}$. Then, R is a $\mathfrak{G}$-weak graded ring (see Example 2). Define

$$M = (\mathbb{Z}_2 \times \mathbb{Z}_2)^+ = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \quad x, y \in \mathbb{Z}_2 \right\}.$$

Then, we have

$$M = M_e \oplus M_{(12)} \oplus M_{(13)} \text{ , where}$$

$$M_e = \begin{bmatrix} \mathbb{Z}_2 \\ 0 \end{bmatrix}, \quad M_{(12)} = \begin{bmatrix} 0 \\ \mathbb{Z}_2 \end{bmatrix} \quad \text{and} \quad M_{(13)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

However, regardless of the trivial case, M is not a $\mathfrak{G}$-weak graded R-module as, for instance, for all $\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \in R_{(13)}$ and $\begin{bmatrix} x \\ 0 \end{bmatrix} \in M_e$, we have $\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ cx \end{bmatrix} \in M_{(12)} \neq M_{(13)} = M_{(13)*e}$.

If we replace the relation $s * t$ by $(s \lhd u) * t$, we obtain $R_s M_t \subseteq M_{(s \lhd u)*t}$ for all $s, t \in \mathfrak{G}$ and for some $u \in H$. Thus, M can be made into a $\mathfrak{G}$-weak graded R-module by proving the inclusion property which can be easily performed by showing that: $R_e M_e \subseteq M_{(e \lhd e)*e} = M_e$, $R_e M_{(12)} \subseteq M_{(e \lhd e)*(12)} = M_{(12)}$, $R_e M_{(13)} \subseteq M_{(e \lhd e)*(13)} = M_{(13)}$, $R_{(12)} M_e \subseteq M_{((12) \lhd e)*e} = M_{(12)}$, $R_{(12)} M_{(12)} \subseteq M_{((12) \lhd e)*(12)} = M_e$, $R_{(12)} M_{(13)} \subseteq M_{\left((12) \lhd (23)\right)*(13)} = M_e$, $R_{(13)} M_e \subseteq M_{\left((13) \lhd (23)\right)*e} = M_{(12)}$, $R_{(13)} M_{(12)} \subseteq M_{\left((13) \lhd (23)\right)*(12)} = M_e$ and $R_{(13)} M_{(13)} \subseteq M_{((13) \lhd e)*(13)} = M_e$.

Therefore, M is a $\mathfrak{G}$-weak graded R-module, which is not full. For instance, $R_{(13)} M_{(13)} \subseteq M_e$ but $M_e \nsubseteq R_{(13)} M_{(13)}$.

4. $\mathfrak{G}$-Weak Graded Rings by a Subset S

In general, $R = \bigoplus_{s \in \mathfrak{G}} R_s$ is not necessarily a non-trivial $\mathfrak{G}$-weak graded ring. However, it can be put into a $\mathfrak{G}$-weak graded ring by choosing a subset S of $\mathfrak{G}$ such that S is a group under the $*$ operation and $R_t = 0$ for all $t \in \mathfrak{G}$ and $t \notin S$.

Definition 5. *Let S be a subset of $(\mathfrak{G}, *)$ such that S is a group under the $*$ operation, where $\mathfrak{G}$ is a fixed set of left coset representatives for the subgroup H of a finite group X. Then, a ring R is called an S-weak graded ring if*

$$R = \bigoplus_{s \in S} R_s \qquad \text{and} \qquad R_s R_{s'} \subseteq R_{s*s'} \text{ for all } s, s' \in S.$$

It can be noted that any ring R can be considered an S-weak graded ring by putting $S = \{e_{\mathfrak{G}}\}$ and $R_t = 0$ for all $e_{\mathfrak{G}} \neq t \in \mathfrak{G}$ in the same way that R can be put into a $\mathfrak{G}$-weak graded ring. In this case, R is called the trivial $\mathfrak{G}$-weak graded ring.

Theorem 4. *Let R be a $\mathfrak{G}$-weak graded ring and let S be a subset of $\mathfrak{G}$ such that S is a group under the $*$ operation. Then, $R_S = \bigoplus_{s \in S} R_s$ is an S-weak graded subring of R.*

Proof. First, since R is a $\mathfrak{G}$-weak graded ring, we have

$$R = \bigoplus_{s \in \mathfrak{G}} R_s \qquad \text{and} \qquad R_s R_t \subseteq R_{s*t} \qquad \text{for all } s, t \in \mathfrak{G}.$$

So, we can rewrite R as

$$R = \bigoplus_{\substack{t \in \mathfrak{G} \\ t \notin S}} R_t \oplus R_S. \qquad \text{(as additive subgroups)}$$

Hence, as $S \subseteq \mathfrak{G}$, we have $R_S = \bigoplus_{s \in S} R_s$ and $R_s R_{s'} \subseteq R_{s*s'}$ for all $s, s' \in S$.

Next, to show that R_S is a subring of R, we use the fact that R_S is an additive subgroup of R and the assumption that S is a group under the binary operation $*$ to have

$$R_s R_{s'} \subseteq R_{s*s'} \qquad \text{where} \qquad s * s' \in S,$$

which means that R_S is closed under multiplications. Moreover, since $(S, *)$ is a group, we have $e_{\mathfrak{G}} \in S$. Thus, $1_R \in R_{e_{\mathfrak{G}}} \subseteq R_S$, as required. $\square$

Definition 6 ([14]). *For a $\mathfrak{G}$-weak graded ring R, a unit $x \in U(R)$ is said to be a weak graded or $\mathfrak{G}$-homogeneous element if $x \in R_s$ for some $s \in \mathfrak{G}$, where $U(R)$ is the group of all units in R.*

Theorem 5. *Let R be a $\mathfrak{G}$-weak graded ring with unity and let S be a subset of $\mathfrak{G}$ such that S is a group under the $*$ operation. Then, $U(R_S) = R_S \cap U(R)$.*

Proof. As R is a $\mathfrak{G}$-weak graded ring, we obtain

$$R = \bigoplus_{t\in\mathfrak{G}} R_t \qquad \text{and} \qquad R_s R_t \subseteq R_{s*t} \qquad \text{for all } s,t \in \mathfrak{G}.$$

In addition, since S is a subset of $\mathfrak{G}$,R can be rewritten as

$$R = \bigoplus_{\substack{t\in\mathfrak{G}\\ t\notin S}} R_t \oplus R_S. \qquad \text{(as additive subgroups)}$$

Moreover, R_S can be written as $R_S = \bigoplus_{s\in S} R_s$ with $R_s R_{s'} \subseteq R_{s*s'}$ for all $s, s' \in S$.

Now, let $x \in R_S \cap U(R)$, which means $x \in R_S$ and $x \in U(R)$. Then, by Theorem 1, there exists an element x^{-1} such that $x^{-1} \in R_{S^L}$.

Since $(S, *)$ is a group, we have

$$x^{-1} \in R_{S^L} = \bigoplus_{s\in S} R_{s^L} = \bigoplus_{s\in S} R_s = R_S.$$

Thus, $x \in U(R_S)$.

On the other hand, let $x \in U(R_S)$. Then, we have $x \in R_S$. Since $U(R_S) \subseteq U(R)$, we obtain $x \in R_S \cap U(R)$.

Therefore, $U(R_S) = R_S \cap U(R)$ as required. $\square$

Example 5. *Let X be the dihedral group D_6 and $H = \{1, x^5y\}$ be a non-normal subgroup of X. We choose $\mathfrak{G} = \{x^5y, y, x^2, x^3, x^4, x^4y\}$ to be the set of left coset representatives. Then, the $*$ operation and the cocycle f as well as the action $\triangleright$ and the coaction $\triangleleft$ are given by the following tables (Tables 17 and 18):*

Table 17. The operation $*$ and the cocycle f.

$*$	x^5y	y	x^2	x^3	x^4	x^4y
x^5y	x^5y	y	x^2	x^3	x^4	x^4y
y	x^4y	x^5y	x^4y	x^2	x^3	x^2
x^2	x^4	x^3	x^4	y	x^5y	y
x^3	x^3	x^2	y	x^5y	x^4y	x^4
x^4	x^2	x^4y	x^5y	x^4y	x^2	x^3
x^4y	y	x^4	x^3	x^4	y	x^5y
f	x^5y	y	x^2	x^3	x^4	x^4y
x^5y	x^5y	x^5y	x^5y	x^5y	x^5y	x^5y
y	x^5y	x^5y	1	x^5y	x^5y	1
x^2	x^5y	x^5y	1	x^5y	x^5y	1
x^3	x^5y	x^5y	x^5y	x^5y	x^5y	x^5y
x^4	x^5y	1	x^5y	x^5y	1	x^5y
x^4y	x^5y	1	x^5y	x^5y	1	x^5y

Table 18. The action $s \triangleright u$ and the coaction $s \triangleleft u$.

$s \triangleright u$	**1**	x^5y
x^5y	1	x^5y
y	1	x^5y
x^2	1	x^5y
x^3	1	x^5y
x^4	1	x^5y
x^4y	1	x^5y
$s \triangleleft u$	**1**	x^5y
x^5y	x^5y	x^5y
y	y	x^4y
x^2	x^2	x^4
x^3	x^3	x^3
x^4	x^4	x^2
x^4y	x^4y	y

Take the ring R to be the matrix ring $R = M_3(T)$, where T is an arbitrary ring. Choose the subset S of $\mathfrak{G}$ to be $S = \{x^5y, x^3\}$, then R is an S-weak graded ring by putting:

$$R_{x^5y} = \begin{bmatrix} T & T & 0 \\ T & T & 0 \\ 0 & 0 & T \end{bmatrix} \quad , \quad R_{x^3} = \begin{bmatrix} 0 & 0 & T \\ 0 & 0 & T \\ T & T & 0 \end{bmatrix}$$

and $R_t = 0$ for all $t \in \mathfrak{G}$, $t \notin S$. Obviously, $R = R_{x^5y} \oplus R_{x^3}$. The inclusion property $R_sR_{s'} \subseteq R_{s*s'}$ can be proved for all $s, s' \in S$ by showing that: $R_{x^5y}R_{x^5y} = R_{x^5y*x^5y} = R_{x^5y}$, $R_{x^5y}R_{x^3} = R_{x^5y*x^3} = R_{x^3}$, $R_{x^3}R_{x^5y} = R_{x^3*x^5y} = R_{x^3}$ and $R_{x^3}R_{x^3} = R_{x^3*x^3} = R_{x^5y}$.

Thus, R is a fully S-weak graded ring.

Example 6. *Take X to be the dihedral group D_6, and $H = \{1, y\}$ to be a non-normal subgroup of X. We choose $\mathfrak{G} = \{1, x^2, x^4, xy, x^5y, x^3\}$ to be the set of left coset representatives. Then, the $*$ operation and the cocycle f as well as the action $\triangleright$ and the coaction $\triangleleft$ are given by the following tables (Tables 19 and 20):*

Table 19. The operation $*$ and the cocycle f.

$*$	**1**	x^2	x^3	x^4	xy	x^5y
1	1	x^2	x^3	x^4	xy	x^5y
x^2	x^2	x^4	xy	1	x^3	xy
x^3	x^3	xy	1	x^5y	x^2	x^4
x^4	x^4	1	x^5y	x^2	x^5y	x^3
xy	xy	x^5y	x^2	x^3	1	x^2
x^5y	x^5y	x^3	x^4	xy	x^4	1
f	**1**	x^2	x^3	x^4	xy	x^5y
1	1	1	1	1	1	1
x^2	1	1	y	1	y	1
x^3	1	y	1	y	y	y
x^4	1	1	y	1	1	y
xy	1	1	y	y	1	1
x^5y	1	y	y	1	1	1

Table 20. The action $s \triangleright u$ and the coaction $s \triangleleft u$.

$s \triangleright u$	**1**	y
1	1	y
x^2	1	y
x^3	1	y
x^4	1	y
xy	1	y
x^5y	1	y
$s \triangleleft u$	**1**	y
1	1	1
x^2	x^2	x^4
x^3	x^3	x^3
x^4	x^4	x^2
xy	xy	x^5y
x^5y	x^5y	xy

Take the ring R to be $R = M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \quad a,b,c,d \in \mathbb{R} \right\}$. Take the subset S of $\mathfrak{G}$ to be $S = \{1, x^2, x^4\}$, then R is an S-weak graded ring by putting:

$$R_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : \quad a,d \in \mathbb{R} \right\}, R_{x^2} = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : \quad b \in \mathbb{R} \right\}, R_{x^4} = \left\{ \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} : \quad c \in \mathbb{R} \right\}$$

and $R_t = 0$ for all $t \in \mathfrak{G}$, $t \notin S$. Hence, $R = R_1 \oplus R_{x^2} \oplus R_{x^4}$. In addition, the inclusion property can be proved easily for all $s, s' \in S$ by showing that: $R_1R_1 \subseteq R_{1*1} = R_1$, $R_1R_{x^2} \subseteq R_{1*x^2} = R_{x^2}$, $R_1R_{x^4} \subseteq R_{1*x^4} = R_{x^4}$, $R_{x^2}R_1 \subseteq R_{x^2*1} = R_{x^2}$, $R_{x^2}R_{x^2} \subseteq R_{x^2*x^2} = R_{x^4}$, $R_{x^2}R_{x^4} \subseteq R_{x^2*x^4} = R_1$, $R_{x^4}R_1 \subseteq R_{x^4*1} = R_{x^4}$, $R_{x^4}R_{x^2} \subseteq R_{x^4*x^2} = R_1$ and $R_{x^4}R_{x^4} \subseteq R_{x^4*x^4} = R_{x^2}$.

Thus, R is an S-weak graded ring which is not a fully. For instance, $R_{x^4}R_{x^4} \neq R_{x^4*x^4}$ as $R_{x^4*x^4} = R_{x^2} \nsubseteq R_{x^4}R_{x^4}$.

Example 7. *Let R be any ring and consider the polynomial ring $R[X]$. Then, $R[X]$ is an S-weak graded ring by choosing $S = \{e_{\mathfrak{G}}, s\}$, where S is a subset of any set of left coset representatives $\mathfrak{G}$ such that S is a group under the $*$ operation, as well as*

$$R_{e_{\mathfrak{G}}} = \left\{ r_0 + r_2x^2 + \ldots + r_{2k}x^{2k} : \qquad r_i \in R,\ k \in \mathbb{N} \cup \{0\} \right\}$$

$$R_s = \left\{ r_1x + r_3x^3 + \ldots + r_{2k+1}x^{2k+1} : \quad r_i \in R,\ k \in \mathbb{N} \cup \{0\} \right\}$$

and $R_t = 0$ for all $t \in \mathfrak{G}$ with $t \neq s \neq e_{\mathfrak{G}}$. Hence, $R = R_{e_{\mathfrak{G}}} \oplus R_s$ as we cannot add two non-zero weak graded elements $p \in R_{e_{\mathfrak{G}}}$ and $q \in R_s$ to obtain zero. Moreover, the inclusion property is satisfied as for any weak graded elements $p \in R_{e_{\mathfrak{G}}}$ and $q \in R_s$, we have $\langle pq \rangle = \langle p \rangle * \langle q \rangle = s = \langle q \rangle * \langle p \rangle = \langle qp \rangle$ and $\langle pp \rangle = \langle p \rangle * \langle p \rangle = e_{\mathfrak{G}} = \langle q \rangle * \langle q \rangle = \langle qq \rangle$, as required.

5. Conclusions

This work shows that the group graded rings and modules which are associated with a factorization of a given finite group may lead to a quantization of the classical results of group graded rings and modules. Moreover, some results in the literature can be generalized by using the concept of the weak $\mathfrak{G}$-graded rings and modules. Grading by a set with a binary operation satisfying specific properties and associated with a factorization of a given finite group is a considerable generalization of the theory of graded rings and modules. The interested reader can check how many results of the classical group-graded rings and modules can carry on in the new setting.

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